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Geometry and Topology of Black Hole Horizons

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Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de Mestre em Geometria e Topologia.

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"Where there is power, there is resistance."

(Michel, Foucault)

RESUMO

A principal motivação deste trabalho se origina com o famoso e fundamental teorema de Hawking sobre a topologia dos buracos negros. O teorema afirma que em um espaço-tempo assintoticamente plano e estacionário de dimensão 4 contendo um buraco negro, e satisfazendo a condição de energia dominante, as seções transversais espaciais do horizonte de eventos são topologicamente esferas de dimensão 2. Neste trabalho, exploramos uma generalização natural do teorema de Hawking para dimensões superiores, obtido por Galloway e Schoen, contendo condições excepcionais, bem como versões posteriores desse resultado, que efetivamente excluem essas condições excepcionais e recuperam o resultado de Hawking em dimensão 4. Em certas dimensões superiores, somos capazes de mostrar que os horizontes de eventos, no caso estacionário, e as *outermost marginally outer trapped surfaces* (MOTSs), no caso geral, admitem uma métrica de curvatura escalar positiva. Essa condição impõe várias restrições conhecidas sobre a topologia e é consistente com exemplos da literatura de espaços-tempos de buraco negro estacionários de cinco dimensões com topologia de horizonte $\mathbb{S}^2 \times \mathbb{S}^1$. A prova desses resultados requer técnicas de geometria diferencial, análise e se tem inspiração na teoria de superfícies mínimas. Portanto, este trabalho tem como objetivo examinar de perto as complexidades do problema e servir como uma introdução amigável ao tópico para leitores que possam não estar familiarizados com técnicas vindas da geometria semi-Riemanniana, relatividade geral e análise geométrica.

Palavras-chave: Superfícies marginalmente aprisionadas exteriormente. Topologia de buracos negros. Curvatura escalar positiva.

RESUMO EXPANDIDO

Introdução

Buracos negros em dimensões superiores têm despertado um interesse significativo nos últimos anos, devido a necessidade de compreender a gravidade dentro do contexto da teoria das cordas. Além disso, na teoria quantica de campos, foram estabelecidas conexões entre as propriedades de buracos negros em dimensões maiores do que quatro $(n > 4)$ e a teoria quântica de campos em $n-1$ dimensões, cuja conexão facilita uma série de cálculos e abre caminho para possíveis avanços na teoria.

Inspirado por esses desenvolvimentos, a estrutura e a topologia de buracos negros em dimensões superiores emergem como objetos intrigantes de investigação. Um ponto de partida seminal para essa linha de pesquisa é o célebre Teorema de Hawking que, de forma geral, afirma que em um espaço-tempo de quatro dimensões as seções transversais dos horizontes de eventos de buracos negros são topologicamente esferas bidimensionais. No entanto, estender esse resultado para espaços-tempo de dimensões superiores não é trivial, uma vez que a prova original se baseia no Teorema de Gauss-Bonnet. A complexidade aumentou quando Emparan and Reall (2002) descobriram um exemplo de um espaço-tempo de buraco negro de cinco dimensões no qual as seções transversais têm topologia $\mathbb{S}^2 \times \mathbb{S}^1$, demonstrando concretamente que a topologia do horizonte não precisa ser esférica em dimensões maiores que quatro.

Mais recentemente, Galloway and Richard Schoen (2006) e Galloway (2008, 2018) conseguiram fornecer uma ampla generalização em dimensões superiores do resultado de Hawking. Essa extensão se refere a uma classe especial de variedades conhecidas como de *tipo Yamabe positivo*, ou seja, variedades que admitem uma métrica de curvatura escalar positiva, o que é consistente com o exemplo fornecido por Emparan and Reall (2002). A generalização desse teorema envolve uma rica interação de técnicas da geometria diferencial, análise e relatividade geral, sendo motivada por ideias e técnicas da teoria de superfícies mínimas.

Objetivos

O principal objetivo deste trabalho é investigar as restrições topológicas associadas aos horizontes de eventos de buracos negros em dimensões superiores e, em particular, entender sob quais condições esses horizontes podem ser topologicamente esféricos ou admitir métricas de curvatura escalar positiva. Além disso, busca-se fornecer uma introdução acessível ao tema para leitores não familiarizados com técnicas de geometria semi-Riemanniana, relatividade geral e análise geométrica.

Metodologia

O trabalho inicia com uma breve revisão dos conceitos fundamentais em geometria semi-Riemanniana e relatividade geral. Isso estabelece as bases necessárias para a compreensão das estruturas e propriedades dos espaços-tempo em questão.

O próximo passo envolve uma exploração aprofundada das hipersuperfícies luminosas e das superfícies marginalmente aprisionadas. Esses objetos desempenham um papel crucial na descrição dos horizontes de eventos de buracos negros em dimensões superiores, e sua análise é fundamental para o desenvolvimento deste trabalho.

Em seguida, prossegue-se com o desenvolvimento da teoria das superfícies marginalmente aprisionadas. Isso inclui a investigação da estabilidade dessas superfícies, incluindo a análise do operador de estabilidade associado. Esse passo é essencial para compreender a dinâmica e as características dessas superfícies em contextos específicos.

Por fim, com base nas técnicas e conceitos estabelecidos nas etapas anteriores, o trabalho investiga a generalização do Teorema de Hawking para dimensões superiores. Esse processo relaciona-se diretamente à busca pela existência de métricas de curvatura escalar positiva em buracos negros nesses espaços-tempo de dimensões superiores.

Resultados e Discussões

Os principais resultados deste trabalho engloba duas versões da generalização do Teorema de Hawking, originalmente demonstradas por Galloway and Richard Schoen (2006) e posteriormente refinadas por Galloway (2008, 2018), aplicadas a dimensões superiores. Essas versões estabelecem que as seções transversais dos horizontes de eventos ou das superfícies marginalmente aprisionadas podem admitir métricas de curvatura escalar positiva, impondo assim restrições significativas à topologia dessas superfícies.

Uma fato relevante deste trabalho é a apresentação dos resultados através reformulação do teorema por meio de um problema de valor inicial. Essa abordagem torna as técnicas de geometria Riemanniana mais acessíveis e destaca a importância das superfícies marginalmente aprisionadas como uma generalização das superfícies mínimas. Tais superfícies emergem como uma ferramenta essencial na análise de horizontes de eventos em dimensões superiores, fornecendo uma perspectiva valiosa para a compreensão desses objetos complexos.

Considerações Finais

De maneira geral, explorou-se a generalização do Teorema de Hawking para dimensões superiores, abordando questões fundamentais na interseção da geometria diferencial, análise e relatividade geral. Foi demonstrado que as seções transversais dos horizontes de eventos e das superfícies marginalmente aprisionadas podem admitir métricas de curvatura escalar positiva em dimensões superiores, ampliando o entendimento de horizonte de eventos de buracos negros. A teoria das superfícies marginalmente aprisionadas emerge como uma valiosa ferramenta de análise nesse contexto. Este trabalho contribui significativamente para a compreensão dos buracos negros em espaços-tempo de dimensões superiores, com implicações que transcendem a matemática pura e alcançam a física teórica.

Palavras-chave: Superfícies marginalmente aprisionadas exteriormente. Topologia de buracos negros. Curvatura escalar positiva.

ABSTRACT

The main motivation of this work stems from a celebrated and fundamental theorem of Hawking on the topology of black holes. The theorem states that in a 4-dimensional asymptotically flat stationary black hole spacetime satisfying the dominant energy condition, the spacelike cross sections of the event horizon are topologically 2-spheres. In this work, we explore a natural generalization of Hawking's theorem to higher dimensions, obtained by Galloway and Schoen, with exceptional conditions, as well as more recent versions of that result, which effectively remove those exceptional conditions and recover Hawking's result in dimension 4. In higher dimensions, we are able to show that event horizons, in the stationary case, and outermost marginally outer trapped surfaces (MOTSs), in the general case, admit a metric of positive scalar curvature. This condition imposes several well-known restrictions on the topology, and it is consistent with examples of five-dimensional stationary black hole spacetimes with horizon topology $\mathbb{S}^2 \times \mathbb{S}^1$. The proof of these results requires techniques from differential geometry, analysis, and draws its motivation from minimal surface theory. Therefore, this work aims to closely examine the intricacies of the problem and serve as a friendly introduction to the topic for readers who may not be familiar with techniques of semi-Riemannian geometry, general relativity and geometric analysis.

Keywords: Marginally outer trapped surfaces. Black hole topology. Positive Scalar Curvature.

CONTENTS

0 INTRODUCTION

The exploration of higher-dimensional black holes has garnered significant interest in recent years, driven by the need to understand gravity within the framework of string theory and the quest for a quantum theory of gravity. In this context, links have been established between properties of black holes in dimensions greater than four ($n > 4$) and quantum field theory in $n-1$ dimensions, leading to potential sources of progress in the theory.

Inspired by these developments, the structure and topology of higher-dimensional black holes have emerged as intriguing objects of investigation. A seminal springboard for this line of research is the celebrated Hawking's black hole topology theorem which, loosely speaking, states that in a 4-dimensional spacetime, cross-sections of black hole event horizons are topologically equivalent to 2-spheres. However, extending this result to higher-dimensional spacetimes is not straightforward, as the original proof relies on the Gauss-Bonnet theorem. The plot thickened when Emparan and Reall (2002) discovered an example of a five-dimensional black hole spacetime where the cross-sections have horizon topology $\mathbb{S}^2 \times \mathbb{S}^1$, thus concretely showing that the horizon topology does not need to be spherical in dimensions larger than four.

More recently, Galloway and Richard Schoen (2006) and Galloway (2008, 2018) were able to provide an ample higher-dimensional generalization of Hawking's result. This extension concerns a special class of manifolds known as of *positive Yamabe type*, i.e., manifolds that admit a metric of positive scalar curvature, which is consistent with the example provided by Emparan and Reall (2002). The generalization of this theorem involves a rich interplay of techniques from differential geometry, analysis, and general relativity, and it is motivated by insights from minimal surface theory.

The objective of this work is to offer a thorough introduction to the topology and rigidity results concerning black hole horizons. To accomplish this aim, we will assume some basic knowledge of Riemannian geometry and our primary focus will be on developing a detailed geometric context for the main results. Differential-geometric prerequisites are therefore developed in great detail, while the analytical aspects will be either assumed or deferred to the appendices. However, we strive to provide technical details whenever possible, addressing intricacies that are often overlooked in the existing literature. Therefore, by reviewing the techniques employed in this field, this work seeks to contribute to the understanding of the subject as well as be a friendly introduction to this thriving theme.

In the Chapter 1, we provide a concise introduction to the prerequisites on semi-Riemannian geometry. This serves to settle our notation and offers an overview for readers who may be less familiar with this subject. At the end of this chapter, we introduce some key notions on variations of a submanifold, aiming to establish connections between the ideas and techniques presented in this work with the minimal surface theory.

The Chapter 2 serves as an introduction to a few very broad aspects of general relativity relevant to our discussion. We outline how we can reformulate the *Einstein field equations* (EFE) as an initial value problem. We also present the best-known black hole spacetime solutions, namely the Schwarzschild and Schwarzschild-Kruskal spacetimes, which provide quintessential examples to illustrate most of the geometric concepts explored here.

In a Lorentzian manifold, hypersurfaces with everywhere-degenerate induced metrics are called *null hypersurfaces*, and serve as models for black hole horizons. In Chapter 3, we provide a detailed exposition of several aspects of the geometry of abstract *null immersed submanifolds*. The importance of this discussion for us here is that one can always view a MOTS as a spacelike cross section of suitable such immersions of codimension one, a fact which motivates and justifies the use of MOTS as quasi-local surrogates of black hole horizons. A result corresponding to the simplest version of the so-called black hole area theorem is presented at the end of the chapter.

In Chapter 4, we delve deeper into the *marginally outer trapped surfaces* (MOTSs). These surfaces naturally arise in black hole spacetimes and can also be seen as a generalization of a minimal surface. We investigate the geometry of MOTSs in the context of a spacetime as well as in terms of initial data. Additionally, we give conditions for spacelike cross-sections of a null hypersurface to be a MOTS. Just as minimal surfaces, MOTSs admit a notion of stability given by the *MOTS stability operator*, an analogue to the second variation for the volume in the minimal surface theory. We exploit certain key analytical aspects of this operator to obtain the desired topological and rigidity results.

Finally, in Chapter 5, the Galloway-Schoen generalization of Hawking's black hole topology theorem is obtained. The result states that, under special conditions, a MOTS admits a metric of positive scalar curvature. We begin the chapter by presenting a short survey on positive scalar curvature which motivates and introduces the topological obstructions associated with the problem of obtaining a metric of positive scalar curvature on Riemannian Manifolds. In the subsequent sections, we establish the main theorems; the detailed proofs involve certain rigidity results that include a local foliation by MOTS and require several interesting ideas and concepts from functional spaces and differential operators on compact manifolds.

This work includes two appendices. In Appendix A, we provide a detailed derivation, in coordinates, of the MOTS stability operator. In Appendix B, a concise discussion regarding differential operators on compact manifolds is presented.

1 PRELIMINARIES

1.1 NOTATION AND REVIEW OF SEMI-RIEMANNIAN GEOMETRY

In this section, we will introduce the notation and terminology used throughout this work. While we will briefly review some of the material on semi-Riemannian geometry commonly covered in textbooks on this topic, it is important to note that this section should be considered as a refresher rather than a comprehensive introduction to the topic. The definitions and results, as well as those that are not explicitly mentioned, are precisely stated in our main resources on semi-Riemannian geometry: (COSTA E SILVA, 2021) and (O'NEILL, 1983). In particular, we shall assume the reader has previous knowledge of the underlying notions in manifold theory; see, for instance, (LEE, J. M., 2012).

1.1.1 CONNECTION AND INDUCED CONNECTION

Let M^n be a smooth manifold of dimension n. For any open set $U \subset M$, we denote as $C^\infty(U)$ the space of smooth functions on U and as $\mathfrak{X}(U)$ the $C^\infty(U)$-module of smooth vector fields defined on U. Moreover, for each point $p \in M$, we denote by T_pM the tangent space of M at p, and by TM the tangent bundle of M.

Recall that a *connection* on the smooth manifold M is a map

$$\nabla : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \to \nabla_X Y \in \mathfrak{X}(M),$$

such that,

1. ∇ is $\mathbb{R}$-bilinear;
2. $\nabla_{f \cdot X} Y = f \cdot \nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M), \forall f \in C^\infty(M)$;
3. $\nabla_X (f \cdot Y) = (Xf) \cdot Y + f \cdot \nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M), \forall f \in C^\infty(M)$.

A connection ∇ on M is said to be *symmetric* if satisfies

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $X, Y \in \mathfrak{X}(M)$ and $[\cdot, \cdot]$ denotes the Lie bracket. When local coordinates $(x^1, \ldots, x^n)$ are chosen, the connection ∇ can be described by the *Christoffel symbols* Γ^i_{jk}, which are defined by the formula[1]

$$\nabla_{\partial_k} \partial_j = \Gamma^i_{jk} \partial_i \quad \forall j, k \in \{1, \ldots, n\}.$$

We define the *curvature tensor* of a connection ∇ on M as follows[2]:

$$R^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

[1] Although the connection is defined on global vector fields, it follows from its properties that it can act on locally defined vector fields.

[2] There is no consensus in the literature regarding the sign convention for the curvature tensor. Additionally, it is worth noting that the definition in this work is opposite to the one used in (O'NEILL, 1983).

for all $X, Y, Z \in \mathfrak{X}(M)$. The curvature tensor is clearly $C^\infty(M)$-trilinear.

We also introduce the notion of *connection over maps*. Let M and N be smooth manifolds and let $F : N \to M$ be a smooth map. A *vector field over* F is a map $V : N \to TM$ for which $F = \pi_M \circ V$ holds, where $\pi_M : TM \to M$ is the standard projection. For any smooth map $F : N \to M$, we define $\mathfrak{X}(F)$ to be the set of smooth vector fields over F, which is a $C^\infty(N)$-module with respect to pointwise operations. For instance, if F is a smooth map, then given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the maps $X \circ F$ and $dF \circ Y$ are smooth vector fields over F where here (and hereafter) dF denotes the derivative of F.

At this point, the definition of connection can be extended to vector fields over maps as follows: Let $F : N \to M$ be a smooth map, a *connection on* F is a map

$$D : (X, V) \in \mathfrak{X}(N) \times \mathfrak{X}(F) \mapsto D_X V \in \mathfrak{X}(F),$$

such that

1. D is $\mathbb{R}$-bilinear;
2. $D_{f\cdot X} Y = f \cdot D_X Y, \quad \forall X \in \mathfrak{X}(N), \forall V \in \mathfrak{X}(F), \forall f \in C^\infty(N);$
3. $D_X(f \cdot Y) = (Xf) \cdot Y + f \cdot D_X Y \quad \forall X \in \mathfrak{X}(N), \forall V \in \mathfrak{X}(F), \forall f \in C^\infty(N).$

We can also define a curvature tensor for the connection D. The *curvature tensor of a connection D on the map $F : N \to M$* is given by

$$R^D(X, Y)V := D_X D_Y V - D_Y D_X V - D_{[X,Y]} V$$

for all $X, Y \in \mathfrak{X}(N)$ and $V \in \mathfrak{X}(F)$. This curvature tensor is $C^\infty(N)$-trilinear. The following results regarding the so-called *induced connection* will be constantly employed in our calculations.

Theorem 1.1.1 (COSTA E SILVA, 2021). *Let ∇ be a connection on the manifold M, let N be any smooth manifold and $F : N \to M$ be a smooth map. Then, there exists a unique connection D^∇ on F such that*

$$D^\nabla_X (V \circ F)(p) = \nabla_{dF_p(X_p)} V(F(p)), \quad \forall p \in N, \forall X \in \mathfrak{X}(N), \forall V \in \mathfrak{X}(M).$$

D^∇ is called the induced connection *on F.*

Proposition 1.1.2 (COSTA E SILVA, 2021). *Let ∇ be a connection on the manifold M, let N be any manifold and let $F : N \to M$ be a smooth map. Finally, let $D = D^\nabla$ be the induced connection on F. Then*

$$R^D_p(x, y)v = R^\nabla_{F(p)}(dF_p(x), dF_p(y))v, \tag{2}$$

for any $p \in N$ and $\forall x, y \in T_pN, \forall v \in T_{F(p)}M$. Moreover, if ∇ is symmetric, then

$$D_X(dF \circ Y) - D_Y(dF \circ X) = dF \circ [X, Y].$$

1.1.2 SEMI-RIEMANNIAN MANIFOLDS

The best-known class of symmetric connections on manifolds, and the only one relevant for us here, is that of *Levi-Civita connections* on manifolds equipped with semi-Riemannian metrics. To define the Levi-Civita connection, we first need to introduce certain requirements.

Definition 1.1.3 (Semi-Riemannian manifold). *Given an integer $0 \leq \nu \leq n$, a* (smooth) semi-Riemannian metric tensor g *of index ν on M is a (smooth) $(0,2)$-tensor field such that $\forall p \in M$,*

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a symmetric non-degenerate bilinear form on T_pM with index ν. A semi-Riemannian manifold *of index ν is a pair (M, g), where M is a smooth manifold and g is a semi-Riemannian metric g of index ν. If $\nu = 0$ [resp. $\nu = 1$ and $n \geq 2$], then g is said to be a* Riemannian (res. Lorentzian) metric, *and (M, g) is a* Riemannian manifold (resp. Lorentzian manifold*).*

Example 1 (Semi-Euclidean Spaces). *Consider $\mathbb{R}^n$ with the standard coordinate system $(x^1, \ldots, x^n)$. We define the semi-Riemannian Euclidean metric η_ν of index ν as follows:*

$$\eta_\nu := -\sum_{i=1}^{\nu} dx^i \otimes dx^i + \sum_{j=\nu+1}^{n} dx^j \otimes dx^j.$$

We shall refer to $\mathbb{R}^n_\nu := (\mathbb{R}^n, \eta_\nu)$ as the semi-Euclidean space of index ν. *When $\nu = 0$ and $n \geq 1$, this space is called the (*standard n-dimensional*) Euclidean space, and we denote $\delta_n := \eta_0$. If $\nu = 1$ and $n \geq 2$, $\mathbb{R}^n_1$ is commonly known as the* Minkowski space*, and we denote $\eta := \eta_1$.*

Let (M^n, g) be a semi-Riemannian manifold. The following notation will be often used: $\langle X, Y \rangle := g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ and $\langle x, y \rangle := g_p(x, y)$ for any $p \in M$ and $x, y \in T_pM$. In the latter case, we often drop the explicit reference to p if there is no risk of confusion.

A *local frame* on (M, g) is a choice of smooth vector fields $V_1, \ldots, V_n$ defined on an open set $U \subset M$ such that at each point $p \in U$, the vectors $V_1(p), \ldots, V_n(p)$ form a basis of the tangent space T_pM. The most common local frames are either coordinate frames or orthonormal frames. Given smooth coordinates $(x^1, \ldots, x^n)$ on U, the vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ form the so-called *coordinate frame*. An *orthonormal frame* is a local frame $\{E_1, \ldots, E_n\}$ defined on an open set $U \subset M$ such that, at each point $q \in U$, the set $\{E_1(q), \ldots, E_n(q)\}$ is an orthonormal basis of the semi-Euclidean vector space (T_qM, g_q). In this case, at any point

$$g(E_i, E_j) = \varepsilon_i \delta_{ij},$$

where $\varepsilon_i = \pm 1$ for $i \in \{1, \ldots, n\}$ represents the signature of the vector space (T_qM, g_q) and δ_{ij} is the Kronecker delta. The frame is always ordered such that the negative-signed vectors come first. In particular, it should be noted that orthonormal frames are not necessarily coordinate frames. However, it can be proven that an orthonormal frame exists around every point on M by using a Gram-Schmidt argument.

Let $V_1, \ldots, V_n$ be a local frame. The metric can be expressed by a symmetric matrix of real functions as follows:

$$g_{ij} = \langle V_i, V_j \rangle, \quad \forall i, j \in \{1, \ldots, n\}.$$

Since g is non-degenerate everywhere, the matrix $[g_{ij}]_{i,j=\{1,\ldots,n\}}$ is invertible at every point. We denote the (i,j)-entry of its inverse matrix by g^{ij}. The following proposition summarizes several useful properties of orthonormal frames.

Lemma 1.1.4 (COSTA E SILVA, 2021). *Let (M^n, g) be a semi-Riemannian manifold, $X \in \mathfrak{X}(M)$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame. Then, the following holds*

1. *$X = \sum_{i=1}^{n} \varepsilon_i \langle X, E_i \rangle E_i$ (everywhere on the open set where the frame is defined).*
2. *For any $Y \in \mathfrak{X}(M)$,*
$$\langle X, Y \rangle = \sum_{i=1}^{n} \varepsilon_i \langle X, E_i \rangle \langle E_i, Y \rangle.$$
3. *Let $(U, (x^1, \ldots, x^n))$ be a local chart defined on the same domain U of the $E_i's$. If we write $E_i = E_i^k \partial_k$, for $i \in \{1, \ldots, n\}$, where E_i^k denote change of basis matrix elements, we have that*
$$E_i^j = g^{jl} \langle \partial_l, E_i \rangle,$$
$$\sum_{i=1}^{n} E_i^k E_i^l = g^{kl}.$$

A well-known fact about smooth manifolds is that every smooth manifold admits a Riemannian metric, but that is not the case for semi-Riemannian metrics of other indices. However, given a semi-Riemannian metric g, we can build the so-called Levi-Civita connection ∇^g as shown in the following key theorem (see, e.g., (COSTA E SILVA, 2021) for a proof).

Theorem 1.1.5 (The fundamental theorem of semi-Riemannian geometry). *Given a semi-Riemannian metric g on M, there exists a unique connection ∇^g on M, called* the Levi-Civita connection *of g (or (M, g)), such that*

1. *∇^g is symmetric;*
2. *∇^g is* compatible with g, *i.e.,*

$$Z(g(X,Y)) = g(\nabla^g_Z X, Y) + g(X, \nabla^g_Z Y), \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Moreover, ∇^g is uniquely characterized by the Koszul formula*:*

$$\begin{aligned} 2g(\nabla^g_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \quad \forall X, Y, Z \in \mathfrak{X}(M). \end{aligned} \tag{3}$$

Example 2 (Flat connection). *Let $M = \mathbb{R}^n$, and consider the standard coordinate system $(x^1, \dots, x^n)$. We define the connection ∇^{flat} as follows: for any $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, we write $X = X^i\partial_i$, $Y = Y^i\partial_i$, where $\{\partial_i\}_{i=1}^n$ are the coordinate fields. Then,*

$$\nabla_X^{flat} Y := X(Y^i)\partial_i.$$

It is straightforward to check that this indeed defines a connection on $\mathbb{R}^n$, known as the standard flat connection. Furthermore, one can observe that this connection is the Levi-Civita connection of the semi-Euclidean space $\mathbb{R}^n_\nu$ of index ν. Additionally, it is worth noting that the curvature tensor of the connection ∇^{flat} vanishes everywhere. This example serves to illustrate that different semi-Riemannian manifolds can share the same Levi-Civita connection.

We shall often drop the superscript "g" from the Levi-Civita connection if there is no risk of confusion. Other basic but important geometric objects are the following.

Definition 1.1.6 (Differential operators). *Let (M^n, g) be a semi-Riemannian manifold. Then*

1. *The* gradient *of a smooth function $f \in C^\infty(M)$ is the vector field ∇f which is metrically related to the differential df, expressed as*

$$g(\nabla f, X) = df(X), \quad \forall X \in \mathfrak{X}(M).$$

2. *The* divergence $\operatorname{div} X$ *of a smooth vector field $X \in \mathfrak{X}(M)$ is the map*

$$\operatorname{div} X := \sum_{i=1}^n \varepsilon_i g(\nabla_{E_i} X, E_i),$$

where $\{E_1, \dots, E_n\}$ is any g-orthonormal local frame.

3. *The* divergence $\operatorname{div} A$ *of a symmetric $(0,2)$ tensor field on M is the one-form defined by*

$$\operatorname{div} A(Z) = \sum_{i=1}^n \varepsilon_i (\nabla_{E_i} A)(E_i, Z), \quad \forall Z \in \mathfrak{X}(M),$$

where $\{E_1, \dots, E_n\}$ is any g-orthonormal local frame.

4. *The* Laplacian *of a smooth function $f \in C^\infty(M)$ is the divergence of its gradient:*

$$\Delta f := \operatorname{div} \nabla f.$$

Note that each of the quantities in the previous definition are initially defined with respect to an orthonormal local frame. It can be shown that they are well-defined and independent of the frame chosen. An important result that involves these quantities is the *divergence theorem*.

Theorem 1.1.7 (ANCIAUX, 2010). *Let (M, g) be an oriented semi-Riemannian manifold with boundary ∂M and let $X \in \mathfrak{X}(M)$ be a smooth vector field on M with compact support. Suppose that ∂M is a semi-Riemannian submanifold (see Section 1.1.3). Then*

$$\int_M (\operatorname{div} X) dV = \varepsilon \int_{\partial M} \langle X, \vec{n} \rangle dV,$$

where $\vec{n}$ is the outward pointing unit normal of ∂M, which is equipped with the induced metric, $\varepsilon := g(\vec{n}, \vec{n})$ and dV denotes the unique volume form associated with g and the given orientation.

Instead of Equation (1), in the context of a semi-Riemannian metric and its Levi-Civita connection on a smooth manifold M^n, it is often convenient to work with the so-called $(0,4)$ *Riemannian curvature tensor*, given by

$$\mathrm{Rm}(W, Z, X, Y) := \langle R^g(X, Y)Z, W\rangle, \quad \forall X, Y, Z, W \in \mathfrak{X}(M), \tag{4}$$

where the superscript g indicates that the curvature is that induced by the Levi-Civita connection. The tensor is antisymmetric in the first pair of indices, and the last pair of indices, and it is symmetric when interchanging those pairs, as stated in the following theorem.

Theorem 1.1.8 (COSTA E SILVA, 2021). *Let (M, g) be a semi-Riemannian manifold. For all $X, Y, Z, W \in \mathfrak{X}(M)$, the Riemannian curvature tensor Rm satisfies*

1. $\mathrm{Rm}(W, Z, X, Y) = -\mathrm{Rm}(Z, W, X, Y)$,
2. $\mathrm{Rm}(W, Z, X, Y) = -\mathrm{Rm}(W, Z, Y, X)$,
3. $\mathrm{Rm}(W, Z, X, Y) = \mathrm{Rm}(X, Y, W, Z)$,

and the curvature tensor R satisfies

4. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$,
5. $\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0$.

The *Ricci tensor* is defined as the trace of the Riemannian curvature tensor over the first and third components. More precisely, given $\{E_1, \ldots, E_n\}$ as a g-orthonormal local frame, the *Ricci tensor* is given by

$$Ric_g(X, Y) := \sum_{i=1}^{n} \varepsilon_i g(R^g(E_i, Y)X, E_i), \quad \forall X, Y \in \mathfrak{X}(M). \tag{5}$$

The trace of the Ricci tensor is called the *scalar curvature* of (M, g):

$$S_g := \sum_{i=1}^{n} \varepsilon_i Ric_g(E_i, E_i). \tag{6}$$

Again, one can easily show that these definitions do not depend on the chosen orthonormal frame.

Let (M, g) be a semi-Riemannian manifold and $p \in M$. A two-dimensional subspace Π of the tangent space T_pM is called a *tangent plane* to M at p. For any tangent vectors $v, w \in T_pM$, we define

$$Q(v, w) = g_p(v, v)g_p(w, w) - g_p(v, w)^2.$$

By (O'NEILL, 1983, Lemma 2.19), a tangent plane Π is a nondegenerate subspace if and only if $Q(v, w) \neq 0$ for one, hence every, basis $\{v, w\}$ of Π. Given a *nondegenerate plane* $\Pi \subset T_pM$, the real number

$$K(\Pi) := \frac{g_p(R^g(w, v)v, w)}{Q(v, w)},$$

where $\{v, w\}$ is a basis of Π, is called the *sectional curvature of the nondegenerate plane* Π.

By (O'NEILL, 1983, Lemma 3.39), we have that $K(\Pi)$ is independent of the choice of basis for Π, so the quantity is well defined. Thus, the sectional curvature K of M is a real-valued function on the set of all nondegenerate tangent planes to M.

1.1.3 SEMI-RIEMANNIAN IMMERSIONS

Let (M, g) be a semi-Riemannian manifold. Given a smooth map $F : N \to M$, the pullback F^*g defines a symmetric $(0, 2)$-type smooth tensor field on N, but it may not necessarily be a semi-Riemannian metric. For instance, the index of F^*g (i.e., the signature of the metric) may fail to be constant or may even be degenerate everywhere, and indeed we will see the latter phenomenon. If the index is constant and the *induced metric* F^*g does not degenerate anywhere, then (N, F^*g) is a semi-Riemannian manifold and F is then said to be a *semi-Riemannian immersion*.

In particular, an *embedded* submanifold $N \subset M$ is referred to as a *semi-Riemannian submanifold* if the inclusion map $i : N \hookrightarrow M$ (which is an embedding) is a semi-Riemannian immersion. In the context of codimension one, when N is an embedded submanifold of M and F is the inclusion, we refer to N as a *semi-Riemannian hypersurface* (see Definition 1.1.13 and Figure 1 for examples of semi-Riemannian hypersurfaces).

There is a particular type of immersion that will consistently appear in this work when (M, g) is a Lorentzian manifold. Given $F : N \to M$ as a semi-Riemannian immersion, we say that F is a *spacelike immersion* if, for all $p \in N$, $dF_p(T_pN)$ is a spacelike subspace of the Lorentz vector space (T_pM, g_p). Here, a spacelike subspace is a subspace that consists only of vectors that are spacelike (i.e., g_p is positive definite on $dF_p(T_pN)$) with respect to the Lorentzian metric. Equivalently, $F : N \to M$ is spacelike if and only if F^*g is a Riemannian metric on N.

In the next part, we fix the following notation: Let $F : N \to M$ be a semi-Riemannian immersion and denote $g_N := F^*g_M$ and g_M as the semi-Riemannian metric on N and M, respectively. The geometric quantities on (N, g_N) [resp. (M, g_M)] are indicated by superscript or subscript "N" [resp. "M"], e.g., ∇^N [resp. ∇^M]. Finally, given any $V, W \in \mathfrak{X}(F)$, we define the smooth real-valued function

$$\ll V, W \gg : p \in N \mapsto (g_M)_{F(p)}(V_p, W_p) \in \mathbb{R}. \tag{7}$$

In the context of semi-Riemannian immersions, any vector can be decomposed into two parts. For any $p \in N$ and $v \in T_{F(p)}M$, there exist unique vectors $v^\top \in dF_p(T_pN)$ and $v^\perp \in dF_p(T_pN)^\perp$, referred to as the *tangent* and *normal* parts of v, respectively, such that

$$v = v^\top + v^\perp.$$

We denote $\mathfrak{X}^\top(F)$ [resp. $\mathfrak{X}^\perp(F)$] as the set of smooth vector fields over F in which each vector is tangent [resp. normal]. The subsets $\mathfrak{X}^\top(F)$ and $\mathfrak{X}^\perp(F)$ are $C^\infty(N)$-submodules of

$\mathfrak{X}(F)$. If N is an embedded submanifold of M and F is the inclusion, we denote $\mathfrak{X}^\perp(F)$ as $\mathfrak{X}^\perp(N)$.

Proposition 1.1.9 (COSTA E SILVA, 2021). *The map*

$$dF : X \in \mathfrak{X}(N) \mapsto dF \circ X \in \mathfrak{X}(F)$$

is a $C^\infty(N)$-module isomorphism.

Indeed, the tangent and normal decomposition can be applied to any smooth vector field $V \in \mathfrak{X}(F)$, as stated in the following lemma.

Lemma 1.1.10 (COSTA E SILVA, 2021). *Given $V \in \mathfrak{X}(F)$, there exists unique $X_V \in \mathfrak{X}(N)$ and $V^\perp \in \mathfrak{X}^\perp(F)$ such that*

$$V = dF \circ X_V + V^\perp.$$

In other words, there is a $C^\infty(N)$-module isomorphism

$$\mathfrak{X}(F) \approx \mathfrak{X}(N) \oplus \mathfrak{X}^\perp(F).$$

Furthermore, the subsequent two technical lemmas will prove to be highly valuable in later discussions.

Lemma 1.1.11 (COSTA E SILVA, 2021). *Let $F : N \to M$ be a smooth immersion into the semi-Riemannian manifold (M, g). Given $V \in \mathfrak{X}(F)$ and $p \in N$, there exist an open set $\mathcal{U} \subset N$ containing p and a vector field $\widetilde{V} \in \mathfrak{X}(M)$ on M such that*

$$\widetilde{V} \circ F\big|_{\mathcal{U}} = V\big|_{\mathcal{U}}.$$

Lemma 1.1.12 (COSTA E SILVA, 2021). *Let $X \in \mathfrak{X}(N)$ and V, $W \in \mathfrak{X}(F)$, and fix $p \in N$. In view of Lemma 1.1.11, let $\widetilde{X}$, $\widetilde{V}$, $\widetilde{W} \in \mathfrak{X}(M)$ and $\mathcal{U} \ni p$ open in N such that*

$$\widetilde{V} \circ F\Big|_{\mathcal{U}} = V|_{\mathcal{U}}, \quad \widetilde{W} \circ F\Big|_{\mathcal{U}} = W|_{\mathcal{U}} \text{ and } \widetilde{X} \circ F\Big|_{\mathcal{U}} = dF \circ X|_{\mathcal{U}}.$$

Then,

1. $\ll V, W \gg|_{\mathcal{U}} = \left(g_M(\widetilde{V}, \widetilde{W}) \circ F\right)\Big|_{\mathcal{U}}$,
2. $D_X V|_{\mathcal{U}} = \left(\nabla^M_{\widetilde{X}} \widetilde{V}\right) \circ F\Big|_{\mathcal{U}}$,
3. $X \ll V, W \gg = \ll D_X V, W \gg + \ll V, D_X W \gg$.

In what follows, we will introduce the so-called *hyperquadrics* which not only serve as illustrations of the concepts of semi-Riemannian immersions/embeddings, but will also give rise to further important examples of semi-Riemannian manifolds. Let $n \geq 2$ and consider the semi-Euclidean space $\mathbb{R}^{n+1}_\nu$. We will use the standard global Cartesian coordinate system $(x^1, \ldots, x^n)$, and the canonical basis $\{e_1, \ldots, e_{n+1}\}$, where $\varepsilon_i := \eta_\nu(e_i, e_i)$ are the signs. Consider the quadratic form $\mathcal{Q} : \mathbb{R}^{n+1}_\nu \to \mathbb{R}$ given by

$$\mathcal{Q}(x) = \langle x, x \rangle_\nu = -\sum_{i=1}^{\nu} (x^i)^2 + \sum_{i=\nu+1}^{n+1} (x^i)^2,$$

and introduce the so-called *position vector field*

$$\mathcal{P}(x) = x^i \frac{\partial}{\partial x^i}\bigg|_x.$$

A simple computation reveals that

$$\nabla \mathcal{Q}(x) = 2x_i \frac{\partial}{\partial x^i}\bigg|_x = 2\mathcal{P}(x),$$

and

$$\eta_\nu(\mathcal{P}, \mathcal{P}) \equiv \mathcal{Q}.$$

Finally, we are now ready to define the so-called *(central) hyperquadrics*, which are a very important class of submanifolds of $\mathbb{R}^{n+1}_\nu$.

Definition 1.1.13 (Hyperquadrics). *The* (central) hyperquadrics *of radius $r > 0$ in $\mathbb{R}^{n+1}_\nu$ are the* pseudospheres

$$\mathbb{S}^n_\nu(r) := \{x \in \mathbb{R}^{n+1}_\nu : \mathcal{Q}(x) = r^2\},$$

and, provided $\nu \geq 1$, the pseudohyperboloids

$$\mathbb{H}^n_{\nu-1}(r) := \{x \in \mathbb{R}^{n+1}_\nu : \mathcal{Q}(x) = -r^2\}.$$

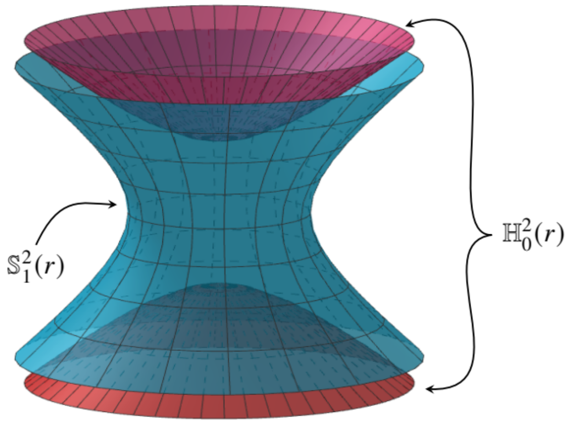


Figure 1 – Pseudosphere and pseudohyperboloid in $\mathbb{R}^3_1$

Source: (ESPINOZA, 2020)

An auxiliary proposition is necessary to analyze and understand these hypersurfaces, which are obtained as the preimage of regular values.

Proposition 1.1.14 (Proposition 4.17 (O'NEILL, 1983)). *Let (M, g) be a semi-Riemannian manifold. Let c be a regular value of $f \in C^\infty(M)$. Then $N = f^{-1}(c)$ is a semi-Riemannian hypersurface of M if and only if $g(\nabla f, \nabla f)$ is strictly positive or strictly negative on N. In this case, $U = \nabla f/|\nabla f|$ is a unit normal vector field on N.*

Notice that $\langle \nabla\mathcal{Q}, \nabla\mathcal{Q}\rangle_\nu = 4\langle \mathcal{P}, \mathcal{P}\rangle_\nu = 4\mathcal{Q}$ on $\mathbb{R}^{n+1}_\nu$. By Proposition 1.1.14, for a fixed $r > 0$, the hypersurfaces $\mathcal{Q}^{-1}(\varepsilon r^2)$ are semi-Riemannian manifolds with the unit vector field $U = \mathcal{P}/r$. Therefore, the hyperquadrics are semi-Riemannian hypersurfaces.

Example 3 (The round sphere $\mathbb{S}^n$). *We define* $\mathbb{S}^n := \mathbb{S}^n_0(1) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1\}$*, for* $n \geq 2$*. If* δ_{n+1} *is the standard* $n+1$*-dimensional Euclidean metric and* $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ *is the inclusion, then* $\omega_n := i^*\delta_{n+1}$ *is the induced Riemannian metric on* $\mathbb{S}^n$*, and* $(\mathbb{S}^n, \omega_n)$ *is called the round* n*-sphere.*

Example 4 (The hyperbolic space). *The* n*-dimensional* hyperbolic space*, for* $n \geq 2$*, is by definition the "upper" component of the hyperquadric* $\mathbb{H}^n_0(1) \subset \mathbb{R}^n_1$*, given by*

$$\mathbb{H}^n_0 := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_1 : -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = -1, x_1 > 0\},$$

endowed with the induced metric by the inclusion map $i : \mathbb{H}^n_0(1) \hookrightarrow \mathbb{R}^{n+1}_1$ *into the* $n+1$*-dimensional Minkowski space.*

Example 5 (De Sitter and anti-de Sitter). *The* n*-dimensional* de Sitter space*, for* $n \geq 2$*, is by definition the pseudosphere* $\mathbb{S}^n_1(1) \subset \mathbb{R}^{n+1}_1$ *endowed with the induced metric by the inclusion map* $i : \mathbb{S}^n_1(1) \hookrightarrow \mathbb{R}^{n+1}_1$*. The* n*-dimensional* anti-de Sitter space *is the pseudohyperboloid* $\mathbb{H}^n_1(1) \subset \mathbb{R}^{n+1}_2$ *endowed with the induced metric by the inclusion map* $i : \mathbb{H}^n_1(1) \hookrightarrow \mathbb{R}^{n+1}_2$*. Both are one-sheeted hyperboloids and hence connected* n*-dimension Lorentzian manifolds.*

1.1.4 THE GAUSS-CODAZZI EQUATIONS

In this section, we fix the following notation. Let (M^n, g) be a semi-Riemannian manifold and $F : N^k \to M^n$ be a semi-Riemannian immersion. The map F induces the connection D. However, for any $X, Y \in \mathfrak{X}(N)$, it is not guaranteed that $D_X(dF \circ Y)$ lies entirely within the tangent space of $dF(TN)$. Consequently, we may ask how to decompose $D_X(dF \circ Y)$ into its tangent and normal parts.

Theorem 1.1.15 (COSTA E SILVA, 2021). *For any* $X, Y \in \mathfrak{X}(N)$

$$D_X(dF \circ Y) = dF\left(\nabla^N_X Y\right) + II(X, Y),$$

where

$$II(X, Y) := \left(D_X(dF \circ Y)\right)^{\perp}.$$

Moreover, the map $II : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}^{\perp}(F)$ *thus defined is* $C^\infty(N)$*-bilinear and symmetric, and is called the* second fundamental form tensor *or* shape tensor *of* F*.*

Given the second fundamental form tensor II of $F : N^k \to M^n$ and $p \in N$, we define the *mean curvature vector of* F *at* p as

$$\vec{H}_p := \sum_{i=1}^{k} \varepsilon_i II_p(e_i, e_i),$$

where $\{e_1, \ldots, e_k\} \subset T_pN$ represents an orthonormal basis at $p \in N$ and $\varepsilon_i = g_N(e_i, e_i)$, for any $i \in \{1, \ldots, k\}$. It is evident that the mean curvature vectors form a smooth normal vector field $\vec{H} \in \mathfrak{X}^{\perp}(F)$ which is the *mean curvature vector field of* F.

We say that the semi-Riemannian immersion $F : N^k \to M^n$ is *totally geodesic* if its second fundamental form vanishes identically; in particular, if N is an (embedded) submanifold of M with F being the inclusion, and F is totally geodesic, then we say that N itself is a *totally geodesic submanifold of* (M, g_M). This has the usual geometric meaning that g_N geodesics are mapped via F into g_M geodesics.

Before introducing the Gauss-Codazzi equations, define the following quantity: for any $X, Y, Z \in \mathfrak{X}(N)$,

$$(D_X II)(Y,Z) := D_X(II(Y,Z)) - II(\nabla^N_X Y, Z) - II(Y, \nabla^N_X Z). \tag{8}$$

We can easily show that the quantity $(D_X II)(Y,Z)$ is $C^\infty(N)$-trilinear in its variables.

Theorem 1.1.16 (COSTA E SILVA, 2021). *Let (N, g_N) and (M, g_M) be semi-Riemannian manifolds, and $F : N \to M$ be a semi-Riemannian immersion. Then, for any $X, Y, Z, W \in \mathfrak{X}(N)$ and any $V \in \mathfrak{X}^\perp(F)$,*

1. *(Gauss Equation)*

$$\begin{aligned} \ll R^D(X,Y)dF \circ Z, dF \circ W \gg = g_N(R^N(X,Y)Z, W) + \ll II(X,Z), II(Y,W) \gg \\ - \ll II(Y,Z), II(X,W) \gg . \end{aligned}$$

2. *(Codazzi Equation)*

$$\ll R^D(X,Y)dF \circ Z, V \gg = \ll (D_X II)(Y,Z) - (D_Y II)(X,Z), V \gg .$$

If $F : N \to M$ is a codimension one semi-Riemannian immersion, it is sometimes possible (and in that case, also very convenient) to choose a distinguished unit normal vector. If there exists an everywhere-nonzero $U \in \mathfrak{X}^\perp(F)$ we say that F is *two-sided*. This motivates the following definition. Given a unit normal vector field $U \in \mathfrak{X}^\perp(F)$,

1. The *second fundamental* form (associated with U) is the symmetric $(0,2)$-tensor on N given by

   $$\mathcal{K}(X,Y) = \mathcal{K}_U(X,Y) := \ll II(X,Y), -U \gg, \quad \forall X, Y \in \mathfrak{X}(N)$$

   where II is the shape tensor of F.

2. The *Weingarten operator* (or *shape operator*) (associated with U) is the $(1,1)$-tensor $\mathcal{S} = \mathcal{S}_U$ on N metrically associated with $\mathcal{K}$, i.e., $\mathcal{S} : \mathfrak{X}(N) \to \mathfrak{X}(N)$ is given by

   $$g_N(\mathcal{S}(X), Y) = \mathcal{K}(X,Y), \quad \forall X, Y \in \mathfrak{X}(N).$$

3. The *mean curvature scalar* (associated with U) is the real-valued function

   $$H = H_U = \ll \vec{H}, -U \gg = \mathrm{tr}_N\, \mathcal{K} = \mathrm{tr}_N\, \mathcal{S}.$$

Remark 1 . *The definitions of the second fundamental form and the Weingarten map with $-U$ may differ from the classical definition that uses U. We specifically choose this unit normal for two main reasons:*

1. *In Euclidean space, for spheres, the second fundamental form $\mathcal{K}$ and the operator $\mathcal{S}$ associated with the outward unit normal are positive definite (see Example 6). Thus, the outward unit normal is the default choice for normal direction;*
2. *The definition of marginally outer trapped surfaces (see Chapter 4) relies on the sign of the so-called null expansion θ and this choice is consistent with the literature.*

The following concept is important for understanding the geometry of warped products, which will be discussed in the next subsection.

Definition 1.1.17 (Totally umbilic submanifolds). *Let (N, g_N) and (M, g_M) be semi-Riemannian manifolds, and $F : N \to M$ be a semi-Riemannian immersion. A point $p \in N$ is said to be* umbilic *(with respect to F) if there exists a normal vector $\vec{z} \in (dF_p(T_pN))^{\perp}$ for which*

$$II_p(x,y) = (g_N)_p(x,y) \cdot \vec{z}, \quad \forall x, y \in T_pN.$$

F is said to be totally umbilic *if every point of N is umbilic. In particular, if N is an embedded submanifold of M, F is the inclusion, and the latter is totally umbilic, then we say that N is a* totally umbilic submanifold*.*

(Note that if F is totally geodesic, then it is in particular totally umbilic.)

For semi-Riemannian immersions of codimension one, the Gauss-Codazzi equations assume a special form.

Proposition 1.1.18 (COSTA E SILVA, 2021). *Let (N^n, g_N) and (M^{n+1}, g_M) be semi-Riemannian manifolds and let $F : N \to M$ be a semi-Riemannian codimension 1 immersion. Given a unit normal vector field $U \in \mathfrak{X}^{\perp}(F)$, then for any $X, Y, Z \in \mathfrak{X}(N)$ the following* Gauss-Codazzi equations *hold*

1. *For $p \in N$,*
$$dF_p(\mathcal{S}_p(v)) = D_vU(p), \quad \forall v \in T_pN.$$
2. *(Gauss Equation)*
$$\begin{aligned}\ll R^D(X,Y)dF \circ Z, dF \circ W \gg\, &= g_N(R^N(X,Y)Z,W) + \varepsilon_N[\mathcal{K}(X,Z)\mathcal{K}(Y,W) \\ &\quad - \mathcal{K}(Y,Z)\mathcal{K}(X,W)],\end{aligned}$$
where $\varepsilon_N :=\ll U, U \gg$.
3. *(Codazzi Equation)*
$$\ll R^D(X,Y)dF \circ Z, V \gg = (\nabla^N_Y \mathcal{K})(X,Z) - (\nabla^N_X \mathcal{K})(Y,Z)$$

By Proposition 1.1.14 and item 1 of Proposition 1.1.18, for hyperquadrics, we obtain that the Weingarten operator associated with the unit normal vector field $U = \mathcal{P}/r$ has the expression $\mathcal{S}_U = \mathbb{I}_{\mathfrak{X}(N)}/r$, where $\mathbb{I}_{\mathfrak{X}(N)}$ is the identity.

Example 6 (Geometry of the hyperquadrics). *In this example, we will compute several geometric quantities for the central hyperquadrics. Let $n > 0$ and $0 \leq \nu \leq n$. Consider the hyperquadric $N = \mathcal{Q}^{-1}(\varepsilon r)$, where $\varepsilon = \pm 1$, and let $r > 0$ be fixed. Denote by $U = \mathcal{P}/r$ the unit normal vector field on N, and let $i : N \hookrightarrow \mathbb{R}^{n+1}_{\nu}$ be the inclusion map of N into $\mathbb{R}^{n+1}_{\nu}$. For any $X, Y \in \mathfrak{X}(N)$, a straightforward computation using $\mathcal{S}_U = \mathbb{I}_{\mathfrak{X}(N)}/r$ reveals that:*

$$\mathcal{K}_U(X, Y) = \frac{1}{r} g_N(X, Y),$$

and

$$H_U = \frac{1}{r}.$$

Since $\vec{H}$ is a normal vector field, the codimension is one, and the unit normal vector field U has sign *ε, we can express $\vec{H}$ and the second fundamental form II as follows:*

$$\vec{H} = -\varepsilon \frac{1}{r} U,$$

and

$$II(X, Y) = -\frac{\varepsilon}{r} g_N(X, Y) U.$$

The information about the Weingarten map, together with (O'NEILL, 1983, Corollary 4.20), enables us to determine the sectional curvature of the hyperquadrics.

Proposition 1.1.19 (Proposition 4.29 (O'NEILL, 1983)). *Let $n \geq 2$ and $0 \leq \nu \leq n$.*

1. *The pseudosphere $\mathbb{S}^n_\nu(r)$ is a semi-Riemannian manifold with constant positive sectional curvature $K = 1/r^2$.*
2. *The pseudohyperboloid $\mathbb{H}^n_{\nu-1}(r)$ is a semi-Riemannian manifold of negative constant sectional curvature $K = -1/r^2$.*

It worth noting that the *round sphere* and the *de Sitter space* have constant positive sectional curvature, while the *hyperbolic space* and the *anti-de Sitter space* have negative sectional curvature.

The following technical lemma elucidates the relationship between two distinct second fundamental forms that arise when there is a composition of immersions. This lemma will be employed in Chapter 4

Lemma 1.1.20 . *Let (M^m, g) be a semi-Riemannian manifold. Let $\phi : S^n \to M^m$ and $\psi : \Sigma^k \to S^n$ be two semi-Riemannian immersions such that $m > n > k > 0$. For any $p \in \Sigma$, we have*

$$II^{\phi\circ\psi}_p(v, w) = d\phi_{\psi(p)}(II^{\psi}_p(v, w)) + II^{\phi}_{\psi(p)}(d\psi_p v, d\psi_p w), \quad \forall v, w \in T_p\Sigma.$$

Proof. Let $p \in \Sigma$ and fix $v, w \in T_p\Sigma$. Let V, W be smooth vector fields on Σ such that $V(p) = v$ and $W(p) = w$. By Lemma 1.1.11, there exists a smooth vector field $\widetilde{W} \in \mathfrak{X}(M)$ and an open set $\mathcal{U} \subset \Sigma$ containing p such that

$$\widetilde{W} \circ (\phi \circ \psi)\Big|_{\mathcal{U}} = d(\phi \circ \psi) \circ W\big|_{\mathcal{U}},$$

Denote by D^{ψ}, D^{ϕ} and $D^{\phi\circ\psi}$ the covariant derivatives induced by the maps ψ, ϕ and $(\phi\circ\psi)$, respectively. Hence, at p, we have

$$\begin{aligned}(D_V^{\phi\circ\psi} d(\phi\circ\psi)\circ W)(p) &= (D_V^{\phi\circ\psi}\widetilde{W}\circ(\phi\circ\psi))(p)\\ &= (\nabla^M_{d(\phi\circ\psi)_p v}\widetilde{W})(\phi\circ\psi(p))\\ &= (D^{\phi}_{d\psi_p v}\widetilde{W}\circ\phi)\psi(p).\end{aligned}$$

As ψ is an immersion, it is locally an embedding. Therefore, by restricting $\mathcal{U}$ if necessary, we can assume that $\psi(\mathcal{U})$ is an open set. Notice that $W\circ\phi|_{\psi(\mathcal{U})}$ is a tangent vector field of $\psi(\mathcal{U})\subset S$ in M, by Lemma 1.1.10, there exists a smooth vector field $\widetilde{Z}\in\mathfrak{X}(S)$ such that

$$d\phi\circ\widetilde{Z}\Big|_{\psi(\mathcal{U})} = \widetilde{W}\circ\phi\Big|_{\psi(\mathcal{U})},$$

thus,

$$\begin{aligned}(D_V^{\phi\circ\psi} d(\phi\circ\psi)\circ W)(p) &= (D^{\phi}_{d\psi_p v}\widetilde{W}\circ\phi)(\psi(p))\\ &= (D^{\phi}_{d\psi_p v} d\phi\circ\widetilde{Z})(\psi(p))\\ &= d\phi(\nabla^S_{d\psi_p v}\widetilde{Z})(\psi(p)) + II^{\phi}(d\psi_p v, \widetilde{Z}_{\psi(p)})\\ &= d\phi_{\psi(p)}(D_V^{\psi}\widetilde{Z}\circ\psi)(p) + II^{\phi}_{\psi(p)}(d\psi_p v, \widetilde{Z}_{\psi(p)}).\end{aligned}$$

Observe that, the vector field $\widetilde{Z}\circ\psi\Big|_{\mathcal{U}}$ is a tangent vector field of $\mathcal{U}\subset\Sigma$ into S. Therefore, there exists a smooth vector field $Z\in\mathfrak{X}(\Sigma)$ such that

$$d\psi\circ Z|_{\mathcal{U}} = \widetilde{Z}\circ\psi\Big|_{\mathcal{U}},$$

moreover, notice that

$$d\phi\circ\widetilde{Z}\circ\psi(p) = \widetilde{W}\circ\phi\circ\psi(p) = d(\phi\circ\psi)\circ W(p),$$

as $d\phi$ is injective, $\widetilde{Z}_{\psi(p)} = d\psi_p w$ and, since $d\psi_p Z_p = \widetilde{Z}_{\psi(p)}$, holds that $Z_p = w$. Decomposing D^{ψ} into tangent and normal parts, we have the following development

$$\begin{aligned}(D_V^{\phi\circ\psi} d(\phi\circ\psi)\circ W)(p) &= d\phi_{\psi(p)}(D_V^{\psi}\widetilde{Z}\circ\psi)(p) + II^{\phi}_{\psi(p)}(d\psi_p w, d\psi_p v)\\ &= d\phi_{\psi(p)}(D_V^{\psi} d\psi\circ Z)(p) + II^{\phi}_{\psi(p)}(d\psi_p w, d\psi_p v)\\ &= d\phi_{\psi(p)}\left[d\psi_p\circ(\nabla^{\Sigma}_V Z)(p) + II^{\psi}_p(v,w)\right]\\ &+ II^{\phi}_{\psi(p)}(d\psi_p w, d\psi_p v)\end{aligned}$$

taking the normal parts in both sides we have

$$II^{\phi\circ\psi}_p(v,w) = d\phi_{\psi(p)} II^{\psi}_p(v,w) + II^{\phi}_{\psi(p)}(d\psi_p w, d\psi_p v).$$

□

1.1.5 WARPED PRODUCTS

The fundamental examples of relativistic spacetimes often depend on a particular class of manifolds obtained via the notion of *warped product*. Before delving into black hole spacetimes in Section 2.3, it is important to establish the definition of warped products.

The warped product generalizes the semi-Riemannian product manifold $M \times N$ by introducing a distortion, that is, each fiber of $p \times N$ is distorted homothetically in order to get a new "warped" metric tensor on the manifold on the manifold $M \times N$. This technique allows a richer class of metrics on the manifold $M \times N$.

Definition 1.1.21 (Warped products). *Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds, and let $f > 0$ be a smooth function on B. The* warped product $M = B \times_f F$ *with* warping function f *is the product manifold $B \times F$ furnished with the metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where π and σ are the natural projections from $B \times F$ to B and F, respectively.

In fact, g is a metric tensor. If $f = 1$, then $B \times_f F$ is just the semi-Riemannian product manifold. B is called the *base* of $M = B \times_f F$, and the F the *fiber*. The *fibers* $p \times F = \pi^{-1}(p)$ and the *leaves* $B \times q = \sigma^{-1}(q)$ are semi-Riemannian submanifolds of M. These submanifolds have particular geometric properties as one can see by the following result.

Corollary 1.1.22 [Corollary 7.36. *(O'NEILL, 1983)*] *The leaves $B \times q$ of a warped product are totally geodesic; the fibers $p \times F$ are totally umbilic.*

In particular, by simple linear algebra, if (B, g_B) is Lorentzian and (F, g_F) is Riemannian, then $B \times_f F$ is Lorentzian. In what follows, we present some examples of warped products.

Example 7 (Riemannian cylinder). *Let $(B, g_B) = (\mathbb{R}, dt^2)$ be the standard positive definite metric on $\mathbb{R}$, and $(F, g_F) = (\mathbb{S}^n, \omega_n)$ the n-sphere. The warped product $(\mathbb{R} \times \mathbb{S}^n, dt^2 + \omega_n)$ with the constant function $f = 1$ on $\mathbb{R}$ is the* standard (Riemannian) cylinder.

Example 8 (Robertson-Walker spacetimes). *Consider the negative definite metric on the open interval $((a, b), -dt^2)$. Let $f : (a, b) \to (0, +\infty)$ be any positive smooth function, and (F, g_F) be an arbitrary Riemannian manifold. The Lorentzian warped product $(a, b) \times_f F$ is referred to as a* generalized Robertson-Walker spacetime. *If (F, g_F) is chosen as either $(\mathbb{R}^n, \eta_0)$, $(\mathbb{S}^n, \omega_n)$, or $\mathbb{H}^n_0$, then the Lorentzian warped product is simply called a* Robertson-Walker spacetime.

See Section 2.3 for the celebrated Schwarzschild and Schwarzschild-Kruskal spacetimes, which are very important examples of warped products.

1.1.6 CAUSALITY THEORY

Let (M^n, g) be a semi-Riemannian manifold with index $0 < \nu < n$, $n \geq 2$. Since the metric g is indefinite, we can define the following classification of vectors, which are particularly of interest in the Lorentzian case. Let $p \in M$, we say that $v \in T_pM$ is

1. *timelike*, if $g_p(v,v) < 0$,
2. *spacelike*, if $g_p(v,v) > 0$ or $v = 0$,
3. *lightlike* (or *null*), if $v \neq 0$ and $g_p(v,v)$,
4. *causal*, if v is either timelike or null.

These notions also extend naturally to other geometric objects. A vector field $X \in \mathfrak{X}(M)$ is said to be *timelike* [resp. *null*, *spacelike* or *causal*] if, for all $p \in M$, we have that X_p is timelike [resp. null, spacelike or causal]. Analogously, a smooth curve $\alpha : I \subseteq \mathbb{R} \to M$ is said to be *timelike* [resp. *null*, *spacelike* or *causal*], for all $t \in I$, if $\alpha'(t)$ is timelike [resp. null, spacelike or causal].

To study the tangent spaces of a Lorentz manifold (M, g) in abstract terms, we define a *Lorentz vector space* to be the a $(\mathbb{V}, g)$ where g is a scalar product space of index 1 and $\dim(\mathbb{V}) \geq 2$. The notion of causal character of vectors has, in this context, a natural generalization to vector subspaces.

Definition 1.1.23 . *Let $(\mathbb{V}, g)$ be a Lorentz vector space, then a vector subspace $\mathbb{W} \subset \mathbb{V}$ is*

1. timelike, *if $g|_{\mathbb{W}\times\mathbb{W}}$ is nondegenerate and has index 1,*
2. spacelike, *if $g\big|_{\mathbb{W}\times\mathbb{W}}$ is positive-definite,*
3. lightlike *(or* null*), if $g|_{\mathbb{W}\times\mathbb{W}}$ is degenerate.*

For any set $X \subset \mathbb{V}$ of a Lorentz vector space $(\mathbb{V}, g)$, we define the vector subspace

$$X^{\perp} := \{v \in \mathbb{V} : g(v,w) = 0, \quad \forall w \in X\}.$$

The following simple results are widely useful to understand the geometry of the Lorentz vector spaces.

Lemma 1.1.24 (O'NEILL, 1983). *Let $(\mathbb{V}, g)$ be a Lorentz vector space. If $v \in \mathbb{V}$ is a timelike vector, then the subspace $v^{\perp}$ is spacelike and $\mathbb{V}$ is the direct sum $\mathbb{R}v \oplus v^{\perp}$.*

This argument demonstrates a more general result: a subspace $\mathbb{W}$ is timelike if and only if $\mathbb{W}^{\perp}$ is spacelike. Since $(\mathbb{W}^{\perp})^{\perp} = \mathbb{W}$, the terms timelike and spacelike can be reversed in this assertion. It follows then that $\mathbb{W}$ is lightlike if and only if $\mathbb{W}^{\perp}$ is lightlike.

Proposition 1.1.25 (COSTA E SILVA, 2021). *Let $(\mathbb{V}, g)$ be a Lorentz vector space. Given a vector subspace $\mathbb{W} \subset \mathbb{V}$, $\dim(\mathbb{W}) \geq 2$, then the following statements are equivalent:*

1. *$\mathbb{W}$ is timelike;*
2. *$\mathbb{W}$ contains two linearly, independent lightlike vectors;*
3. *$\mathbb{W}$ contains a timelike vector.*

Moreover, the following statements also are equivalent:

1. *$\mathbb{W}$ is lightlike;*
2. *$\mathbb{W}$ contains a lightlike but not a timelike vector;*
3. $\dim(\mathbb{W} \cap \mathbb{W}^{\perp}) = 1$.

Therefore, any nondegenerate subspace of a Lorentz vector space is either timelike or spacelike. Let $\mathcal{T}$ be the set of all timelike vectors in a Lorentz vector space $(\mathbb{V}, g)$. Given any timelike vector $u \in \mathcal{T}$, we define the *timecone* of u as the set

$$C(u) = \{v \in \mathcal{T} : \langle u, v \rangle < 0\}$$

of $\mathbb{V}$. The *opposite timecone* of u is

$$C(-u) = -C(u) = \{v \in \mathcal{T} : \langle u, v \rangle > 0\}.$$

Since $u^\perp$ is spacelike, $\mathcal{T}$ is the disjoint union of these two timecones.

Lemma 1.1.26 (O'NEILL, 1983). *Timelike vectors v and w in a Lorentz vector space $(\mathbb{V}, g)$ are in the same timecone if and only if $g(v,w) < 0$.*

Let again $(\mathbb{V}^n, g)$ be a Lorentz vector space. As any finite-dimensional real vector space, $\mathbb{V}$ can be equipped with the natural topology of $\mathbb{R}^n$. With respect to this topology, the set $\mathcal{T}$ of timelike vectors in $(\mathbb{V}, g)$ is open, and has exactly two connected components, which are precisely the two timecones (see Figure 2). A *time-orientation* on $(\mathbb{V}, g)$ is a choice of one of these two timecones. $(\mathbb{V}, g)$ is said to be *time-oriented* if such a choice has been made. In this case, the distinguished timecone is said to be the *future timecone*, often denoted as τ^+, while the opposite timecone is the *past timecone*, denoted by τ^-. The *future causal cone* is $\mathcal{C}^+ = \overline{\tau^+} \setminus \{0_{\mathbb{V}}\}$ (where the overbar indicates topological closure), and the *past causal cone* $\mathcal{C}^-$ is defined analogously from the past timecone. It is easily shown that any causal vector is in one of these two causal cones. Specifically, the null vectors are either in the *future null cone* $\Lambda^+ := \mathcal{C}^+ \setminus \tau^+$ or in the analogously defined *past null cone*. Noncollinear causal vectors $v, w \in \mathbb{V}$ are in the same causal cone if and only if $g(v, w) < 0$.

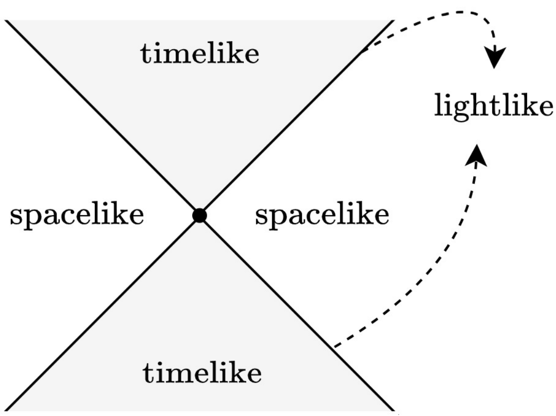


Figure 2 – Cones in Minkowski $\mathbb{R}^2_1$.

Let (M, g) be a Lorentzian manifold. For each $p \in M$, we have that T_pM is Lorentzian vector space, and as such, it contains two timecones. We are interested in developing a notion of time-orientability on each tangent space which is suitably continuous.

Consider a function τ defined on a Lorentzian manifold (M, g) that associates to each point $p \in M$ a timecone $\tau_p^+ \subset T_pM$. We say that τ is smooth if, for every $p \in M$, there exists a smooth vector field V defined on some neighborhood $\mathcal{U}$ of p such that $V_q \in \tau_q^+$, for all $q \in \mathcal{U}$. This smooth function is a *time-orientation* of (M, g). If (M, g) admits a time-orientation, we say that it is *time-orientable*. A time-orientable Lorentzian manifold (M, g) is *time-oriented* if a choice of a time-orientation thereon has been made.

Lemma 1.1.27 (O'NEILL, 1983). *A Lorentzian manifold (M, g) is time-orientable if and only if there exists a timelike vector field $X \in \mathfrak{X}(M)$.*

If there exists a vector field $X \in \mathfrak{X}(M)$ as described in the lemma above, the time-orientation defined by X is referred to as the *induced time-orientation*. It is worth noting that, in the context of a Lorentzian manifold, orientability and time-orientability are logically independent.

Definition 1.1.28 . *Let (M, g) be a time-orientable Lorentzian manifold with induced time-orientation by the timelike vector field $X \in \mathfrak{X}(M)$. For $p \in M$, a causal vector $v \in T_pM$ is* future-directed *[resp.* past-directed*] if $g_p(v, X_p) < 0$ [res. $g_p(v, X_p) > 0$]. Analogously, we say that a causal curve $\alpha : I \subseteq \mathbb{R} \to M$ is* future-directed *[resp.* past-directed*] if, for all $t \in I$, $\alpha'(t)$ is future-directed [resp. past-directed].*

Definition 1.1.29 (Spacetime). *A* spacetime *is a connected, time-oriented Lorentzian manifold.*

The time-orientability is not as restrictive as one might expect. For instance, for every connected Lorentzian manifold it is possible to construct a double covering that is a connected, time-orientable and Lorentzian manifold. Consequently, this manifold becomes a spacetime locally isometric to M (see discussion in Beem, Ehrlich, and Easley (1996) after Definition 3.1).

Example 9 (Minkowski spacetime). *Minkowski space $\mathbb{R}_1^n$ is time-orientable, with time-orientation defined by the timecones containing the coordinate vector field ∂_1 in the natural coordinates $(x^1, \dots, x^n)$. When endowed with this standard time-orientation, $\mathbb{R}_1^n$ is referred to as the (n-dimensional)* Minkowski spacetime.

Example 10 (De Sitter spacetime). *The de Sitter spaces $\mathbb{S}_1^n(1)$ are time-orientable, because if ∂_1 is the (timelike) standard coordinate vector field of $\mathbb{R}_1^{n+1}$, then $X := \tan(\partial_1)$, i.e., the tangent part of ∂_1, is a timelike vector field on $\mathbb{S}_1^n(1)$. Therefore, $\mathbb{S}_1^n(1)$ endowed with this time-orientation is known as the (n-dimensional)* de Sitter spacetime.

Example 11 (Anti-de Sitter spacetime). *The anti-de Sitter spaces $\mathbb{H}_1^n(1) \subset \mathbb{R}_2^{n+1}$ are time-orientable. In fact, since $\mathcal{P} = x^i \frac{\partial}{\partial x^i}$ is normal to $\mathbb{H}_1^n(1)$, it is possible to check that $x^2\partial_1 - x^1\partial_2$ is tangent to $\mathbb{H}_1^n(1)$ and timelike. Therefore, $\mathbb{H}_1^n(1)$ endowed with this time-orientation is known as the (n-dimensional)* anti-de Sitter spacetime.

1.2 THE VARIATION FORMULA FOR THE VOLUME

This section will follow the approach outlined in the first chapter of (ANCIAUX, 2010). We will now introduce the concept of a variation of a submanifold, which can be understood as a "curve of submanifolds". Variations of a submanifold will provide the foundation for understanding minimal surfaces, which in turn will serve both as a motivation for and a means of interpreting certain results for the marginally trapped surfaces defined in Chapter 4. Furthermore, these also provide a nice geometric interpretation to the mean curvature of semi-Riemannian submanifolds.

1.2.1 VARIATION OF A SUBMANIFOLD

Let $\phi : S^n \to M^m$ be an immersion where S has a (possibly empty) boundary ∂S, and with $0 < n < m$. A *variation* of ϕ is a smooth map $\Phi : S \times (-t_0, t_0) \to M$ with $t_0 > 0$, such that $\Phi(x, 0) = \phi(x)$. The vector field $V := \frac{\partial \Phi}{\partial t}(x, 0)$ is called the variation vector field on ϕ associated with the variation Φ. For later convenience, we shall also impose two technical requirements on Φ: (i) the variation fixes the boundary ∂S, and (ii) it is *compactly supported*, i.e., there exists a relatively compact open subset U of S such that $U \cap \partial S = \emptyset$, and

$$\Phi(x, t) = \phi(x), \quad \forall x \in S \backslash U.$$

The assumption of compact support implies that the associated vector field V is compactly supported and that V and all its derivatives vanish on ∂S.

It is possible to show that given a variation $\Phi : S^k \times (-t_0,t_0) \to M^n$ of an immersion [resp. semi-Riemannian immersion of induced index ν] $\phi : S \to M$, each map $\phi_t : x \in S \mapsto \Phi(x, t) \in M$ is also an immersion [resp. semi-Riemannian immersion of index ν] for small enough t. We shall, henceforth, implicitly assume that $t_0 > 0$ has been chosen small enough so that this occurs. If (M, g) is a semi-Riemannian manifold and $\phi : S \to M$ is a semi-Riemannian immersion, then we denote by S_t the manifold S endowed with the induced semi-Riemannian metric $g_t := \phi_t^* g$. If S is oriented, then we shall denote by Vol_t the associated volume n-form of S_t.

Definition 1.2.1 (Minimal submanifold). *Let (M,g) be a semi-Riemannian manifold. A semi-Riemannian immersion $\phi : S \to M$ is said to be* minimal *if its mean curvature vector field $\vec{H} \in \mathfrak{X}^{\perp}(\phi)$ vanishes identically.*

If (M, g) is a semi-Riemannian manifold, then given an embedded submanifold $S \subset M$, we shall say that S is minimal if its inclusion is a minimal embedding. The word *minimal* comes from Lagrange's tradition and it could be particularly misleading: a minimal submanifold does not necessarily minimize the induced volume, even in the Riemannian case. Indeed, in some situations, the submanifold actually maximizes the volume. We shall see, however, that given certain technical conditions, being minimal means that the submanifold manifold (or immersion) is a critical point for the volume functional.

1.2.2 THE FIRST VARIATION FORMULA

Theorem 1.2.2 (First Variation formula). *Let $\Phi : S^n \times (-t_0, t_0) \to M^m$ be a variation of a semi-Riemannian immersion $\phi : S \to M$ of the compact oriented manifold S into the semi-Riemannian manifold (M, g), where $m > n > 0$. Then we have*

$$\frac{d}{dt} Vol(S_t)\bigg|_{t=0} = \int_S \theta dV, \tag{9}$$

where $\theta = g(\vec{H}, -V)$, the vector field V is the variation vector field on Φ and $\vec{H}$ is the mean curvature vector of S.

Corollary 1.2.3 . *Let (M, g) be a semi-Riemannian manifold, and let $\phi : S \to M$ be a semi-Riemannian immersion, with S compact and oriented. Then ϕ is minimal if and only if with respect to any variation we have*

$$\frac{d}{dt} Vol(S_t)\bigg|_{t=0} = 0.$$

Theorem 1.2.4 . *A compact submanifold S^n of a semi-Euclidean space $\mathbb{R}^m_\nu$ is not minimal.*

Proof. Let $\phi : S^n \to \mathbb{R}^m$ be a parametrization of S and define

$$\begin{aligned} \Phi : \quad S^n \times (-t_0, t_0) \quad &\to \mathbb{R}^m \\ (x, t) \quad &\mapsto (1 + t) \cdot \phi(x). \end{aligned}$$

The compactness assumption ensures that the variation Φ satisfies the required conditions (i) and (ii). On the other hand, an easy computation shows that $Vol(S_t) = (1 + t)^n Vol(S)$, so that

$$\frac{d}{dt} Vol(S_t)\bigg|_{t=0} = nVol(S) \neq 0.$$

□

It is worth noting that this corollary establishes a link between two distinct fields: the calculus of variations (optimization of volume) and the field of differential geometry (geometric quantities, mean curvature vector). The proof of Theorem 1.2.2 will follow from the two subsequent lemmas.

Lemma 1.2.5 (Jacobi's formula). *Let $t \to A(t)$ be a smooth curve of $n \times n$ real (or complex) invertible matrices, where $t \in (-t_0, t_0)$. Then*

$$\frac{d}{dt} \det A(t)\bigg|_{t=0} = \det A(0) \cdot \operatorname{tr}(A^{-1}(0)A'(0)). \tag{10}$$

Proof. Let A and B be invertible $n \times n$ real matrices, and let I denote the identity matrix. We can differentiate the $\det$ function using the definition of a directional derivative. Then, we have:

$$\det{}'(I)(B) = \lim_{h \to 0} \frac{\det(I + hB) - \det(I)}{h},$$

observing that $\det(I+hB)$ is a polynomial in h of order n with the constant term equal to 1, while the linear term in h is $\operatorname{tr} B$, then

$$\det'(I)(B) = \operatorname{tr} B.$$

Considering $\det$ as a function of X, we have the following:

$$\det(X) = \det(AA^{-1}X) = \det(A)\det(A^{-1}X),$$

then calculating the derivative in X and applying the chain rule at the point B yields

$$\det'(X)(B) = \det(A)\det'(A^{-1}X)(A^{-1}B),$$

and evaluating it at $X = A$

$$\det'(A)(B) = \det(A)\det'(A^{-1}A)(A^{-1}B) = \det(A)\operatorname{tr}(A^{-1}B),$$

Finally, given a family of invertible matrices parameterized by t, denoted by $A(t)$, we have

$$\frac{d}{dt}\det A(t)\bigg|_{t=0} = \det'(A_0)(\frac{dA}{dt}) = \det A(0)\operatorname{tr}(A^{-1}(0)A'(0)).$$

□

Lemma 1.2.6 . *Let $\Phi : S^n \times (-t_0, t_0) \to M^m$ be a variation of a semi-Riemannian immersion $\phi : S \to M$ into the semi-Riemannian manifold (M, g), where $m > n > 0$. Then follows that*

$$\frac{d}{dt}g_t(X,Y)\bigg|_{t=0} = g(D_X V, d\phi(Y)) + g(d\phi(X), D_Y V), \tag{11}$$

where V is the variation vector field of the variation and D is the induced connection on ϕ. Moreover, if V is a normal vector field then holds that

$$\frac{d}{dt}g_t(X,Y)\bigg|_{t=0} = 2g(II(X,Y), -V), \tag{12}$$

where II is the second fundamental form tensor of ϕ.

Proof. Let $\Phi : S \times (-t_0, t_0) \to M$ be the smooth map associated with the variation of the immersion ϕ, and let $\phi(x) := \Phi(x, 0)$ and $\Phi_t(x) := \Phi(x,t)$. We define $V \in \mathfrak{X}(\phi)$ as the variation vector field of the variation, i.e., $V := \frac{\partial \Phi}{\partial t}(x, 0)$. Let (U, x^a) be a coordinate chart on S and let $(\widetilde{U}, \zeta^i)$ be a coordinate chart on M such that $\Phi_t(U) \subset \widetilde{U}$ for t small enough. From now on, we denote the entries of coordinates charts S by alphabetical letters such as a, b, c, d, and the entries of M by alphabetical letters such as i, j, k, l. Denoting $(g_t)_{cd} = g_t(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d})$, then

$$(g_t)_{cd} = g(d\Phi_t(\frac{\partial}{\partial x^c}), d\Phi_t(\frac{\partial}{\partial x^d})) = (g_{ij} \circ \Phi_t)\frac{\partial \Phi_t^i}{\partial x^c}\frac{\partial \Phi_t^j}{\partial x^d},$$

$$\frac{d}{dt}(g_t)_{cd}\bigg|_{t=0} = (\frac{\partial g_{ij}}{\partial \zeta^k} \circ \phi)V^k\frac{\partial \phi^i}{\partial x^c}\frac{\partial \phi^j}{\partial x^d} + (g_{ij} \circ \phi)(\frac{\partial V^i}{\partial x^c}\frac{\partial \phi^j}{\partial x^d} + \frac{\partial \phi^i}{\partial x^c}\frac{\partial V^j}{\partial x^d}),$$

note that the first term in the previous equation can be written as

$$\frac{\partial g_{ij}}{\partial \zeta^k} \circ \phi = (g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik}) \circ \phi, \tag{13}$$

using Equation (13), changing the indexes and denoting by D^ϕ the induced connection on ϕ, we have

$$\begin{aligned}
\frac{d}{dt}(g_t)_{cd}\Big|_{t=0} &= ((g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik}) \circ \phi)V^k \frac{\partial \phi^i}{\partial x^c}\frac{\partial \phi^j}{\partial x^d} + (g_{ij} \circ \phi)(\frac{\partial V^i}{\partial x^c}\frac{\partial \phi^j}{\partial x^d} + \frac{\partial \phi^i}{\partial x^c}\frac{\partial V^j}{\partial x^d}) \\
&= (g_{ij} \circ \phi)\left[(\Gamma^j_{lk} \circ \phi)V^k \frac{\partial \phi^l}{\partial x^d} + \frac{\partial V^j}{\partial x^d}\right]\frac{\partial \phi^i}{\partial x^c} \\
&\quad + (g_{ij} \circ \phi)\left[(\Gamma^i_{lk} \circ \phi)V^k \frac{\partial \phi^l}{\partial x^c} + \frac{\partial V^i}{\partial x^c}\right]\frac{\partial \phi^j}{\partial x^d} \\
&= g(d\phi(\frac{\partial}{\partial x^c}), D^\phi_{\frac{\partial}{\partial x^d}}V) + g(D^\phi_{\frac{\partial}{\partial x^c}}V, d\phi(\frac{\partial}{\partial x^d})),
\end{aligned}$$

so, for any $X, Y \in \mathfrak{X}(S)$, follows the first assertion:

$$\frac{d}{dt}g_t(X,Y)\Big|_{t=0} = g(D^\phi_X V, d\phi(Y)) + g(d\phi(X), D^\phi_Y V). \tag{14}$$

If $V \in \mathfrak{X}^\perp(\phi)$ follows that $g(d\phi(\frac{\partial}{\partial x^c}), V) = 0$ and applying the metric compatibility with the induced connection

$$\frac{\partial}{\partial x^d}g(d\phi(\frac{\partial}{\partial x^c}), V) = g(D^\phi_{\frac{\partial}{\partial x^d}}(d\phi(\frac{\partial}{\partial x^c})), V) + g(d\phi(\frac{\partial}{\partial x^c}), D^\phi_{\frac{\partial}{\partial x^d}}V) = 0,$$

which implies that

$$\begin{aligned}
g(d\phi(\frac{\partial}{\partial x^c}), D^\phi_{\frac{\partial}{\partial x^d}}V) &= -g(D^\phi_{\frac{\partial}{\partial x^d}}(d\phi(\frac{\partial}{\partial x^c})), V) \\
&= -g(d\phi(\nabla^S_{\frac{\partial}{\partial x^c}}(\frac{\partial}{\partial x^d})) + II^\phi(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d}), V) \\
&= -g(II^\phi(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d}), V),
\end{aligned}$$

where ∇^S is the Levi-Civita connection on S, therefore,

$$\frac{d}{dt}g_t(X,Y)\Big|_{t=0} = 2g(II^\phi(X,Y), -V). \tag{15}$$

□

Proof of Theorem 4. Let $\Phi : S \times (-t_0, t_0) \to M$ be the smooth map associated with the variation of the immersion ϕ, and let $\phi(x) := \Phi(x, 0)$ and $\Phi_t(x) := \Phi(x, t)$. We define $V \in \mathfrak{X}(\phi)$ as the variation vector field of the variation, i.e., $V := \frac{\partial \Phi}{\partial t}(x, 0)$. By the assumption that V is compactly supported, there exists t_0 such that $\forall t$, $|t| < t_0$ the map $\phi_t(x) := \Phi(x, t)$ defines a immersion of S. As ϕ_t is a immersion of S, for each t, $|t| < t_0$, we have a induced metric on S denoted g_t, and a volume form denoted dV_t. The space of n-forms on a manifold of dimension n is one, therefore, they are proportional. Thus, there exists a map $v(x,t)$ on $S\times(-t_0,t_0)$ such that $dV_t = v(x,t)dV$, and therefore

$$\frac{d}{dt}Vol(S_t)\Big|_{t=0} = \int_S \frac{\partial v}{\partial t}(x,0)dV.$$

In order to calculate $v(x,t)$, we introduce a local frame $\{E_1,\ldots,E_n\}$ of S such that it is orthonormal for the induced metric $\phi^* g_0$. We then introduce the matrix

$$M(t) := [g_t(E_i,E_j)]_{1\le i,j\le n},$$

where $g_t(E_i,E_j) := g(d\phi_t\circ E_i, d\phi_t\circ E_j)$ and claim that $v(x,t) = |\det M(t)|^{1/2}$. In fact

$$\begin{aligned} dV_t(E_1,\ldots,E_n) &= |\det[g_t(E_i,E_j)]_{1\le i,j\le n}|^{1/2} \\ &= |\det M(t)|^{1/2} \\ &= |\det M(t)|^{1/2}\cdot dV(E_1,\ldots,E_n), \end{aligned}$$

We are now in position to compute the first variation formula: by Lemma 1.2.5 and using the formula $\frac{d}{dt}|u|^{1/2} = \frac{u'u}{2|u|^{3/2}}$,

$$\frac{\partial v}{\partial t}(x,0) = \frac{\operatorname{tr}(M^{-1}(0)M'(0))\cdot\det M(0)^2}{2|\det M(0)|^{3/2}}.$$

Since $\{E_1,\ldots,E_n\}$ is orthonormal with respect to $\phi^* g_0$, $M^{-1}(0) = M(0) = diag(\varepsilon_1,\ldots,\varepsilon_n)$, so that

$$\frac{\partial v}{\partial t}(x,0) = \frac{1}{2}\sum_{i=1}^n \varepsilon_i \frac{d}{dt}g_t(E_i,E_i)\big|_{t=0},$$

It remains to compute $\frac{d}{dt}g_t(E_i,E_i)\big|_{t=0}$. From Lemma 1.2.6, we obtain that

$$\begin{aligned} \frac{d}{dt}g_t(E_i,E_i)\Big|_{t=0} &= \frac{d}{dt}g(d\phi_t\circ E_i, d\phi_t\circ E_i)\Big|_{t=0} \\ &= 2g(D^\phi_{E_i}V, d\phi\circ E_i) \\ &= 2g(D^\phi_{E_i}V^\top, d\phi\circ E_i) + 2g(D^\phi_{E_i}V^\perp, d\phi\circ E_i) \\ &= 2g(D^\phi_{E_i}V^\top, d\phi\circ E_i) - 2g(V^\perp, II(E_i,E_i)), \end{aligned}$$

it follows that

$$\begin{aligned} \frac{\partial v}{\partial t}(x,0) &= \sum_{i=1}^n \varepsilon_i\left(g(d\phi\circ E_i, D^\phi_{E_i}V^\top) - g(V^\perp, II(E_i,E_i))\right) \\ &= \operatorname{div}(V^\top) + g(-V^\perp,\vec{H}). \end{aligned}$$

To conclude, we can use the divergence theorem and the fact that V vanishes on ∂S to obtain

$$\frac{d}{dt}Vol(S_t)\Big|_{t=0} = \int_S \operatorname{div}(V^\top) + g(-V^\perp,\vec{H})dV = \int_S g(-V,\vec{H})dV.$$

Hence, the proof is complete. □

2 GENERAL RELATIVITY

2.1 EINSTEIN EQUATIONS

The theory of general relativity, first published by Einstein (1915), provides a geometric description of the phenomenon of gravity. More precisely, this means that in general relativity the portion of the universe one wishes to describe is modeled via a (4-dimensional) spacetime (M,g), and gravity is a manifestation of its underlying geometry, rather then forces between massive objects as in Newtonian mechanics. Therefore, general relativity can be understood as a geometric theory of gravity.

The specific dynamical content of the theory is described by the *Einstein Field Equations* (EFE). Once a coordinate system is chosen, the EFE become a set of second-order nonlinear partial equations. These equations associate the geometry of spacetime to the distribution of matter and energy in the universe. It is important to note that a distribution of matter only makes sense in the context of a background spacetime, and therefore, the distribution of matter and the geometry of spacetime must be solved simultaneously. This fact presents a significant challenge when attempting to solve the EFE.

Setting the discussion in a more precise mathematical language, the universe is modeled as a manifold M endowed with a Lorentzian metric g, which represents the gravitational field. We assume that the pair (M,g) is a connected, time-oriented Lorentzian manifold, already referred to as *spacetime* in this text. The gravitational field is described by the pair (M,g), while the distribution of matter and energy is represented by a symmetric $(0,2)$-tensor field T, known as the *stress-energy tensor*, rather than a mass density function as in Newtonian mechanics. Additionally, the model also incorporates a constant called *cosmological constant* $\Lambda \in \mathbb{R}$ that represents a form of energy that is inherent to space itself. The relationship between the spacetime (M,g), the tensor T and the constant Λ is established via the EFE, as defined below.

Definition 2.1.1 (Einstein field equations). *Let (M,g) be a spacetime and T be a symmetric $(0,2)$-tensor field on M. In this context, we say that (M,g)* solves the Einstein field equations*, with* stress-energy tensor T *and* cosmological constant Λ*, if g satisfies*

$$Ric_g - \frac{1}{2}S_g \cdot g + \Lambda \cdot g = T, \tag{16}$$

where Ric_g and S_g are the Ricci and scalar curvature, respectively. Alternatively, the equations can be written as $G = T - \Lambda g$, where G is the Einstein tensor *of g defined by*

$$G := Ric_g - \frac{1}{2}S_g \cdot g.$$

Remark 2 . *The EFE only acquire a physical meaning when we understand that T is actually dependent on g and additional matter fields such as the electromagnetic field. The EFE thus describe a specific dynamical coupling of gravity and matter fields. Additional dynamical* *equations for such matter fields must also be separately provided. Since our interest here is purely geometric, we shall take the EFE as a mere* definition *of the stress-energy tensor* $T = T(g, \Lambda)$ *for a given spacetime* (M,g) *and cosmological constant* Λ*, which we shall often take to be zero, so that* $T = G$*, i.e., the stress-energy is just the Einstein tensor.*

Remark 3 . *The Einstein tensor of* g *is divergence free, i.e.,* $\operatorname{div} G = 0$*, this fact is a consequence of the identity* 5 *of the Theorem 1.1.8. Therefore, the stress-energy tensor* T *must satisfy* $\operatorname{div} T = 0$*. This means, intuitively, that the matter modeled by* T *does not have any sources or sinks. In other words, matter does not spontaneously appear or disappear.*

The simplest version of the EFE is the *vacuum* case, which is obtained when T is identically zero and $\Lambda = 0$. In this scenario, the Minkowski space satisfies the EFE and it is considered as a fundamental model of empty spacetime. Furthermore, the vacuum case also includes other non-trivial solutions of interest, such as the Schwarzschild spacetime, which will be discussed later. In particular, a spacetime (M,g) is a *vacuum solution of the EFE* if and only if it is *Ricci-flat*, i.e., $Ric_g = 0$.

2.2 THE INITIAL VALUE FORMULATION OF GENERAL RELATIVITY

It is natural to ask whether it is possible to formulate an initial value problem for the EFE, i.e., specify an initial metric at some fixed time and a stress-energy tensor and use the EFE to evolve the system forward in time. However, the issue here becomes much more subtle than in usual PDE theory for the simple fact that the notion of "time", with respect to which one might contemplate evolution, has no independent meaning in general relativity. The notion of time only acquires some observer-dependent significance *after* fixing a spacetime, which nevertheless is precisely the variable in the EFE! Amazingly, not only these issues can be circumvented, but the approach of initial value formulation proves highly valuable for fields such as numerical relativity, which aims to simulate the dynamics of strong gravitational fields and various astrophysical phenomena.

In order to understand the initial value formulation of EFE, we need to first understand the so-called *constraint equations*. To investigate the origins of the constraint equations, a natural approach is to assume a spacetime solution and study the induced data on spacelike hypersurfaces. Before proceeding, we start by defining two quantities: the energy density and energy-momentum current.

Definition 2.2.1 (Energy density and energy-momentum current). *Let* (M, g) *be a spacetime and* $u \in TM$ *be any unit future-directed timelike vector. We define the* energy density $\rho_u := G(u, u)$ *and the one-form* energy-momentum current *on the vector space* $u^{\perp}$ *as* $J_u(\cdot) := -G(u, \cdot)$ *associated with* u*, where* G *denotes the Einstein tensor of* g*.*

Particularly, if we are concerned about vacuum spacetimes, where the stress-energy tensor T is identically zero and $\Lambda = 0$, then both the energy density ρ_u and the energy-momentum current density J_u vanish identically for any timelike vector u.

Theorem 2.2.2 (Constraint Equations on Spacelike Hypersurfaces). *Let (M^{n+1}, g) be a spacetime, and let $S^n \subset M$ be an embedded spacelike hypersurface in (M,g) with induced (Riemannian) metric h. Let $\mathcal{K} : \mathfrak{X}(S) \times \mathfrak{X}(S) \to C^\infty(S)$ be the second fundamental form associated with the unique unit future-directed timelike vector field $U \in \mathfrak{X}^\perp(S)$ normal to S. Therefore, the following* constraint equations *hold on S:*

1. (Hamiltonian Constraint Equation)

$$S_h - |\mathcal{K}|_h^2 + (\operatorname{tr}_h \mathcal{K})^2 = 2\rho_U,$$

2. (Momentum Constraint Equation)

$$\operatorname{div}_h \mathcal{K} - d(\operatorname{tr}_h \mathcal{K}) = J_U,$$

where ρ_U and J_U are the energy density and energy-momentum current density associated with U, respectively.

Proof. See Section 2.2.1. □

When considering a vacuum spacetime, it becomes possible to express both constraint equations in a simplified form,

$$S_h - |\mathcal{K}|_h^2 + (\operatorname{tr}_h \mathcal{K})^2 = 0,$$

$$\operatorname{div}_h \mathcal{K} - d(\operatorname{tr}_h \mathcal{K}) = 0.$$

The constraint equations above could be viewed as restrictions on *abstract* data $(S, h, \mathcal{K})$, where (S, h) is a Riemannian manifold and $\mathcal{K}$ is a symmetric $(0, 2)$-tensor field on S. These conditions are *necessary* for the data to describe a spacelike hypersurface in a vacuum spacetime. What is surprising is that these conditions are also *sufficient*: given the data $(S, h, \mathcal{K})$, there exists a vacuum spacetime (M, g) where the Riemannian manifold (S, h) is a spacelike hypersurface of (M, g) endowed with the induced metric. This assertion was shown by the seminal work of Y. Choquet-Bruhat, which is summarized in the following theorem.

Theorem 2.2.3 (FOURÈS-BRUHAT, 1952; CHOQUET-BRUHAT; GEROCH, 1969; LEE, D. A., 2019). *Let (S^n, h) be a Riemannian manifold and let $\mathcal{K}$ be a smooth symmetric $(0, 2)$-tensor field on S. Suppose that the following equations hold:*

$$S_h + (\operatorname{tr}_h \mathcal{K})^2 - |\mathcal{K}|_h^2 = 0,$$

$$(\operatorname{div}_h \mathcal{K} - d \operatorname{tr}_h \mathcal{K}) = 0.$$

Then there exists a vacuum spacetime (M^{n+1}, g) such that (S^n, h) isometrically embeds into (M^{n+1}, g) as a hypersurface with second fundamental form $\mathcal{K}$. Furthermore, there is a unique (up to isometry) maximal *such solution in the sense that any other solution satisfying these conditions can be isometrically embedded therein.*

The aforementioned theorem has two important points that deserve mention. First, the second fundamental form $\mathcal{K}$ is associated with the unit future-directed timelike normal vector field U normal to S, i.e., $\mathcal{K} = g(II, -U)$. Second, the uniqueness of the (maximal) solution means that (M^{n+1}, g) does not lie inside any larger globally hyperbolic spacetime. Therefore, (M^{n+1}, g) is referred to as *the Cauchy (vacuum) development* of the initial data $(S^n, h, \mathcal{K})$.

Although in the Theorem 2.2.3 one wishes to find a vacuum spacetime, i.e. T, Λ are identically zero, similar theorems can also be derived for the EFE with various matter fields. For a comprehensive discussion on this topic, we refer to the book (CHOQUET-BRUHAT, 2008).

From now on, we will assume $\Lambda = 0$ for convenience. It is worth noting that the ρ_U and J_U can be expressed solely in terms of initial data, without referring to the unit future-directed timelike vector field U, as can be observed in Theorem 2.2.2. Thus, we say that the pair (ρ, J) constrains the pair $(h, \mathcal{K})$, leading to a general definition of initial data.

Definition 2.2.4 (Initial Data Set). *An* initial data set *is a triple* $(S, h, \mathcal{K})$ *where* (S, h) *is an* n*-dimensional Riemannian manifold and* $\mathcal{K}$ *is a symmetric* $(0, 2)$*-tensor field on* S*. Given an initial data* $(S, h, \mathcal{K})$*, we define a function* $\rho \in C^\infty(S)$ *and a one-form* $J \in \Omega^1(S)$*, called respectively the* energy density *and* energy-momentum current *associated with the data by*

$$\rho = \frac{1}{2}\left[S_h - |\mathcal{K}|_h^2 + (\mathrm{tr}_h\, \mathcal{K})^2\right],$$
$$J = \mathrm{div}_h\, \mathcal{K} - d(\mathrm{tr}_h\, \mathcal{K}).$$

If the tensor $\mathcal{K}$ vanishes identically, then the associated initial data is said to be *symmetric*. Symmetric initial data sets are basically the same as Riemannian manifolds.

The non-vanishing case of the stress-energy tensor T can lead to various possible solutions, including non-physical ones. In order to control the solutions, it is natural to impose certain physically motivated constraints. One such constraint is the *Dominant Energy Condition* (DEC), which is stated as follows.

Definition 2.2.5 . *Let* (M, g) *be a spacetime and* $p \in M$*. We say that* (M, g) *satisfies the* dominant energy condition *at* p*, if for any future-pointing unit timelike vector* $v \in T_pM$ *the covector* $-G(v, \cdot)$ *is either zero or future-pointing causal. If the DEC holds for each point* $p \in M$*, then the spacetime* (M, g) *itself is said to satisfy the DEC.*

The covector $-G(v, \cdot)$ can be interpreted, in the context of the EFE, as the energy-momentum density of the gravitational sources seen by an observer v with 4-velocity at $p \in M$ given by $v \in T_pM$. The DEC ensures that the energy density should be non-negative and that the energy does not flow faster than the speed of light as measured by any such observer preserving, at least locally, the causality in transmission of energy and matter in the theory.

The DEC statement can be conveniently reformulated in order to be applied in initial data sets. Let (M^n, g) be a spacetime and let v be any unit future-pointing timelike vector at $p \in M$. Consider an orthonormal basis $\{v, e_2, \ldots, e_n\}$ of T_pM. Note that $J_v = -G(v, \cdot)$ is the associated energy-momentum density and $\rho_v = G(v, v)$ is the energy density. For convenience, from now on we drop the subscript v. If X_J is the metrically related vector field to the one-form J, we have $\rho = -\langle X_J, v\rangle$, and we can decompose X_J as

$$X_J = \widetilde{X_J} + \rho v,$$

where $\widetilde{X_J}$ is the spacelike component of X_J. Supposing that the DEC holds at p, X_J is causal, then

$$\langle X_J, X_J\rangle = \langle \widetilde{X_J}, \widetilde{X_J}\rangle - \rho^2 \leq 0.$$

Since X_J and v are future-pointing, we have that $\rho > 0$ and conclude that

$$\rho \geq |J|_{v^\perp},$$

where $|J|_{v^\perp} := \sqrt{\langle \widetilde{X_J}, \widetilde{X_J}\rangle}$ is the norm of the spacelike part of the one-form J. These computations motivate an abstract definition of the DEC for initial data.

Definition 2.2.6 (Dominant energy condition in an initial data). *Let $(S, h, \mathcal{K})$ be an initial data set, then we say that $(S, h, \mathcal{K})$ satisfies the* dominant energy condition *if it holds that $\rho \geq |J|_h$, where ρ and J are as in Definition 2.2.4:*

$$\rho := \frac{1}{2}\left[S_h - |\mathcal{K}|_h^2 + (\operatorname{tr}_h \mathcal{K})^2\right],$$

$$J := \operatorname{div}_h \mathcal{K} - d(\operatorname{tr}_h \mathcal{K}).$$

Another constraint on the Einstein tensor is the Null Energy Condition (NEC), which focuses on the sign of the Ricci tensor along null vectors. The NEC is employed in foundational theorems, such as Proposition 3.1.9. In particular, the NEC is weaker than the DEC, as can be seen from the definition below.

Definition 2.2.7 (Null energy condition). *We say that a spacetime (M, g) satisfies the* null energy condition *(NEC) if $Ric_g(v, v) \geq 0$ for all null vectors $v \in TM$, or equivalently, if for all such vectors the Einstein tensor G satisfies $G(v,v) \geq 0$.*

2.2.1 CONSTRAINT EQUATIONS ON SPACELIKE HYPERSURFACES

In order to prove Theorem 2.2.2, some machinery from the theory of semi-Riemannian submanifolds will be required. These include the induced curvature tensor and the Gauss-Codazzi equations, which were already discussed in the previous chapter.

Proof. Let (M^{n+1}, g) be a spacetime, and let $S^n \subset M^{n+1}$ be an embedded spacelike hypersurface in (M,g) with induced (Riemannian) metric h and $\phi : S \hookrightarrow M$ denoting its inclusion. Since S^n is a spacelike hypersurface, let $U \in \mathfrak{X}(M)$ be the unique unit future-directed timelike vector field, and define the normal vector field $V := U \circ \phi \in \mathfrak{X}^\perp(\phi)$.

We denote by $D = D^\nabla : \mathfrak{X}(S) \times \mathfrak{X}(\phi) \to \mathfrak{X}(\phi)$ the unique induced connection on ϕ. It is important to recall that given any vector field $Z \in \mathfrak{X}(\phi)$, it can be decomposed into tangent and normal parts, i.e., $Z = d\phi \circ X_Z + Z^\perp$, for some $X_Z \in \mathfrak{X}(S)$. Additionally, for any $X, Y \in \mathfrak{X}(S)$, there is a corresponding decomposition for the induced connection:

$$D_X(d\phi \circ Y) = d\phi(\nabla^M_X Y) + II(X, Y),$$

where ∇^M is the Levi-Civita connection in the spacetime M and $II(X,Y) = (D_X(d\phi \circ Y))^\perp \in \mathfrak{X}^\perp(\phi)$ is the normal part of the induced connection. With these notions in place, we can proceed with the proof.

1. (Hamiltonian Constraint Equation)

Fix $p \in S$ and let $\{E_1, \ldots, E_n\}$ be a h-orthonormal frame defined in some open set $\mathcal{U}$ containing p. Denote $E_j^* := d\phi \circ E_j \in \mathfrak{X}(\phi)$ for each $j = 1, \ldots, n$. It is evident that the frame $\{E_1^*(q), \ldots, E_n^*(q), V(q)\}$ forms a g-orthonormal basis on $\phi(q)$ for each $q \in \mathcal{U}$. By applying the Gauss equation, Proposition 1.1.18, and taking into account $\varepsilon_S = \ll V, V \gg = -1$, we can conclude that

$$\begin{aligned}
\sum_{i,j=1}^n \ll R^D(E_i, E_j)E_j^*, E_i^* \gg &= \sum_{i,j=1}^n h(R^S(E_i, E_j)E_j, E_i) - [\mathcal{K}(E_i, E_j)\mathcal{K}(E_i, E_j) \\
&\quad - \mathcal{K}(E_j, E_j)\mathcal{K}(E_i, E_i)] \\
&= S_h - |\mathcal{K}|_h^2 + (\operatorname{tr}_h \mathcal{K})^2,
\end{aligned}$$

where $\mathcal{K}$ is the second fundamental form associated with V. By applying Proposition 1.1.2, we have

$$\ll R^D(E_i, E_j)E_j^*, E_i^* \gg_p = \langle R^g_{\phi(p)}(E_i^*(p), E_j^*(p))E_j^*(p), E_i^*(p)\rangle_{\phi(p)}.$$

Now, omitting the evaluated point p and proceeding, we obtain

$$\begin{aligned}
\sum_{i,j=1}^n \ll R^D(E_i, E_j)E_j^*, E_i^* \gg &= \sum_{i,j=1}^n \langle R^g(E_i^*, E_j^*)E_j^*, E_i^*\rangle \\
&= \sum_{j=1}^n Ric_g(E_j^*, E_j^*) + \sum_{i=1}^n \langle R^g(E_i^*, V)V, E_i^*\rangle \\
&= \sum_{j=1}^n Ric_g(E_j^*, E_j^*) + Ric_g(V, V) \\
&= S_g + 2Ric_g(V, V) \\
&= 2(Ric_g - \frac{1}{2}S_g \cdot g)(V, V) \\
&= 2G(V, V) \\
&= 2\rho_U,
\end{aligned}$$

where ρ_U is the energy density associated with the unit future-directed timelike vector field U (see Definition 2.2.1). Finally,

$$S_h + (\operatorname{tr}_h \mathcal{K})^2 - |\mathcal{K}|_h^2 = 2\rho_U. \tag{17}$$

2. (Momentum Constraint Equation)

Fixing $p \in S$, let $X \in \mathfrak{X}(S)$ and consider a h-orthonormal frame $\{E_1, \ldots, E_n\}$ in some open set containing p. According to the Codazzi equation, Proposition 1.1.18, we have

$$\sum_{i=1}^{n} \ll R^D(E_i, X)E_i^*, V \gg = \sum_{i=1}^{n} (\nabla_X^S \mathcal{K})(E_i, E_i) - (\nabla_{E_i}^S \mathcal{K})(X, E_i).$$

Employing the definition of tensor derivation and the fact that $\operatorname{div}_h A(Z) = \sum_{i=1}^{n} (\nabla_{E_i} A)(E_i, Z)$ for any $(0,2)$-symmetric tensor field A on S and any vector field $Z \in \mathfrak{X}(S)$ (see Definition 1.1.6), we can see that

$$\begin{aligned}\sum_{i=1}^{n} (\nabla_X^S \mathcal{K})(E_i, E_i) - (\nabla_{E_i}^S \mathcal{K})(X, E_i) &= \sum_{i=1}^{n} [X(\mathcal{K}(E_i, E_i)) - 2\mathcal{K}(\nabla_X^S E_i, E_i)] - \operatorname{div}_h \mathcal{K}(X) \\ &= (d \operatorname{tr}_h \mathcal{K} - \operatorname{div}_h \mathcal{K})(X),\end{aligned}$$

where the quantity $\mathcal{K}(\nabla_X^S E_i, E_i)$ vanishes identically. The proof of this fact is quite involved and it is a consequence of the symmetry of the $\mathcal{K}$ and the anti-symmetry of $h(\nabla_X^S E_i, E_j) = -h(E_i, \nabla_X^S E_j)$. For each pair of integers $i, j \in \{1, \ldots, n\}$, we denote the family of smooth functions $a_{ij} \in C^\infty(S)$ that satisfies $\nabla^S E_i = a_{ij} E_j$, then

$$h(\nabla_X^S E_i, E_j) = h(a_{ik} E_k, E_j) = a_{ij},$$

and,

$$a_{ij} = h(\nabla_X^S E_i, E_j) = -h(\nabla_X^S E_j, E_i) = -a_{ji},$$

therefore $a_{ij} = -a_{ji}$. Additionally, with the symmetry of $\mathcal{K}$, we have that

$$\begin{aligned}\sum_{i=1}^{n} \mathcal{K}(\nabla_X^S E_i, E_i) &= \sum_{i,j=1}^{n} a_{ij} \mathcal{K}(E_j, E_i) \\ &= -\sum_{i,j=1}^{n} a_{ji} \mathcal{K}(E_j, E_i) \\ &= -\sum_{i=1}^{n} \mathcal{K}(E_i, \nabla_X^S E_i),\end{aligned}$$

which implies that $\mathcal{K}(\nabla_X^S E_i, E_i) = 0$. Returning to the curvature term, we show that this quantity can be related with energy-momentum current J_U, where $U \in \mathfrak{X}(M)$ is a fixed unit future-directed field. Omitting the evaluation at p and using the notation of part 1, we have

$$\begin{aligned}\sum_{i=1}^{n} \ll R^D(E_i, X)E_i^*, V \gg &= -\sum_{i=1}^{n} \langle R^g(E_i^*, d\phi \circ X)V, E_i^* \rangle \\ &= -Ric_g(d\phi \circ X, V) - \langle R^g(V, d\phi \circ X)V, V \rangle,\end{aligned}$$

since the curvaturelike functions are antisymmetric the second term is zero. Employing that $g(d\phi \circ X, V) = 0$, because $d\phi \circ X$ is tangent and V normal, we have

$$\begin{aligned}\sum_{i=1}^{n} \ll R^D(E_i, X)E_i^*, V \gg &= -Ric_g(d\phi \circ X, V) = -(Ric_g - \frac{1}{2} S_g g)(d\phi \circ X, V) \\ &= -G(V, d\phi \circ X) = J_U(d\phi \circ X).\end{aligned}$$

Finally, omitting the composition with $d\phi$, we arrive at our result:

$$(\operatorname{div}_h \mathcal{K} - d\operatorname{tr}_h \mathcal{K})(X) = J(X), \tag{18}$$

for any $X \in \mathfrak{X}(S)$. □

2.3 BLACK HOLES

In this section, we will briefly introduce a special class of solutions to the EFE describing *black holes*. We shall refer to these generally as *black hole spacetimes*, although the general mathematical description of black holes is far more complicated, and outside of our scope here. The main goal of our discussion is to give a physical context to the geometric ideas that will be studied later on, such as marginally outer trapped surfaces (which will be discussed in Chapter 4).

In sequence, we will cover two special black hole solutions to the EFE: the Schwarzschild spacetime and the Kruskal spacetime. The Schwarzschild spacetime describes the interior region and the exterior region of a black hole, while Kruskal spacetime allows us to connect these regions of the Schwarzschild in a single spacetime. Through this joining process, we can investigate the formation and properties of the *event horizon*, which will be a natural motivation for the notion of MOTS.

2.3.1 SCHWARZSCHILD SPACETIME

The first black hole solution was discovered by the physicist and astronomer Karl Schwarzschild in late 1915 when he was trying to describe the gravitational field outside a static and perfectly spherical body, such as a star. At first, only half of the solution, i.e. the exterior, seemed to have physical significance. However, further investigations utilized the neglected half, i.e. the interior, with the exterior to elaborate the simplest model of a black hole. The Schwarszchild spacetime emerges naturally when the star is assumed to be static and spherically symmetric. Its mathematical formulation is constructed using warped products, as discussed in Section 1.1.5.

Definition 2.3.1 (Schwarzschild spacetime). *Given a number $M > 0$, let P_I and P_{II} be the regions $r > 2M$ and $0 < r < 2M$ in the tr-half-plane $\mathbb{R} \times \mathbb{R}^+$, each furnished with the line element $-V(r)dt^2 + V(r)^{-1}dr^2$, where $V(r) = 1 - (2M/r)$. If $\mathbb{S}^2$ is the unit sphere, then the warped product $N = P_I \times_r \mathbb{S}^2$ is called* Schwarzschild exterior spacetime *and $B = P_{II} \times_r \mathbb{S}^2$ the* Schwarzschild black hole *(or Schwarzschild interior spacetime), both of* mass M.

Regarding the geometric properties of the warped products, see Figure 3 for a schematically description and notice that

1. For each $(t, r) \in P_I$, the fiber $\pi^{-1}(t, r)$ is isometric to the 2-sphere $\mathbb{S}^2(r)$ of radius $r > 2M$, in the rest-space of Schwarzschild time t. By Corollary 1.1.22, this sphere is totally umbilic in N, and σ maps the fiber homothetically onto $\mathbb{S}^2$.

2. For each $q \in \mathbb{S}^2$, the leaf $\sigma^{-1}(q) = P_I \times q$ is, by Corollary 1.1.22, totally geodesic in N. Moreover, the projection π an is isometry from the leaf to the Schwarzschild half-plane P_I.
3. The time-orientation on the spacetimes is defined as follows. Observe that lift ∂_t to N of the coordinate vector field $\partial/\partial t$ is timelike; we thus define it to be future-directed on N. But on B it is the lift ∂_r of $\partial/\partial r$ that becomes timelike. We then define $-\partial_r$ to be future-directed (The reason for the minus sign in the latter case will be clear later on).

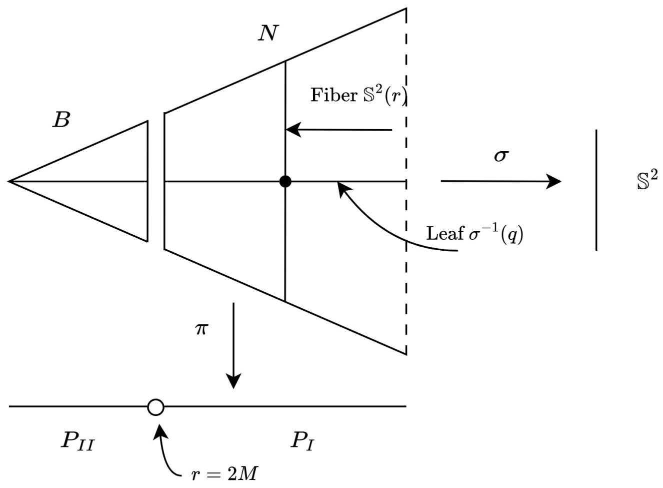


Figure 3 – Schwarzschild warped product structure (schematic).

These geometric remarks above continue to hold for the black hole B although some physical interpretations will change.

It is worth noting that the metric provides a reasonable approximation for an isolated spherical body. For instance, as r grows large the metric converges to the flat metric, which is expected for a gravitationally isolated massive object.

The function $V(r)$ in the Schwarzschild metric has two different singularities if one attempts to define it everywhere, at $r = 2M$ and $r = 0$, and these points carry a physical meaning. The singularity $r = 2M$ represents a null surface (which will be introduced in the following section) called the event horizon of the black hole. But, at $r = 0$, there is a true singularity since the curvature of spacetime becomes infinite. The region B within the event horizon is one for which not even light can escape because of the strength of the gravitational field.

2.3.2 KRUSKAL SPACETIME

The Schwarzschild spacetime consists of two distinct versions, namely N and B. On the other hand, the *Kruskal spacetime* K is an alternative version which joins these components and yields a connected Lorentzian manifold.

Fix a number $M > 0$ and consider the smooth function

$$f : r \in (0, +\infty) \mapsto (r - 2M)e^{(r/2M)-1} \in \mathbb{R}.$$

Since $f' > 0$ on $\mathbb{R}$, f is a diffeomorphism onto its image $(-2M/e, +\infty)$. Now, define the following set

$$Q := \{(u, v) \in \mathbb{R}^2 \mid \quad uv > -2M/e\},$$

in other words, the region in the uv-plane given by $uv > -2M/e$ (See Figure 4). If f^{-1} is the inverse of f, then define the smooth positive function

$$r : (u, v) \in Q \mapsto f^{-1}(uv) \in (0, +\infty),$$

which is characterized implicitly by the equation $f(r) = uv$. The rectangular hyperbolas $uv =$ constant give us the regular level curves of r in Q, except when the constant $= 0$, i.e., r equals $2M$, in which case the coordinate axes are described. The value of function $r(u, v)$ approaches 0 at the boundary of the rectangular hyperbola $uv = -2M/e$, which lies outside of Q.

Define the metric

$$g_Q = F(u, v)(du \otimes dv + dv \otimes du), \quad \text{where } F(u, v) = (8M^2/r(u, v))e^{1-(r(u,v)/2M)}$$

on Q. The region Q in the uv-plane with this line element is called the *Kruskal plane* of mass M.

Let $Q_I, \dots, Q_{IV}$ denote the four *open quadrants* obtained by removing the coordinate axes (see Figure 4)S. It can be shown that the even quadrants, as well as the odds quadrants, are isometric. Furthermore, according to Proposition 13.24 (O'NEILL, 1983), $Q_I \cup Q_{II}$ is isometric to the Schwarzschild strips $P_I \cup P_{II}$. As a result, the essential problem of joining N and B is solved, where they fit together naturally in Q along the positive v axis. In light of this, we can provide the following definition of the Kruskal spacetime as an extension of the Schwarzschild spacetime.

Definition 2.3.2 (Kruskal spacetime). *Let Q be a Kruskal plane of mass $M > 0$, and let $\mathbb{S}^2$ be the unit 2-sphere. The* Kruskal spacetime *of mass M is the warped product $K = Q \times_r \mathbb{S}^2$, where r is the function on Q characterized by $f(r) = uv$.*

Bear in mind the following geometric information: In the natural coordinates u and v, a null coordinate system arises on Q, because $g_Q(\partial_u, \partial_u) = g_Q(\partial_v, \partial_v) = 0$, and $g_Q(\partial_u, \partial_v) = F(r) > 0$. From Lemma 13.23 (O'NEILL, 1983), it holds that $\nabla r = (1/4M)(u\partial_u + v\partial_v)$. The Kruskal spacetime is time-orientable, since, for example $\partial_v - \partial_u$ is a timelike vector field. In particular, the null vector fields $-\partial_u$ and ∂_v are future-directed.

Let π and σ be the projections of K onto Q and $\mathbb{S}^2$, respectively. We can observe the following properties:

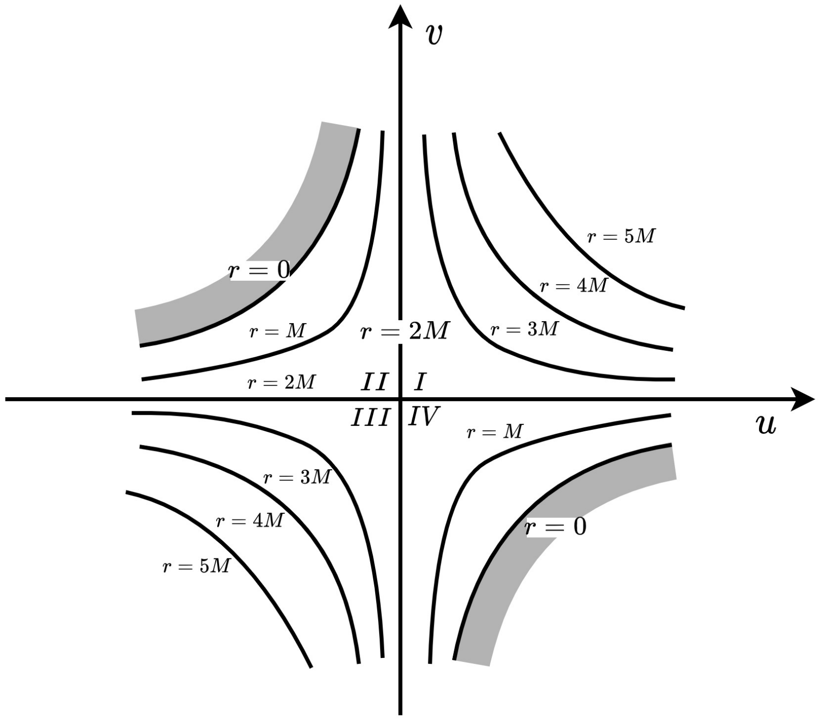


Figure 4 – The Kruskal plane Q.

1. For each $(u, v) \in Q$ the fiber $\pi^{-1}(u, v)$ is the sphere $\mathbb{S}^2(r(u, v))$. According to Corollary 1.1.22, this sphere is totally umbilic in K, and the projection σ maps the fiber homothetically onto $\mathbb{S}^2$.
2. For each $q \in \mathbb{S}^2$ the leaf $\sigma^{-1}(q) = Q \times q$, is totally geodesic in K due to the properties of warped products. Moreover, the projection π is an isometry from the leaf to the Kruskal plane Q.

The spacetime K contains a black hole event horizon. For each $n \in N := \{I, II, III, IV\}$, we define the open submanifold $K_n := \pi^{-1}(Q_n)$ over the quadrant Q_n of Q. The *event horizon* $\mathcal{H}$ is defined as

$$\mathcal{H} := K / \bigcup_{n \in N} K_n,$$

which consists of all points over the coordinate axes of Q. The event horizon $\mathcal{H}$ contains four hypersurfaces which are diffeomorphic to $\mathbb{R}^+ \times \mathbb{S}^2$ and these hypersurfaces can be obtained by removing the central sphere $\pi^{-1}(0, 0)$ from $\mathcal{H}$.

As discussed before, the open upper and lower regions of the Kruskal spacetime are isometric. Therefore, it is convenient to focus on only one portion of this spacetime. We refer to this region as the *truncated Kruskal spacetime* K', which is defined as the subset of K where $v > 0$. Correspondingly, if Q' denotes the region of Q such that $v > 0$, then we have

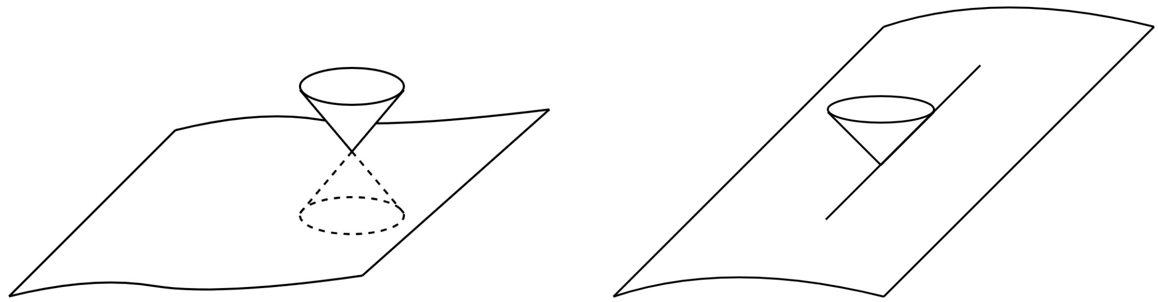
$$K' = \pi^{-1}(Q') = Q' \times_r \mathbb{S}^2.$$

The spacetime K' is composed of the Schwarzschild exterior $K_I \approx N$ $(r > 2M)$ and the black hole $K_{II} \approx B$ $(r < 2M)$ connected along the horizon $\mathcal{H}' = \mathcal{H} \cap K'$ $(r = 2M)$. In matter of fact, according to Proposition 13.30 (O'NEILL, 1983), any geodesic within the black hole region $K_{II} \approx B$ moves inward and eventually ending (on a finite parameter interval) at the central singularity $r = 0$, if not before. Therefore, we have causal geodesic incompleteness in this connected spacetime, and any geodesic crossing the event horizon $\mathcal{H}$ will inevitably encounter a singularity.

3 NULL HYPERSURFACES

In this section we will be interested in a special class of hypersurfaces in spacetime, the so-called *null* (or *lightlike*) hypersurfaces, which are very important in applications. The study of null hypersurfaces is a specialized topic, for more details, see (GALLOWAY, 2000).

A null hypersurface is a hypersurface where the null cones are tangent to it at every point (see Figure 5). These surfaces play a significant role in general relativity, particularly in representing event horizons. For example, in Schwarzschild and Kerr[1] spacetimes, which are the most important solutions to the EFE describing black holes, the associated event horizons are null hypersurfaces.



(a) Spacelike hypersurface (b) Null hypersurface

Figure 5 – Example of causal submanifolds.

Since a null hypersurface can represent event horizons, the main goal of this section is to introduce the geometry of null hypersurfaces in a general setting by utilizing null immersions. The theory will be developed to derive a generalized version of the Riccati equation that establishes relationships between some geometric operators. This equation assumes a special form along null geodesics, enabling us to prove a version of Hawking's *black hole area theorem*. Finally, the geometric interpretation of these operators will be provided.

3.1 CODIMENSION ONE NULL IMMERSIONS

Definition 3.1.1 (Null immersion and null embedding). *Let (M^n, g) be a Lorentzian manifold. For any integer $0 < k < n$, a map $\phi : \Sigma^k \to M^n$ is a* null immersion *[resp.* null embedding*] if it satisfies the following properties.*

1. *ϕ is an immersion [resp.* embedding*];*
2. *For all $p \in \Sigma^k$, $d\phi_p(T_p\Sigma)$ is a degenerate subspace of $(T_{\phi(p)}M, g_{\phi(p)})$.*

In other words, the pullback $\phi^* g$ is degenerate everywhere. If a null immersion $\phi : \Sigma \to M$ is one-to-one, then the immersed submanifold $\widetilde{\Sigma} := \phi(\Sigma)$ is called a null immersed submanifold*. An embedded submanifold $\Sigma \subset M$ is a* null submanifold *if the inclusion map $i : \Sigma \hookrightarrow M$ is a null embedding. Finally, a* null hypersurface *is a null submanifold of codimension one.*

[1] In the case of Kerr spacetimes, which are not presented in this work, they represent rotating black holes, a generalization of the non-rotating Schwarzschild black holes.

Example 12 (Null hypersurface in Kruskal spacetime). *Let $K' = Q' \times_r \mathbb{S}^2$ be the truncated Kruskal spacetime, i.e., the region $v > 0$. Let $\mathcal{H}' = \mathcal{H} \cap K'$ $(r = 2M)$ be the black hole horizon which is diffeomorphic to $\mathbb{R}^+ \times \mathbb{S}^2$. By the geometric properties of the Kruskal spacetime (refer to the discussion after Definition 2.3.2), we have $\nabla r = (1/4M)(u\partial_u + v\partial_v)$. However, on $\mathcal{H}'$, we have $u = 0$ and $g_Q(\partial_v, \partial_v) = 0$, so the vector field $\nabla r = (1/4M)(v\partial_v)$ is null on $\mathcal{H}'$. Therefore, each subspace is a degenerate subspace in the induced metric, i.e., the null cone is tangent to $\mathcal{H}'$ at each point. As a result, $\mathcal{H}'$ is a null hypersurface.*

Let $\phi : \Sigma^k \to M^n$ be a null immersion. For each $p \in \Sigma$, the bilinear product $g_{\phi(p)}(\cdot, \cdot)$ is degenerate on the subspace $d\phi_p(T_p\Sigma)$ of $T_{\phi(p)}M$, i.e., there exists a nonzero vector $\widetilde{K_p} \in \widetilde{T_p\Sigma} := d\phi_p(T_p\Sigma) \subset T_{\phi(p)}M$ such that

$$g_{\phi(p)}(\widetilde{K_p}, X) = 0, \quad \forall X \in \widetilde{T_p\Sigma}.$$

i.e. $\widetilde{K_p} \in \widetilde{T_p\Sigma} \cap \widetilde{T_p\Sigma}^{\perp}$. Clearly, $\widetilde{K_p}$ is a null vector; moreover, since (M, g) is a Lorentzian manifold, any null vector in T_pM is orthogonal to $\widetilde{K_p}$ if and only if it is parallel to the latter, then $\dim(\widetilde{T_p\Sigma} \cap \widetilde{T_p\Sigma}^{\perp}) = 1$ (see Proposition 1.1.25). In particular, if ϕ has codimension one, then $[\widetilde{K_p}]^{\perp} = \widetilde{T_p\Sigma}$ and $\widetilde{K_p}$ can be chosen future-directed if (M, g) is time-oriented. From now on, our focus will be on the study of null immersions of codimension one, which includes in particular (embedded) null hypersurfaces.

Recall that any manifold admits a Riemannian metric. This fact has a nice consequence for null immersions.

Proposition 3.1.2 . *Let (M^{n+1}, g) be a time-oriented Lorentzian manifold, fix a background Riemannian metric h_0 on M, and let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. Then, there exists a unique $K \in \mathfrak{X}(\Sigma)$ such that, for all $p \in \Sigma$, the vector $d\phi_p(K_p)$ is null, future-directed and $h_0(d\phi_p K_p, d\phi_p K_p) = 1$.*

Proof. Fixing the point $p \in \Sigma$, since $\dim(\widetilde{T_p\Sigma} \cap \widetilde{T_p\Sigma}^{\perp}) = 1$, there is a unique future-directed null vector $\widetilde{K_p} \in \widetilde{T_p\Sigma} \cap \widetilde{T_p\Sigma}^{\perp}$ such that $h_0(\widetilde{K_p}, \widetilde{K_p}) = 1$. Since ϕ is an immersion, $d\phi_p$ is one-to-one, there exists a unique $K_p \in T_p\Sigma$ such that $d\phi_p(K_p) = \widetilde{K_p}$. Therefore, the map $K : p \in \Sigma \mapsto K_p \in T_p\Sigma$ defines a vector field on Σ, and we need to demonstrate that K is smooth.

Let $p \in \Sigma$. Since ϕ is an immersion, there exists an open set U containing p such that $\phi\big|_U : U \subset \Sigma \to M$ is an smooth embedding. By (LEE, J. M., 2012, Proposition 5.16), there exists a neighborhood $V \subset M$ of $\phi(p)$ such that $V \cap \phi(U)$ is a level set of a smooth function $f : V \subset M \to \mathbb{R}$. By (LEE, J. M., 2012, Proposition 5.38), we have that the tangent space of $V \cap \phi(U)$ is characterized by the kernel of df. Let $W \subset \Sigma$ be an open set such that $\phi(W) \subset V$. This implies that $df_{\phi(q)}(d\phi_q v) = \ll \nabla f_{\phi(q)}, d\phi_q v \gg = 0$, for all $q \in W$ and for all $v \in T_q\Sigma$. Thus, $\nabla f_{\phi(q)}$ is a normal vector.

For $q \in W$, define $Z_{\phi(q)} = \nabla f_{\phi(q)} / \sqrt{h_0(\nabla f_{\phi(q)}, \nabla f_{\phi(q)})}$. As a result, there exists $a(q) \in \mathbb{R}$ such that $Z_{\phi(q)} = a(q)\widetilde{K_q}$, since the codimension is one, it holds that $\widetilde{T_q\Sigma}^{\perp} = \mathbb{R}\widetilde{K_q}$.

Assuming that $Z_{\phi(q)}$ is future-directed together with $h_0(Z_{\phi(q)}, Z_{\phi(q)}) = 1$, we obtain, necessarily, that $Z_{\phi(q)} \equiv \widetilde{K}_q$. Therefore, we obtained that $Z \circ \phi\big|_W$ is a normal smooth vector field and $Z \circ \phi\big|_W(q) = \widetilde{K}_q$, for any $q \in W$. Hence, the vector field K is smooth. □

Although we used a background Riemannian metric to fix each K_p in the previous proof, one can easily see that for the condition that $\widetilde{K}$ is future-directed and null fixes K up to rescaling by a positive function $f \in C^\infty(\Sigma)$.

Definition 3.1.3 (Null section). *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion, we say that an everywhere-nonzero vector field $K \in \mathfrak{X}(\Sigma)$ is a* null section *for ϕ if, for all $p \in \Sigma$, $d\phi_p(K_p)$ is future-directed null.*

Since a global null section $K \in \mathfrak{X}(\Sigma)$ for a codimension one null immersion $\phi : \Sigma \to M$ is unique up to a positive rescaling, it defines a unique global foliation of Σ by immersed submanifolds of dimension one. Then, given any integral curve $\alpha : I \to \Sigma$ of a null section K for ϕ, the curve $\phi \circ \alpha$ is actually a null pregeodesic in (M,g). This property will be fundamental in further investigations. This fact follows from the results that will be presented next.

Proposition 3.1.4 . *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. Given any null section $K \in \mathfrak{X}(\Sigma)$ for ϕ, then*

$$D_K d\phi \circ K = f_K d\phi \circ K, \tag{19}$$

for some $f_K \in C^\infty(\Sigma)$.

Proof. Let $X \in \mathfrak{X}(\Sigma)$. The orthogonality between $D_K d\phi \circ K$ and the tangent space of Σ comes from

$$\begin{aligned} \ll D_K d\phi \circ K, d\phi \circ X \gg &= - \ll d\phi \circ K, D_K d\phi \circ X \gg \\ &= - \ll d\phi \circ K, D_X d\phi \circ K + d\phi \circ [K, X] \gg \\ &= -\frac{1}{2} X \ll d\phi \circ K, d\phi \circ K \gg = 0. \end{aligned}$$

where we employed Proposition 1.1.2 in the second line. Recall that in codimension one we have $\widetilde{T_p\Sigma}^\perp = \mathbb{R}\widetilde{K_p}$ for each $p \in \Sigma$, which implies that $D_K d\phi \circ K = f_K d\phi \circ K$ for some $f_K \in C^\infty(\Sigma)$. □

Corollary 3.1.5 . *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion, and let $\alpha : I \subseteq \mathbb{R} \to \Sigma$ denote an integral curve of a null section $K \in \mathfrak{X}(\Sigma)$, i.e., $\alpha' = K \circ \alpha$. Then*

$$(\phi \circ \alpha)'' = \frac{D}{dt}(\phi \circ \alpha)' = (f_K \circ \alpha)(\phi \circ \alpha)',$$

where $f_K \in C^\infty(\Sigma)$, so $\phi \circ \alpha$ is a null pregeodesic on (M, g). Therefore, given any maximal integral curve α of K, the curve $\phi \circ \alpha$ can be reparametrized as a null geodesic, called a null (geodesic) generator of ϕ

Proof. Let t_0 be a point in I and fix $p = \alpha(t_0)$. Since $d\phi \circ K \in \mathfrak{X}(\phi)$, by Lemma 1.1.11, there is a vector field $V \in \mathfrak{X}(M)$ and an open set $\mathcal{U} \subset \Sigma$ containing p such that $V \circ \phi\big|_{\mathcal{U}} = d\phi \circ K\big|_{\mathcal{U}}$.

Notice that, for t close enough to t_0, we have

$$(\phi \circ \alpha)'(t) = d\phi_{\alpha(t)} \circ \alpha'(t) = d\phi_{\alpha(t)} \circ K_{\alpha(t)} = V \circ (\phi \circ \alpha)(t),$$

thus, $(\phi \circ \alpha)'(t) = V \circ (\phi \circ \alpha)(t)$ for all $t \in U \subset I$ where U is an open set containing t_0 and such that $\alpha(U) \subset \mathcal{U}$. Let D/dt be the covariant derivative on the curve $\phi \circ \alpha$ and t any point in $U \subset I$. Then

$$\begin{aligned}(\phi \circ \alpha)''(t) &= \frac{D}{dt}(\phi \circ \alpha)'(t) = \frac{D}{dt}V(\phi \circ \alpha)(t) \\ &= \left(\nabla^M_{(\phi\circ\alpha)'(t)}V\right)(\phi \circ \alpha(t)) = (D_K V \circ \phi)(\alpha(t)) \\ &= (D_K d\phi \circ K)(\alpha(t)) = ((f_K \circ \alpha)(\phi \circ \alpha)')(t),\end{aligned}$$

as a consequence of Proposition 3.1.4. Then $(\phi \circ \alpha)'' = (f \circ \alpha)(\phi \circ \alpha)'$ and, consequently, by (O'NEILL, 1983, Exercise 3.19), $\phi \circ \alpha$ is a null pregeodesic on (M, g). □

The existence of null sections $K \in \mathfrak{X}(\Sigma)$ provides means to define more sophisticated geometric structures in codimension one, even in the presence of the degenerate induced metric. For the sake of clarity and consistency, we will adopt the following notation throughout our discussion: Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion, K be any null section for ϕ and f_K denote the smooth function in Proposition 3.1.4. Define the map $\widetilde{\mathcal{S}} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ as

$$\widetilde{\mathcal{S}}(X) := d\phi^{-1}(D_X d\phi \circ K) \quad \forall X \in \mathfrak{X}(\Sigma).$$

The operator $\widetilde{\mathcal{S}}$ is well-defined because ϕ is an immersion and $D_X(d\phi \circ K)$ is a tangent vector field in the codimension one case, as demonstrated in the proof of Proposition 3.1.2. Furthermore, $\widetilde{\mathcal{S}}$ is $C^\infty(\Sigma)$-linear, a fact that can be easily established using the properties of the covariant derivative and the linearity of $d\phi$.

Since ϕ induces a degenerate metric, there is no induced Levi-Civita connection on Σ associated with this metric. Nevertheless, the following notion of derivation can be established: define the map $\widetilde{\nabla}^K : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ as

$$\widetilde{\nabla}^K(X) := d\phi^{-1}(D_K d\phi \circ X), \quad \forall X \in \mathfrak{X}(\Sigma),$$

which is well-defined since $(D_K d\phi \circ X)(p)$ is tangent for all $p \in \Sigma$. This is a consequence of $d\phi \circ K$ being orthogonal to $d\phi \circ X$ and

$$\ll D_K d\phi \circ X, d\phi \circ K \gg_p = K_p \ll d\phi \circ X, d\phi \circ K \gg = 0.$$

Moreover, the following property holds

$$\widetilde{\nabla}^K(fX) = KfX + f\widetilde{\nabla}^K(X), \quad \forall X \in \mathfrak{X}(\Sigma), \forall f \in C^\infty(\Sigma).$$

Thus, $\widetilde{\nabla}^K$ is well-defined and satisfies Leibniz's rule for smooth functions on Σ. These two maps, $\widetilde{\mathcal{S}}$ and $\widetilde{\nabla}^K$, are related by a Lie bracket. For any smooth vector field X on Σ, we can compute their Lie bracket as follows:

$$\widetilde{\mathcal{S}}(X) - \widetilde{\nabla}^K(X) = d\phi^{-1}(D_X d\phi \circ K - D_K d\phi \circ X) = [X, K]. \tag{20}$$

As $\widetilde{\nabla}^K$ is a derivation, it induces a tensor derivation of $\widetilde{\mathcal{S}}$ which is naturally defined by

$$(\widetilde{\nabla}^K \widetilde{\mathcal{S}})(X) := \widetilde{\nabla}^K(\widetilde{\mathcal{S}}(X)) - \widetilde{\mathcal{S}}(\widetilde{\nabla}^K X), \quad \forall X \in \mathfrak{X}(\Sigma).$$

Finally, we can establish that these maps satisfy an important type of Riccati equation which is described below.

Proposition 3.1.6 . *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. Given any null section $K \in \mathfrak{X}(\Sigma)$ for ϕ and $f_K \in C^\infty(\Sigma)$, as defined in Proposition 3.1.4, the following Riccati equation holds:*

$$\widetilde{\nabla}^K \widetilde{\mathcal{S}} + \widetilde{\mathcal{S}}^2 + \widetilde{R}_K = df_K \otimes K + f_K \widetilde{\mathcal{S}}, \tag{21}$$

where the map $\widetilde{R}_K : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is called curvature endomorphism and it is defined by $\widetilde{R}_K(X) := d\phi^{-1}(R(X, K)d\phi \circ K)$, where R is the curvature tensor of the induced connection on the map ϕ.

Proof. Let X be a smooth vector field on Σ, then

$$\begin{aligned}
(\widetilde{\nabla}^K \widetilde{\mathcal{S}})(X) &= \widetilde{\nabla}^K(\widetilde{\mathcal{S}}(X)) - \widetilde{\mathcal{S}}(\widetilde{\nabla}^K X) \\
&= d\phi^{-1}(D_K d\phi \circ \widetilde{\mathcal{S}}(X) - D_{\widetilde{\nabla}^K X} d\phi \circ K) \\
&= d\phi^{-1}(D_K D_X d\phi \circ K - D_{\widetilde{\mathcal{S}}(X) - [X,K]} d\phi \circ K) \\
&= d\phi^{-1}(D_K D_X d\phi \circ K + D_{[X,K]} d\phi \circ K) - \widetilde{\mathcal{S}}^2(X),
\end{aligned}$$

and some of these quantities are related by the curvature tensor,

$$R(X, K)d\phi \circ K = D_X D_K d\phi \circ K - D_K D_X d\phi \circ K - D_{[X,K]} d\phi \circ K,$$

which is orthogonal to $d\phi \circ K$ by the symmetries of the curvature (see Theorem 1.1.8), and thus a tangent vector field. Together with Proposition 3.1.4, one computes

$$\begin{aligned}
(\widetilde{\nabla}^K \widetilde{\mathcal{S}})(X) &= d\phi^{-1}(D_X D_K d\phi \circ K - R(X, K)d\phi \circ K) - \widetilde{\mathcal{S}}^2(X) \\
&= d\phi^{-1}(D_X f_K d\phi \circ K - R(X, K)d\phi \circ K) - \widetilde{\mathcal{S}}^2(X) \\
&= d\phi^{-1}(X f_K \cdot d\phi \circ K + f_K D_X d\phi \circ K - R(X, K)d\phi \circ K) - \widetilde{\mathcal{S}}^2(X) \\
&= X f_K \cdot K + f_K \widetilde{\mathcal{S}}(X) - \widetilde{R}_K(X) - \widetilde{\mathcal{S}}^2(X),
\end{aligned}$$

which directly implies Equation (21). □

Let (M^{n+1}, g) be a spacetime and let $\phi : \Sigma^n \to M^{n+1}$ be a fixed codimension one null immersion with associated null section $K \in \mathfrak{X}(\Sigma)$.

We introduce the following equivalence relation on tangent vectors: for $x, y \in T_p\Sigma$,

$$x \sim y \iff x - y = \lambda K_p \text{ for some } \lambda \in \mathbb{R}.$$

The set of all equivalence classes induced by $\sim$ is denoted by $T_p\Sigma/K$, and $\overline{x}$ denotes the equivalence class of the tangent vector x. Similarly, we define the induced tangent bundle as

$$T\Sigma/K = \bigcup_{p\in\Sigma} T_p\Sigma/K,$$

which is a smooth rank $n-1$ vector bundle over Σ called the *screen bundle* over Σ induced by ϕ. This vector bundle does not depend on the particular choice of the null section K and we denote by $\overline{\mathfrak{X}}(\Sigma)$ the $C^\infty(\Sigma)$-module of smooth sections of the screen bundle and its elements by $\overline{X}$.

There is a positive definite fiber metric h on $T\Sigma/K$ induced from the metric g. For each point $p \in \Sigma$, we define the bilinear form $h_p : T_p\Sigma/K \times T_p\Sigma/K \longrightarrow \mathbb{R}$ by

$$h_p(\overline{X_p}, \overline{Y_p}) := (\phi^* g)_p(X_p, Y_p) = g_{\phi(p)}(d\phi_p \circ X_p, d\phi_p \circ Y_p) = \ll d\phi \circ X, d\phi \circ Y \gg_p, \quad (22)$$

where $\overline{X}, \overline{Y} \in \overline{\mathfrak{X}}(\Sigma)$. This metric is well-defined: Let $X, X' \in \overline{X}$, $Y, Y' \in \overline{Y}$ and $\lambda_X, \lambda_X \in C^\infty(\Sigma)$ such that $X' = X + \lambda_X K$ and $Y' = Y + \lambda_X K$ then,

$$\begin{aligned}(\phi^* g)_p(X', Y') &= (\phi^* g)_p(X + \lambda_X K, Y + \lambda_X K)\\ &= (\phi^* g)_p(X, Y).\end{aligned}$$

There are two remarkable maps induced by $\widetilde{\mathcal{S}}$ and $\widetilde{\nabla}^K$ over $\overline{\mathfrak{X}}(\Sigma)$. We define the *null Weingarten map* of Σ associated with K by $\mathcal{S} : \overline{\mathfrak{X}}(\Sigma) \to \overline{\mathfrak{X}}(\Sigma)$ such that

$$\mathcal{S}(\overline{X}) := \overline{\widetilde{\mathcal{S}}(X)}, \quad (23)$$

and the map $\nabla^K : \overline{\mathfrak{X}}(\Sigma) \to \overline{\mathfrak{X}}(\Sigma)$ defined by

$$\nabla^K(\overline{X}) := \overline{\widetilde{\nabla}^K(X)}.$$

We need to verify that these operations are well-defined. Let $\overline{X} \in \overline{\mathfrak{X}}(\Sigma)$ and let X and $X' \in \overline{X}$ be two equivalent vector fields, i.e., $X - X' = \lambda_X K$ for some smooth function $\lambda_X \in C^\infty(\Sigma)$. Applying Proposition 3.1.4, we can show that $\mathcal{S}$ is well-defined:

$$\begin{aligned}\widetilde{\mathcal{S}}(X) &= d\phi^{-1}(D_X d\phi \circ K)\\ &= d\phi^{-1}(D_{X' + \lambda_X K} d\phi \circ K)\\ &= d\phi^{-1}(D_{X'} d\phi \circ K) + \lambda_X d\phi^{-1}(D_K d\phi \circ K)\\ &= \widetilde{\mathcal{S}}(X') + \lambda_X f_K K,\end{aligned}$$

thus $\widetilde{\mathcal{S}}(X)$ and $\widetilde{\mathcal{S}}(X')$ lie in $\overline{\widetilde{\mathcal{S}}(X)}$. For ∇^K the process is similar,

$$\begin{aligned}\nabla^K(X) &= \nabla^K(X' + \lambda_X K)\\ &= \nabla^K(X') + (K\lambda_X)K + \lambda_X \nabla^K K\\ &= \nabla^K(X') + (K\lambda_X + \lambda_X f_K)K,\end{aligned}$$

where $f_K \in C^\infty(\Sigma)$ is given by Proposition 3.1.4 and Leibnitz's rule was applied. Additionally, ∇^K induces a tensor derivation which is defined as

$$(\nabla^K \mathcal{S})(\overline{X}) := \nabla^K(\mathcal{S}(\overline{X})) - \mathcal{S}(\nabla^K(\overline{X})). \tag{24}$$

As $d\phi \circ K$ is null, we can show that $\mathcal{S}$ is self-adjoint with respect to h and ∇^K is compatible with h.

Proposition 3.1.7 . *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. Let $K \in \mathfrak{X}(\Sigma)$ be any null section for ϕ and let h be the induced metric on the vector bundle $T\Sigma/K$ by the metric g. Then the null Weingarten map $\mathcal{S} : \overline{\mathfrak{X}}(\Sigma) \to \overline{\mathfrak{X}}(\Sigma)$ is self-adjoint with respect to h, i.e.,*

$$h(\mathcal{S}(\overline{X}), \overline{Y}) = h(\overline{X}, \mathcal{S}(\overline{Y})), \quad \forall \overline{X}, \overline{Y} \in \overline{\mathfrak{X}}(\Sigma),$$

and the map $\nabla^K : \overline{\mathfrak{X}}(\Sigma) \to \overline{\mathfrak{X}}(\Sigma)$ is compatible with the metric h, i.e.,

$$h(\nabla^K(\overline{X}), \overline{Y}) + h(\overline{X}, \nabla^K(\overline{Y})) = Kh(\overline{X}, \overline{Y}), \quad \forall \overline{X}, \overline{Y} \in \overline{\mathfrak{X}}(\Sigma).$$

Proof. To show that $\mathcal{S}$ is self-adjoint, it is sufficient to follow these steps:

$$\begin{aligned} h(\mathcal{S}(\overline{X}), \overline{Y}) &= g(d\phi \circ \widetilde{\mathcal{S}}(X), d\phi \circ Y) = g(D_X d\phi \circ K, d\phi \circ Y) \\ &= -g(d\phi \circ K, D_X d\phi \circ Y) = -g(d\phi \circ K, D_Y d\phi \circ X + d\phi \circ [X, Y]) \\ &= g(D_Y d\phi \circ K, d\phi \circ X) = h(\overline{X}, \mathcal{S}(\overline{Y})), \end{aligned}$$

and, in order to establish the compatibility, notice that

$$\begin{aligned} Kh(\overline{X}, \overline{Y}) &= Kg(d\phi \circ X, d\phi \circ Y) \\ &= g(D_K d\phi \circ X, d\phi \circ Y) + g(d\phi \circ X, D_K d\phi \circ Y) \\ &= h(\nabla^K(\overline{X}), \overline{Y}) + h(\overline{X}, \nabla^K(\overline{Y})). \end{aligned} \tag{25}$$

□

Proposition 3.1.8 . *Let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. Given any null section $K \in \mathfrak{X}(\Sigma)$ for ϕ and $f_K \in C^\infty(\Sigma)$, as defined in Proposition 3.1.4, then the following equation holds*

$$\nabla^K \mathcal{S} + \mathcal{S}^2 + R_K = f_K \mathcal{S}, \tag{26}$$

where the map $R_K : \overline{\mathfrak{X}}(\Sigma) \to \overline{\mathfrak{X}}(\Sigma)$ is called the curvature endomorphism and it is defined by $R_K(\overline{X}) := \overline{\widetilde{R}_K(X)} = \overline{d\phi^{-1}(R(X, K)d\phi \circ K)}$. Tracing the above expression with respect to the metric h, we obtain

$$K\theta + \sigma^2 + \frac{1}{n-1}\theta^2 + Ric_g(d\phi \circ K, d\phi \circ K) \circ \phi = f_K \theta, \tag{27}$$

where $\theta = \mathrm{tr}_h \mathcal{S}$ is the null mean curvature *(or* null expansion*), $\sigma := \sqrt{(\mathrm{tr}_h \mathring{\mathcal{S}})^2}$ is the* shear scalar *and $\mathring{\mathcal{S}} := \mathcal{S} - \frac{1}{n-1}\theta \cdot \mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)}$ is the trace free part of the Weingarten map $\mathcal{S}$.*

Proof. In order to show Equation (26), let any smooth vector field $\overline{X} \in \overline{\mathfrak{X}}(\Sigma)$. By Equations (21) and (24), we obtain

$$\begin{aligned}
(\nabla^K \mathcal{S})(\overline{X}) &= \nabla^K(\mathcal{S}(\overline{X})) - \mathcal{S}(\nabla^K(\overline{X})) \\
&= \nabla^K(\overline{\widetilde{\mathcal{S}}(X)}) - \mathcal{S}(\overline{\widetilde{\nabla}^K(X)}) \\
&= \overline{\widetilde{\nabla}^K(\widetilde{\mathcal{S}}(X))} - \overline{\widetilde{\mathcal{S}}(\widetilde{\nabla}^K(X))} \\
&= \overline{(\widetilde{\nabla}^K\widetilde{\mathcal{S}})(X)} \\
&= \overline{X f_K \cdot K} + \overline{f_K \widetilde{\mathcal{S}}(X)} - \overline{\widetilde{R}_K(X)} - \overline{\widetilde{\mathcal{S}}^2(X)}, \\
&= f_K \mathcal{S}(\overline{X}) - R_K(\overline{X}) - \mathcal{S}^2(\overline{X}),
\end{aligned}$$

where $R_K(\overline{X})$ is the curvature endomorphism and it is well-defined. Let X and $X' \in \overline{X}$ be smooth vector fields on Σ, i.e., $X - X' = \lambda K$ for some $\lambda \in C^\infty(\Sigma)$. Thus

$$\begin{aligned}
d\phi \circ \widetilde{R}_K(X) &= d\phi \circ \widetilde{R}_K(X' + \lambda K) \\
&= R(X' + \lambda K, K) d\phi \circ K = R(X', K) d\phi \circ K \\
&= d\phi \circ \widetilde{R}_K(X'),
\end{aligned} \tag{28}$$

where the curvature antisymmetry on the first two entries was applied in the second line. Rearranging the expression obtained above, we arrive at

$$(\nabla^K \mathcal{S})(\overline{X}) + \mathcal{S}^2(\overline{X}) + R_K(\overline{X}) = f_K \mathcal{S}(\overline{X}), \quad \forall \overline{X} \in \overline{\mathfrak{X}}(\Sigma),$$

showing that the first equation holds. The second equation, which involves the trace, is more involved. Let $\{\overline{E}_1, \ldots, \overline{E}_{n-1}\}$ be an orthonormal frame in the induced metric h. Then

$$\theta = \sum_{i=1}^{n-1} h(\overline{E}_i, \mathcal{S}(\overline{E}_i)).$$

As a starting point, the contribution of $\nabla^K \mathcal{S}$ will be determined. Observe that

$$\begin{aligned}
(\nabla^K \mathcal{S})(\overline{X}) &= \nabla^K(\mathcal{S}(\overline{X})) - \mathcal{S}(\nabla^K(\overline{X})) \\
&= \nabla^K((\mathring{\mathcal{S}} + \frac{1}{n-1}\theta \cdot \mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)})(\overline{X})) - (\mathring{\mathcal{S}} + \frac{1}{n-1}\theta \cdot \mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)})(\nabla^K(\overline{X})) \\
&= \nabla^K(\mathring{\mathcal{S}}(\overline{X})) - \mathring{\mathcal{S}}(\nabla^K(\overline{X})) + \frac{1}{n-1}(\nabla^K(\theta \cdot \overline{X}) - \theta \cdot \nabla^K(\overline{X})) \\
&= (\nabla^K \mathring{\mathcal{S}})(\overline{X}) + \frac{1}{n-1} K\theta \cdot \overline{X},
\end{aligned}$$

then $(\nabla^K \mathcal{S}) = (\nabla^K \mathring{\mathcal{S}}) + \frac{1}{n-1} K\theta \cdot \mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)}$. Taking the trace of $\nabla^K \mathring{\mathcal{S}}$ we have

$$\begin{aligned}
\operatorname{tr}_h(\nabla^K \mathring{\mathcal{S}}) &= \sum_{i=1}^{n-1} h(\overline{E}_i, (\nabla^K \mathring{\mathcal{S}})(\overline{E}_i)) \\
&= \sum_{i=1}^{n-1} h(\overline{E}_i, \nabla^K (\mathring{\mathcal{S}}(\overline{E}_i))) - h(\overline{E}_i, \mathring{\mathcal{S}}(\nabla^K (\overline{E}_i))) \\
&= \sum_{i=1}^{n-1} K h(\overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)) - h(\nabla^K \overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)) - h(\overline{E}_i, \mathring{\mathcal{S}}(\nabla^K (\overline{E}_i))) \\
&= -2 \sum_{i=1}^{n-1} h(\nabla^K \overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)),
\end{aligned}$$

where we used the fact that $\mathring{\mathcal{S}}$ is trace-free and self-adjoint, notice that it is the sum of two self-adjoint operators. Now, we will show that the last quantity vanishes identically. Let i be an integer, and, for each integer $j \in \{1, \ldots, n-1\}$, denote the smooth functions $a_{ij} \in C^\infty(\Sigma)$ which satisfies $\nabla^K \overline{E}_i = a_{ij} \overline{E}_j$, then

$$h(\nabla^K \overline{E}_i, \overline{E}_j) = h(a_{ik} \overline{E}_k, \overline{E}_j) = a_{ij},$$

and

$$a_{ij} = h(\nabla^K \overline{E}_i, \overline{E}_j) = -h(\overline{E}_i, \nabla^K \overline{E}_j) = -a_{ji},$$

so we have $a_{ij} = -a_{ji}$. Employing that $\mathring{\mathcal{S}}$ is self-adjoint, since is the sum of two self-adjoint operators, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} h(\nabla^K \overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)) &= \sum_{i,j=1}^{n-1} a_{ij} h(E_j, \mathring{\mathcal{S}}(\overline{E}_i)) \\
&= -\sum_{i,j=1}^{n-1} a_{ji} h(\mathring{\mathcal{S}}(\overline{E}_j), \overline{E}_i) \\
&= -\sum_{i=1}^{n-1} h(\nabla^K \overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)),
\end{aligned}$$

consequently,

$$\operatorname{tr}_h(\nabla^K \mathring{\mathcal{S}}) = -2 \sum_{i=1}^{n-1} h(\nabla^K \overline{E}_i, \mathring{\mathcal{S}}(\overline{E}_i)) = 0,$$

and we discover that $(\nabla^K \mathcal{S})$ has the following contribution

$$\operatorname{tr}_h(\nabla^K \mathcal{S}) = K\theta. \tag{29}$$

The trace of $\mathcal{S}^2$ can be obtained straightforwardly by using the trace-free decomposition,

$$\begin{aligned}
\operatorname{tr}_h(\mathcal{S}^2) &= \operatorname{tr}_h\left((\mathring{\mathcal{S}} + \frac{1}{n-1}\theta \cdot \mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)})^2 \right) \\
&= \operatorname{tr}_h(\mathring{\mathcal{S}}^2) + \frac{2}{n-1}\theta \operatorname{tr}_h(\mathring{\mathcal{S}}) + \frac{1}{(n-1)^2}\theta^2 \operatorname{tr}_h(\mathbb{1}_{\overline{\mathfrak{X}}(\Sigma)}) \qquad (30) \\
&= \sigma^2 + \frac{1}{n-1}\theta^2,
\end{aligned}$$

notice that $\mathrm{tr}_h(\mathring{\mathcal{S}})^2$ is always non-negative,

$$\mathrm{tr}_h(\mathring{\mathcal{S}})^2 = h(\overline{E}_i, \mathring{\mathcal{S}}^2(\overline{E}_i)) = h(\mathring{\mathcal{S}}(\overline{E}_i), \mathring{\mathcal{S}}(\overline{E}_i)) \geq 0.$$

In order to analyze the curvature endomorphism, we introduce the notation: Fix $p \in \Sigma$, $\widetilde{K} = d\phi \circ K(p)$ and $e_i = d\phi \circ E_i(p)$ where E_i lies in the equivalence class of $\overline{E}_i$ from the orthonormal frame. Define $\widetilde{N} \in T_{\phi(p)}M/d\phi(T_p\Sigma)$ a lightlike vector such that $g_{\phi(p)}(\widetilde{K}, \widetilde{N}) = -1$ and orthogonal to $\{e_1, \dots, e_{n-1}\}$, finally, introduce the orthonormal vectors

$$e_0 = \frac{\widetilde{K} + \widetilde{N}}{\sqrt{2}}, \qquad e_n = \frac{\widetilde{K} - \widetilde{N}}{\sqrt{2}},$$

as a result, the frame $\{e_0, \dots, e_n\}$ is an g-orthonormal frame in M^{n+1} with

$$g_{\phi(p)}(e_0, e_0) = -1, \qquad g_{\phi(p)}(e_n, e_n) = 1, \qquad g_{\phi(p)}(e_0, e_n) = 0,$$

thus,

$$\begin{aligned} \mathrm{tr}_h\, R_K^{(p)} &= \sum_{i=1}^{n-1} h_p(\overline{E}_i, R_K(\overline{E}_i)) \\ &= \sum_{i=1}^{n-1} g_{\phi(p)}(e_i, R_{\phi(p)}(e_i, \widetilde{K})\widetilde{K}) \\ &= Ric_g^{\phi(p)}(\widetilde{K}, \widetilde{K}) + g_{\phi(p)}(e_0, R_{\phi(p)}(e_0, \widetilde{K})\widetilde{K}) - g_{\phi(p)}(e_n, R_{\phi(p)}(e_n, \widetilde{K})\widetilde{K}), \end{aligned}$$

the difference between the last two term is zero, since

$$\begin{aligned} g_{\phi(p)}(\widetilde{K} \pm \widetilde{N}, R_{\phi(p)}(\widetilde{K} \pm \widetilde{N}, \widetilde{K})\widetilde{K}) &= g_{\phi(p)}(\widetilde{K} \pm \widetilde{N}, R_{\phi(p)}(\pm\widetilde{N}, \widetilde{K})\widetilde{K}) \\ &= g_{\phi(p)}(\pm\widetilde{N}, R_{\phi(p)}(\pm\widetilde{N}, \widetilde{K})\widetilde{K}) \\ &= g_{\phi(p)}(\widetilde{N}, R_{\phi(p)}(\widetilde{N}, \widetilde{K})\widetilde{K}), \end{aligned}$$

where the curvature antisymmetry was used above. As both terms yields the same quantity, their difference is zero as expected and this implies that

$$\mathrm{tr}_h\, R_K = Ric_g(d\phi \circ K, d\phi \circ K) \circ \phi, \tag{31}$$

and with Equations (29) to (31) we have shown that

$$K\theta + \sigma^2 + \frac{1}{n-1}\theta^2 + Ric_g(d\phi \circ K, d\phi \circ K) \circ \phi = f_K\theta.$$

□

Let $\alpha : I \subseteq \mathbb{R} \to \Sigma$ be a integral curve of K, i.e., $\alpha'(t) = K_{\alpha(t)}$ for $t \in I$, and, by Corollary 3.1.5, the smooth curve $(\phi \circ \alpha)$ is a pregeodesic on M. Therefore, there exists a diffeomorphism $h : J \subseteq \mathbb{R} \to I$ such that $\gamma := (\phi \circ \alpha \circ h)$ is a geodesic on M. According to Exercise 3.19 (O'NEILL, 1983), denoting $\beta := (\alpha \circ h)$, the reparametrization h satisfies the subsequent expression

$$h'' + (f_K \circ \beta)(h')^2 = 0. \tag{32}$$

Let s be a point in the open interval J. Multiplying Equation (27) by $h'(s)^2$ and evaluating at the point $\beta(s) \in \Sigma$ we obtain,

$$h'(s)^2[K\theta(\beta(s)) + \sigma^2(\beta(s)) + \frac{\theta^2(\beta(s))}{n-1} + Ric_g(\widetilde{K}, \widetilde{K}) \circ (\phi \circ \beta(s))] = h'(s)^2(f_K\theta)(\beta(s)).$$

First, notice that

$$\begin{aligned} h'(s)(K\theta)(\beta(s)) &= h'(s)(K\theta)(\alpha \circ h(s)) = h'(s)K_{\alpha \circ h(s)}\theta \\ &= h'(s)\alpha'_{h(s)}\theta = (\alpha \circ h)'_s\theta \\ &= \beta'_s\theta = (\theta \circ \beta)'(s). \end{aligned} \tag{33}$$

Second, denoting $\widetilde{K} := d\phi \circ K$, we have

$$\begin{aligned} (\phi \circ \beta)'(s) &= (\phi \circ \alpha \circ h)'(s) = (\phi \circ \alpha)'_{h(s)}h'(s) = (d\phi_{\alpha \circ h(s)} \circ \alpha'_{h(s)})h'(s) \\ &= (d\phi_{\alpha \circ h(s)} \circ K_{\alpha \circ h(s)})h'(s) = (d\phi \circ K)_{\beta(s)}h'(s) \\ &= \widetilde{K}_{\beta(s)}h'(s). \end{aligned} \tag{34}$$

Applying Equations (32) to (34) and rearranging some terms yields the following expression

$$(h'(\theta \circ \beta)')(s) + (\sigma \circ \beta)^2(s) + \frac{(h'\theta \circ \beta)^2(s)}{n-1} + Ric_g((\phi \circ \beta)', (\phi \circ \beta)')(s) = -h''(s)\theta(\beta(s)).$$

Moreover, employing the identity

$$(h'(\theta \circ \beta))' = h''(\theta \circ \beta) + h'(\theta \circ \beta)',$$

and defining the quantities $\hat{\theta} := h'(\theta \circ \beta)$ and $\hat{\sigma}^2 := (h'(\sigma \circ \beta))^2$, we arrive at the following equation

$$\hat{\theta}'(s) + \hat{\sigma}^2(s) + \frac{1}{n-1}\hat{\theta}^2(s) + Ric_g(\gamma'(s), \gamma'(s)) = 0. \tag{35}$$

Equation (35) plays a fundamental role in general relativity, and it is called the *Raychaudhuri equation* for null geodesics. This equation reveals the influence of the Ricci curvature of spacetime into the null mean curvature of a null hypersurface. By employing the Raychaudhuri equation, we are able to demonstrate the following remarkable proposition.

Proposition 3.1.9 (GALLOWAY, 2014). *Let (M^{n+1}, g) be a spacetime which obeys the null energy condition (NEC), $Ric(v, v) \geq 0$ for all null vectors v, and let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion. If the null generators of ϕ in M are future geodesically complete, then ϕ has nonnegative null expansion*[2]*, $\theta \geq 0$.*

Proof. Suppose $\theta < 0$ at $p \in \Sigma$. Let $\alpha : I \subseteq \mathbb{R} \to \Sigma$ be a maximal integral curve of some fixed null section $K \in \mathfrak{X}(\Sigma)$ for ϕ such that $\alpha(0) = p$ and let $h : J \subset \mathbb{R} \to I$ be a diffeomorphism such that $(\phi \circ \beta) = (\phi \circ \alpha \circ h)$ is a future-directed null geodesic in M. Let $\hat{\theta}(s) = (h'(\theta \circ \beta))$ be

[2] Although the expansion θ depends on the choice of some null section K, given any smooth positive function $f > 0$ on Σ, we have $\theta_{fK} = f\theta_K$, so the statement of the nonnegativity of θ - or more generally any sign - is actually independent of the choice of K.

as in Equation (35) and by the invariance of sign under scaling and $h' > 0$, one has $\hat{\theta}(0) < 0$. Raychaudhuri's equation and the NEC imply that $\hat{\theta}(s)$ obeys the inequality,

$$\hat{\theta}'(s) \leq -\frac{1}{n-1}\hat{\theta}^2(s),$$

and hence $\hat{\theta} < 0$ for all $s > 0$. Dividing through by $-\hat{\theta}^2$ then gives

$$\frac{d}{ds}\left(\frac{1}{\hat{\theta}}\right) \geq \frac{1}{n-1},$$

which implies $1/\hat{\theta} \to 0$, i.e., $\hat{\theta} \to -\infty$ in finite affine parameter time, contradicting the smoothness of $\hat{\theta}$. □

Therefore, by applying Proposition 3.1.9 and its assumptions to the case of an embedded null hypersurface, we can conclude that as one moves towards the future, spacelike cross sections of the null hypersurface are nondecreasing in area (see Theorem 1.2.2). This observation will be further clarified in the next subsection. This statement is commonly referred to as the simplest version of Hawking's black hole area theorem (HAWKING; ELLIS, 1973).

3.2 GEOMETRIC INTERPRETATION

Let $\phi : N^k \to M^n$ be a smooth immersion into a semi-Riemannian manifold, not necessarily semi-Riemannian or degenerate everywhere. For each point $p \in N$, we define the quotient subspace $[T_{\phi(p)}M] := T_{\phi(p)}M/d\phi_p(T_pN)$. Observe that this defines a rank $n-k$ vector bundle $\nu(\phi)$ over N whose fibers are these quotients. This is called the *quotient normal bundle* of ϕ. If $v \in T_{\phi(p)}M$, we denote by $[v]$ the equivalence class of vectors in $[T_{\phi(p)}M]$ that contains v. With this, we can introduce a generalized notion of second fundamental $II : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \nu(\phi)$ form as follows: For any $X, Y \in \mathfrak{X}(N)$, we define the operator $II(X,Y) := [D_X d\phi \circ Y]$ and it holds that II is symmetric and $C^\infty(N)$-bilinear. Furthermore, if $II = 0$, then there exists a map $\widetilde{\nabla}^N : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$ on N defined by

$$\widetilde{\nabla}^N_X Y := d\phi^{-1}(D_X d\phi \circ Y), \quad \forall X, Y \in \mathfrak{X}(N).$$

Since $II = 0$, the covariant derivatives are tangent, and thus the map is well-defined. Moreover, one can easily check that $\widetilde{\nabla}^N$ satisfies the properties of a connection, as stated in the beginning of Chapter 1. Therefore, $\widetilde{\nabla}^N$ is, in fact, a *connection* on the smooth manifold N.

Let $\alpha : I \subseteq \mathbb{R} \to N$ be a smooth curve on N and denote by $\widetilde{D}/dt$ the unique connection on α induced by the connection $\widetilde{\nabla}^N$. Fixed $t_0 \in I$, then, by (O'NEILL, 1983, Proposition 3.19), given $\widetilde{v} \in T_{\alpha(t_0)}N$ there exists a unique vector field $\widetilde{V} \in \mathfrak{X}(\alpha)$ such that $\widetilde{V}(t_0) = \widetilde{v}$ and

$$\frac{\widetilde{D}}{dt}\widetilde{V} = 0.$$

Analogously, let $\phi \circ \alpha$ be a smooth curve on M and denote by D/dt the unique connection on $\phi \circ \alpha$ induced by the connection ∇^M. Given $v = d\phi_{\alpha(t_0)}\widetilde{v} \in T_{\phi\circ\alpha(t_0)}M$, there is a unique vector field $V \in \mathfrak{X}(\phi \circ \alpha)$ such that $V_{t_0} = v$ and

$$\frac{D}{dt}V = 0.$$

Notice that $d\phi_{\alpha(t_0)}(\widetilde{V}_{\alpha(t_0)}) = d\phi_{\alpha(t_0)}\widetilde{v} = v = V_{t_0}$. Since V is unique, we have that $d\phi_{\alpha(t)}(\widetilde{V}_{\alpha(t)}) = V_t$ for all $t \in I$. To sum up, in the context of $II = 0$, we conclude that if a tangent vector $v \in TN$ is parallel translated along a curve α, then $d\phi(v)$ is parallel transported along the curve $\phi \circ \alpha$. Consequently, if α is a geodesic on N, then $\phi \circ \alpha$ is a geodesic in M. Working out these results, we have the following equivalent properties

1. $II = 0$;
2. If $v \in TN$ is a tangent vector parallel translated along a curve α, then the vector $d\phi(v)$ is parallel transported along the curve $\phi \circ \alpha$;
3. If α geodesic on N, then, $\phi \circ \alpha$ is a geodesic on M.

We shall say that the immersion ϕ is *totally geodesic* if one - and hence all - of these properties holds. If ϕ is a semi-Riemannian immersion, then this notion of totally geodesic agrees with the usual one and we are allowed to understand II as a second-fundamental form.

Returning to our main context, let $\phi : \Sigma^n \to M^{n+1}$ be a codimension one null immersion in the spacetime (M, g). We can define the operator II similarly, as in the previous paragraph. Since ϕ is a null immersion, there exists a null section $K \in \mathfrak{X}(\Sigma)$ for ϕ and we introduce the tensor $\mathcal{K} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$ such that

$$\mathcal{K}(X,Y) := \ll II(X,Y), -d\phi \circ K \gg = \ll D_X d\phi \circ Y, -d\phi \circ K \gg,$$

where $D_X d\phi \circ Y$ is an element of the class $[D_X d\phi \circ Y]$. This tensor is well-defined: Let $V \in \mathfrak{X}(\phi)$, V is a tangent smooth vector field, then $D_X d\phi \circ Y + V$ is contained in the equivalence class $[D_X d\phi \circ Y]$ and

$$\ll D_X d\phi \circ Y + V, -d\phi \circ K \gg = \ll D_X d\phi \circ Y, -d\phi \circ K \gg ,$$

since $d\phi \circ K$ is orthogonal to any tangent vector field.

Surprisingly, the tensor $\mathcal{K}$ has a clear relation with the Riemannian metric h induced by $K \in \mathfrak{X}(\Sigma)$, which was defined by Equation (22). Let $X, Y \in \mathfrak{X}(\Sigma)$ and let $D_X d\phi \circ Y$ be an element of the equivalence class of $[D_X d\phi \circ Y]$, then we have

$$\begin{aligned}
\mathcal{K}(X,Y) &= \ll II(X,Y), -d\phi \circ K \gg \\
&= \ll D_X d\phi \circ Y, -d\phi \circ K \gg \\
&= \ll d\phi \circ Y, d\phi \circ \widetilde{\mathcal{S}}(X) \gg \\
&= h(\mathcal{S}(\overline{X}), \overline{Y}),
\end{aligned}$$

to sum up, we arrived at

$$\mathcal{K}(X,Y) = \ll II(X,Y), -d\phi \circ K \gg = h(\mathcal{S}(\overline{X}), \overline{Y}) \quad \forall X, Y \in \mathfrak{X}(\Sigma). \tag{36}$$

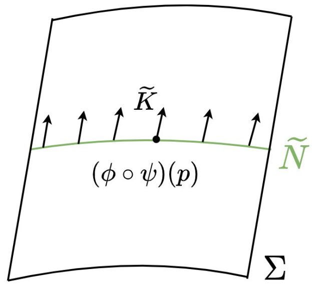


Figure 6 – Intersection between ϕ and ψ in M which is transverse to $\widetilde{K}$ in M.

Therefore, the tensor $\mathcal{K}$ can be interpreted as the *second-fundamental form* associated with the normal vector field $d\phi \circ K$. Consequently, the null Weingarten map $\mathcal{S}$, previously defined in Equation (23), is the Weingarten map associated with the normal vector field $d\phi \circ K$. Observe, in addition, that since the normal bundle of ϕ has rank 1, $\mathcal{K}$ gives all the extrinsic geometric information in the second fundamental form tensor II apart from positive rescaling.

The *null mean curvature* (or *null expansion scalar*) of Σ with respect to K is a smooth function $\theta \in C^\infty(\Sigma)$ and it can be written as

$$\theta = \sum_{i=1}^{n-1} h(\overline{E}_i, \mathcal{S}(\overline{E}_i)) = \operatorname{tr}_h \mathcal{K}.$$

Furthermore, θ has a natural geometric interpretation. Let $\psi : N^{n-1} \to \Sigma^n$ be a codimension one immersion into Σ such that the composition $(\phi \circ \psi)$ is a spacelike immersion, then N is transverse to the null section $d\phi \circ K$ on M. Let $q \in N$ and $\{E_1, \ldots, E_{n-1}\}$ be an orthonormal local frame near q in the induced metric $((\phi \circ \psi)^* g)$. Then $\{\overline{d\psi \circ E_1}, \ldots, \overline{d\psi \circ E_{n-1}}\}$ is a h-orthonormal frame of $T_{\psi(q)}\Sigma/K$. Let $V \in \mathfrak{X}(M)$ be a smooth vector field such that $V \circ \phi = d\phi \circ K$ in an open set near $\psi(q)$. Hence, at $\psi(q)$,

$$\begin{aligned}
\theta = \operatorname{tr}_h \mathcal{K} &= \sum_{i=1}^{n-1} h(\mathcal{S}(\overline{d\psi \circ E_i}), \overline{d\psi \circ E_i}) \\
&= \sum_{i=1}^{n-1} g_{\phi\circ\psi}((D^{\phi}_{d\psi\circ E_i} d\phi \circ K) \circ \psi, d\phi \circ d\psi \circ E_i) \\
&= \sum_{i=1}^{n-1} g_{\phi\circ\psi}((\nabla^M_{d(\phi\circ\psi)\circ E_i} V)(\phi \circ \psi), d(\phi \circ \psi) \circ E_i) \\
&= \sum_{i=1}^{n-1} g_{\phi\circ\psi}(D^{\phi\circ\psi}_{E_i}(d\phi \circ K \circ \psi), d(\phi \circ \psi) \circ E_i) \\
&= \operatorname{div}_{\widetilde{N}} \widetilde{K},
\end{aligned}$$

where $\operatorname{div}_{\widetilde{N}} \widetilde{K}$ is the divergence of $\widetilde{K} := d\phi \circ K$ along $\widetilde{N} := (\phi \circ \psi)(N)$. Thus, θ measures the overall expansion of the null generators of ϕ towards the future (see Figure 6).

If $K_f = fK$, $f \in C^\infty(\Sigma)$, is any other null section for the null immersion, then $II_{K_f} = fII_K$, and hence, $\tilde{\theta} = f\theta$. Therefore, the null mean curvature inequalities $\theta \geq 0$, $\theta \leq 0$, are invariant under positive rescaling of K.

4 MARGINALLY OUTER TRAPPED SURFACES

This chapter introduces the concept of marginally outer trapped surface (MOTS) and explores their relation with the geometry of null hypersurfaces. Furthermore, we present a reformulation of this concept in terms of the initial data formulation of the Einstein field equations. MOTSs naturally arise in black hole spacetime solutions, for instance, in spacetimes where the black hole horizon $\mathcal{H}$ is a *Killing horizon*, that is, it is a null hypersurface and there exists a Killing vector field X which becomes a null section upon restriction to $\mathcal{H}$, then spacelike surfaces on the event horizon $\mathcal{H}$ are MOTSs. Thus, by studying MOTSs, we gain insights into the nature of black hole event horizons. Defining precisely the notion of event horizon $\mathcal{H}$, typically, requires global information of the spacetime. In contrast, MOTSs offer an alternative "quasi-local" approach to the study of black holes which is more amenable to numerical simulations. Additionally, this chapter presents the stability operator for MOTS, which can be viewed as a generalization of the so-called *stability operator* for minimal surfaces. The properties of this operator play a crucial role in our later investigations regarding the topology of black holes.

4.1 MOTS

In this section, we will investigate the geometric properties of certain spacelike submanifolds of codimension two in a spacetime and introduce the concept of *trapped surfaces* and, of course, *marginally outer trapped surfaces*. We will also discuss these surfaces solely in terms of initial data, without reference to an underlying spacetime.

4.1.1 GEOMETRY OF CODIMENSION TWO SPACELIKE IMMERSIONS

Before proceeding with our study, let us introduce the notion of *normal bundle* of a semi-Riemannian immersion. Let $\phi : \Sigma^k \to M^n$ be a semi-Riemannian immersion of codimension $n-k > 0$. For each $p \in \Sigma$, we define the fiber $N_{\phi(p)} := d\phi_p(T_p\Sigma)^\perp$ and the projection $\pi_N : v \in N_{\phi(p)} \to p \in \Sigma$. Observe that this defines a vector bundle $N(\phi)$ of rank $n-k$ over Σ whose fiber are these subspaces $N_{\phi(p)}$. This is called the *normal bundle of* ϕ, which is a particular type of *pullback bundle* (see, e.g., (HUSEMOLLER, 1994), for more details on fiber bundles). One can show that $\Gamma(N(\phi)) \approx \mathfrak{X}^\perp(\phi)$, i.e., the sections of $N(\phi)$ are isomorphic to the normal smooth vector fields over ϕ. Furthermore, since ϕ is a semi-Riemannian immersion, one can shows that each fiber of the vector bundle $N(\phi)$ is isomorphic to the fibers of the quotient normal bundle $\nu(\phi)$. Consequently, the two notions of normal bundle essentially agrees in this context. We say that the normal bundle of ϕ is *trivial* if it admits $n-k$ normal vector fields $X_1, \dots, X_{n-k} \in \mathfrak{X}^\perp(\phi)$ such that the vectors $(X_1(p), \dots, X_{n-k}(p))$ are a linearly independent tuple in $N_{\phi(p)}$ and span the latter for each $p \in \Sigma$.

In particular, we have a convenient criterion for a codimension two semi-Riemannian immersion to have a trivial normal bundle.

Proposition 4.1.1 . *Let $\psi : \Sigma^{n-1} \to M^{n+1}$ be a codimension two spacelike immersion. Then, the normal bundle of ψ is trivial if and only if there exists $\ell \in \mathfrak{X}^{\perp}(\psi)$ null future-directed vector field.*

Proof. ($\Longleftarrow$) Suppose that there exists $\ell \in \mathfrak{X}^{\perp}(\psi)$ null future-directed. For any $p \in \Sigma$, there exists a unique $K_p \in (T_{\psi(p)}\Sigma)^{\perp}$ null such that $g_{\psi(p)}(K_p, \ell_p) = -1$. The map $K : p \in \Sigma \mapsto K_p \in (T_{\psi(p)}\Sigma)^{\perp}$ defines a normal vector field. It remains to show that K is smooth.

Let $p \in \Sigma$ and let $\{E_1, \dots, E_{n-1}\}$ be an adapted orthonormal frame on an open set $U \subset M$ containing $\psi(p)$ such that $\operatorname{span}\{E_1 \circ \psi(q), E_2 \circ \psi(q)\} = (T_{\psi(q)}\Sigma)^{\perp}$, for all $q \in V := \psi^{-1}(U) \subset \Sigma$. We can assume that E_1 is a timelike future-directed vector field and E_2 is a spacelike vector field. Let $q \in V$ and let $\alpha_{\pm} \in C^{\infty}(V)$ be such that

$$\ell_q = \alpha_+(q)E_1 \circ \psi(q) + \alpha_-(q)E_2 \circ \psi(q).$$

Since ℓ is null, we obtain the following restriction on these functions:

$$0 = \ll \ell_q, \ell_q \gg = -\alpha_+(q)^2 + \alpha_-(q)^2,$$

that is, $\alpha_-(q)/\alpha_+(q) = \pm 1$. Because ℓ is future-directed, thus, necessarily $\alpha_+ > 0$. Let $\beta_{\pm}$ be functions on V, not necessarily smooth, and write the null vector K_q as

$$K_q = \beta_+(q)E_1 \circ \psi(q) + \beta_-(q)E_2 \circ \psi(q).$$

By the same arguments, $\beta_+ > 0$ and again $\beta_-(q)/\beta_+(q) = \pm 1$. The condition $g_{\psi(p)}(K_p, \ell_p) = -1$ implies that

$$-\alpha_+\beta_+ + \alpha_-\beta_- = -1.$$

Rearranging the terms

$$\beta_+ = \frac{1 + \alpha_-\beta_-}{\alpha_+},$$

and squaring the expression and using that $\beta_+^2 = \beta_-^2$ yields

$$\beta_-^2 = \frac{1 + 2\alpha_-\beta_- + \alpha_-^2\beta_-^2}{\alpha_+^2}.$$

Using that the ration between the $\alpha_{\pm}$ and $\beta_{\pm}$ equals ± 1, we find that $\beta_- = -1/2\alpha_-$, i.e. β_- and β_+ are smooth. Therefore, K is a smooth vector field.

($\Longrightarrow$) Suppose that the normal bundle of ψ is trivial. Then, there exist $X_1, X_2 \in \mathfrak{X}^{\perp}(\psi)$ such that, $\{X_1, X_2\}$ forms a basis of $(T_{\psi(p)}\Sigma)^{\perp}$, for all $p \in \Sigma$. Define the symmetric matrix

$$M := \begin{bmatrix} \ll X_1, X_1 \gg & \ll X_1, X_2 \gg \\ \ll X_1, X_2 \gg & \ll X_2, X_2 \gg \end{bmatrix},$$

and denote its entries by $a := M_{11}$, $b := M_{22}$ and $c := M_{12} = M_{21}$. Since each of these normal subspaces is a 2-dimensional Lorentzian vector space, the matrix is nondegenerate and its determinant is strictly negative. Our goal is to find $\alpha, \beta \in C^\infty(\Sigma)$ such that the vector field

$$\ell = \alpha X_1 + \beta X_2$$

is null. Therefore, we require that

$$\alpha^2 \ll X_1, X_1 \gg +2\alpha\beta \ll X_1, X_2 \gg +\beta^2 \ll X_2, X_2 \gg = 0,$$

which is equivalent to

$$\begin{bmatrix}\alpha & \beta\end{bmatrix} \cdot M \cdot \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = 0. \tag{37}$$

Since M is a symmetric invertible matrix, its eigenvalues are real and nonzero. The eigenvalues of M can be determined by

$$(\lambda - a)(\lambda - b) - c^2 = 0,$$

or equivalently,

$$\lambda^2 - (a+b)\lambda + ab - c^2 = 0.$$

Noting that $ab - c^2 = \det M < 0$, this second-order equation has two roots, which are given by

$$\lambda_\pm = \frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 + (-\det M)}.$$

It is worth noting that $\lambda_+ > 0$ and $\lambda_- < 0$. Let $Z_+ = [Z_+^1, Z_+^2]$ and $Z_- = [Z_-^1, Z_-^2]$ be vectors, where $Z_\pm^1, Z_\pm^2$ are functions on Σ, such that

$$MZ_+ = \lambda_+ Z_+,$$

$$MZ_- = \lambda_- Z_-,$$

holding that $\langle Z_+, Z_+\rangle = \langle Z_-, Z_-\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product, and $\langle Z_+, Z_-\rangle = 0$. Finally, we define

$$Z := \frac{1}{\sqrt{\lambda_+}} Z_+ + \frac{1}{\sqrt{-\lambda_-}} Z_-,$$

then we apply the matrix M to obtain

$$MZ = \frac{\lambda_+}{\sqrt{\lambda_+}} Z_+ + \frac{\lambda_-}{\sqrt{-\lambda_-}} Z_-,$$

and, writing Equation (37) in a more compact form, we have

$$\begin{aligned}\langle Z, MZ\rangle &= \langle \frac{1}{\sqrt{\lambda_+}} Z_+ + \frac{1}{\sqrt{-\lambda_-}} Z_-, \frac{\lambda_+}{\sqrt{\lambda_+}} Z_+ + \frac{\lambda_-}{\sqrt{-\lambda_-}} Z_-\rangle \\ &= \langle Z_+, Z_+\rangle - \langle Z_-, Z_-\rangle \\ &= 0.\end{aligned}$$

Therefore, writing $Z = [Z^1, Z^2]$ where Z^1 and Z^2 are functions on Σ, we find that $\ell = Z^1 X_1 + Z^2 X_2$ is a null vector field, as shown in the computation above. The smoothness of Z^1, Z^2 can be assured by the systems $MZ_+ = \lambda_+ Z_+$ and $MZ_- = \lambda_- Z_-$. Hence, since Z^1, Z^2 are smooth, we obtain that ℓ is a null smooth vector field and it can be chosen future-directed. □

Corollary 4.1.2 . *Let (M^{n+1}, g) be a spacetime and let $\psi : \Sigma^{n-1} \to M^{n+1}$ be a codimension two spacelike immersion. Assume that the normal bundle of ψ is trivial. Then, there exist two linearly independent future-directed null vector fields $\ell_+, \ell_- \in \mathfrak{X}^\perp(\psi)$ that can be chosen such that $\ll \ell_+, \ell_- \gg = -1$.*

Proof. According to Proposition 4.1.1, there exists a future-directed null vector field $\ell \in \mathfrak{X}^\perp(\psi)$. In this proposition, we have constructed another future-directed null vector field $K \in \mathfrak{X}^\perp(\psi)$ such that they are linearly independent and $\ll \ell, K \gg = -1$. Therefore, by defining $\ell_+ := \ell$ and $\ell_- := K$, the result follows. □

Definition 4.1.3 (Cross section). *Let (M^{n+1}, g) be a spacetime and let $\phi : H^n \to M^{n+1}$ be a null immersion. A codimension one immersion $\psi : \Sigma^{n-1} \to H^n$ is a* cross section *of ϕ if $\phi \circ \psi$ is a spacelike immersion.*

Corollary 4.1.4 . *Let (M^{n+1}, g) be a spacetime. Given a codimension one null immersion $\phi : H^n \to M^{n+1}$ and let $\psi : \Sigma^{n-1} \to H^n$ be a cross section of ϕ. Then, the normal bundle of $\phi \circ \psi$ is trivial.*

Proof. Consider any null section $K \in \mathfrak{X}(H)$ for ϕ, and define the future-directed null vector field $\ell_+ := d\phi(K) \circ \psi \in \mathfrak{X}^\perp(\phi \circ \psi)$. According to Proposition 4.1.1, the normal bundle of $\phi \circ \psi$ is trivial. □

Remark 4 . *In all that follows, we shall fix a background Riemannian metric h_0 on M. Therefore, given any null immersion $\phi : H^n \to M^{n+1}$ and a cross section $\psi : \Sigma^{n-1} \to H^n$ of ϕ, we shall adopt $K \in \mathfrak{X}(H)$ as the unique null section for ϕ established in Proposition 3.1.2, and we always choose $\ell_\pm \in \mathfrak{X}^\perp(\phi \circ \psi)$ as constructed from K in Corollary 4.1.4.*

In studying codimension two spacelike immersions, it is also convenient to introduce certain geometric quantities related to its second fundamental form tensor.

Definition 4.1.5 (Null second fundamental forms). *Let (M^{n+1}, g) be a spacetime and let $\psi : \Sigma^{n-1} \to M^{n+1}$ be a spacelike immersion with trivial normal bundle. Fix the two null future-directed vector fields $\ell_\pm \in \mathfrak{X}(\psi)$ as in Corollary 4.1.2. We define the* null second fundamental forms $\chi_\pm$ *associated with $\ell_\pm$ as*

$$\chi_\pm(X, Y) := \ll D_X \ell_\pm, d\psi \circ Y \gg, \quad \forall X, Y \in \mathfrak{X}(\Sigma). \tag{38}$$

Additionally, we introduce the null mean curvatures *(or* null expansion scalars*) $\theta_\pm$ as*

$$\theta_\pm := \operatorname{tr}_h \chi_\pm = \operatorname{div}_h \ell_\pm, \tag{39}$$

where h is the induced Riemannian metric on Σ.

It is important to bear in mind that these quantities depend on the choices of $\ell_{\pm}$. However, as previously discussed after Proposition 3.1.9, multiplying the null vectors by positive functions is equivalent to multiplying the null second fundamental form and the null expansion by the same function. As a result, the signs of these quantities have intrinsic geometric meaning, in the sense that they remain unchanged upon rescaling of $\ell_{\pm}$ by a positive function.

We will see in Proposition 4.1.7 that if we have $\Sigma^{n-1} \subset M^{n+1}$ a compact embedded submanifold where $i_\Sigma : \Sigma \hookrightarrow M$ is a spacelike semi-Riemannian immersion with trivial normal bundle, then we can always find two null hypersurfaces associated with Σ (see Figure 7). The quantities $\chi_{\pm}$ and $\theta_{\pm}$ will be closely related to θ and $\mathcal{K}$ of the null hypersurfaces.

Lemma 4.1.6 . *Let (M^{n+1}, g) be a spacetime, $\phi : H^n \to M^{n+1}$ be a null immersion and $\psi : \Sigma^{n-1} \to H^n$ be a cross section of ϕ. Fix $\ell_+ \in \mathfrak{X}^{\perp}(\phi \circ \psi)$ the future-directed null vector field, as in Remark 4, then*

$$\chi_+(v, w) = \mathcal{K}(d\psi_p v, d\psi_p w), \quad \forall v, w \in T_p\Sigma, \tag{40}$$

where $\mathcal{K}$ is the second-fundamental associated with the null section for ϕ (see Section 3.2).

Proof. By Remark 4, we have that $\ell_+ = (d\phi \circ K) \circ \psi$. Moreover, by Lemma 1.1.11, there exists a smooth vector field $V \in \mathfrak{X}(M)$ and an open set $\mathcal{U} \subset \Sigma$ containing p such that

$$V \circ (\phi \circ \psi)\big|_{\mathcal{U}} = \ell_+\big|_{\mathcal{U}} = (d\phi \circ K) \circ \psi\big|_{\mathcal{U}}.$$

Since ψ is an immersion, it is locally an embedding. Therefore, by restricting $\mathcal{U}$, if necessary, we can assume that $\psi(\mathcal{U})$ is an open set and

$$V \circ \phi\big|_{\psi(\mathcal{U})} = d\phi \circ K\big|_{\psi(\mathcal{U})},$$

where it should be noted that ψ is locally bijective onto its image. Let v, $w \in T_p\Sigma$ and denote $\hat{p} = \phi \circ \psi(p)$, so

$$\begin{aligned} \ll D_v \ell_+(p), d(\phi \circ \psi) w \gg &= g_{\hat{p}}\left((D_v V \circ \phi \circ \psi)(p), d(\phi \circ \psi)_p w\right) \\ &= g_{\hat{p}}\left((\nabla^M_{d(\phi\circ\psi)_v} V)(\hat{p}), d(\phi \circ \psi)_p w\right) \\ &= g_{\hat{p}}\left((D^{\phi}_{d\psi(v)} V \circ \phi) \circ \psi(p), d(\phi \circ \psi)_p w\right) \\ &= g_{\hat{p}}\left((D^{\phi}_{d\psi(v)} d\phi \circ K) \circ \psi(p), d(\phi \circ \psi)_p w\right), \end{aligned}$$

where we have employed the identities for each induced connection on a smooth map. Using the definition of the null Weingarten map $\mathcal{S}$ and the positive definite fiber metric h induced by the null immersion ϕ, we obtain the following

$$\begin{aligned} \ll D_v \ell_+(p), d(\phi \circ \psi) w \gg &= g_{\hat{p}}\left((D^{\phi}_{d\psi(v)} d\phi \circ K) \circ \psi(p), d(\phi \circ \psi)_p w\right) \\ &= (\phi^* g)_{\psi(p)}\left(\widetilde{\mathcal{S}}(d\psi_p v), d\psi_p w\right) \\ &= h_{\psi(p)}\left(\mathcal{S}(\overline{d\psi_p v}), \overline{d\psi_p w}\right) \\ &= \mathcal{K}(d\psi_p v, d\psi_p w). \end{aligned}$$

where $\mathcal{K}$ is the second-fundamental associated with the null section K. □

Proposition 4.1.7 . *Let (M^{n+1}, g) be a spacetime and $\Sigma^{n-1} \subset M^{n+1}$ be a compact embedded submanifold. Assume that $i_\Sigma : \Sigma \hookrightarrow M$ is a spacelike semi-Riemannian immersion and the normal bundle of i_Σ is trivial. Let $\ell_\pm \in \mathfrak{X}^\perp(i_\Sigma)$ be the future-directed null vector fields as in Corollary 4.1.2. Then, there exist $\mathcal{H}^n_\pm \subset M^{n+1}$ null hypersurfaces containing Σ^{n-1} and associated null sections $K_\pm \in \mathfrak{X}(\mathcal{H}_\pm)$ such that for each $p \in \Sigma$ we have*

$$\chi_\pm(v, w) = \mathcal{K}_{\mathcal{H}_\pm}(v, w) \quad \forall v, w \in T_p\Sigma, \tag{41}$$

where $\mathcal{K}_{H_\pm}$ is the second fundamental form associated with $K_\pm$.

Proof. By Corollary 4.1.2, we can choose $\ell_\pm \in \mathfrak{X}^\perp(\Sigma)$ future-directed null vector fields. As every semi-Riemannian embedded submanifold has a normal neighborhood in M, according to (O'NEILL, 1983, Proposition 7.26), there exists $\mathcal{O}$ a normal neighborhood of Σ such that $\mathcal{O}$ is the diffeomorphic image under $\exp^\perp$ of a neighborhood Z in $N\Sigma$. As Σ is compact, for $\varepsilon \in \mathbb{R}$ small enough, we can define the map $\mathcal{J}_\pm : (-\varepsilon, \varepsilon) \times \Sigma \to Z \subset N\Sigma$ such that $\mathcal{J}_\pm(t, p) = t\ell_\pm(p)$, for all $t \in (-\varepsilon, \varepsilon)$. Therefore, we define the following diffeomorphism under its image by $\Psi_\pm : (-\varepsilon, \varepsilon) \times \Sigma^{n-1} \to M^{n+1}$ such that

$$\Psi_\pm(t, p) := \exp^\perp \circ \mathcal{J}_\pm(t, p).$$

The embedded hypersurface $\mathcal{H}_\pm := \Psi_\pm(\Sigma \times (-t, t))$ is a null hypersurface of M since the null geodesic with velocity $\ell_\pm$ in Σ remains in $\mathcal{H}_\pm$. Therefore, by Corollary 4.1.4, the result follows. □

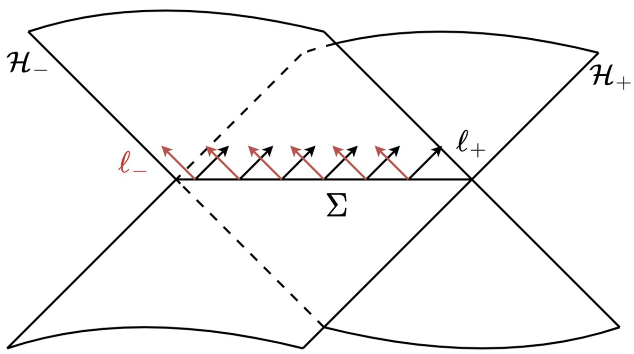


Figure 7 – Null hypersurfaces $\mathcal{H}_\pm$ associated with the spacelike compact embedded submanifold Σ and its future-directed null vector fields $\ell_\pm$.

Corollary 4.1.8 . *Let $\phi : H^n \to M^{n+1}$ be a totally geodesic null immersion (in the sense of Section 3.2) and let $\psi : \Sigma^{n-1} \to H^n$ be a cross section of ϕ. Fix $\ell_+ \in \mathfrak{X}^\perp(\phi \circ \psi)$ the future-directed null vector field as in Remark 4 (or any strictly positive rescale of ℓ_+), then $\chi_+ = 0$.*

In sequence, we will see a criteria that asserts that a null immersion is totally geodesic.

Definition 4.1.9 (Killing vector field). *A* Killing vector field *on a semi-Riemannian manifold is a vector field X for which the Lie derivative of the metric tensor vanishes: $\mathcal{L}_X g = 0$.*

Definition 4.1.10 (Killing horizon). *Let (M^{n+1}, g) be a spacetime. A map $\phi : H^n \to M^{n+1}$ is a* Killing immersion *if ϕ is a null immersion and there exists a Killing vector field $X \in \mathfrak{X}(M)$ such that $X \circ \phi$ is a null section. An embedded submanifold $H \subset M$ is a* Killing horizon *if the inclusion map $i : H \hookrightarrow M$ is a Killing immersion.*

There are several solutions to the Einstein field equations that admit a Killing horizon, e.g. Schwarzschild and Kerr. Finally, we can relate the Killing immersions with totally geodesic immersions.

Proposition 4.1.11 . *Let (M^{n+1}, g) be a spacetime. If $\phi : H^n \to M^{n+1}$ is a Killing immersion, then*

1. *ϕ is totally geodesic,*
2. *for any $\psi : \Sigma^{n-1} \to H^n$ cross section of ϕ, the null second fundamental form χ_+ vanishes identically.*

Proof. Let Y, $Z \in \mathfrak{X}(H)$, $K \in \mathfrak{X}(H)$ be any null section for ϕ, and X be a Killing vector field on M. Invoking the definition of the tensor $\mathcal{K} : \mathfrak{X}(H) \times \mathfrak{X}(H) \to C^\infty(H)$ associated with K and denoting by h the positive definite fiber metric induced by K, we have

$$\mathcal{K}(Y, Z) = h(\mathcal{S}(\overline{Y}), \overline{Z}).$$

Applying the definition of the map $\mathcal{S} : \overline{\mathfrak{X}}(H) \to \overline{\mathfrak{X}}(H)$ and the item 3 of Lemma 1.1.12, we obtain

$$\mathcal{K}(Y, Z) = \ll D_Y d\phi \circ K, d\phi \circ Z \gg = - \ll d\phi \circ K, D_Y d\phi \circ Z \gg .$$

Employing Proposition 1.1.2 and that $X \circ \phi = \lambda_X d\phi \circ K$ for some nonvanishing function $\lambda_X \in C^\infty(\Sigma)$, we proceed with the following computations:

$$\begin{aligned}\mathcal{K}(Y, Z) &= - \ll d\phi \circ K, D_Z d\phi \circ Y + d\phi \circ [Y, Z] \gg = - \ll d\phi \circ K, D_Z d\phi \circ Y \gg \\ &= -\frac{1}{\lambda_X} \ll X \circ \phi, D_Z d\phi \circ Y \gg = \frac{1}{\lambda_X} \ll D_Z X \circ \phi, d\phi \circ Y \gg .\end{aligned}$$

By (O'NEILL, 1983, Proposition 9.25), if X is a Killing vector field, then

$$\ll D_Z(X \circ \phi), d\phi \circ Y \gg + \ll D_Y(X \circ \phi), d\phi \circ Z \gg = 0.$$

This proposition implies that

$$\begin{aligned}\mathcal{K}(Y, Z) &= -\frac{1}{\lambda_X} \ll D_Y X \circ \phi, d\phi \circ Z \gg = - \ll D_Y d\phi \circ K, d\phi \circ Z \gg \\ &= -h(\mathcal{S}(\overline{Y}), \overline{Z}) = -\mathcal{K}(Y, Z).\end{aligned}$$

Therefore, $\mathcal{K} \equiv 0$, i.e., $\mathcal{K}$ vanishes identically and ϕ is totally geodesic. The item 2 is an immediate consequence of Corollary 4.1.8. □

Example 13 (Killing horizon in Kruskal spacetime). *We have already shown in Example 12 that the event horizon $\mathcal{H}'$ of the Kruskal spacetime of mass $M > 0$ is a null hypersurface with associated null section $K = (1/4M)(v\partial_v)$, where v is the coordinate function and ∂_v is the coordinate null vector field of v. The result (O'NEILL, 1983, Corollary 13.27) establishes that the Kruskal spacetime admits a Killing vector field X such that $X\big|_{\mathcal{H}'} = (1/4M)(v\partial_v)$. Consequently, by definition, $\mathcal{H}'$ is a Killing horizon, and, by Proposition 4.1.11, the event horizon $\mathcal{H}'$ is totally geodesic. In particular, given any $v_0 \in \mathbb{R}^+$, the cross section $\psi_{v_0} : p \in \mathbb{S}^2 \to ((0, v_0), p) \in \mathcal{H}'$ is the round-sphere $\mathbb{S}^2(2M)$ of radius $2M$ has vanishing null second fundamental form.*

4.1.2 TRAPPED SURFACES

Definition 4.1.12 (Trapped Surfaces). *Let (M^{n+1}, g) be a spacetime. A map $\psi : \Sigma^{n-1} \to M^{n+1}$ is said to be a* trapped immersion[1] *if ψ is a spacelike immersion and the mean curvature vector field $\vec{H}$ is past-directed timelike. An embedded submanifold $\Sigma \subset M$ is a* trapped surface *if $i : \Sigma \hookrightarrow M$ is a trapped immersion.*

A trapped immersion $\psi : \Sigma^{n-1} \to M^{n+1}$ with trivial normal bundle can be interpreted in terms of $\theta_\pm$ when the normal bundle of ψ is trivial. In this sense, suppose that the normal bundle of ψ is trivial. By Corollary 4.1.2, there exist linearly independent future-directed null smooth vector fields $\ell_\pm \in \mathfrak{X}^\perp(\psi)$ such that $\ll \ell_+, \ell_- \gg = -1$. The mean curvature vector $\vec{H} \in \mathfrak{X}^\perp(\psi)$ can be written as

$$\vec{H} = a\ell_+ + b\ell_-,$$

where $a, b \in C^\infty(\Sigma)$. Employing $\ll \ell_+, \ell_- \gg = -1$ and the definition of the expansion scalars $\theta_\pm$ associated with $\ell_\pm$, respectively (see Definition 4.1.5), we can show that

$$\vec{H} = \theta_-\ell_+ + \theta_+\ell_-.$$

Since $\vec{H}$ is timelike, we can obtain the following condition involving the null mean curvatures:

$$\ll \vec{H}, \vec{H} \gg = -2\theta_+\theta_- < 0.$$

This condition implies that $\theta_+\theta_- > 0$, and, together with the condition that $\vec{H}$ is past-directed and $\ell_\pm$ are future-directed, we have that $\theta_+ < 0$ and $\theta_- < 0$. In particular, if ψ is an embedding, then $\Sigma^{n-1} \subset M^{n+1}$ is a trapped surface then $\theta_\pm < 0$. When this condition does not hold, we say that Σ is an *untrapped surface*.

The signs of the quantities $\theta_\pm$ have a clear meaning within our framework. Recall that, from Theorem 1.2.2, $\theta_\pm$ measure how the volume form on Σ changes when we consider a variation in the $\ell_\pm$ direction. This justifies why they are referred to as null expansions. The null vectors $\ell_\pm$ represent two normal directions that null geodesics (or, physically, light rays)

[1] Technically, we are defining here what is known in the literature as a *future-trapped* immersion. However, since we shall never deal with the so-called *past-trapped* immersions in this work, we omit the adjective "future" throughout.

can escape from Σ. For instance, if $\Sigma \subset M$ is an embedded submanifold, ℓ_+ can be thought of as the outgoing (ℓ_- the ingoing) light rays emanating from Σ.

Therefore, a trapped surface exhibits the property that any family of light rays emanating from $\Sigma \subset M$ decreases their volume (see Figure 8). Physically, this phenomenon is not expected when light rays are shot outward, which is associated with θ_+. The behavior of decreasing volume indicates regions of spacetime where the gravitational field is intense enough to significantly affect the propagation of light.

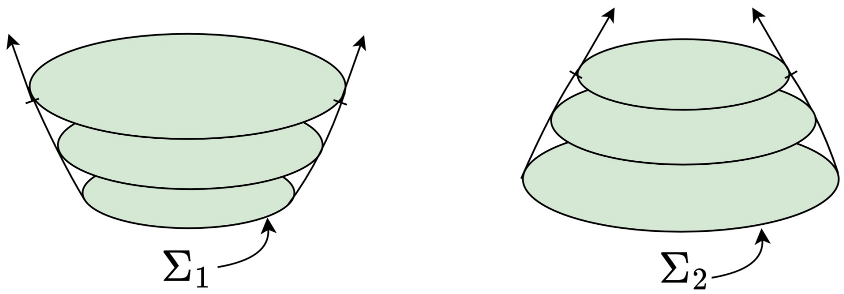


Figure 8 – Typically, the outgoing null geodesics (or, physically, light rays) from a null surface exhibit a positive null expansion (θ_+) that increases as they move away from the surface, as exemplified by the surface Σ_1. In contrast, a trapped surface, like Σ_2, demonstrates the opposite behavior with a negative and decreasing null expansion (θ_+). This phenomenon indicates the presence of a strong gravitational field that causes the light rays to converge and results in a reduction in volume.

Example 14 (Untrapped surfaces in Minkowski). *In $\mathbb{R}^4_1$, consider the spacelike sphere $\mathbb{S}^2(1) := \{(t, x, y, z) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = 1, t = t_0\}$ of radius 1 in $\mathbb{R}^4_1$ at $t = t_0 \in \mathbb{R}$. In the standard coordinate system, we can define the normal future-directed null vector fields $\ell_\pm = (\partial_t \pm x\partial_x \pm y\partial_y \pm z\partial_z)/\sqrt{2}$. Let $V = f_x\partial_x + f_y\partial_y + f_x\partial z$ be any tangent vector, where $f_x, f_y, f_z \in C^\infty(\mathbb{S}^2(1))$. Denote by ∇ the Levi-Civita connection on $\mathbb{R}^4_1$, which is the flat connection (see Example 2). The covariant derivative of $\ell_\pm$ with respect to any tangent vector V is*

$$\nabla_V \ell_\pm = V(\ell^i_\pm)\partial_i = \pm f_x\partial_x \pm f_y\partial_y \pm f_x\partial z = \pm\frac{1}{\sqrt{2}}\mathbb{I}(V),$$

where $\mathbb{I}$ is the identity operator. Using an orthonormal frame and tracing null second fundamental forms $\chi_\pm$ associated with the future-directed null vectors $\ell_\pm$ we obtain that $\theta_+ > 0$ and $\theta_- < 0$. Therefore, codimension 2 spacelike spheres in Minkowski are untrapped surfaces*.*

Example 15 (Trapped surfaces in the black hole region). *Let K be the Kruskal spacetime of mass $M > 0$ (see Definition 2.3.2). According to (O'NEILL, 1983, Proposition 13.4), the sphere $\mathbb{S}^2(r)$ in the restspace t constant has mean curvature vector field $\vec{H} = -1/r\nabla r$ (This result follows from the geometry of warped products, see (O'NEILL, 1983, Proposition 7.35)). The time-orientation of the Kruskal spacetime is given by $\partial_v - \partial_u$, where $-\partial_u$ and ∂_v are future-directed null vector fields and both are normal vector fields to $\mathbb{S}^2(r)$. As a reminder, Q_1 is the black hole region, i.e., $0 < r(u,v) < 2M$ with $u < 0$ and $v > 0$, and $F(r)$ is a strictly* *positive function. Now, since* $\nabla r = (1/4M)(u\partial_u + v\partial_v)$*, we have*

$$\langle \nabla r, \nabla r\rangle = 2uvF(r) < 0, \quad \text{on } Q_1,$$

and

$$\langle \nabla r, \partial_v - \partial_u\rangle = \frac{(1/4M)}{r}(u-v)F(r)) < 0, \quad \text{on } Q_1.$$

Hence, ∇r *is timelike future-pointing on* Q_1*. Since* $r > 0$*, we obtain that* $\vec{H}$ *is timelike past-directed. Therefore,* $\mathbb{S}^2(r)$ *is a trapped surface for any* $0 < r < 2M$*.*

4.1.3 NULL EXPANSION - INITIAL DATA VERSION

As briefly discussed, the global structure of the spacetime must be known in order to properly define a black hole and its event horizon. Instead of relying on the global geometry of spacetime, MOTS provides an alternative method to study black holes through local geometry from the perspective of the initial data setting. To investigate MOTS in initial data, some terminologies are introduced. First, we show that we can describe the null expansion solely in terms of spacelike immersions, or an initial data set, as stated in the following proposition.

Proposition 4.1.13 . *Let* (M^{n+1}, g) *be a spacetime,* $\phi : S^n \to M^{n+1}$ *be a spacelike immersion with* $\vec{u} \in \mathfrak{X}^{\perp}(\phi)$ *the unique unit future-directed timelike normal vector field and* $\psi : \Sigma^{n-1} \to S^n$ *a two-sided immersion with* $\vec{n} \in \mathfrak{X}^{\perp}(\psi)$ *a unit normal. Defining the the future-directed null vector fields as* $\ell_{\pm} := (\vec{u} \circ \psi \pm d\phi \circ \vec{n})/\sqrt{2} \in \mathfrak{X}^{\perp}(\phi \circ \psi)$*, it follows that* $\phi \circ \psi$ *is a spacelike immersion of codimension two with trivial normal bundle and the null expansion given by*

$$\theta_{\pm} = \mathrm{tr}_{\Sigma}\, \mathcal{K} \circ \psi \pm H, \tag{42}$$

where H *is the mean curvature scalar of* ψ *with respect to the normal* $\vec{n}$ *and* $\mathcal{K}$ *is the second fundamental form of* ϕ *with respect to the normal* $\vec{u}$ *and the partial trace is with respect to the induced metric.*

Proof. Let $\{E_1, \dots, E_{n-1}\}$ be an orthonormal frame on Σ in the induced metric $(\widetilde{\phi}^* g)$, where $\widetilde{\phi} := (\phi \circ \psi)$ is the immersion from Σ into M. Fixing $p \in \Sigma$, we have

$$\begin{aligned}
\theta_{\pm}(p) &= tr_{\Sigma}\chi_{\pm}(p), \\
&= \sum_{i=1}^{n-1} \ll D_{E_i}\ell_{\pm}, d\widetilde{\phi} \circ E_i \gg_p, \\
&= \sum_{i=1}^{n-1} \ll -\ell_{\pm}, D_{E_i} d\widetilde{\phi} \circ E_i \gg_p, \\
&= \sum_{i=1}^{n-1} \ll -\vec{u} \circ \psi, II^{\widetilde{\phi}}(E_i, E_i) \gg_p \pm \ll -d\phi \circ \vec{n}, II^{\widetilde{\phi}}(E_i, E_i) \gg_p .
\end{aligned}$$

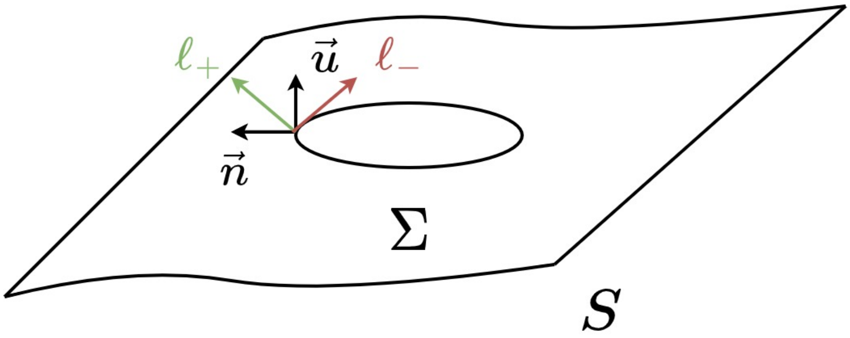


Figure 9 – A spacelike surface S^n with a closed surface Σ^{n-1} containing the two special vectors $\vec{n}$ and $\vec{u}$.

For any $i \in \{1, \dots, n-1\}$, we denote $e_i := E_i(p)$, then, by Lemma 1.1.20, we have

$$\theta_{\pm}(p) = \sum_{i=1}^{n-1} \ll -\vec{u} \circ \psi(p), II^{\phi}_{\psi(p)}(d\psi_p e_i, d\psi_p e_i) \gg \pm \ll -d\phi_{\psi(p)}\vec{n}, d\phi_{\psi(p)}(II^{\psi}_p(e_i, e_i)) \gg$$
$$= tr_{\Sigma}\mathcal{K} \circ \psi(p) \pm H(p).$$

□

Remark 5 . *Given a two-sided immersion $\psi : \Sigma^{n-1} \to S^n$ in an initial data, by convention, we always implicitly choose a unit normal vector field $\vec{n} \in \mathfrak{X}(\psi)$ and refer to it as the outward pointing unit normal vector field (of ψ), and $-\vec{n}$ as the inward pointing unit normal vector field (of ψ).*

Finally, we can define the null expansion in terms of the initial data set.

Definition 4.1.14 (Null expansion - Initial data version). *Let $(S^n, h, \mathcal{K})$ be an initial data set and $\psi : \Sigma^{n-1} \to S^n$ be a two-sided immersion with $\vec{n} \in \mathfrak{X}^{\perp}(\psi)$ as the outward pointing unit normal vector field of ψ. The outward* null expansion *θ_+ [resp. inward* null expansion *θ_-] of Σ is defined as*

$$\theta_{\pm} := \mathrm{tr}_{\Sigma}\,\mathcal{K} \circ \psi \pm H, \tag{43}$$

where H is the mean curvature scalar of ψ with respect to the normal $\vec{n}$ and the partial trace is respect to the induced metric.

The signs of $\theta_{\pm}$ will be used in our discussion, therefore, they require a proper definition for each case in terms of initial data.

Definition 4.1.15 . *Let $(S^n, h, \mathcal{K})$ be an initial data set and $\psi : \Sigma^{n-1} \to S^n$ be a two-sided immersion with $\vec{n} \in \mathfrak{X}^{\perp}(\psi)$ as the outward pointing unit normal vector field of ψ. We say that Σ is*

outer trapped,	*if $\theta_+ < 0$,*
weakly outer trapped,	*if $\theta_+ \leq 0$,*
marginally outer trapped,	*if $\theta_+ = 0$,*

Example 16 (MOTS in Kruskal Spacetime). *Example 13 shows that the round spheres $\mathbb{S}^2$ on the event horizon are MOTS, since its second fundamental form vanishes identically.*

Definition 4.1.16 (Homologous Surfaces). *Let $(S^n, h, \mathcal{K})$ be an initial data. A pair of codimension one surfaces Σ and Σ' in S are said to be* homologous *if there exists a smooth map $\Phi : (a, b) \times \Sigma \to S$ satisfying*

1. $[0, 1] \subset (a, b)$,
2. *for each $t \in (a, b)$, the map $\phi_t : x \in \Sigma \mapsto \Phi(t, x) \in S$ is an embedding,*
3. $\phi_0 = id_\Sigma$ *and* $\phi_1(\Sigma) = \Sigma'$.

We say that Σ and Σ' are outward homologous *if the variation vector field $V := \frac{\partial \Phi}{\partial t}\big|_{t=0}$ is equals to $V = f \cdot \vec{n}$, where $\vec{n}$ the outward pointing unit normal vector field of ϕ_0, for some strictly positive function $f \in C^\infty(\Sigma)$.*

Definition 4.1.17 (Outermost MOTS). *Let Σ be a MOTS in an initial data set $(S, h, \mathcal{K})$ with $\vec{n}$ an outward pointing unit normal vector field of Σ in S.*

1. *We say that Σ is* outermost MOTS *in S if there are no outer trapped $(\theta_+ < 0)$ or marginally outer trapped $(\theta_+ = 0)$ surfaces outward homologous to Σ.*
2. *We say that Σ is a* weakly outermost MOTS *in S provided there are no outer trapped surfaces $(\theta_+ < 0)$ outward homologous to Σ.*

Figure 10 illustrates the idea of outermost and weakly outermost MOTS.

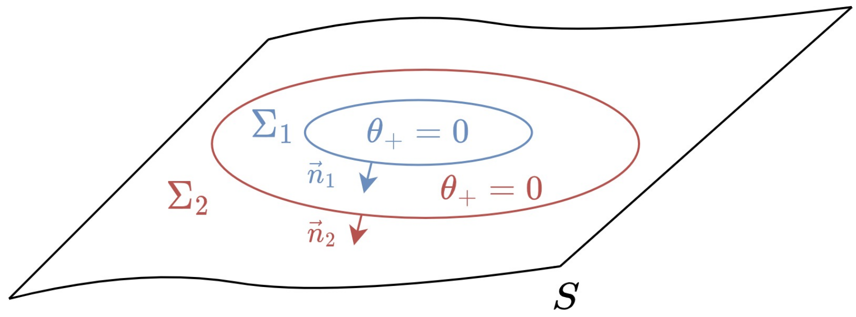


Figure 10 – Let Σ_1 and Σ_2 be homologous surfaces with unit normal outward pointing $\vec{n}_1$ and $\vec{n}_2$, respectively. If only homologous surfaces with $\theta_+ = 0$ lie in the region between Σ_1 and Σ_2, then Σ_1 is a weakly outermost MOTS. If there are no marginally outer trapped surfaces outside of Σ_2, then is Σ_2 outermost MOTS.

4.2 MOTS STABILITY OPERATOR

MOTS can be seen as a generalization of minimal surfaces. For a MOTS $(\theta_+ = 0)$ in a time-symmetric initial data set $(\mathcal{K} = 0)$, from Equation (43), the mean curvature H vanishes identically, then, in this particular case, a MOTS is a minimal surface. Minimal surfaces and MOTS share several similarities; however, while minimal surfaces can be described via a variational formulation, no such description is known for MOTS. A powerful tool for studying minimal surfaces is the notion of *stability*, which comes from the sign of the second variation of the volume measure; this concept of stability can be generalized to the setting of MOTS through the linearization of the null expansion θ_+.

The notion of stability is grounded in the variations of the null expansion, which were introduced and discussed in (ANDERSSON; MARS; SIMON, 2008). The expression that defines the linearization of θ_+ is quite involved and challenging to obtain. Therefore, in this context, we briefly introduce the normal variation of a MOTS in the initial data set and defer the complete proof to Appendix A.

For the sake of notational simplicity, we shall assume from now on that all submanifolds we consider are embedded. Let (M^{n+1}, g) be a spacetime and let $\Sigma^{n-1} \subset M$ be a smooth closed (i.e., compact without boundary) codimension two spacelike submanifold with trivial normal bundle. We shall fix, as per Corollary 4.1.2, two future-directed normal null vector fields $\ell_\pm$ such that $\ll \ell_+, \ell_- \gg = -1$. We shall assume that Σ is a MOTS with respect to ℓ_+, i.e., $\theta_+ = \operatorname{div}_\Sigma \ell_+ = 0$. For convenience, we also define on Σ the normal unit timelike vector field $\vec{u} := \frac{\ell_+ + \ell_-}{\sqrt{2}}$ and normal unit spacelike vector field $\vec{n} := \frac{\ell_+ - \ell_-}{\sqrt{2}}$. Finally, let $\Phi : (-t_0, t_0) \times \Sigma \to M$ be a smooth variation of Σ in M with a normal variation vector field V. As a result, the normal vector field V can be decomposed into

$$V = \frac{\partial \Phi}{\partial t}\bigg|_{t=0} = \phi \ell_+ + \psi \vec{n}, \quad \phi,\, \psi \in C^\infty(\Sigma).$$

We shall in addition assume that a smooth choice was made on each $\Sigma_t := \phi_t(\Sigma)$ of future-directed null normal vector fields $\ell_\pm(t)$ so that $\ell_\pm(0) = \ell_\pm$, and $\ll \ell_+(t), \ell_-(t) \gg = -1$. Thus, denote by $\theta_+(t)$ the null expansion with respect to ℓ_+. With this convention, we compute the linearization of the null expansion θ_+.

Proposition 4.2.1 . *Let Σ^{n-1} be a MOTS within a spacetime (M^{n+1}, g). Let $\Phi : (-t_0, t_0) \times \Sigma \to M$ be a variation with normal variation vector field $V = \phi \ell_+ + \psi \vec{n}$ with the conventions described above. Then, the variation of the null expansion scalar $\theta_+(t)$ on Σ in the direction of the variation vector field V is given by*

$$\theta'_+(0) = -(\|\chi_+\|^2 + Ric_g(\ell_+, \ell_+)) \cdot \phi + L(\psi), \tag{44}$$

where

$$L(\psi) = -\Delta \psi + 2\langle X, \nabla \psi \rangle + (Q + \operatorname{div} X - \|X\|^2)\psi, \tag{45}$$

$$Q := \frac{1}{2} S_\Sigma - [J(\vec{n}) + \rho] - \frac{1}{2}\|\chi_+\|^2. \tag{46}$$

In these expressions, Δ, ∇ and div are the Laplacian, gradient and divergence operators, respectively, on Σ, while S_Σ is the scalar curvature of Σ all computed with respect to the induced metric $\langle , \rangle$ on Σ, X is the vector field on Σ metrically dual - also with the induced metric - to the one-form $\mathcal{K}_{\vec{u}}(\vec{n}, \cdot)|_{T\Sigma}$ and where ρ and J are defined as in Definition 2.2.1 associated with the timelike vector field $\vec{u}$.

Proof. See Appendix A. $\square$

The Proposition 4.2.1 allows us to define a linear second-order elliptic differential operator L, called the *MOTS stability operator*. Furthermore, although we have considered a MOTS in a spacetime, we can give a purely initial-data description via the following simple key observation. Suppose that, in addition to the conventions leading to Proposition 4.2.1, we have $\Sigma^{n-1} \subset S^n$, where is a S spacelike hypersurface in the spacetime (M^{n+1}, g), with the unique unit normal future-directed timelike normal vector field U, induced metric h and second fundamental form $\mathcal{K}$ with respect to U. Assume also that an h-unit normal vector field $\vec{N}$ is chosen on Σ, so that $\ell_\pm = \frac{U|_\Sigma \pm N}{\sqrt{2}}$. Then $\vec{u} \equiv U|_\Sigma$ and $\vec{n} \equiv \vec{N}$ and noting that $\chi_+ = \mathcal{K}_{\vec{n}} + \mathcal{K}_{\vec{u}}$, where $\mathcal{K}_{\vec{n}}$ is the second-fundamental form of Σ associated with the normal vector field $\vec{n}$. We motivate thus the following definition.

Definition 4.2.2 (MOTS Stability Operator - Initial Data Version). *Let Σ^{n-1} be a closed MOTS (compact without boundary) within an initial data $(S^n, h, \mathcal{K})$. We define the* MOTS stability operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ *to be*

$$L(\psi) := -\Delta\psi + 2\langle X, \nabla\psi\rangle + (Q + \operatorname{div} X - \|X\|^2)\psi, \tag{47}$$

$$Q := \frac{1}{2}S_\Sigma - [J(\vec{n}) + \rho] - \frac{1}{2}\|\mathcal{K}_{\vec{n}} + \mathcal{K}\|^2, \tag{48}$$

where the geometric quantities are defined on Σ, $\vec{n}$ is the outward pointing unit normal vector field on Σ, X is the vector field dual to the one-form $\mathcal{K}(\vec{n}, \cdot)$ along Σ and where ρ and J are defined as in Definition 2.2.4.

Remark 6 . *One may be interested in evaluating $\theta'_+(t)$ for $t \neq 0$, but the previous proof would fail, since there is no guarantee that the others hypersurfaces Σ_t in the foliation will be MOTS (this fact was explicitly used in Equations (85) and (86)). However, a similar expression can be obtained. First, notice that the null vector fields $\ell_\pm(t)$ are still null and orthogonal to each other. Therefore, with a analogous proof, without assuming $\theta_+ = 0$, we obtain the expression*

$$\theta'_+(t) = -(\|\chi_+(t)\|_t^2 + Ric_g(\ell_+, \ell_+)) \cdot v_t + L_{\Sigma_t}(\psi_t), \tag{49}$$

$$Q_t := \frac{1}{2}S_t - [J_t(\vec{n_t}) + \rho_t] - \frac{1}{2}\|\chi_+(t)\|_t^2, \tag{50}$$

where Q_t and L_t are defined as

$$L_{\Sigma_t}(\psi_t) := -\Delta_t\psi_t + 2\langle X_t, \nabla_t\psi_t\rangle_t + (Q_t + \frac{1}{2}\theta_+(t)[\theta_-(t) + 2\mathcal{K}_{\vec{u}}(\vec{n}, \vec{n})] + \operatorname{div}_t X_t - \|X_t\|_t^2)\psi_t. \tag{51}$$

It is important to note that each quantity depends on t since each sheet Σ_t has its own induced metric and vector fields, which are functions of t. Moreover, two new geometric quantities appeared in the expression.

It is worth mentioning that the MOTS stability operator can be derived from normal variations in the initial data. In the case of time-symmetric initial data ($\mathcal{K} = 0$), the operator L reduces to the self-adjoint, classic stability (or Jacobi) operator of the minimal surface theory, which consists of the second variation of the volume. Although the operator L is not self-adjoint in general, the operator possesses crucial properties as stated below.

Lemma 4.2.3 (GALLOWAY, 2018; ANDERSSON; MARS; SIMON, 2008). *Let Σ^{n-1} be a closed MOTS (compact without boundary) within an initial data set $(S^n, h, \mathcal{K})$. The following statements hold for the MOTS stability operator L.*

1. *There is a real eigenvalue $\lambda_1 = \lambda_1(L)$, called the* principal eigenvalue *of L, such that for any other eigenvalue μ, $Re(\mu) \geq \lambda_1$. The associated eigenfunction $\phi \in C^\infty(\Sigma)$, $L\phi = \lambda_1\phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.*
2. *$\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if only if there exist some $\psi \in C^\infty(\Sigma), \psi > 0$, such that $L(\psi) \geq 0$ (resp., $L(\psi) > 0$).*

An eigenvalue λ is called *simple* if its algebraic multiplicity is equal to one, i.e., the associated eigenfunction is unique up to a multiplicative constant. The principal eigenvalue allows us to define the notion of stability of a MOTS.

Definition 4.2.4 (Stability of MOTS). *We say that a closed MOTS is* stable *provided that the principal eigenvalue $\lambda_1(L)$ is nonnegative.*

Indeed, weakly outermost MOTS are necessarily stable. If the principal eigenvalue $\lambda_1(L)$ of the MOTS stability operator L is negative, and ϕ is a positive eigenfunction associated with $\lambda_1(L)$, then the outward deformation of Σ using the function ϕ, or in other words, using a normal variation with variation vector field $\phi\vec{n}$, where $\vec{n}$ is the outward pointing unit normal vector field on Σ inside the initial data, would result in an outer trapped surface. This contradicts the definition of a weakly outermost MOTS (see Definition 4.1.17). More generally, the negativity of $\lambda_1(L)$ implies that Σ can be deformed outward to an outer trapped surface.

We will be interested in comparing L with the so-called *symmetrized operator stability operator* $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$, obtained by setting $X = 0$,

$$L_0(\psi) = -\Delta\psi + Q\psi,$$

where Q is defined as before. The operator L_0 share some properties with L, such as the fact that Lemma 4.2.3 holds for L_0, allowing us to study its principal eigenvalue $\lambda_1(L_0)$. The following lemma shows that if $\lambda_1(L) \geq 0$ then $\lambda_1(L_0) \geq 0$.

Lemma 4.2.5 (GALLOWAY, 2008). *Let Σ^{n-1} be a closed MOTS (compact without boundary) in an initial data set $(S^n, h, \mathcal{K})$. The principal eigenvalues $\lambda_1(L)$ and $\lambda_1(L_0)$ satisfy $\lambda_1(L_0) \geq \lambda_1(L)$. In particular, if Σ is stable, then for any $\phi \in C^\infty(\Sigma)$ it holds that*

$$\langle L_0(\phi), \phi\rangle = \int_\Sigma |\nabla\phi|^2 + Q\phi^2 \geq 0.$$

Proof. Let λ_1 be the principal eigenvalue of L and $\psi \in C^\infty(\Sigma)$ its respective positive eigenfunction. Then, given that $L(\psi) = \lambda_1\psi$, we have

$$-\Delta\psi + 2\langle X, \nabla\psi\rangle + (Q + \operatorname{div} X - |X|^2)\psi = \lambda_1\psi.$$

Employing the identities $\nabla \ln \psi = \frac{1}{\psi}\nabla\psi$ and $|X - \nabla \ln \psi|^2 = |X|^2 - 2\langle X, \nabla \ln \psi\rangle + |\nabla \ln \psi|^2$ it follows that

$$-\Delta\psi + (Q + \operatorname{div} X - |X|^2 + 2\langle X, \nabla \ln \psi\rangle)\psi = \lambda_1\psi,$$

$$-\Delta\psi + (Q + \operatorname{div} X + |\nabla \ln \psi|^2 - |X - \nabla \ln \psi|^2)\psi = \lambda_1\psi,$$

$$-\frac{1}{\psi}\Delta\psi + Q + \operatorname{div} X + |\nabla \ln \psi|^2 - |X - \nabla \ln \psi|^2 = \lambda_1.$$

Notice that $\Delta \ln \psi = \frac{1}{\psi}\Delta\psi - |\ln \nabla\psi|^2$; thus, setting $u = \ln\psi$, we obtain

$$-\Delta u + Q + \operatorname{div} X - |X - \nabla u|^2 = \lambda_1,$$

$$\operatorname{div}(X - \nabla u) + Q - |X - \nabla u|^2 = \lambda_1.$$

Denoting $Y = X - \nabla u$, we arrive at the equality,

$$\operatorname{div}(Y) + Q - |Y|^2 = \lambda_1.$$

Given any $\phi \in C^\infty(\Sigma)$ and multiplying through by ϕ^2 the previous equation we derive

$$\lambda_1\phi^2 - Q\phi^2 + |Y|^2\phi^2 = \operatorname{div}(Y)\phi^2,$$

and observing that

$$|(|\nabla\phi| - |\phi||Y|)|^2 = |\nabla\phi|^2 - 2|\phi||Y||\nabla\phi| + |\phi|^2|Y|^2 \geq 0,$$

and $\operatorname{div}(\phi^2 Y) = \phi^2 \operatorname{div} Y + 2\phi\langle\nabla\phi, Y\rangle$, it follows that

$$\lambda_1\phi^2 - Q\phi^2 + |Y|^2\phi^2 = \operatorname{div}(\phi^2 Y) - 2\phi\langle\nabla\phi, Y\rangle,$$

$$\lambda_1\phi^2 - Q\phi^2 + |Y|^2\phi^2 \leq \operatorname{div}(\phi^2 Y) + 2|\phi||\nabla\phi||Y|,$$

$$\lambda_1\phi^2 - Q\phi^2 \leq \operatorname{div}(\phi^2 Y) + |\nabla\phi|^2,$$

Integrating the last inequality yields

$$\int_\Sigma |\nabla\phi|^2 + Q\phi^2 \geq \int_\Sigma \lambda_1\phi^2,$$

and using that $\operatorname{div}(\phi\nabla\phi) = \phi\Delta\phi + |\nabla\phi|^2$, we have

$$\langle L_0(\phi), \phi\rangle \geq \int_\Sigma \lambda_1\phi^2, \quad \text{for all } \phi \in C^\infty(\Sigma), \tag{52}$$

because Σ is closed, i.e., the term with the divergence drops out by the divergence theorem.

Taking $\phi \in C^\infty(\Sigma)$ as the positive eigenfunction associated with the principal eigenvalue μ_1 of L_0 in Equation (52) and multiplying by a appropriate constant we obtain that $\mu_1 \geq \lambda_1$, in other words, $\lambda_1(L_0) \geq \lambda_1(L)$. □

5 TOPOLOGY OF BLACK HOLES

A fundamental result in the theory of black holes was given by Hawking in his celebrated theorem, known as Hawking's black hole topology theorem (see Theorem 5.0.1). This result provides important restrictions on the possible shapes of black hole horizons predicted by general relativity, which can be then tested in actual astrophysical context.

Theorem 5.0.1 (HAWKING; ELLIS, 1973). *Suppose (M, g) is a $(3+1)$-dimensional asymptotically flat stationary black hole spacetime obeying the dominant energy condition. Then cross sections of the event horizon are topologically 2-spheres.*

In other words, if a 4-dimensional spacetime contains an approximately isolated black hole, such as in the case of Kruskal spacetime or Kerr spacetime, under reasonable extra conditions on the matter content, then, the 2-dimensional spacelike hypersurfaces of the event horizon are necessarily spherical. This result is proved using a variational argument that leads to the existence of an outer trapped surface outside of the event horizon, which is forbidden by a well-known result (see Proposition 9.2.8 (HAWKING; ELLIS, 1973)).

The question then arises: Is there a similar version of the Hawking's black hole topology theorem in higher dimensions? The proof of Theorem 5.0.1 relies on the Gauss-Bonnet Theorem (see Theorem 5.1.2), as a result, it cannot be directly extended to higher dimensions. The problem becomes more intricate in higher dimensions, as will be evidenced by the discussion of scalar curvature and its relation to topology in Section 5.1.2. As previously discussed by (EMPARAN; REALL, 2008), there has been a growing interest in classical general relativity in higher dimensions in recent years. There are several reasons for this interest, particularly in solutions of the EFE such as higher-dimensional black holes. Some of the key motivations include:

1. String theory incorporates gravity and demands more than four dimensions. For instance, calculations involving five-dimensional black holes in string theory appeared on Strominger and Vafa (1996). This example provided a validation of the theory, and the study of higher dimensional black holes can contribute to the development of a quantum theory of gravity.

2. The AdS/CFT correspondence (AHARONY et al., 2000) relates properties of black holes in $n > 4$ dimensions to quantum field theory in $n-1$ dimensions. This provides a more straightforward approach to compute certain field theory quantities compared to conventional techniques.

3. There is the possibility of producing tiny higher-dimensional black holes at colliders (CAVAGLIÀ, 2003; KANTI, 2004)

Due to the significance of these potential higher-dimensional black holes, it is natural to inquire which of their properties remain in higher-dimensions. Therefore, an possible extension of Hawking's theorem on the topology of black holes to higher dimensions becomes a natural logical step in this direction. Furthermore, although it is true the above-listed motivations remain largely speculative from a physical point of view, this problem nevertheless has a definite geometric significance. Indeed, for the purposes of the rest of this work we adopt the "conceptual equation" black hole horizon = closed MOTS. But one regards MOTS as natural spacetime generalizations of minimal hypersurfaces. The obtained results, when reinterpreted in this light, place in particular definite restriction on the possible topologies of minimal hypersurfaces under suitable restriction on the scalar curvature.

Emparan and Reall (2002) discovered a remarkable example of a 4+1 asymptotically flat stationary vacuum black hole spacetime with a horizon topology $\mathbb{S}^2 \times \mathbb{S}^1$, known as a *black ring*. This finding demonstrated that the horizon topology is not necessarily spherical in higher dimensions. As a result, there is a need to investigate an analogue of Hawking's theorem that might conceivably restrict the allowable topologies for higher dimensional black holes.

We shall see that the black ring falls into a special class of manifolds, those that admit a metric of positive scalar curvature, as in Theorem 5.1.5, the same class of the spherical case. Therefore, a natural version of Hawking's theorem in higher-dimensions is that each spacelike cross section of the event horizon (or MOTS) is of *positive Yamabe type*, i.e., admits a metric of positive scalar curvature.

The previous statement in higher dimensions holds under certain physical restrictions, such as outermost MOTS. We provide the proof in two distinct versions: the more restrictive version, Theorem 5.2.4, and the initial data version without exceptional cases, Theorem 5.3.4. Before delving into our main results, we will offer a concise introduction to the relationship between positive scalar curvature and topology in the upcoming section.

5.1 POSITIVE SCALAR CURVATURE

This initial section of the chapter is devoted to briefly discuss the topological obstructions to defining a metric of positive scalar curvature on Riemannian manifolds. These will serve to give context to the main results. We will begin by introducing the problem for compact two-dimensional manifolds using the Gauss-Bonnet theorem. Subsequently, we will present the compact n-dimensional case with $n \geq 3$, through the *Kazdan and Warner's classification theorem*, which serves as an essential tool in our subsequent results in Chapter 5.

5.1.1 GAUSSIAN CURVATURE

A fundamental problem in Riemannian geometry is to understand which kind of topological restrictions arise due to certain features of the curvature. In the case of a two-dimensional manifold, there exists only one notion of curvature, as demonstrated by the following result.

Proposition 5.1.1 (COSTA E SILVA, 2021). *If (M, g) is a two-dimensional semi-Riemannian manifold, then there exist a (unique) smooth function $K \in C^{\infty}(M)$ for which the curvature* *has the form*

$$R(X,Y)Z = K \cdot [g(Y,Z)X - g(X,Z)Y], \quad \forall X,Y,Z \in \mathfrak{X}(M). \tag{53}$$

The function K is called Gaussian curvature *of the (M,g).*

In other words, in dimension two, the whole information about the curvature is completely captured by the scalar curvature S_g or Gaussian curvature K since Equation (53) implies that $Ric_g = K \cdot g$, resulting in $S_g = 2K$. Consequently, our problem becomes that of describing the set of Gaussian curvature functions. The most notable condition on curvature for compact two-dimensional manifolds is the global condition given by the Gauss-Bonnet theorem.

Theorem 5.1.2 (Gauss-Bonnet). *Let (M,g) be a compact oriented two-dimensional Riemannian manifold and let K be the Gaussian curvature of M. Then*

$$\int_M K dA_g = 2\pi\chi(M), \tag{54}$$

where dA_g is the element of area with respect to g and the underlying orientation, and $\chi(M)$ is the Euler characteristic of M.

The theorem clearly imposes sign conditions on K depending on $\chi(M)$ which can be stated as:

$$\begin{aligned} &1.\ \chi(M) > 0 : K \text{ is positive somewhere,} \\ &2.\ \chi(M) = 0 : K \text{ changes sign (unless } K{=}0), \\ &3.\ \chi(M) < 0 : K \text{ is negative somewhere.} \end{aligned} \tag{55}$$

Hence, the only two-dimensional *oriented compact*[1] manifolds that can possess metrics with positive scalar curvature are those with a positive Euler characteristic. For example, the sphere $\mathbb{S}^2$ with $\chi(\mathbb{S}^2) = 2$, and the real projective plane $\mathbb{R}P^2$ with $\chi(\mathbb{R}P^2) = 1$. Furthermore, these surfaces do not admit metrics with $K \leq 0$. On the other hand, the 2-torus $\mathbb{T}^2$ with $\chi(\mathbb{T}^2) = 0$ does not have any metric with positive scalar curvature. Therefore, in this context, there are topological restrictions on the existence of a metric with positive scalar curvature.

This situation naturally leads to the following question, which can be regarded as a converse to the Gauss-Bonnet theorem: Do the sign conditions given in Equation (55) serve as sufficient conditions for some $K \in C^\infty(M)$ on a two-dimensional manifold to be the Gaussian curvature of some Riemannian metric g on M? Jerry L. Kazdan and Warner (1974) successfully answered this question with the following theorem.

Theorem 5.1.3 (KAZDAN, Jerry L.; WARNER, 1974). *On a compact M^2, a function $K \in C^\infty(M)$ is the Gaussian curvature of some Riemannian metric g if and only if K satisfies the obvious Gauss-Bonnet sign condition in Equation (55).*

[1] The result still holds for the non-orientable case via orientable coverings. manifolds that can possess metrics with positive scalar curvature are those with a positive Euler characteristic

The aforementioned theorem solved a problem known as the *Prescribed Gaussian Curvature* for a compact two-dimensional manifold, i.e., given a function K, find a metric g whose Gaussian curvature is K. Additionally, this theorem offers a description regarding the set of curvatures and topological obstructions for the two-dimensional case.

5.1.2 SCALAR CURVATURE

On a Riemannian manifold (M^n, g) with $n \geq 3$, the Proposition 5.1.1 does not hold as stated. For example, any odd-dimensional round sphere $\mathbb{S}^{2n-1}$ has zero Euler characteristic, but its standard metric has constant positive curvature. In this higher-dimensional setting, the simplest measure of curvature on M is the scalar curvature S_g.

We saw above how obstructions to the sign of the scalar curvature were obtained in two dimensions via Gauss-Bonnet theorem when Gaussian curvature was prescribed. However, the scalar curvature is a weak geometric invariant (it is obtained as a double average of the full sectional curvature), so it is not clear that any similar obstructions exist if the dimension is $n \geq 3$.

Employing the theory of minimal surfaces R. Schoen and Yau (1979) showed that several three-dimensional manifolds, including 3-torus $\mathbb{T}^3$, do not have metrics with positive scalar curvature. Thereafter, Gromov and Lawson (1980), with another approach, proved that for all n, $\mathbb{T}^n$ has no metric with $S_g > 0$. Furthermore, there are manifolds which do not carry any metric with zero scalar curvature either, cf. Jerry L. Kazdan and Warner (1975b).

However, these results do not provide obstructions to negative scalar curvature, only for non-negative scalar curvature. In fact, Jerry L. Kazdan and Warner (1975c) proved that every compact manifold M admits a metric whose scalar curvature is negative somewhere on M. By combining these restrictions with existence theorems for metrics with constant scalar curvature, Jerry L. Kazdan and Warner (1975a) obtained the following classification theorem, which was later improved by Bérard Bergery (1981).

Theorem 5.1.4 (BESSE, 2007). *Compact manifolds of dimension $n \geq 3$ can be divided into three classes:*

(A) Any function on M is the scalar curvature of some metric;

(B) A function on M is the scalar curvature of some metric if and only if either it is identically zero or negative somewhere; furthermore, any metric with vanishing scalar curvature on M is Ricci-flat;

(C) A function on M is a scalar curvature if only if it is negative somewhere.

For instance, in the context of the classification theorem mentioned, the n-dimensional sphere $\mathbb{S}^n$ belongs to class (A), the n-dimensional torus $\mathbb{T}^n$ falls into class (B), while the connected sum of two n-dimensional tori $\mathbb{T}^n \# \mathbb{T}^n$ is in class (C) (BÉRARD BERGERY, 1981).

Consequently, we have topological obstructions for the existence of metrics with positive scalar curvature in higher dimensions, and a result with several important applications. For instance, the positive mass problem in general relativity (SCHOEN, Richard; YAU, 1979) is intimately related to the question of the existence of a metric with positive scalar curvature, which was resolved using minimal surfaces (KAZDAN, Jerry L, 1985). Furthermore, compact orientable 3-manifolds that admit a metric with positive scalar curvature can be classified into three distinct classes, as demonstrated in the following result:

Theorem 5.1.5 (GALLOWAY, 2012; GROMOV; LAWSON, 1983). *If M is a compact orientable 3-manifold and admits a metric of positive scalar curvature then M must be diffeomorphic to:*

1. *a spherical space (i.e. a quotient of the 3-sphere by a discrete group of isometries).*
2. *$\mathbb{S}^1 \times \mathbb{S}^2$, or*
3. *a connected sum of the previous two types.*

5.2 MOTS TOPOLOGY: A FIRST THEOREM

Some analytical preliminaries are necessary to establish the main theorems of this section. We will see that the eigenfunction of a certain version of the stability operator for MOTS can be used to conformally rescale a metric to a metric of positive scalar curvature under certain conditions. To explore this topic, a generalized stability operator and its properties will be introduced.

Definition 5.2.1 (Generalized Stability Operator). *Let (Σ, γ) be a compact Riemannian manifold. Given any $\mathcal{Q} \in C^\infty(\Sigma)$ and any a smooth vector field X on Σ, the* generalized stability operator *$\mathcal{L} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ has the form*

$$\mathcal{L}(\psi) := -\Delta\psi + 2\langle X, \nabla\psi\rangle + (\mathcal{Q} + \operatorname{div} X - \|X\|^2)\psi, \tag{56}$$

where $\langle \cdot, \cdot \rangle = \gamma$.

The reason to introduce this more abstract concept is that the operators L and L_0, introduced in the previous section, become simply specific cases of the operator $\mathcal{L}$, and one can show that the principal eigenvalue properties (Lemma 4.2.3) still holding for this the generalized"version. This abstract approach has been championed by Galloway (2008).

Lemma 5.2.2 (GALLOWAY, 2008). *Let (Σ^{n-1}, γ) be a compact Riemannian manifold, $n \geq 4$. Consider the generalized stability operator $\mathcal{L}$ with $\mathcal{Q} \in C^\infty(\Sigma)$ such that*

$$\mathcal{Q} = \frac{1}{2}S_\gamma - P, \tag{57}$$

where S_γ is the scalar curvature of Σ and $P \geq 0$. If $\lambda_1(\mathcal{L}) \geq 0$, then Σ admits a metric of positive scalar curvature, unless $\lambda_1(\mathcal{L}) = 0$, $P = 0$ and (Σ, γ) is Ricci-flat.

Proof. The proof starts by observing that it is possible to conformally rescale the metric using the positive eigenfunction such that the scalar curvature of the conformally rescaled metric is non-negative.

Indeed, consider the symmetrized generalized stability operator $\mathcal{L}_0 = -\Delta + \mathcal{Q}$, obtained by taking $X = 0$. Let $f \in C^\infty(\Sigma)$ be a positive eigenfunction associated to the eigenvalue $\mu_1 = \mu_1(\mathcal{L}_0)$. The scalar curvature $S_{\hat{\gamma}}$ of Σ^{n-1} in the conformally rescaled metric $\hat{\gamma} = f^{2/(n-2)}\gamma$ is given by:

$$S_{\hat{\gamma}} = \Omega^{-2}[S_\gamma - 2(n-2)\Delta\phi - (n-2)(n-3)|\nabla\phi|^2], \tag{58}$$

where $\Omega = f^{1/(n-2)}$ and $\phi = \ln\Omega$. We can express $S_{\hat{\gamma}}$ in a more practical form by noting that

$$\begin{aligned}\Delta\phi = \Delta \ln f^{1/(n-2)} &= \mathrm{div}(\nabla \ln f^{1/(n-2)}),\\ &= \frac{1}{n-2}\mathrm{div}(\frac{\nabla f}{f}),\\ &= \frac{1}{n-2}(\frac{1}{f}\Delta f - \frac{1}{f^2}|\nabla f|^2).\end{aligned}$$

Now, using $\nabla\phi = \frac{1}{n-2}\frac{\nabla f}{f}$, the previous identity and substituting into Equation (58) it follows that

$$S_{\hat{\gamma}} = f^{-2/(n-2)}\left[S_\gamma - 2(\frac{1}{f}\Delta f - \frac{1}{f^2}|\nabla f|^2) - \frac{(n-3)}{(n-2)}\frac{1}{f^2}|\nabla f|^2\right],$$

finally,

$$S_{\hat{\gamma}} = f^{-2/(n-2)}\left[S_\gamma - 2\frac{\Delta f}{f} + \frac{(n-1)}{(n-2)}\frac{|\nabla f|^2}{f^2}\right]. \tag{59}$$

As f is the positive eigenfunction associated to the eigenvalue μ_1, we have

$$\mathcal{L}_0(f) = -\Delta f + \mathcal{Q}f = \mu_1 f, \tag{60}$$

and, substituting Equation (57) into Equation (59), we arrive at the equality,

$$S_{\hat{\gamma}} = f^{-2/(n-2)}\left[2\mu_1 + 2P + \frac{(n-1)}{(n-2)}\frac{|\nabla f|^2}{f^2}\right]. \tag{61}$$

Notice the following fact: Lemma 4.2.5 is independent of $Q \in C^\infty(M)$, consequently, that lemma holds for the generalized version $\mathcal{L}$. Therefore, we can conclude that $\mu_1 \geq \lambda_1 \geq 0$.

Since all terms in the brackets above are non-negative, Equation (61) implies that $S_{\hat{\gamma}} \geq 0$. According to Theorem 5.1.4, if $S_{\hat{\gamma}} > 0$ at some point, then Σ belongs to class (A), and as a result, Σ carries a metric of strictly positive scalar curvature.

On the other hand, if $S_{\hat{\gamma}}$ vanishes identically, then by Equation (61) we infer that $\mu_1 \equiv 0$, $P \equiv 0$, and f is a positive constant. Furthermore, the inequality $\mu_1 \geq \lambda_1 \geq 0$ implies that $\lambda_1 = 0$. From these facts and Equation (60), we find that $\mathcal{Q}$ vanishes identically. Moreover, as $P \equiv 0$ we have

$$\mathcal{Q} = \frac{1}{2}S_\gamma - P = \frac{1}{2}S_\gamma,$$

so S_γ vanishes identically. Then, Σ admits a metric whose scalar curvature is vanishes identically. By Theorem 5.1.4, we conclude that Σ falls into either class (A) or (B). If (Σ, γ) is not Ricci-flat, then Σ is in class (A) and admits a metric of positive scalar curvature. However, if (Σ, γ) is Ricci-flat, it falls into class (B). From this, the statement is proven. $\square$

Corollary 5.2.3 . *Lemma 5.2.2 is applicable for the stability operator L, as in Definition 4.2.2, with Q given by Equation (48).*

First, let us recall some definitions. Let Σ^{n-1} be a closed MOTS (compact without boundary) in an initial data set $(S^n, h, \mathcal{K})$. We say that an initial data set obeys the DEC if holds that $\rho \geq |J|_h$, as defined on Definition 2.2.6. Let $\vec{n}$ denote the outward pointing unit normal vector field of Σ in S. In this context, the null expansion θ_+ is explicitly defined by Equation (42). For convenience, we will henceforth omit the positive sign in the notation. With these definitions in place, we are now prepared to state our first theorem.

Theorem 5.2.4 (GALLOWAY; SCHOEN, Richard, 2006; GALLOWAY, 2018). *Let $(S^n, h, \mathcal{K})$, $n \geq 3$, be an initial data obeying the DEC. If Σ^{n-1} is a weakly outermost closed MOTS in S^n then Σ is of positive Yamabe type, unless Σ is Ricci-flat (flat if $n = 3, 4$) in the induced metric, $(J(\vec{n}) + \rho)$ vanishes identically on Σ and $\lambda_1(L) = 0$.*

Proof. Let $\vec{n}$ be the outward pointing unit normal vector field of Σ in S, and let $i : \Sigma^{n-1} \hookrightarrow S^n$ be the inclusion map of Σ into S. Let $\psi \in C^\infty(\Sigma)$, and define the normal variation of i by the smooth map $\Phi(x,t) : \Sigma \times (-t_0, t_0) \to S$, where $t_0 > 0$, given by $\Phi(x,t) = \exp_x^\perp(t\psi(x)\vec{n}_x)$. As Σ is compact, there is a tubular neighborhood containing Σ and the map is well-defined for t_0 small enough. In particular, the normal variation vector field of Φ is given by $V = \psi\vec{n}$.

For any $t \in (-t_0, t_0)$, we denote $\Sigma_t := \Phi_t(\Sigma)$ and let $\theta(t)$ denote the null expansion of the hypersurface Σ_t, i.e., $\theta(t) := H_{\Sigma_t} + \operatorname{tr}_{\Sigma_t}\mathcal{K}$, where H_{Σ_t} is the mean curvature of Σ_t in S associated with the outward pointing unit normal vector field $\vec{n}_t(x) = \frac{1}{\psi}\frac{\partial\Phi}{\partial t}(x,t)$ of Σ_t and $\operatorname{tr}_{\Sigma_t}\mathcal{K}$ is the trace of $\mathcal{K}$ on Σ_t in the induced metric $(\Phi_t^* h)$. As Proposition 4.2.1 shows, we have

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L(\psi) = -\Delta\psi + 2\langle X, \nabla\psi\rangle + (Q + \operatorname{div} X - |X|^2)\psi, \tag{62}$$

$$Q = \frac{1}{2}S_\Sigma - (\rho + J(\vec{n})) - \frac{1}{2}\|\mathcal{K}_{\vec{n}} + \mathcal{K}\|^2, \tag{63}$$

where $\left.\frac{\partial}{\partial t}\right|_{t=0} := V$, S_Σ is the scalar curvature of Σ, $\mathcal{K}_{\vec{n}}$ is the second-fundamental form of Σ associated with $\vec{n}$, X is a specific vector field on Σ and $\langle\cdot,\cdot\rangle$ denotes the induced metric on Σ.

Let λ_1 be the principal eigenvalue of L as in Lemma 4.2.3, which recall, is real. We can choose the function ψ above to be a strictly positive principal eigenfunction. Using such an eigenfunction ψ to define our variation, we have from Equation (62),

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L(\psi) = \lambda_1\psi. \tag{64}$$

The eigenvalue λ_1 cannot be negative; otherwise, Equation (64) would imply $\frac{\partial\theta}{\partial t} < 0$ on Σ. Since we have $\theta = 0$ on Σ, this would mean that, for $t > 0$ sufficiently small, Σ_t would be outer trapped and homologous to Σ, contrary to our assumptions of Σ being a weakly outermost MOTS (see Definition 4.1.17). Hence, $\lambda_1 \geq 0$, and, consequently, Σ is stable (see Definition 4.2.4). By applying Lemma 5.2.2 to L with

$$P = (\rho + J(\vec{n})) + \frac{1}{2}\|\mathcal{K}_{\vec{n}} + \mathcal{K}\|^2 \geq 0,$$

which is greater than zero by the DEC hypothesis, we show that Σ admits a metric of positive scalar curvatures unless $\rho + J(\vec{n}) = 0$, $\lambda_1 = 0$ on Σ and Σ is Ricci-flat in the induced metric. □

As one may observe, under special conditions in four-dimensional spacetimes, the above theorem still permits the existence of a toroidal topology for Σ^2. In other words, Σ may fall into class (B) as defined in Theorem 5.1.4, and consequently, it may possess toroidal topology. Typically, one would like to rule out the existence of black hole spacetimes with toroidal topology in dimension four, which suggests that improvements to the theorem may be desirable.

5.3 MOTS TOPOLOGY: SECOND THEOREM

In this section, we will study the consequences of $\lambda_1(L) = 0$, eliminate the exceptional circumstance of the Theorem 5.2.4 and reformulate it assuming that DEC holds only on the initial data. The key element for the proof of the initial version in Theorem 5.3.4 below is the splitting Lemma 5.3.2 together with the following analytical result.

Lemma 5.3.1 . *Let Σ be a MOTS within an initial data set $(S^n, h, \mathcal{K})$. If $\lambda_1(L) = 0$, then the adjoint operator L^* (with respect to the L^2 product) also has a simple eigenvalue $\lambda_1(L^*) = 0$. Furthermore, there exists a smooth positive eigenfunction ϕ^* of L^* associated with this eigenvalue.*

Before we proceed with the proof, let us recall some important concepts. A bounded linear operator $T : X \to Y$ between Banach space is called *Fredholm* if it has closed range and both its kernel and cokernel (defined as $\operatorname{coker}(T) := Y/\operatorname{Im}(T)$) are finite-dimensional. The *index* of a Fredholm operator is defined as the difference between the dimensions of its kernel and cokernel, denoted as

$$\operatorname{ind} T := \dim\ker T - \dim\operatorname{coker} T.$$

Proof of Lemma 5.3.1. Firstly, we redefine the MOTS stability operator in the context of Hilbert spaces, viewing it as a bounded linear map $L : W^{2,2}(M) \to L^2(M)$ (see (HEBEY, 2000) or (AUBIN, 1998) for the definition of functional spaces on manifolds).

From (LEE, D. A., 2019, Corollary A.9), it follows that the operator $L : W^{2,2}(M) \to L^2(M)$ is a Fredholm operator with index 0. By the definition of index, we have $\dim(\ker L) = \dim\operatorname{coker}(L)$. In the case of Hilbert spaces, we can show that $\operatorname{coker}(L) \approx \operatorname{Im}(L)^{\perp}$ and it is straightforward to see that $\operatorname{Im}(L)^{\perp} = \ker L^*$. Therefore,

$$\dim\ker(L) = \dim\operatorname{coker}(L) = \dim\operatorname{Im}(L)^{\perp} = \dim\ker(L^*).$$

Since $\lambda_1(L) = 0$ is a simple eigenvalue, we conclude that $\dim\ker(L^*) = 1$. As a result, there exists a eigenfunction $\phi^* \in \ker(L^*)$, unique up to multiplication by a constant, associated with the eigenvalue $\lambda_1(L^*) = 0$. Furthermore, L^* is also a elliptic operator, by the elliptic regularity (LEE, D. A., 2019, Theorem A.4), as $L^*\phi^* = 0$ and L has smooth coefficients, then ϕ^* is smooth.

To show that we can choose the eigenfunction ϕ^* to be positive, we write $\phi^* = \phi^*_+ - \phi^*_-$, where $\phi^*_\pm \geq 0$ denote the positive and negative parts of ϕ^*, respectively. Next, we choose a large enough constant $C > 0$ such that the strong maximum principle can be applied to the operator $L^* - C$ (see (GILBARG; TRUDINGER, 2001, Theorem 3.5) for the classical version or (LEE, D. A., 2019, Theorem A.2) for the Riemannian version). Since $(L^* - C)\phi^*_\pm = -C\phi^*_\pm \leq 0$, the strong maximum principle implies that if ϕ^*_+ attains a nonpositive minimum[2] on Σ, then ϕ^*_+ must be constant on all Σ. Consequently, either $\phi^*_+ > 0$ on Σ or $\phi^*_+ \equiv 0$ on Σ. Similar arguments and consequences apply to ϕ^*_-. However, it is not possible for both ϕ^*_+ and ϕ^*_- to be positive simultaneously. Therefore, we can choose $\phi^* > 0$ as the positive eigenfunction. □

Lemma 5.3.2 (GALLOWAY, 2008). *Let (Σ, γ) be a MOTS within an initial data set $(S^n, h, \mathcal{K})$. If $\lambda_1(L) = 0$, where L is the MOTS stability operator, then up to isometry, there exists a neighborhood W of Σ such that:*

1. *$W = (-t_0, t_0) \times \Sigma$ and $h\big|_W$ has the orthogonal decomposition,*

$$h\big|_W = \phi^2 dt^2 + \gamma_t$$

   where $\phi = \phi(t, x)$ and γ_t is the induced metric on $\Sigma_t = \{t\} \times \Sigma$.

2. *The outward null expansion $\theta(t)$ of each Σ_t is constant, with respect to $\vec{n}_t$, where $\vec{n}_t = \frac{1}{\phi}\frac{\partial}{\partial t}$ is the outward pointing unit normal vector field on Σ_t.*

Proof. Let $\vec{n}$ be the outward pointing unit normal vector field on Σ in S. For each smooth function $u \in C^\infty(\Sigma)$, we denote by $\Sigma[u]$ the image of map $F_u(x) = \exp^\perp_x(u(x)\vec{n}_x)$, for all $x \in \Sigma$ such that $|u(x)|$ is small enough throughout Σ. The map is well-defined and $F_u : \Sigma \to S$ describes an embedding such that the hypersurface $\Sigma[u] = F_u(\Sigma)$ is a hypersurface close to $\Sigma = \Sigma[0]$.

In the same sense of the Proof of Theorem 5.2.4, denote by $\theta(u)$ the null expansion of the hypersurface $\Sigma[u]$ with respect to the (suitably normalized) outward pointing unit normal vector field of $\Sigma[u]$. For a small ball $\mathcal{U} \subset C^\infty(\Sigma) \times \mathbb{R}$, we introduce the operator $\Theta : \mathcal{U} \subset C^\infty(\Sigma) \times \mathbb{R} \to C^\infty(\Sigma) \times \mathbb{R}$, such that

$$\Theta(u, k) = \left(\theta(u) - k, \int_\Sigma u\right), \quad (u, k) \in \mathcal{U}. \tag{65}$$

First, we calculate the directional derivative of Θ at $(0, 0)$ using the definitions regarding analysis in Banach Spaces (see Appendix B.1). Let $(u, k) \in \mathcal{U}$, then

$$\delta\Theta((0,0);(u,k)) = \left(\frac{\partial}{\partial t}\bigg|_{t=0} \theta(tu) - k, \int_\Sigma u\right).$$

[2] Remember that, for a smooth function u, if we replace the usual assumption $Lu \geq 0$ by $Lu \leq 0$, then the theorem still holds, with the "nonnegative maximum" replaced by "nonpositive minimum".

In particular, the variation of the family of smooth functions $\theta(tu)$ can be obtained by the variation $\Phi : (-t_0, t_0) \times \Sigma \to S$, $t_0 > 0$ small enough, such that $\Phi(x,t) := \exp_x^{\perp}(tu(x)\vec{n}_x)$ which has the variation vector field $V = u\vec{n}$. We already have determined the variation of this family a one parameter of the null expansion by Proposition 4.2.1, therefore, its linearization at $(0,0)$ is

$$\delta\Theta((0,0);(u,k)) = \left(L(u) - k, \int_{\Sigma} u\right). \tag{66}$$

In order to construct the desired neighborhood, our goal is to apply the inverse function theorem to the function Θ. However, before doing so, we need to address the regularity of this operator and ensure that the appropriate Banach spaces are being considered.

At this point, it is important to note that $\theta(u)$, in a coordinate frame, gives rise to a quasi-linear second-order differential operator. This operator can be extended globally to a differential operator $\theta : u \in C^{\infty}(\Sigma) \mapsto \theta(u) \in C^{\infty}(\Sigma)$. By the discussion in Appendix B.4, we can redefine the domain and codomain to Hölder spaces on manifolds, specifically $\theta : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$, where $\alpha \in (0,1]$. We can then conclude that the operator θ and $\Theta : \mathcal{U} \subset C^{2,\alpha}(\Sigma) \times \mathbb{R} \to C^{0,\alpha}(\Sigma) \times \mathbb{R}$ has regularity C^1, as explained in the appendix, where $\mathcal{U}$ is an appropriate open set. Additionally, it is worth noting that the Fréchet derivative coincides with the directional derivative. Henceforth, we will work with the redefined version of Θ on suitable Hölder spaces.

The kernel of $\theta'(0) = L$ consists only of constant multiples of the eigenfunction ϕ associated with the principal eigenvalue $\lambda_1(L)$, because, by hypothesis, $\lambda_1(L) = 0$, and the eigenfunction is unique up to a multiplicative constant, as stated in Lemma 4.2.3.

From Lemma 5.3.1, let ϕ^* be the smooth eigenfunction of the adjoint operator L^* associated with the simple eigenvalue $\lambda(L^*) = 0$. As the kernel of L and L^* has dimension one, we can use the following standard result known as *solvability criterion*: the equation $Lu = f$ is solvable if and only if $\int f\phi^* = 0$ (AUBIN, 1998, p. 126). From these facts, it will follow that Θ has an invertible derivative at $(0,0)$.

Injectivity: Let $(u,k) \in \mathcal{U}$ such that $(L(u) - k, \int_{\Sigma} u) = (0,0)$. By the solvability criterion, we have $\int_{\Sigma} k\phi^* = 0$, where $\phi^* > 0$ is the positive eigenfunction of L^* associated with $\lambda_1(L^*) = 0$. Thus, $k = 0$ since k is a constant and $\phi^* > 0$. As a consequence, $u \in \ker L$ because $Lu = 0$. Furthermore, since $\lambda_1(L) = 0$ is a simple eigenvalue, there exists some constant $c \in \mathbb{R}$ such that $u = c\phi$, where ϕ is the eigenfunction associated with $\lambda_1(L)$. However, since $\int_{\Sigma} u = 0$ and $\phi > 0$, it follows that $c = 0$. Therefore, we conclude that $(u,k) = (0,0)$.

Surjectivity: Given $(f,s) \in C^{0,\alpha}(\Sigma) \times \mathbb{R}$, our goal is to find $(u,k) \in \mathcal{U} \subset C^{2,\alpha}(\Sigma) \times \mathbb{R}$ such that $(L(u) - k, \int_{\Sigma} u) = (f,s)$. We choose the constant $k \in \mathbb{R}$ as

$$k := -\frac{\int_{\Sigma} f\phi^*}{\int_{\Sigma} \phi^*},$$

noting that $\int_{\Sigma}(f + k)\phi^* = 0$. From the solvability condition, there exists $u_0 \in C^{2,\alpha}(\Sigma)$ such that

$$L(u_0) = f + k \iff \int_\Sigma (f+k)\phi^* = 0.$$

We define the solution $u := u_0 + c\phi$ where

$$c = \frac{(s - \int_\Sigma u_0)}{\int_\Sigma \phi}.$$

It is important to observe that $\int_\Sigma u = s$ and $L(u) = L(u_0) = f + k$, since $\phi \in \ker L$ and L are linear. Therefore, with the chosen pair (u, k), the surjectivity of the operator is established.

Since Θ it is a map between Banach spaces, it is a linear homeomorphism by the open mapping theorem, we are thus allowed to apply the inverse function theorem as desired (see Theorem B.1.4). Therefore, combining that $\Theta'(0,0)$ is invertible and Θ has regularity C^1, the inverse function theorem guarantees that Θ is local diffeomorphism. Consequently, there exists $\varepsilon > 0$ and a smooth map $(u,k) : (-\varepsilon, \varepsilon) \to C^{2,\alpha}(\Sigma) \times \mathbb{R}$ such that

$$\Theta(u(t), k(t)) = \left(\theta(u(t)) - k(t), \int_\Sigma u(t)\right) = (0,t) \quad \forall t \in (-\varepsilon, \varepsilon) \tag{67}$$

The equation $\Theta(u(t), k(t)) = (0,t)$ implies that each surface $\Sigma_t := \Sigma[u(t)]$ has constant null expansion. Applying the chain rule, we have $(\theta \circ u(t))'(0) = \theta'(0)(u'(0)) = L(u'(0)) = k'(0)$. By the solvability condition, $k'(0)$ is orthogonal to ϕ^* which implies that $k'(0) = 0$. Furthermore, $u'(0) \in \ker \theta'(0) = \ker L$ and from $\int_\Sigma u(t) = t$, follows that $\int_\Sigma u'(0) = 1$. Hence, we conclude that $u'(0) = \text{const} \cdot \phi > 0$, and, in particular, $u(t)$ must be a smooth family of functions for t small enough.

This means that, for t sufficiently small, the hypersurfaces Σ_t form a smooth foliation of a neighborhood of Σ in S^n by hypersurfaces of constant null expansion. In particular, by taking ε sufficiently small, we can identify a tubular neighborhood of Σ in S^n with $\Sigma \times (-\varepsilon, \varepsilon)$, via the map $\Psi : \Sigma \times (-\varepsilon, \varepsilon) \to S^n$ defined by $\Psi(x,t) = F_{u(t)}(x)$.

Under the diffeomorphism Ψ, one can introduce coordinates (t, x^i) in a neighborhood W of Σ in S, such that, with respect to these coordinates, $W = (-t_0, t_0) \times \Sigma$, and for each $t \in (-t_0, t_0)$, the t-slice Σ_t has constant null expansion $\theta(t)$ with respect to $\vec{n}_t$, where $\vec{n}_t$ is the outward pointing unit normal vector field to each Σ_t in S. In addition, the coordinates (t, x^i) can be chosen so that $\frac{\partial}{\partial t} = \phi \vec{n}_t$, for some positive function $\phi = \phi(t, x_i)$ on W. □

The lemma above does not provide information about the sign of each $\theta(t)$, but it remains constant on each leaf. However, assuming that the initial data set satisfies the DEC and that Σ is weakly outermost MOTS, we can achieve a stronger version, where each slice Σ_t has identically vanishing constant expansion, i.e., each Σ_t is a MOTS, and immediately implies Theorem 5.3.4.

Theorem 5.3.3 . *(GALLOWAY, 2018) Let $(S^n, h, \mathcal{K})$, $n \geq 3$, be an initial data set satisfying the DEC, $\rho \geq |J|_h$. Suppose Σ is a weakly outermost MOTS in S^n that does not admit a metric of positive scalar curvature. Then, there exists an outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$*

of Σ such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \varepsilon)$ is a MOTS. In fact, each such slice has vanishing outward null second fundamental form, and is Ricci-flat.

Proof. Since Σ is a weakly outermost MOTS in S^n, Σ is stable (see proof of Theorem 5.2.4) and then $\lambda_1(L) \geq 0$. Given the hypothesis that Σ does not admit a metric of positive scalar curvature and applying Lemma 5.2.2 to L, with $P = (\rho + J(\vec{n})) + \frac{1}{2}\|\mathcal{K}_{\vec{n}} + \mathcal{K}\|^2$, leads to $\lambda_1 = 0$. Therefore, there exists a neighborhood $W = (-t_0, t_0) \times \Sigma$ of Σ with the properties specified in Lemma 5.3.2. For all values of t in the interval $(-t_0, t_0)$, the outward null expansion $\theta_+ = \theta_+(t)$ of the surface Σ_t has constant value. By Remark 6, the null expansion $\theta_+ = \theta_+(t)$ of the foliation can be expressed by

$$\frac{\partial \theta_+}{\partial t}(t) = L_{\Sigma_t}(\psi_t) = -\Delta\psi_t + 2\langle X, \nabla\psi_t\rangle + (Q + \frac{1}{2}\theta_+[\theta_- + 2\mathcal{K}(\vec{n}, \vec{n})] + \operatorname{div} X - |X|^2)\psi_t, \quad (68)$$

where it is to be understood that, for each t, the above terms are defined on Σ_t. For example, $\Delta = \Delta_t$ is the Laplacian on Σ_t, $Q = Q_t$ is now defined on Σ_t, and so on.

The constancy of each $\theta_+(t)$, together with the assumption that Σ is weakly outermost, implies that $\theta_+(t) \geq 0$ for all $t \in [0, t_0)$. Let $\varepsilon \in (0, t_0)$, we aim to demonstrate that $\theta_+(t) = 0$ for all $t \in [0, \varepsilon)$. For this purpose, we will rewrite the Equation (68) as follows,

$$L_{\Sigma_t}(\psi_t) - \frac{1}{2}\theta_+[\theta_- + 2\mathcal{K}(\vec{n}, \vec{n})]\psi_t = L^0_{\Sigma_t}(\psi_t), \quad (69)$$

where

$$L^0_{\Sigma_t}(\psi_t) = -\Delta\psi_t + 2\langle X, \nabla\psi_t\rangle + (Q + \operatorname{div} X - |X|^2)\psi_t. \quad (70)$$

Now, on $[0, \varepsilon] \times \Sigma$, we choose a constant C such that $\frac{1}{2}[\theta_- + 2\mathcal{K}(\vec{n}, \vec{n})]\psi_t \leq C$. Therefore, combining Equation (69) and the nonnegativity of θ_+, we obtain that

$$L^0_{\Sigma_t}(\psi_t) \geq L_{\Sigma_t}(\psi_t) - C\theta_+ = e^{Ct}\frac{\partial}{\partial t}F(t), \quad \text{for all } t \in [0, \varepsilon), \quad (71)$$

where $F(t) = e^{-Ct}\theta_+(t)$. We have that $F(0) = 0$ and $F(t) \geq 0$ on $[0, \varepsilon)$. In order to demonstrate that $F(t) = 0$ on $[0, \varepsilon)$, we need to show that $F'(t) \leq 0$ for all $t \in [0, \varepsilon)$.

Firstly, suppose there exists $t \in [0, \varepsilon)$, such that $F'(t) > 0$. Then, the Equation (71) implies that $L^0_{\Sigma_t}(\psi_t)$ is greater than zero, consequently, by Lemma 4.2.3, $\lambda_1(L^0_{\Sigma_t}) > 0$. Applying Lemma 5.2.2 to the operator $L^0_{\Sigma_t}$, where, in this case, $P = P_t = \rho_t + J(\vec{n_t}) + \frac{1}{2}\|\mathcal{K}_{\vec{n}_t} + \mathcal{K}\|_t^2 \geq 0$, Σ_t carries a metric of positive scalar curvature. Note that Σ and Σ_t are diffeomorphic and denote by $\Psi : \Sigma \to \Sigma_t$ the diffeomorphism. Let $\widetilde{\gamma}_t$ be the metric of positive scalar curvature in Σ_t, then the semi-Riemannian manifolds $(\Sigma_t, \widetilde{\gamma}_t)$ and $(\Sigma, \Psi^*\widetilde{\gamma}_t)$ are isometric, as a result, $(\Sigma, \Psi^*\widetilde{\gamma}_t)$ admits a metric of positive scalar curvature, which contradicts the hypothesis.

Therefore, we have that $F(t) = 0$, and hence, $\theta_+(t) = 0$ for all $t \in [0, \varepsilon)$, i.e., each Σ_t is a MOTS and does not admits a metric of positive scalar curvature. Since, by Equation (69), $L^0_{\Sigma_t}(\psi_t) = 0$, Lemma 4.2.3 implies $\lambda_1(L^0_{\Sigma_t}) \geq 0$ for each $t \in [0, \varepsilon)$. Finally, by Lemma 5.2.2, we obtain that for each $t \in [0, \varepsilon)$, $\chi_t := (\mathcal{K}_{\vec{n}_t} + \mathcal{K}) = 0$ and Σ_t is Ricci-flat. $\square$

Theorem 5.3.4 . *Let $(S^n, h, \mathcal{K})$, $n \geq 3$ be an initial data set satisfying the DEC. If Σ^{n-1} is an outermost MOTS in $(S^n, h, \mathcal{K})$ then Σ admits a metric of positive scalar curvature.*

Proof. Suppose that Σ does not admit a metric of positive scalar curvature. In this case, Theorem 5.3.3 holds and there exists an outer neighborhood of Σ in S such that each leaf is a MOTS. However, this contradicts the fact that Σ is an outermost MOTS. $\square$

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Appendix

APPENDIX A – STABILITY OPERATOR

Proof of Proposition 4.2.1. Let $\phi : \Sigma^{n-1} \to M^{n+1}$ be a smooth spacelike immersion in the spacetime (M,g) with trivial normal bundle[1]. Then there exist $K_+, K_- \in \mathfrak{X}^\perp(\phi)$ which are future directed null and we can choose $g(K_+, K_-) = -2$. Furthermore, the outward null expansion scalar is defined as $\theta = -(n-1)g(K_+, H^\phi)$ where H^ϕ is the mean curvature vector field of Σ and we shall assume that $\theta = 0$.

We now consider a variation of ϕ in M. Let $\Phi : (-\delta, \delta) \times \Sigma \to M$ be a variation where $\Phi(0,x) = \phi(x)$, denote $\Phi_t(x) := \Phi(t,x)$ and take V as the associated normal variation vector field.

As discussed in Theorem 1.2.2, suppose that Φ_t is a spacelike immersion $\forall t \in (-\delta, \delta)$ and $\exists\ l_t \in \mathfrak{X}^\perp(\Phi_t)$ which is null future-pointing and $l_0 = K_+$. Analogously, define $\theta(t) := -(n-1)g(l_t, H^{\Phi_t})$ for each Σ_t.

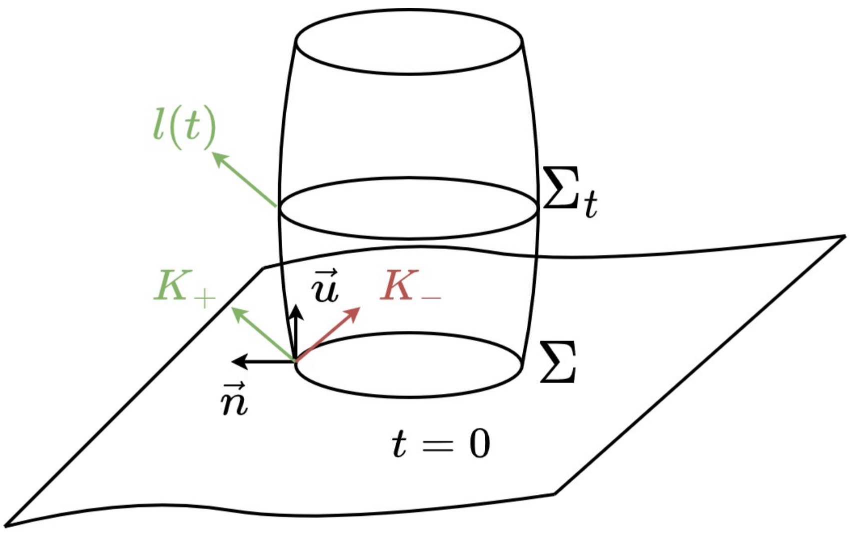

Figure A.1 – Variation Σ_t of a MOTS.

Fix $x \in \Sigma$, (U, x^a) chart on Σ and $(\widetilde{U}, \zeta^i)$ chart on M such that $\Phi_t(U) \subset \widetilde{U}$ and let $p := \Phi_t(x)$. In coordinates, it holds that $l_t = l^i(t)\frac{\partial}{\partial \zeta^i} \circ \Phi_t$ and $h = \phi^* g$ is the induced Riemannian metric on Σ and $h_t = \Phi_t^* g$ is the induced metric on Σ_t. Therefore, writing $\theta(t)$ in coordinates:

$$
\begin{aligned}
\theta(t) &= -(n-1)g_p(l_t, H^{\Phi_t}),\\
&= -(n-1)g_p(l^i(t)\frac{\partial}{\partial \zeta^i} \circ \Phi_t, \frac{1}{n-1}h_t^{ab} II^{\Phi_t}(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})),\\
&= -g_p(l^i(t)\frac{\partial}{\partial \zeta^i} \circ \Phi_t, h_t^{ab} D^{\Phi_t}_{\frac{\partial}{\partial x^a}}(d\Phi_t(\frac{\partial}{\partial x^b}))),
\end{aligned}
$$

[1] What follows is a very long calculation. We carry it out for the more general case of immersions, but to avoid notational conflict with earlier versions of the text, for simplicity we have changed here our conventions regarding the normal null fields and normalization of the mean value curvature vector. Namely, comparing with the main text we take here $K_\pm := \sqrt{2}\ell_\pm$ and $H^\phi := \frac{\vec{H}^\phi}{n-1}$, where the right-hand side of these equations are the quantities as defined in the main text, while the left-hand side are the versions we adopt in this appendix.

since $l_t \in \mathfrak{X}^{\perp}(\Phi_t)$. Now, writing the covariant derivative in theses coordinates:

$$D^{\Phi_t}_{\frac{\partial}{\partial x^b}}(d\Phi_t(\frac{\partial}{\partial x^b})) = (\frac{\partial^2 \Phi^j_t}{\partial x^a \partial x^b} + \Gamma^j_{lk} \circ \Phi_t \frac{\partial \Phi^l_t}{\partial x^a}\frac{\partial \Phi^k_t}{\partial x^b})\frac{\partial}{\partial \zeta^j} \circ \Phi_t,$$

and working on the coordinate expression of $\theta(t)$ again we arrive at

$$\theta(t) = -h^{ab}_t (g_{ij} \circ \Phi_t) l^i(t) \cdot (\frac{\partial^2 \Phi^j_t}{\partial x^a \partial x^b} + \Gamma^j_{lk} \circ \Phi_t \frac{\partial \Phi^l_t}{\partial x^a}\frac{\partial \Phi^k_t}{\partial x^b}).$$

Define $A^j_{ab}(t) := (\frac{\partial^2 \Phi^j_t}{\partial x^a \partial x^b} + \Gamma^j_{lk} \circ \Phi_t \frac{\partial \Phi^l_t}{\partial x^a}\frac{\partial \Phi^k_t}{\partial x^b})$, taking the derivative of the previous equation and applying the Leibniz's rule:

$$\begin{aligned}
\theta'(0) = &- \left.\frac{\partial h^{ab}_t}{\partial t}\right|_{t=0} (g_{ij} \circ \Phi_t) l^i(0) \cdot A^j_{ab}(0) \\
&- h^{ab}_t \left.\frac{\partial (g_{ij} \circ \Phi_t)}{\partial t}\right|_{t=0} l^i(0) \cdot A^j_{ab}(0) \\
&- h^{ab}_t (g_{ij} \circ \Phi_t) \left.\frac{\partial l_i}{\partial t}\right|_{t=0} \cdot A^j_{ab}(0) \\
&- h^{ab}_t (g_{ij} \circ \Phi_t) l_i(0) \cdot \left.\frac{\partial A^j_{ab}}{\partial t}\right|_{t=0},
\end{aligned}$$

from previous definitions and the chain-rule the following holds:

$$\begin{aligned}
\theta'(0) = &- \left.\frac{\partial h^{ab}_t}{\partial t}\right|_{t=0} g_{\phi(x)}(K_+, II^{\phi}(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})) \\
&- h^{ab}\left(\frac{\partial g_{ij}}{\partial \zeta^k} \circ \phi\right) V^k l^i(0) \cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j \\
&- h^{ab}(g_{ij} \circ \phi) \left.\frac{\partial l_i}{\partial t}\right|_{t=0} \cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j \\
&- h^{ab}(g_{ij} \circ \phi) K^i_+ \cdot \left.\frac{\partial A^j_{ab}}{\partial t}\right|_{t=0}.
\end{aligned}$$

There are some more involved terms that will be worked out one by one. Differentiation of $A^j_{ab}(t)$ at $t = 0$ is easily handled using the definition of the variation vector field of Φ_t and that $\Phi(x, 0) = \phi(x)$ yielding

$$\left.\frac{\partial A^j_{ab}}{\partial t}\right|_{t=0} = \frac{\partial^2 V^j}{\partial x^a \partial x^b} + (\frac{\partial \Gamma^j_{lk}}{\partial \zeta^m} \circ \phi) V^m \frac{\partial \phi^k}{\partial x^a}\frac{\partial \phi^l}{\partial x^b} + (\Gamma^j_{lk} \circ \phi)\left(\frac{\partial V^l}{\partial x^a}\frac{\partial \phi^k}{\partial x^b} + \frac{\partial \phi^l}{\partial x^a}\frac{\partial V^k}{\partial x^b}\right). \quad (72)$$

Proceeding with $\left.\frac{\partial h^{ab}_t}{\partial t}\right|_{t=0}$, notice that given an invertible matrix M(t), i.e. $M(t)M^{-1}(t) = I_{n-1}$, we have $\frac{dM^{-1}}{dt} = -M^{-1}\frac{dM}{dt}M^{-1}$, then

$$-\left.\frac{\partial h^{ab}_t}{\partial t}\right|_{t=0} = h^{ac}\left.\frac{\partial (h_t)_{cd}}{\partial t}\right|_{t=0} h^{db},$$

noticing that $V \in \mathfrak{X}^{\perp}(\phi)$ and taking the coordinate version of Lemma 1.2.6 then

$$\frac{\partial (h_t)_{cd}}{\partial t}\bigg|_{t=0} = -2g(II^{\phi}(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d}),V)$$

and,

$$-\frac{\partial h_t^{ab}}{\partial t}\bigg|_{t=0} = -2h^{ac}h^{db}g(II^{\phi}(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d}),V)$$

The normal bundle of ϕ has rank two, then $K_{\pm}$ is a basis for each tangent space. Applying this fact and $g(K_+,K_-)=-2$ to the second fundamental form

$$II^{\phi}(X,Y) = -\frac{1}{2}g(II^{\phi}(X,Y),K_-)K_+ - \frac{1}{2}g(II^{\phi}(X,Y),K_+)K_- \quad \forall X,Y\in\mathfrak{X}(\Sigma), \tag{73}$$

and let $\chi_{\pm}(X,Y) = -g(K_{\pm},II^{\phi}(X,Y))\ \forall X,Y\in\mathfrak{X}(\Sigma)$ the second fundamental form associated with $K_{\pm}$ then

$$-\frac{\partial h_t^{ab}}{\partial t}\bigg|_{t=0} = -h^{ac}h^{db}\left(\chi_+(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d})g(K_-,V) + \chi_-(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d})g(K_+,V)\right),$$

with (73), note that

$$-\frac{\partial h_t^{ab}}{\partial t}\bigg|_{t=0} g(K_+,II^{\phi}(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})) = \frac{\partial h_t^{ab}}{\partial t}\bigg|_{t=0}\chi_+(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b}).$$

Define the following two quantities

$$(\chi_+,\chi_-)_h = h^{ab}h^{cd}\chi_+(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})\chi_-(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d}), \tag{74}$$

$$\|\chi_{\pm}\|_h^2 = h^{ab}h^{cd}\chi_{\pm}(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})\chi_{\pm}(\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^d}), \tag{75}$$

then

$$-\frac{\partial h_t^{ab}}{\partial t}\bigg|_{t=0} g(K_+,II^{\phi}(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})) = \|\chi_+\|_h^2\cdot g(K_-,V) + (\chi_+,\chi_-)_h\cdot g(K_+,V), \tag{76}$$

finally, with (13), (76) and $\frac{\partial g_{ij}}{\partial \zeta^k}\circ\phi = (g_{im}\Gamma^m_{jk}+g_{jm}\Gamma^m_{ik})\circ\phi$ we arrive at

$$\begin{aligned}\theta'(0) =& \|\chi_+\|_h^2\cdot g(K_-,V) + (\chi_+,\chi_-)_h\cdot g(K_+,V)\\ &- h^{ab}(g_{im}\Gamma^m_{jk}+g_{jm}\Gamma^m_{ik})\circ\phi\cdot V^k l^i(0)\cdot\left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j\\ &- h^{ab}(g_{ij}\circ\phi)\frac{\partial l_i}{\partial t}\bigg|_{t=0}\cdot\left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j\\ &- h^{ab}(g_{ij}\circ\phi)K_+^i\cdot\frac{\partial A^j_{ab}}{\partial t}\bigg|_{t=0}.\end{aligned}$$

We now work out the other terms. First, we change some indexes and separate terms in a convenient way

$$\begin{aligned}
\theta'(0) =& \|\chi_+\|_h^2 \cdot g(K_-,V) + (\chi_+,\chi_-)_h \cdot g(K_+,V) \\
&- h^{ab}(g_{ij}\circ\phi)\frac{\partial l_i}{\partial t}\bigg|_{t=0} \cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j \\
&- h^{ab}(g_{ij}\circ\phi)(\Gamma^i_{mk}\circ\phi)\cdot V^k l^m(0)\cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^j \\
&- h^{ab}(g_{ij}\circ\phi)(\Gamma^j_{mk}\circ\phi)V^k K^i_+ \cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^m \\
&- h^{ab}(g_{ij}\circ\phi)K^i_+ \cdot \frac{\partial A^j_{ab}}{\partial t}\bigg|_{t=0},
\end{aligned}$$

and, in coordinates, the covariant derivative components from Φ has the following expression

$$\begin{aligned}
D^{\Phi}_{\frac{\partial}{\partial t}}(l)\Big|_{t=0} &= \left[\frac{\partial l^i}{\partial t}(0) + (\Gamma^i_{mk}\circ\Phi)(0)\frac{\partial\Phi^k}{\partial t}(0)l^m(0)\right]\frac{\partial}{\partial\zeta^i}\circ\Phi(0), \\
&= \left[\frac{\partial l^i}{\partial t}(0) + (\Gamma^i_{mk}\circ\phi)V^k l^m(0)\right]\frac{\partial}{\partial\zeta^i}\circ\phi,
\end{aligned}$$

then, we re-express $\theta'(0)$ as

$$\begin{aligned}
\theta'(0) =& \|\chi_+\|_h^2 \cdot g(K_-,V) + (\chi_+,\chi_-)_h \cdot g(K_+,V) \\
&- h^{ab} g(D^{\Phi}_{\frac{\partial}{\partial t}} l\Big|_{t=0}, D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b})) \\
&- h^{ab}(g_{ij}\circ\phi)(\Gamma^j_{mk}\circ\phi)V^k K^i_+ \cdot \left(D^{\phi}_{\frac{\partial}{\partial x^a}}(d\phi(\frac{\partial}{\partial x^b}))\right)^m \\
&- h^{ab}(g_{ij}\circ\phi)K^i_+ \cdot \frac{\partial A^j_{ab}}{\partial t}\bigg|_{t=0}.
\end{aligned}$$

Developing the covariant derivative of ϕ, we can relate the previous equation with the curvature tensor, $\frac{\partial A^j_{ab}}{\partial t}\Big|_{t=0}$ and some other terms, but we need some massaging first. Notice that

$$\begin{aligned}
D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V &= D^{\phi}_{\frac{\partial}{\partial x^a}}\left[\frac{\partial V^j}{\partial x^b} + (\Gamma^j_{lk}\circ\phi)V^l\frac{\partial\phi^k}{\partial x^b}\right]\frac{\partial}{\partial\zeta^j}\circ\phi, \\
&= \left[\frac{\partial^2 V^j}{\partial x^a\partial x^b} + (\Gamma^j_{lk}\circ\phi)\frac{\partial\phi^l}{\partial x^a}\frac{\partial V^k}{\partial x^b}\right]\frac{\partial}{\partial\zeta^j}\circ\phi \\
&+ \left[(\frac{\partial\Gamma^j_{lm}}{\partial\zeta^k}\circ\phi)\frac{\partial\phi^k}{\partial x^a}\frac{\partial\phi^l}{\partial x^b}V^m + (\Gamma^j_{lk}\circ\phi)\frac{\partial V^l}{\partial x^a}\frac{\partial\phi^k}{\partial x^b} + (\Gamma^j_{nm}\circ\phi)V^m\frac{\partial^2\phi^n}{\partial x^a\partial x^b}\right]\frac{\partial}{\partial\zeta^j}\circ\phi \\
&+ \left[(\Gamma^j_{nm}\circ\phi)(\Gamma^n_{lk}\circ\phi)V^k\frac{\partial\phi^l}{\partial x^b}\frac{\partial\phi^m}{\partial x^a}\right]\frac{\partial}{\partial\zeta^j}\circ\phi,
\end{aligned}$$

combining Equation (72) with the previous equation, and changing some indices, we obtain

$$\begin{aligned} D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V = & \left[\left. \frac{\partial A^j_{ab}}{\partial t} \right|_{t=0} - \left(\frac{\partial \Gamma^j_{lk}}{\partial \zeta^m} \circ \phi\right) \frac{\partial \phi^k}{\partial x^a} \frac{\partial \phi^l}{\partial x^b} V^m \right] \frac{\partial}{\partial \zeta^j} \circ \phi \\ & + \left[\left(\frac{\partial \Gamma^j_{lm}}{\partial \zeta^k} \circ \phi\right) \frac{\partial \phi^k}{\partial x^a} \frac{\partial \phi^l}{\partial x^b} V^m + (\Gamma^j_{nm} \circ \phi) V^m \frac{\partial^2 \phi^n}{\partial x^a \partial x^b} \right] \frac{\partial}{\partial \zeta^j} \circ \phi \\ & + \left[(\Gamma^n_{lm} \circ \phi)(\Gamma^j_{nk} \circ \phi) \frac{\partial \phi^k}{\partial x^a} \frac{\partial \phi^l}{\partial x^b} V^m \right] \frac{\partial}{\partial \zeta^j} \circ \phi. \end{aligned}$$

Now, remember that,

$$R^j_{lkm} \partial_j = R(\partial_k, \partial_m) \partial_l, \text{ and } \quad R^j_{lkm} = \frac{\partial \Gamma^j_{lm}}{\partial \zeta^k} - \frac{\partial \Gamma^j_{lk}}{\partial \zeta^m} + \Gamma^n_{lm} \Gamma^j_{nk} - \Gamma^n_{lk} \Gamma^j_{nm}, \tag{77}$$

the curvature in coordinates into the second covariant derivative term yields:

$$\begin{aligned} D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V = & \left[\left. \frac{\partial A^j_{ab}}{\partial t} \right|_{t=0} + (\Gamma^j_{nm} \circ \phi) V^m \frac{\partial^2 \phi^n}{\partial x^a \partial x^b} \right] \frac{\partial}{\partial \zeta^j} \circ \phi \\ & + \left[\left(R^j_{lkm} + (\Gamma^n_{lk} \circ \phi)(\Gamma^j_{nm} \circ \phi) \right) \frac{\partial \phi^k}{\partial x^a} \frac{\partial \phi^l}{\partial x^b} V^m \right] \frac{\partial}{\partial \zeta^j} \circ \phi, \\ = & \left[\left. \frac{\partial A^j_{ab}}{\partial t} \right|_{t=0} + V^m (\Gamma^j_{nm} \circ \phi) \left(D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b}) \right)^n \right] \frac{\partial}{\partial \zeta^j} \circ \phi \\ & + \left[R(d\phi(\frac{\partial}{\partial x^a}), V) d\phi(\frac{\partial}{\partial x^b}) \right] \frac{\partial}{\partial \zeta^j} \circ \phi, \end{aligned}$$

that becomes, after rearranging and changing indices,

$$D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V + R(V, d\phi(\frac{\partial}{\partial x^a})) d\phi(\frac{\partial}{\partial x^b}) = \left[\left. \frac{\partial A^j_{ab}}{\partial t} \right|_{t=0} + V^k (\Gamma^j_{mk} \circ \phi) \left(D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b}) \right)^m \right] \frac{\partial}{\partial \zeta^j} \circ \phi.$$

Therefore, we end up with the more concise expression

$$\begin{aligned} \theta'(0) = & \|\chi_+\|^2_h \cdot g(K_-, V) + (\chi_+, \chi_-)_h \cdot g(K_+, V) \\ & - h^{ab} g(D^{\Phi}_{\frac{\partial}{\partial t}} l \Big|_{t=0}, D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b})) \\ & - h^{ab} g(D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V + R(V, d\phi(\frac{\partial}{\partial x^a})) d\phi(\frac{\partial}{\partial x^b}), K_+). \end{aligned}$$

Henceforth, geometric quantities will be identified and rewritten looking for dependencies on the metric h. Let $\vec{u}, \vec{n}$ be vector fields defined as $\vec{u} := \frac{K_+ + K_-}{2}$ and $\vec{n} := \frac{K_+ - K_-}{2}$. Clearly, $\vec{u}$ is timelike, $g(\vec{u}, \vec{u}) = -1$, and $\vec{n}$ spacelike, $g(\vec{n}, \vec{n}) = 1$, even more, $\vec{u}$ and $\vec{n}$ are orthogonal. Let $\{E_i\}$ a h-orthornomal frame and $E^a_i \in C^\infty(\Sigma)$ such that $h^{ab} = \sum_{i=1}^{n-1} E^a_i E^b_i$, the curvature term can be identified as Ricci, actually giving the Ricci tensor, since

$$\begin{aligned}
h^{ab} g(R(V, d\phi(\frac{\partial}{\partial x^a}))d\phi(\frac{\partial}{\partial x^b}), K_+) &= \sum_{i=1}^{n-1} g(R(V, d\phi(\frac{\partial}{\partial x^a}))d\phi(\frac{\partial}{\partial x^b}), K_+) \\
&= \sum_{i=1}^{n-1} E_i^a E_i^b g(R(V, d\phi(E_i))d\phi(E_i), K_+) \\
&= Ric(V, K_+) + g(R(V, \vec{u})\vec{u}, K_+) - g(R(V, \vec{n})\vec{n}, K_+) \\
&= Ric(V, K_+) - \frac{1}{2} g(R(V, K_+)K_+, K_-)
\end{aligned}$$

so,

$$\begin{aligned}
\theta'(0) =& \|\chi_+\|_h^2 \cdot g(K_-, V) + (\chi_+, \chi_-)_h \cdot g(K_+, V) \\
&- h^{ab} g(D^{\Phi}_{\frac{\partial}{\partial t}} l \Big|_{t=0}, D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b})) \\
&- h^{ab} g(D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V, K_+) \\
&- Ric(V, K_+) + \frac{1}{2} g(R(V, K_+)K_+, K_-).
\end{aligned}$$

Decomposing the covariant derivative of ϕ into tangent and normal parts with (73) yields

$$\begin{aligned}
D^{\phi}_{\frac{\partial}{\partial x^a}} d\phi(\frac{\partial}{\partial x^b}) &= d\phi\left(\nabla^{\Sigma}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b}\right) + \frac{1}{2}\chi_+(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})K_- + \frac{1}{2}\chi_-(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})K_+, \\
&= d\phi\left((\Gamma^{\Sigma})^c_{ab} \frac{\partial}{\partial x^c}\right) + \frac{1}{2}\chi_+(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})K_- + \frac{1}{2}\chi_-(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})K_+,
\end{aligned}$$

and focusing on the covariant derivative of Φ and on the tangent quantity, since $l \in \mathfrak{X}^{\perp}(\Phi)$, then

$$\begin{aligned}
g(D^{\Phi}_{\frac{\partial}{\partial t}} l \Big|_{t=0}, d\phi(\frac{\partial}{\partial x^c})) &= g(D^{\Phi}_{\frac{\partial}{\partial t}} l, d\Phi(\frac{\partial}{\partial x^c}))\Big|_{t=0}, \\
&= -g(l, D^{\Phi}_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial x^c}))\Big|_{t=0}, \\
&= -g(l, II(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial t}))\Big|_{t=0} \\
&= -g(l, D^{\Phi}_{\frac{\partial}{\partial x^c}} d\Phi(\frac{\partial}{\partial t}))\Big|_{t=0}, \\
&= -g(K_+, D^{\Phi}_{\frac{\partial}{\partial x^c}} V),
\end{aligned}$$

and in order to see that the normal components do not give any contribution let $\lambda_+, \lambda_- \in C^{\infty}(M)$ and denote $\chi_{\pm}(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = (\chi_{\pm})_{ab}$, so

$$g(D^{\Phi}_{\frac{\partial}{\partial t}} l \Big|_{t=0}, K_-) = g(\lambda_+ K_+, K_-) = -2\lambda_+,$$

$$g(D^{\Phi}_{\frac{\partial}{\partial t}} l \Big|_{t=0}, K_+) = g(\lambda_- K_-, K_+) = -2\lambda_-,$$

thus,

$$
\begin{aligned}
g(D^{\Phi}_{\frac{\partial}{\partial t}}l\Big|_{t=0}, \frac{1}{2}(\chi_+)_{ab}K_- + \frac{1}{2}(\chi_-)_{ab}K_+) &= g(\lambda_-K_- + \lambda_+K_+, \frac{1}{2}(\chi_+)_{ab}K_- + \frac{1}{2}(\chi_-)_{ab}K_+) \\
&= -\lambda_+(\chi_+)_{ab} - \lambda_-(\chi_-)_{ab} \\
&= -g(D^{\Phi}_{\frac{\partial}{\partial t}}l\Big|_{t=0}, \frac{1}{2}(\chi_+)_{ab}K_- + \frac{1}{2}(\chi_-)_{ab}K_+)
\end{aligned}
$$

which implies that

$$
g(D^{\Phi}_{\frac{\partial}{\partial t}}l\Big|_{t=0}, \frac{1}{2}(\chi_+)_{ab}K_- + \frac{1}{2}(\chi_-)_{ab}K_+) = 0.
$$

Employing the lasts three developments into the second line of $\theta'(0)$, then

$$
\begin{aligned}
\theta'(0) =& \|\chi_+\|^2_h \cdot g(K_-, V) + (\chi_+, \chi_-)_h \cdot g(K_+, V) \\
&+ h^{ab}(\Gamma^{\Sigma})^c_{ab} g(K_+, D^{\phi}_{\frac{\partial}{\partial x^c}}V) \\
&- h^{ab} g(D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V, K_+) \\
&- Ric(V, K_+) + \frac{1}{2} g(R(V, K_+)K_+, K_-),
\end{aligned}
\tag{78}
$$

The next step is to obtain a more convenient expression for $g(D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V, K_+)$. In order to reach this goal, define $\psi_{\pm} \in C^{\infty}(\Sigma)$ and $\omega \in \Omega^1(\Sigma)$ by

$$
V = \psi_+K_+ + \psi_-K_-,
$$

$$
\omega(X) := -\frac{1}{2} g(K_+, D^{\phi}_X K_-), \quad \forall X \in \mathfrak{X}(\Sigma),
$$

first, compute the following quantity with the new expression for V:

$$
\begin{aligned}
g(K_+, D^{\phi}_X V) &= g(K_+, [X(\psi_+)K_+ + \psi_+ D^{\phi}_X K_+] + [X(\psi_-)K_- + \psi_- D^{\phi}_X K_-]), \\
&= -2X(\psi_-) - 2\psi_-\omega(X) + \psi_+ g(K_+, D^{\phi}_X K_+), \\
&= -2X(\psi_-) - 2\psi_-\omega(X),
\end{aligned}
\tag{79}
$$

second, we develop the following expression:

$$
\begin{aligned}
g(K_+, D^{\phi}_X D^{\phi}_Y V) &= g(K_+, D^{\phi}_X \Big[Y(\psi_+)K_+ + \psi_+ D^{\phi}_Y K_+ + Y(\psi_-)K_- + \psi_- D^{\phi}_Y K_-\Big]), \\
&= g(K_+, [XY(\psi_+)K_+ + Y(\psi_+)D^{\phi}_X K_+] + [X(\psi_+)D^{\phi}_Y K_+ + \psi_+ D^{\phi}_X D^{\phi}_Y K_+] \\
&+ [XY(\psi_-)K_- + Y(\psi_-)D^{\phi}_X K_-] + [X(\psi_-)D^{\phi}_Y K_- + \psi_- D^{\phi}_X D^{\phi}_Y K_-]), \\
&= g(K_+, \psi_+ D^{\phi}_X D^{\phi}_Y K_+ \\
&+ [XY(\psi_-)K_- + Y(\psi_-)D^{\phi}_X K_-] + [X(\psi_-)D^{\phi}_Y K_- + \psi_- D^{\phi}_X D^{\phi}_Y K_-], \\
&= -2XY(\psi_-) - 2Y(\psi_-)\omega(X) - 2X(\psi_-)\omega(Y) \\
&+ \psi_+ g(K_+, D^{\phi}_X D^{\phi}_Y K_+) + \psi_- g(K_+, D^{\phi}_X D^{\phi}_Y K_-),
\end{aligned}
$$

then, with the metric compatibility in the last line we have

$$
\begin{aligned}
g(K_+, D^{\phi}_X D^{\phi}_Y V) &= -2XY(\psi_-) - 2Y(\psi_-)\omega(X) - 2X(\psi_-)\omega(Y) \\
&- \psi_+ g(D^{\phi}_X K_+, D^{\phi}_Y K_+) + \psi_-(2X(\omega(Y)) + g(D^{\phi}_X K_+, D_Y K_-)),
\end{aligned}
\tag{80}
$$

taking $\{d\phi(\frac{\partial}{\partial x^a}), K_+, K_-\}$ with $a = 1, \ldots, n-1$ as a basis for $D^\phi_X K_\pm$, with $A^a_\pm$ as coefficients in the tangent terms, then,

$$\begin{aligned} D^\phi_X K_\pm &= A^a_\pm(X) d\phi(\frac{\partial}{\partial x^a}) - \frac{1}{2} g(K_+, D^\phi_X K_\pm) K_- - \frac{1}{2} g(K_-, D^\phi_X K_\pm) K_+, \\ &= A^a_\pm(X) d\phi(\frac{\partial}{\partial x^a}) \mp \omega(X) K_\pm, \end{aligned}$$

looking closely at the coefficients $A^a_\pm(X)$, we observe that

$$\begin{aligned} g(d\phi(\frac{\partial}{\partial x^a}), D^\phi_X K_\pm) &= -g(D^\phi_X d\phi(\frac{\partial}{\partial x^a}), K_\pm), \\ &= -g(d\phi(\nabla^\Sigma_X \frac{\partial}{\partial x^a}) + II(X, \frac{\partial}{\partial x^a}), K_\pm), \\ &= -g(II(X, \frac{\partial}{\partial x^a}), K_\pm), \\ &= -\frac{1}{2} g(\chi_-(X, \frac{\partial}{\partial x^a}) K_+ + \chi_+(X, \frac{\partial}{\partial x^a}) K_-, K_\pm), \\ &= -\frac{1}{2} \chi_+(X, \frac{\partial}{\partial x^a}) g(K_\mp, K_\pm), \\ &= \chi_\pm(X, \frac{\partial}{\partial x^a}), \end{aligned}$$

then,

$$g(d\phi(\frac{\partial}{\partial x^b}), D^\phi_X K_\pm) = g(d\phi(\frac{\partial}{\partial x^b}), A^a_\pm(X) d\phi(\frac{\partial}{\partial x^a})) = A^a_\pm(X) h_{ab},$$

$$\implies A^a_\pm(X) = h^{ab} \chi_\pm(X, \frac{\partial}{\partial x^b}),$$

$$D^\phi_X K_\pm = h^{ab} \chi_\pm(X, \frac{\partial}{\partial x^b}) d\phi(\frac{\partial}{\partial x^a}) \mp \omega(X) K_\pm,$$

and notice that

$$g(D^\phi_X K_+, D^\phi_Y K_+) = h^{ef} h^{lm} h_{el} \chi_+(X, \frac{\partial}{\partial x^f}) \chi_+(Y, \frac{\partial}{\partial x^m}),$$

$$g(D^\phi_X K_+, D^\phi_Y K_-) = h^{ef} h^{lm} h_{el} \chi_+(X, \frac{\partial}{\partial x^f}) \chi_-(Y, \frac{\partial}{\partial x^m}) + 2\omega(X)\omega(Y).$$

Now, returning to Equation (80), and changing the vector fields X and Y by $\frac{\partial}{\partial x^a}$ and $\frac{\partial}{\partial x^b}$ and contracting h^{ab}:

$$
\begin{aligned}
h^{ab} g(K_+, D^{\phi}_{\frac{\partial}{\partial x^a}} D^{\phi}_{\frac{\partial}{\partial x^b}} V) = {} & h^{ab} \left[-2 \frac{\partial^2 \psi_-}{\partial x^a \partial x^b} - 2 \frac{\partial \psi_-}{\partial x^b} \omega_a - 2 \frac{\partial \psi_-}{\partial x^a} \omega_b - 2 \psi_- \frac{\partial \omega_b}{\partial x^a} \right. \\
& \left. - \psi_+ g(D^{\phi}_{\frac{\partial}{\partial x^a}} K_+, D^{\phi}_{\frac{\partial}{\partial x^b}} K_+) - \psi_- g(D^{\phi}_{\frac{\partial}{\partial x^a}} K_+, D_{\frac{\partial}{\partial x^b}} K_-) \right], \\
= {} & h^{ab} \left[-2 \frac{\partial^2 \psi_-}{\partial x^a \partial x^b} - 2 \frac{\partial \psi_-}{\partial x^b} \omega_a - 2 \frac{\partial \psi_-}{\partial x^a} \omega_b - 2 \psi_- \frac{\partial \omega_b}{\partial x^a} \right. \\
& - \psi_+ \left(h^{ef} h^{lm} h_{el} \chi_+ (\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^f}) \chi_+ (\frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^m}) \right) \\
& \left. - \psi_- \left(h^{ef} h^{lm} h_{el} \chi_+ (\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^f}) \chi_- (\frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^m}) + 2 \omega_a \omega_b \right) \right], \\
= {} & -2 h^{ab} \frac{\partial^2 \psi_-}{\partial x^a \partial x^b} - 4 h^{ab} \frac{\partial \psi_-}{\partial x^a} \omega_b - 2 h^{ab} \psi_- \frac{\partial \omega_b}{\partial x^a} \\
& - \psi_+ \|\chi_+\|_h^2 - \psi_- (\chi_+, \chi_-)_h - 2 \psi_- \|\omega\|_h^2 .
\end{aligned}
\tag{81}
$$

Next, insert Equations (79) and (81) and $V = \psi_+ K_+ + \psi_- K_-$ in Equation (78):

$$
\begin{aligned}
\theta'(0) = & - \psi_+ \|\chi_+\|_h^2 - \psi_- (\chi_+, \chi_-)_h \\
& - 2 h^{ab} (\Gamma^{\Sigma})^c_{ab} \frac{\partial \psi_-}{\partial x^c} - 2 h^{ab} (\Gamma^{\Sigma})^c_{ab} \psi_- \omega_c \\
& + 2 h^{ab} \frac{\partial^2 \psi_-}{\partial x^a \partial x^b} + 4 h^{ab} \frac{\partial \psi_-}{\partial x^a} \omega_b + 2 h^{ab} \psi_- \frac{\partial \omega_b}{\partial x^a} + 2 \psi_- \|\omega\|_h^2 \\
& - \psi_+ Ric(K_+, K_+) - \psi_- Ric(K_-, K_+) + \psi_- \frac{1}{2} g(R(K_-, K_+) K_+, K_-).
\end{aligned}
$$

Recall that

$$
\omega(\nabla_\Sigma f) = \omega(h^{ab} \frac{\partial f}{\partial x^b} \frac{\partial}{\partial x^a}) = h^{ab} \omega_a \frac{\partial f}{\partial x^b}, \tag{82}
$$

$$
div_h \omega = h^{ab} (\frac{\partial \omega_a}{\partial x^b} - (\Gamma^{\Sigma})^c_{ab} \omega_c), \tag{83}
$$

$$
\Delta_\Sigma f = h^{ab} Hess_\Sigma f(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = h^{ab} \left(\frac{\partial^2 f}{\partial x^a \partial x^b} - (\Gamma^S)^c_{ab} \frac{\partial f}{\partial x^c} \right), \tag{84}
$$

and applying the previous definitions leads to

$$
\begin{aligned}
\theta'(0) = & - \psi_+ \|\chi_+\|_h^2 \cdot - \psi_- (\chi_+, \chi_-)_h \\
& + 2 \Delta_\Sigma \psi_- + 2 div_h \omega \cdot \psi_- + 4 \omega(\nabla_\Sigma \psi_-) + 2 \psi_- \|\omega\|_h^2 \\
& - \psi_+ Ric(K_+, K_+) - \psi_- Ric(K_-, K_+) + \psi_- \frac{1}{2} g(R(K_-, K_+) K_+, K_-).
\end{aligned}
$$

From now on, our goal is to develop the Ricci and curvature quantities in $\theta'(0)$. The Gauss Equation in Theorem 1.1.16 establishes the relationship between curvature on M and on Σ: □

$$
\begin{aligned}
g(R_M(d\phi(X), d\phi(Y))d\phi(Z), d\phi(W)) = h(R_\Sigma(X,Y)Z, W)& \\
+ g(II^\phi(X,Z), II^\phi(Y,W)) - g(II^\phi(Y,Z), II^\phi(X,W))&,
\end{aligned}
$$

for all $X, Y\mathfrak{X}(\Sigma)$, and with (73),

$$
\begin{aligned}
g(R_M(d\phi(X), d\phi(Y))d\phi(Z), d\phi(W)) = h(R_\Sigma(X,Y)Z, W)& \\
- \frac{1}{2}\chi_+(X,Z)\chi_-(Y,W) - \frac{1}{2}\chi_-(X,Z)\chi_+(Y,W)& \\
+ \frac{1}{2}\chi_+(Y,Z)\chi_-(X,W) + \frac{1}{2}\chi_-(Y,Z)\chi_+(X,W)&,
\end{aligned}
$$

take $\{E_1, \ldots, E_{n-1}\}$ local h-orthonormal frame, then,

$$
\begin{aligned}
\sum_{i=1}^{n-1} g(R_M(d\phi(E_i), d\phi(Y))d\phi(Z), d\phi(E_i)) = Ric_\Sigma(Y,Z)& \\
- \frac{1}{2}\sum_{i=1}^{n-1}\left(\chi_+(E_i,Z)\chi_-(Y,E_i) + \chi_-(E_i,Z)\chi_+(Y,E_i)\right)& \\
+ \frac{1}{2}\sum_{i=1}^{n-1}\left(\chi_+(Y,Z)\chi_-(E_i,E_i) + \chi_-(Y,Z)\chi_+(E_i,E_i)\right)&, \\
= Ric_\Sigma(Y,Z) + \frac{1}{2}\theta_-\chi_+(Y,Z) + \frac{1}{2}\ \chi_-(Y,Z)\theta_+& \\
- \frac{1}{2}\sum_{i=1}^{n-1}\left(\chi_+(E_i,Z)\chi_-(Y,E_i) + \chi_-(E_i,Z)\chi_+(Y,E_i)\right)&,
\end{aligned}
$$

remember that $\theta_\pm = tr_h\chi_\pm$. Now, reintroduce the vectors

$$
\vec{u} := \frac{K_+ + K_-}{2},
$$

$$
\vec{n} := \frac{K_+ - K_-}{2},
$$

and, consequently, $\{\vec{u}, \vec{n}, d\phi(E_1), \ldots, d\phi(E_{n-1})\}$ g-orthonormal basis which allow us to restate Ric_M as

$$
\begin{aligned}
Ric_M(d\phi(Y), d\phi(Z)) = -g(R_M(\vec{u}, d\phi(Y))d\phi(Z), \vec{u}) + g(R_M(\vec{n}, d\phi(Y))d\phi(Z), \vec{n})& \\
+ Ric_\Sigma(Y,Z) + \frac{1}{2}\theta_-\chi_+(Y,Z) + \frac{1}{2}\ \chi_-(Y,Z)\theta_+& \\
- \frac{1}{2}\sum_{i=1}^{n-1}\left(\chi_+(E_i,Z)\chi_-(Y,E_i) + \chi_-(E_i,Z)\chi_+(Y,E_i)\right)&.
\end{aligned} \tag{85}
$$

Turning now to contracting the previous equation and using curvature scalar definition and the curvature properties, we have

$$
\begin{aligned}
S_g = -2Ric_M(\vec{u},\vec{u}) + 2Ric_M(\vec{n},\vec{n}) + 2g(R_M(\vec{u},\vec{n})\vec{n},\vec{u})& \\
+ S_h + \theta_-\theta_+& \\
- \frac{1}{2}\sum_{i,j=1}^{n-1}\left(\chi_+(E_i,E_j)\chi_-(E_j,E_i) + \chi_-(E_i,E_j)\chi_+(E_j,E_i)\right)&,
\end{aligned}
$$

since Σ is a MOTS, i.e. $\theta_+ = 0$, and from the $(\chi_+, \chi_-)_h$ definition, the scalar curvature of the metric g is expressed as

$$S_g = -2Ric_M(\vec{u}, \vec{u}) + 2Ric_M(\vec{n}, \vec{n}) + 2g(R_M(\vec{u}, \vec{n})\vec{n}, \vec{u}) + S_h - (\chi_+, \chi_-)_h, \tag{86}$$

writing again with K_+ and K_- we arrive at

$$(\chi_+, \chi_-)_h = -2Ric(K_+, K_-) + \frac{1}{2}g(R_M(K_-, K_+)K_+, K_-) + S_h - S_g.$$

Finally, substituting last equation into $\theta'(0)$ we get

$$\begin{aligned}\theta'(0) = &- \psi_+(\|\chi_+\|_h^2 + Ric(K_+, K_+))\\ &+ \psi_-[Ric(K_+, K_-) + S_g - S_h] + 2\Delta_S\psi_-\\ &+ 2div_h\omega \cdot \psi_- + 4\omega(\nabla_S\psi_-) + 2\psi_-\|\omega\|_h^2.\end{aligned}$$

The energy dominant condition, Definition 2.2.6, gives us a useful way to relate scalar curvature and Ricci curvature. Define the following quantities: $J(\vec{n}) := G(\vec{u}, \vec{n})$ and $\rho := G(\vec{u}, \vec{u})$, then,

$$J(\vec{n}) + \rho = G(\vec{u}, K_+) = \frac{1}{2}\left(Ric_M(K_+, K_+) + Ric_M(K_+, K_-) + S_g\right), \tag{87}$$

writing $v := \psi_+ + \psi_-$ and $\psi := -2\psi_-$, which implies that $V = vK_+ + \psi\vec{n}$. Therefore,

$$\begin{aligned}\theta'(0) = &- (v + \frac{\psi}{2})(\|\chi_+\|_h^2 + Ric(K_+, K_+))\\ &- \frac{\psi}{2}[Ric(K_+, K_-) + S_g - S_h] - \Delta_\Sigma\psi\\ &- div_h\omega \cdot \psi - 2\omega(\nabla_\Sigma\psi) - \psi\|\omega\|_h^2\end{aligned}$$

$$\begin{aligned}\theta'(0) = &- v(\|\chi_+\|_h^2 + Ric(K_+, K_+))\\ &- \psi\left[J(\vec{n}) + \rho - \frac{1}{2}S_h + \frac{1}{2}\|\chi_+\|_h^2\right] - \Delta_\Sigma\psi\\ &- div_h\omega \cdot \psi - 2\omega(\nabla_\Sigma\psi) - \psi\|\omega\|_h^2\end{aligned}$$

Defining $Z \in \mathfrak{X}(\Sigma)$ such that $\omega(X) = -h(X, Z)$ for all $X \in \mathfrak{X}(\Sigma)$ and noticing that $\|\omega\|_h^2 = \|Z\|_h^2$ and $div_h\omega = -div_h Z$, thus

$$\begin{aligned}\theta'(0) = &- v(\|\chi_+\|_h^2 + Ric(K_+, K_+))\\ &- \psi[J(\vec{n}) + \rho - \frac{1}{2}S_h + \frac{1}{2}\|\chi_+\|_h^2] - \Delta_\Sigma\psi\\ &+ div_h Z \cdot \psi + 2h(Z, \nabla_\Sigma\psi) - \psi\|Z\|_h^2.\end{aligned}$$

After rearranging terms we arrive at

$$\begin{aligned}\theta'(0) = &- (\|\chi_+\|_h^2 + Ric(K_+, K_+)) \cdot v\\ &- \Delta_\Sigma\psi + 2h(Z, \nabla_\Sigma\psi) + ([\frac{1}{2}S_h - J(\vec{n}) - \rho - \frac{1}{2}\|\chi_+\|_h^2] + div_h Z - \|Z\|_h^2)\psi,\end{aligned}$$

and defining the quantities

$$Q := \frac{1}{2}S_h - [J(\vec{n}) + \rho] - \frac{1}{2}\|\chi_+\|_h^2, \tag{88}$$

$$L(\psi) = -\Delta_\Sigma\psi + 2h(Z, \nabla_\Sigma\psi) + (Q + div_h Z - \|Z\|_h^2)\psi, \tag{89}$$

we finally get

$$\theta'(0) = -(\|\chi_+\|_h^2 + Ric(K_+, K_+)) \cdot v + L(\psi). \tag{90}$$

APPENDIX B – REGULARITY OF THE NULL EXPANSION OPERATOR

B.1 ANALYSIS IN BANACH SPACES

This subsection is devoted to briefly presenting notations related to analysis in Banach spaces and stating the inverse function theorem in this context. For a comprehensive treatment, we recommend (DRÁBEK; MILOTA, 2013), which inspired this subsection.

In infinite-dimensional, it is often the case that no natural basis exists, consequently there is no way of a straightforward generalization of partial derivatives. In this context, we define the directional derivative as an alternative approach.

Definition B.1.1 (Directional derivative). *Let X, Y be normed spaces and let $f : X \to Y$. If for a, $h \in X$ the limit (in the norm of Y)*

$$\lim_{t \to 0} \frac{f(a + th) - f(a)}{t}$$

exists, then its value is called the derivative of f at the point a and in the direction of h *(or* directional derivative *or* Gâteaux variation*) and is denoted by $\delta f(a; h)$.*

If $\delta f(a; h)$ exists for all $h \in X$ and the mapping $Df(a) : h \mapsto \delta f(a; h)$ is linear and continuous, then $Df(a)$ is called the Gâteaux derivative of f at the point a.

For the most interesting results, a stronger notion of differentiability is required. Therefore, a generalization of the differential of a function of two variables is given as follows.

Definition B.1.2 (Fréchet derivative). *Let X, Y be normed spaces. A mapping $f : X \to Y$ is said to be Fréchet differentiable at a point $a \in X$ if there exists a linear and continuous operator $A : X \to Y$ such that*

$$\lim_{h \to 0_X} \frac{\|f(a + h) - f(a) - Ah\|_Y}{\|h\|_X}$$

In this case, A is called Fréchet derivative of f at the point a and is denoted by $f'(a)$.

Remark 7 . *Given $a \in X$, if we have that $f'(a)$ exists, then also $Df(a)$ exists and $f'(a)h = Df(a)h$, for all $h \in X$. Moreover, it is easy to see that if f is Fréchet differentiable, then f is continuous.*

Theorem B.1.3 (Chain rule). *Let X, Y, Z be normed linear spaces and suppose that there exist $\delta g(a; h) : X \to Y$. If $g(a) = b$ and for $f : Y \to Z$ the* Fréchet derivative *$f'(b)$ exists, then*

$$\delta(f \circ g)(a; h) = f'(b)[\delta(g(a; h)]$$

Proof. See (DRÁBEK; MILOTA, 2013, Theorem 3.2.12) □

Remark 8 . *A similar result holds if $Dg(a)$ exists or if g'.*

After these preliminary considerations, we finally state the main theorem of this subsection. Notice that we need to formulate it for general Banach spaces instead of normed linear spaces.

Theorem B.1.4 (Inverse Function Theorem). *Let X, Y be Banach spaces, U an open set in X, $f : U \subset X \to Y$ of class C^1. Let the derivative $f'(a)$ be an isomorphism of X onto Y for $a \in U$. Then there exist neighborhoods $\mathcal{U}$ of a, $\mathcal{V}$ of $f(a)$ such that*

1. *$\mathcal{V} = f(\mathcal{U})$, and*
2. *The restriction $f\big|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is a bijection with continuously differentiable inverse.*

Proof. See (DRÁBEK; MILOTA, 2013, Theorem 4.1.1). □

B.2 SECOND-ORDER DIFFERENTIAL OPERATORS

In this section, we will be interested in discussing the Fréchet differentiability of second-order differential operators on open sets in $\mathbb{R}^n$. A general second-order equation, on a domain $\Omega \subseteq \mathbb{R}^n$ can be written in the form,

$$P_F[u] = F(x, u, Du, D^2u), \tag{91}$$

where F is a real function on the set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, where $\mathbb{R}^{n\times n}$ denotes the $n(n+1)/2$ dimensional space of real symmetric $n \times n$ matrices. Points in Γ are typically denoted by $\gamma = (x, z, p, r)$ where $x \in \Omega$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $r \in \mathbb{R}^{n\times n}$. If F is differentiable with respect to the r variables, its partial derivatives are denoted by

$$F_{ij}(\gamma) = \frac{\partial F}{\partial r_{ij}}(\gamma), \quad i,j = 1, \ldots ,n,$$

and the notation is similar for the variables p and z.

Before proceeding, we need to define a Banach space of interest. Let k be a nonnegative integer, $\alpha \in (0, 1]$ and $\Omega \subset \mathbb{R}$ be an open set. The Hölder space $C^{k,\alpha}(\Omega)$ is the space of functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ of class C^k such that the norm

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \max_{|\beta|=k} |D^\beta f|_{C^\alpha}$$

is finite, where $\beta \in \mathbb{Z}_+$ are multi-indices and

$$\|f\|_{C^k} = \max_{|\beta|\leq k} \sup_{x\in\Omega} \left|D^\beta f(x)\right|,$$

and

$$|f|_{C^\alpha} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}.$$

If Ω is open and bounded, then $C^{k,\alpha}(\overline{\Omega})$ is a Banach space with respect to the norm $\|\cdot\|_{C^{k,\alpha}}$. In particular, there is an obvious inclusion map between Hölder spaces:

$$i : C^{0,\beta}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega}),$$

which is continuous and compact, as long as $0 < \alpha < \beta \leq 1$. Moreover, for domains of interest, i.e., bounded convex, we have that the following inclusion

$$i : C^{k+1,\alpha}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega}), \tag{92}$$

is also continuous and compact. The same result holds under a condition weaker than convexity, as described in (DRÁBEK; MILOTA, 2013, Definition 8.3.3). In particular, for a bounded convex set, the mean value theorem guarantees that the following map

$$i : C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega}) \tag{93}$$

is continuous. Finally, we can state the main lemma of this subsection.

Lemma B.2.1 (GILBARG; TRUDINGER, 2001). *The operator P_F given by (91) is Fréchet differentiable as a mapping from $C^{2,\alpha}(\overline{\Omega})$ into $C^{0,\alpha}(\overline{\Omega})$, for any $\alpha \in (0,1]$, if the function $F \in C^{2,\alpha}(\overline{\Gamma})$. Furthermore, let u, $h \in C^{2,\alpha}(\overline{\Omega})$, then the Fréchet derivative F_u is expressed as follows*

$$L_u[h] = {P_F}'(u)[h] = F_{ij}(x)D_{ij}h + b^i(x)D_ih + c(x)h, \tag{94}$$

where

$$F_{ij}(x) = F_{ij}(x, u, Du, D^2u),$$
$$b_i(x) = F_{p_i}(x, u, Du, D^2u),$$
$$c(x) = F_z(x, u, Du, D^2u).$$

Idea of the Proof. We make the assumption that the open set Ω is bounded and convex, which guarantees that the map stated in Equation (92) is continuous. Firstly, recall that for any $\alpha \in (0,1]$, the inequality $\|fg\|_{C^{0,\alpha}} \leq \|f\|_{C^{0,\alpha}}\|g\|_{C^{0,\alpha}}$ holds. Now, we apply the Taylor formula to the function F and observe that, in this context, the Hessian of F is a continuous bilinear operator. Applying the norm definitions to the expression $\|P_F[u+h] - P_F[u] - L_u[h]\|_{C^{0,\alpha}}$, substituting the Taylor formula of F and employing the continuity and bilinearity of the Hessian of F, thus, together with the inequality and the continuity of the map $i : C^{k+1,\alpha}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$, we can see that the expression has order $\|h\|^2_{C^{2,\alpha}}$, which proves the assertion. □

B.3 DIFFERENTIAL OPERATOR

The main objective of this subsection is to provide a criterion for demonstrating the Fréchet differentiability of a global differential operator $P : C^\infty(\Sigma) \to C^\infty(\Sigma)$ on a compact Riemannian manifold which is defined locally by a family of suitable differential operators.

Consider a finite covering family of coordinate charts $(U_\alpha, \phi_\alpha)_{\alpha=1,\ldots,N}$ for Σ. Let $V_\alpha := \phi_\alpha(U_\alpha)$ be the corresponding open sets, and for any $u \in C^\infty(\Sigma)$, denote the coordinate expression of u by $u_\alpha = u \circ \phi_\alpha^{-1}$. Now, suppose we have a family of operators $P_\alpha : C^\infty(V_\alpha) \to C^\infty(V_\alpha)$ defined on each V_α. Furthermore, for any $u \in C^\infty(\Sigma)$ and $\alpha, \beta \in \{1, \ldots, N\}$, the family of operators satisfy the following condition

$$P_\alpha[u_\alpha](\phi_\alpha(x)) = P_\beta[u_\beta](\phi_\beta(x)), \quad \forall x \in U_\alpha \cap U_\beta.$$

Let $(\eta_\alpha)_{\alpha=1,\ldots,N}$ be a smooth partition of unity subordinate to the covering $(U_\alpha)_{\alpha=1,\ldots,N}$. We define the global operator $P : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by

$$P[u] := \sum_{\alpha=1}^{N} T_\alpha(\eta_\alpha P_\alpha[u_\alpha] \circ \phi_\alpha), \quad \forall u \in C^\infty(\Sigma). \tag{95}$$

Here, the operator T_α should be understood as the extension operator, which allow us to extend functions to the full domain, more precisely,

$$T_\alpha(\eta_\alpha P_\alpha[u_\alpha] \circ \phi_\alpha) = \begin{cases} \eta_\alpha P_\alpha[u_\alpha] \circ \phi_\alpha & \text{on } \operatorname{supp} \eta_\alpha \subseteq U_\alpha, \\ 0 & \text{on } \Sigma \backslash \operatorname{supp} \eta_\alpha, \end{cases}$$

where $\operatorname{supp} \eta_\alpha$ is the support of the bump functions η_α.

By restricting the operators $P_\alpha : X \to Y$ to suitable normed linear spaces X and Y, we can investigate whether the global operator P is Fréchet-differentiable. Suppose that each $P_\alpha : X \to Y$ is Fréchet differentiable, for a given $u \in X$, each term in the sum of Equation (95) can be written as

$$T_\alpha(B_{\eta_\alpha} \cdot R_{\phi_\alpha} \circ P_\alpha \circ R_{\phi_\alpha^{-1}})[u]$$

where R_{ϕ_α} and $R_{\phi_\alpha^{-1}}$ denote the composition on the right by the subscript function, and B_{η_α} represents the multiplication by the bump function η_α. All these three mentioned operators are linear and continuous on X and are of class C^∞. Similarly, the extension operator T_α is linear and continuous, so is also smooth. Consequently, the smoothness of P depends solely on the family P_α. If each P_α is Fréchet-differentiable, then P is also Fréchet-differentiable.

B.4 NULL EXPANSION

This section is dedicated to studying the null expansion operator and demonstrating its Fréchet differentiability. Furthermore, the discussion will involve the definition of Hölder spaces on compact Riemannian manifolds. For a detailed discussion on this topic, we refer to (AUBIN, 1998) or (HEBEY, 2000).

Let $(M^n, g, \mathcal{K})$ be an initial data set, and Σ^{n-1} be a compact embedded hypersurface in M^n with a global unit normal vector field $\vec{n}$. Using the normal exponential map, we can obtain a neighborhood U of Σ in M such that $U = \Sigma \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let $x \in \Sigma$, within this neighborhood U, we can choose a coordinate system $(x^1, \ldots, x^{n-1}, t)$, where $(x^1, \ldots, x^{n-1})$ are coordinates in Σ centered at x. In this coordinate system, the metric has the following coordinate expression

$$g = (dt)^2 + \sum_{i,j=1}^{n-1} g_{ij}(x^1, \ldots, x^{n-1}, t) dx^i dx^j.$$

Hypersurfaces close enough to Σ can be parametrized by smooth functions. Let u be a function near the origin in $\mathbb{R}^{n-1}$ and define the map F_u such that

$$F_u(x^1,\dots,x^{n-1}) = (x^1,\dots,x^{n-1}, u(x^1,\dots,x^{n-1})).$$

This function parametrizes a smooth hypersurface Σ_u in $\Sigma \times (-\varepsilon,\varepsilon)$. Denote by G the induced metric on Σ_u, then G has the form

$$G_{ij}(x) = \frac{\partial u}{\partial x^i}(x)\frac{\partial u}{\partial x^j}(x) + g_{ij}(x,u(x)).$$

Moreover, we have that

$$X_i := \frac{\partial}{\partial x^i} + \partial_i u \frac{\partial}{\partial t}$$

is a basis for the tangent space of Σ_u, for $i \in \{1,\dots,n-1\}$. The normal unit vector $\vec{n}_u$ of Σ_u is defined as

$$\vec{n}_u := \frac{1}{W}\left(\frac{\partial}{\partial t} + \sum_{ij}^{n-1} g^{ij}\partial_i u \frac{\partial}{\partial x^j}\right), \quad W := \left(1 + \sum_{i,j=1}^{n-1} g^{ij}\partial_i u \partial_j u\right)^{1/2}.$$

where $\partial u_i = \frac{\partial u}{\partial x^i}$. First, we will proceed computing the mean curvature $H(u)$ of Σ_u in coordinates. To begin with, notice that

$$H(u) = \operatorname{div}_g \vec{n}_u = \operatorname{div}_g\left(\frac{\nabla^g u}{\sqrt{1+|\nabla^g u|^2}}\right),$$

where, in coordinates, $\nabla^g u = g^{ij}\partial_j u \frac{\partial}{\partial x^i}$ and $|\nabla^g u|^2 = g(\nabla^g u, \nabla^g u)$. Computing the divergence in coordinates, we have

$$H(u) = \sum_{i=1}^{n-1} \frac{\partial}{\partial x^i}\left(\frac{1}{W} g^{ij}\partial_j u\right) + \sum_{i,j=1}^{n-1} \Gamma^i_{ij}\left(\frac{1}{W} g^{jk}\partial_k u\right),$$

and following the computation, we obtain the following expression

$$H(u) = \sum_{i,j=1}^{n-1} = a^{ij}(x,u,\partial u)\partial_{ij} u + b(x,u,\partial u),$$

where $\partial u = (\partial u_1, \dots, \partial u_{n-1})$, $\partial_{ij} u = \frac{\partial^2 u}{\partial x^i x^j}$ and $a^{ij}(x,u,\partial u)$ is given by

$$a^{ij}(x,u,\partial u) = \frac{1}{W}\left(g^{ij} - \frac{1}{W^2}(\nabla^g u)^i (\nabla^g u)^j\right), \tag{96}$$

while $b(x,u,\partial u)$ is a polynomial expression in $\partial_i u$, Γ^i_{ij}, $\partial_k g^{ij}$ and W. From the form is this operator, $H(u)$ is a second order quasi-linear operator. The second part of $\theta(u)$ involves the trace of the tensor $\mathcal{K}$, so we have that

$$\operatorname{tr}_{\Sigma_u} \mathcal{K} = \sum_{i,j=1}^{n-1} G^{ij}\mathcal{K}(X_i, X_j)$$

where G^{ij} denotes the inverse of the metric G_{ij}. Clearly, it is a polynomial expression on $\partial_i u$, G^{ij}, and the smooth function $\mathcal{K}_{ij}$ and $\mathcal{K}_{it}$. We now introduce the following quantity

$$b_1(x,u,\partial u) = b(x,u,\partial u) + \sum_{i,j=1}^{n-1} G^{ij}\mathcal{K}(X_i,X_j). \tag{97}$$

Therefore, $\theta(u) = H(u) + \operatorname{tr}_{\Sigma_u}\mathcal{K}$ can be written as

$$\theta(u) = \sum_{i,j=1}^{n-1} = a^{ij}(x,u,\partial u)\partial_{ij}u + b_1(x,u,\partial u),$$

where a^{ij} and b_1 are as in (96) and (97). Therefore, $\theta(u)$ is a second order quasi-linear operator.

Finally, we can see the null expansion as a differential operator $\theta : C^\infty(\Sigma) \to C^\infty(\Sigma)$. Let $(U_k,\phi_k)_{k=1,\dots,N}$ be finite covering family of coordinate charts for Σ and let $V_k := \phi_k(U_k)$ be the corresponding open sets. In coordinates, we define the local operator $\widetilde{\theta}_k : C^\infty(U_k) \to C^\infty(U_k)$ as

$$\widetilde{\theta}_k[u] = \sum_{i,j=1}^{n-1} = a^{ij}(x,u,\partial u)\partial_{ij}u + b_1(x,u,\partial u), \quad \forall u \in C^\infty(U_k),$$

where a^{ij} and b_1 are as in (96) and (97). This operator induces a version defined on function on V_k. We define the operator $\theta_k : C^\infty(V_k) \to C^\infty(V_k)$ as follows

$$\theta_k[v] := R_{\phi_k^{-1}} \circ \widetilde{\theta}_k \circ R_{\phi_k}(v), \quad \forall v \in C^\infty(V_k).$$

For any $u \in C^\infty(\Sigma)$ and any $k \in \{1,\dots,N\}$, denote by $u_k := u \circ \phi_k^{-1}$ the coordinate expression of u. Since $\theta(u)$ does not depend on the coordinate expression, for each $u \in C^\infty(\Sigma)$, and any $k,m \in \{1,\dots,N\}$ we have

$$\theta_k[u_k](\phi_k(x)) = \theta_m[u_m](\phi_m(x)), \quad \forall x \in U_k \cap U_m.$$

The operator θ_k can be written in the form of Equation (91), that is, $\theta_k[u] = F_k(x,u,\partial u,\partial^2 u)$ where is a real function on the set $\Gamma_k = V_k \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1\times n-1}$ and $\partial^2 u$ is the symmetric matrix of $[\partial_{ij}u]_{i,j=1,\dots,n-1}$.

Without loss of generality, we can choose a covering family such that each U_k is a regular coordinate ball, thus V_k is a ball in the Euclidean space. Recall that the functions $a^{ij}(x,u,\partial u) \in C^\infty(U_k)$ and $b_1(x,u,\partial u) \in C^\infty(U_k)$. Furthermore, in coordinates, as discussed before Equation (93), for any $\alpha \in (0,1]$, these functions, in coordinates, are in $C^{2,\alpha}(V_k)$ and they can be extended to $C^{2,\alpha}(\overline{V_k})$. In coordinates, since a^{ij} and b_1 are polynomial expressions of smooth functions on $x \in \overline{V_k}$ and $\partial_i u$, consequently, we have that $F_k \in C^{2,\alpha}(\overline{\Gamma_k})$. Furthermore, by Lemma B.2.1, each $\theta_k : C^{2,\alpha}(\overline{V_k}) \to C^{0,\alpha}(\overline{V_k})$ is Fréchet differentiable. Therefore, $\theta : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ is Fréchet differentiable, as discussed at the end of Appendix B.3.

We actually have even higher regularity. Due to the smoothness of the functions, we can conclude that $F_k \in C^{l,\alpha}(\overline{\Gamma_k})$ for any l nonnegative integer. As a result, the C^1 (or C^l) regularity can be established through a bootstrap argument employing the Lemma B.2.1, because we can apply a similar argument for the linearization of F_k, as given in Equation (94), it is Fréchet differentiable, and hence continuous.