UNIVERSIDADE FEDERAL DE SANTA CATARINA
CENTRO DE CIÊNCIAS FÍSICAS E MATEMÁTICAS
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA PURA E APLICADA

Paula Savana Estácio Moreira

Ideals of partial skew groupoid rings and primeness of groupoid graded rings

Florianópolis
2022

Paula Savana Estácio Moreira

Ideals of partial skew groupoid rings and primeness of groupoid graded rings

Tese submetida ao Programa de Pós-Graduação em Matemática Pura e Aplicada da Universidade Federal de Santa Catarina para a obtenção do título de doutora em Matemática.
Orientador: Prof. Daniel Gonçalves, Dr.

Florianópolis
2022

Ficha de identificação da obra elaborada pelo autor,
através do Programa de Geração Automática da Biblioteca Universitária da UFSC.

Moreira, Paula Savana Estácio
Ideals of partial skew groupoid rings and primeness of groupoid graded rings / Paula Savana Estácio Moreira ; orientador, Daniel Gonçalves, 2022.
84 p.

Tese (doutorado) - Universidade Federal de Santa Catarina, Centro de Ciências Físicas e Matemáticas, Programa de Pós-Graduação em Matemática Pura e Aplicada, Florianópolis, 2022.

Inclui referências.

1. Matemática Pura e Aplicada. 2. Anel skew parcial de grupóide. 3. Anel graduado por grupóide. 4. Primalidade. 5. Álgebra de ultragrafo. I. Gonçalves, Daniel. II. Universidade Federal de Santa Catarina. Programa de Pós Graduação em Matemática Pura e Aplicada. III. Título.

Paula Savana Estácio Moreira[1]

Ideals of partial skew groupoid rings and primeness of groupoid graded rings

O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

Prof. Dirceu Bagio, Dr.
Universidade Federal de Santa Catarina

Prof. Gilles Gonçalves de Castro, Dr.
Universidade Federal de Santa Catarina

Prof. Johan Öinert, Dr.
Blekinge Tekniska Högskola

Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de doutora em Matemática.

ICP-Edu
Documento assinado digitalmente
Daniel Goncalves
Data: 24/11/2022 14:07:02-0300
CPF: ***.022.459-**
Verifique as assinaturas em https://v.ufsc.br

Coordenação do Programa de
Pós-Graduação

ICP-Edu
Documento assinado digitalmente
Daniel Goncalves
Data: 24/11/2022 14:06:42-0300
CPF: ***.022.459-**
Verifique as assinaturas em https://v.ufsc.br

Prof. Daniel Gonçalves, Dr.
Orientador

Florianópolis, 2022.

[1] Bolsista CAPES/Brasil. O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

AGRADECIMENTOS

Agradeço aos meus pais Adriano e Ângela por toda a dedicação e generosidade e ao meu irmão Paulo Sérgio pelo incentivo. Agradeço ao Natã pelo companheirismo durante esses anos de casamento e doutorado.

Agradeço ao meu orientador Daniel Gonçalves que mesmo durante a pandemia conseguiu ser presente. Além disso, me apoiou para que o doutorado sanduíche fosse possível e foi um ótimo orientador também no aspecto acadêmico.

Agradeço aos professores Dirceu Bagio, Gilles G. de Castro, Ruy Exel, Alcides Buss e Eliezer Batista pelos ensinamentos durante o doutorado. Agradeço também ao professor Danilo Royer que foi meu orientador de mestrado e muito importante nessa trajetória.

Agradeço ao professor Johan Öinert pela oportunidade de pesquisar na BTH, por todos os conselhos acadêmicos e pela cordialidade. Agradeço a hospitalidade de todos na BTH, principalmente à Bruna e ao Vinicius.

Agradeço à CAPES pelo financiamento durante o doutorado e no programa PRINT. Agradeço também a todos os funcionários do Departamento de Matemática.

RESUMO

Dada uma ação parcial α de um grupóide G em um anel R, estudamos o anel skew parcial de grupóide $R \rtimes_{\alpha} G$ associado. Mostramos que há uma correspondência entre os ideais G-invariantes de R e os ideais graduados do anel G-graduado $R \rtimes_{\alpha} G$. Fornecemos condições suficientes para que o anel skew parcial de grupóide $R \rtimes_{\alpha} G$ seja primo e condições necessárias e suficientes para a simplicidade do mesmo. Provamos que todo ideal de $R \times_{\alpha} G$ é graduado se, e somente se, α possui a propriedade da interseção residual. Além disso, se a ação algébrica α é induzida por uma ação parcial topológica θ, mostramos que θ é minimal se e somente se o anel R é G-simples, θ é topologicamente transitiva se e somente se o anel R é G-primo e θ é topologicamente livre em todo subconjunto fechado invariante do espaço topológico se e somente se α possui a propriedade da interseção residual. Como aplicação, caracterizamos a Condição (K) para ultragrafos por meio das propriedades algébrica e topológicas das álgebras de ultragrafos associadas. Além disso, investigamos condições para que anéis graduados por grupóides sejam primos. Provamos uma equivalência para que um anel quase epsilon-fortemente graduado por um grupóide seja primo. Aplicamos nossos resultados para o caso de anel skew parcial de grupóide e obtemos uma caracterização de primalidade para essa classe de anéis.

Palavras-chave: Anel skew parcial de grupóide, propriedade da interseção residual, primalidade, liberdade topológica, transitividade topológica, Condição (K), álgebra de ultragrafo, anel quase epsilon-fortemente graduado por grupóide.

RESUMO EXPANDIDO

INTRODUÇÃO

Ações parciais de grupos foram introduzidas em [26] no estudo de C^*-álgebras como uma forma de realizar exemplos importantes de C^*-álgebras como produtos cruzados parciais. Em 2005, o estudo de ações algébricas parciais de grupos, e seus anéis skew parciais de grupo associados, foi formalizado em [21]. Desde então, as aplicações e generalizações da teoria aumentaram consistentemente. Mencionaremos alguns exemplos. Em [32], as álgebras de caminhos de Leavitt foram realizadas como anéis skew parciais de grupo. Conexões com dinâmicas topológicas no contexto de anéis skew parciais de grupo foram descritas em [31], no contexto (global) de álgebras skew de categorias em [45] e [48], para C^*-sistemas dinâmicos em [29] e para anéis skew parciais de semigrupos inversos em [6]. Uma perspectiva sobre a evolução desta área de pesquisa foi dada em [20].

Entre as generalizações de anéis skew parciais de grupo, distinguimos duas: Anéis skew parciais de semigrupos inversos e anéis skew parciais de grupóides. O primeiro foi introduzido em [11] e o último em [4]. A principal diferença entre as duas construções é que ao definir um anel skew parcial de semigrupo inverso um certo quociente é tomado para descartar as chamadas *redundâncias* (ver [6]). Este quociente, embora necessário para realizar certas álgebras como anéis skew parciais de semigrupos inversos, traz uma complexidade extra para o estudo destes anéis. Em contraste, a definição de um anel skew parcial de grupóide não requer tal quociente (ver Definição 1.13).

Ainda que o quociente mencionado na definição de anel skew parcial de semigrupo inverso não esteja presente na definição de anel skew parcial de grupóide, álgebras importantes, como por exemplo, álgebras de caminho de Leavitt, ainda podem ser realizadas como anéis skew parciais de grupóides (ver [33]). Além disso, há um crescente interesse na dinâmica de ações parciais de grupóides. No artigo [27], por exemplo, um estudo puramente dinâmico foi realizado, sem menção às propriedades dos anéis skew parciais de grupóides associados. Neste trabalho, veremos que no caso de um grupóide discreto, a definição ação parcial de grupoide (topológica) em [27] é um caso particular da Definição 1.8.

Todo anel skew parcial de grupóide $R \rtimes_\alpha G$ possui uma G-graduação natural. Mencionamos alguns exemplos da importância das graduações no estudo dos anéis: A conjectura do monóide talentoso de Hazrat para álgebras de caminho de Leavitt (ver [36, 35, 19]). Ideais e ideais graduados de álgebras associadas a estruturas combinatórias, são estudados em [23] e [59]. Ideais graduados também possuem conexão com a teoria de representações de álgebras (ver [60]).

Anéis quase epsilon-fortemente graduados por grupos foram introduzidos em [47]. Condições necessárias e suficientes para que anéis quase epsilon-fortemente graduados por grupos não sejam primos foram apresentadas em [40]. Generalizando o resultado de Passman em [52], que estabeleceu uma equivalência para que anéis unitais fortemente graduados não sejam primos.

Ainda em [40] foi observado que anéis skew parciais de grupo s-unitais são exemplos de anéis quase epsilon-fortemente graduados por grupos. Sendo assim, os resultados sobre primalidade podem ser aplicados neste contexto, motivando o estudo da primalidade de anéis quase epsilon-fortemente graduados por grupóides e as possíveis aplicações para anéis skew parciais de grupóide.

OBJETIVOS

Dado $R \rtimes_\alpha G$ um anel skew parcial de grupóide, investigar a relação entre os ideais G-invariantes de R e os ideais G-graduados de $R \rtimes_\alpha G$. Caracterizar as propriedades de anéis skew parciais de grupóides, cuja ação associada possua a propriedade da interseção e a propriedade da interseção residual. Estabelecer critérios para primalidade, primalidade graduada, simplicidade e simplicidade graduada do anel skew parcial de grupóide.

Descrever a relação entre as propriedades de ações topológicas de grupóides e ações algébricas induzidas por estas. Definir o grupóide de transformação associado a uma ação topológica de grupóide e caracterizar as propriedades correspondentes em relação à ação. Aplicar resultados de primalidade e simplicidade no contexto de anel skew parcial de grupóide associado a uma ação topológica.

Estudar o anel skew parcial de grupo associado à ação parcial topológica para ultragrafos via espaços etiquetados definida em [14, Section 4.1]. Encontrar a propriedade topológica desta ação que corresponde à Condição (K) no ultragrafo. Aplicar os resultados algébricos neste contexto.

Descrever propriedades básicas de anéis quase epsilon-fortemente graduados por grupóides. Encontrar condições necessárias e suficientes para primalidade e primalidade graduada de anéis quase epsilon-fortemente graduados por grupóides.

Observar se anéis skew parciais de grupóide são exemplos de anéis quase epsilon-fortemente graduados por grupóides, generalizando o caso de grupo. Aplica os resultados de primalidade neste contexto. Investigar a relação entre ações parciais de grupóide tipo-grupo e a primalidade de anéis skew parciais de grupóide. Descrever a primalidade dos casos particulares de anéis skew de grupóide e anéis de grupóide.

METODOLOGIA

Pesquisa bibliográfica, por meio da revisão de artigos científicos relacionados ao tema de interesse. Além disso, foram realizadas reuniões frequentes com o orientador e demais pesquisadores para a elaboração do trabalho.

RESULTADOS E DISCUSSÃO

Considere $R \rtimes_\alpha G$ um anel skew parcial de grupóide. Mostramos no Theorem 2.10 que, sob certas hipóteses, há uma correspondência entre os ideais G-invariantes de R e os ideais G-graduados de $R \rtimes_\alpha G$. Provamos que a ação parcial de grupóide α possui a propriedade da interseção residual se, e somente se, todo ideal de $R \rtimes_\alpha G$ é G-graduado (ver Proposition 2.20).

Demonstramos que $R \rtimes_\alpha G$ é primo graduado se e só se o anel R é G-primo e que $R \rtimes_\alpha G$ é simples graduado se e somente se R é G-simples. Além disso, caracterizamos a simplicidade de $R \rtimes_\alpha G$ e demos condições suficientes para a primalidade de $R \rtimes_\alpha G$ (ver Seção 2.3).

Provamos correspondências entre propriedades topológicas de uma ação parcial de um grupóide G sobre um espaço localmente compacto, Hausdorff, zero-dimensional X e as propriedades algébricas da ação parcial induzida do grupóide G sobre o anel das funções localmente constantes $\mathcal{L}_c(X, \mathbb{K})$. Especificamente, a ação é topologicamente transitiva se e somente se $\mathcal{L}_c(X, \mathbb{K})$ é G-primo e a ação é topologicamente livre em todo fechado G-invariante de X se e só se a ação algébrica induzida possui a propriedade da interseção residual.

Definimos o grupóide de transformação associado a uma ação topológica de grupóide e provamos que a ação é topologicamente livre em todo fechado G-invariante se, e somente se, o grupóide de transformação é fortemente efetivo e que a ação é topologicamente transitiva se, e somente se, o grupóide de transformação é topologicamente transitivo. Na Seção 4.3, utilizamos propriedades topológicas da ação partical para caracterizar propriedades algébricas do anel skew parcial de grupóide associado.

Mostramos que um ultragrafo satisfaz a Condição (K) se e somente se a ação parcial topológica definida em [14, Section 4.1] é topologicamente livre em todo fechado invariante do espaço topológico associado (ver Theorem 5.28). Utilizamos esse resultado para caracterizar os ideais graduados no skew parcial de grupo correspondente a esta ação.

Provamos condições necessárias e suficientes para que anéis quase epsilon-fortemente graduados por grupóides sejam primos (em Theorem 6.31). Mostramos uma equivalência para a primalidade de anéis skew parciais de grupóide associados a uma ação parcial de grupóide do tipo-grupo (ver Theorem 7.14).

CONSIDERAÇÕES FINAIS

Resultados interessantes sobre a estrutura de anéis skew parciais de grupóide foram obtidos neste trabalho. Uma continuação natural da pesquisa seria a caracterização de outras propriedades de anéis quase epsilon-fortemente graduados por grupóides, utilizando como inspiração os resultados conhecidos para anéis skew parciais de grupóide ou para anéis quase epsilon-fortemente graduados por grupos.

Palavras-chave: Anel skew parcial de grupóide, propriedade da interseção residual, primalidade, liberdade topológica, transitividade topológica, Condição (K), álgebra de ultragrafo, anel quase epsilon-fortemente graduado por grupóide.

ABSTRACT

Given a partial action α of a groupoid G on a ring R, we study the associated partial skew groupoid ring $R \rtimes_\alpha G$ which carries a natural G-grading. We show that there is a one-to-one correspondence between the G-invariant ideals of R and the graded ideals of the G-graded ring $R \rtimes_\alpha G$. We provide sufficient conditions for primeness, and necessary and sufficient conditions for simplicity of $R \rtimes_\alpha G$. We show that every ideal of $R \rtimes_\alpha G$ is graded if, and only if, α has the so-called residual intersection property. Furthermore, if α is induced by a topological partial action θ, then we prove that minimality of θ is equivalent to G-simplicity of R, topological transitivity of θ is equivalent to G-primeness of R, and topological freeness of θ on every closed invariant subset of the underlying topological space is equivalent to α having the residual intersection property. As an application, we characterize condition (K) for ultragraphs by means of algebraic and topological properties of their associated ultragraph algebras. Futhermore, We investigate primeness of groupoid graded rings. We provide necessary and sufficient conditions for primeness of a general nearly epsilon-strongly groupoid graded ring. Moreover, we apply our results to partial skew groupoid rings and get a characterization of primeness for that class of rings.

Keywords: Partial skew groupoid ring, residual intersection property, primeness, topological freeness, topological transitivity, Condition (K), ultragraph algebra, nearly epsilon-strongly groupoid graded ring.

CONTENTS

INTRODUCTION

This thesis consists of two parts. In Part 1, we study the structure of partial skew groupoid rings, applications to topological dynamics and applications to ultragraph algebras. In Part 2, we characterize primeness of nearly epsilon-strongly groupoid graded rings and the applications to partial skew groupoid rings. In Chapter 1, we present some definitions that will be used in both parts.

PART 1

Partial actions and their associated structures arose in the study of C^*-algebras as a way to realize important classes of C^*-algebras as partial crossed products (see [26]). In 2005, the study of algebraic partial actions of groups, and their associated partial skew group rings, was formalized (see [21]). Since then, the applications and generalizations of the theory have increased steadily. We will mention a few examples. In [32], Leavitt path algebras were realized as partial skew group rings and in [22] cohomology of partial actions was developed. Topological dynamics associated with partial skew group rings were described in [31], with (global) skew category algebras in [45, 48], with C^*-dynamical systems in [29], and with skew inverse semigroup rings in [6]. A comprehensive overview of the evolution of the research area was given in [20].

Among the generalizations of partial skew group rings, we distinguish two: Partial skew inverse semigroup rings and partial skew groupoid rings. The former was introduced in [11] and the latter in [4]. The key difference between the two constructions is that when defining a partial skew inverse semigroup ring a certain quotient is taken, to get rid of so-called *redundancies* (see [6]). This quotient, although necessary to realize known algebras as partial skew inverse semigroup rings, brings an extra layer of complexity to the study of partial skew inverse semigroup rings. In contrast, the definition of a partial skew groupoid ring does not require such a quotient (see Definition 1.13 below).

Although the aforementioned quotient used in the definition of partial skew inverse semigroups is not present in the definition of partial skew groupoid rings, many important algebras, such as e.g. Leavitt path algebras (see [33]), can still be realized as partial skew groupoid rings. Furthermore, there is a growing interest in the dynamics of groupoid partial actions. In the recent paper [27], a purely dynamical study of groupoid partial actions was conducted, without mentioning any interplay with the associated partial skew groupoid rings. In the case of discrete groupoids, the definition of a (topological) groupoid partial action in [27] becomes a particular case of Definition 1.8.

Every partial skew groupoid ring $R \rtimes_\alpha G$ carries a natural G-grading, and we will make use of this insight. We mention a few examples of the importance of gradings in the study of rings: Hazrat's talented monoid conjecture for Leavitt path algebras states that the talented monoid of a Leavitt path algebra is a complete invariant for graded Morita equivalence of the algebras (see [36, 35, 19]). The ideals, and graded ideals, of algebras associated with combinatorial structures, were studied in [23] and [59]. Graded ideals are also in close connection with the representation theory of the algebra (see [60]).

Next, we describe our goals and give an outline of the first part of the thesis.

In Chapter 1, we recall key definitions and the connection between groupoid fibred actions and global groupoid actions on sets, which was established in Proposition 4.1 of [28]. We also ensure that the partial skew groupoid rings that we will be working with are indeed associative rings (see Remark 1.14).

In Chapter 2, we study the algebraic structure of partial skew groupoid rings and pursue two goals. The first goal is to reach a better understanding of the (graded) ideal structure of partial skew groupoid rings. To that end, we show that there is a one-to-one correspondence between G-graded ideals of $R \rtimes_\alpha G$ and G-invariant ideals of R (see Theorem 2.10). We also show that every ideal of $R \rtimes_\alpha G$ is G-graded if, and only if, α has the residual intersection property (see Theorem 2.21). It is worth pointing out that the aforementioned result is new even for partial skew group rings. Nevertheless, a similar result has been proved in the context of partial actions of groups on C^*-algebras (see [29]).

The second goal is to describe primeness and simplicity for partial skew groupoid rings. We show that, if the action α has the intersection property, then the partial skew groupoid ring $R \rtimes_\alpha G$ is prime if, and only if, the ring R is G-prime (see Theorem 2.28 (ii)). Furthermore, we show that $R \rtimes_\alpha G$ is simple if, and only if, R is G-simple and α has the intersection property (see Theorem 2.28 (iii)), thereby generalizing [31] and exemplifying [51].

In Chapter 3, we show that any partial action of a groupoid on a torsion-free, unital, commutative algebra generated by its idempotents, actually corresponds to a partial action of the same groupoid on the ring of locally constant functions with compact support over a Stone space (see Proposition 3.2). Therefore, given an algebraic groupoid partial action as above, we can use our results of Section 2 and topological results of Section 4 to characterize the primeness, the simplicity and the graded ideals of the partial skew groupoid ring associated to it.

In Chapter 4, we turn our focus to partial skew groupoid rings associated with (groupoid) topological dynamical systems, and study how properties of the dynamical system are reflected in the associated partial skew groupoid ring. To be precise, we start out with a field $\mathbb{K}$ and a topological partial action θ of a groupoid G on a zero-dimensional, locally compact, Hausdorff space X, and consider the associated partial skew groupoid ring $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$. We show e.g. that every ideal of $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ is G-graded if, and only if, the partial action θ of G on X is topologically free on every closed invariant subset of X, which occurs if, and only if, the associated transformation groupoid $G \ltimes_\theta X$ is strongly effective (see Theorem 4.16). We point out that the dynamical properties appearing in Theorem 4.16 have analogues in the literature on Steinberg algebras (see e.g. [10, 17, 55]).

In Chapter 5, we apply our results to ultragraphs and their associated algebras. Using the characterization of an ultragraph Leavitt path algebra as a partial skew group ring associated with a certain topological partial action (see [8, 14, 37]), we show that given a field $\mathbb{K}$ and an ultragraph $\mathcal{G}$ with associated tight spectrum T, one has e.g. that $\mathcal{G}$ satisfies Condition (K) if, and only if, every ideal of $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{Z}$-graded (see Theorem 5.28). We point out that some of the equivalences appearing in Theorem 5.28 are already known in the context of Leavitt path algebras of ultragraphs, but that several of them are new even in the context of Leavitt path algebras of graphs.

PART 2

Nearly epsilon-strongly group graded rings were introduced in [47]. Subsequently, in [40] the authors established necessary and sufficient conditions for non-primeness of these rings ([40, Theorem 1.3]) and proved that s-unital partial skew group ring are examples of nearly epsilon-strongly group graded ring. This motivates our study on primeness of nearly epsilon-strongly groupoid graded rings (see [41]) with a view towards applications for partial skew groupoid rings.

A (non-necessarily unital) ring S is prime if there are no nonzero ideals I, J of S such that $IJ = \{0\}$. Recall that if G is a group and S is a G-graded ring, then S is *strongly G-graded* if $S_g S_h = S_{gh}$ for all $g,h \in G$. The aforementioned Theorem 1.3 from [40] generalizes the following theorem due to Passman.

Theorem ([52, Theorem 1.3])**.** *Let G be a group. Suppose that S is a unital and strongly G-graded ring. Then, S is not prime if and only if there exist:*

(i) subgroups $N \lhd H \subseteq G$ with N finite,

(ii) an H-invariant ideal I of S_e such that $I^g I = \{0\}$ for all $g \in G \setminus H$, and

(iii) nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.

See Definition 6.7 for more details about the statements in the theorem above.

Here, we will go one step further. We present the notion of a nearly epsilon-strongly groupoid graded ring and show necessary and sufficient conditions for primeness for this class of rings. Our main result reduces the primeness of a groupoid graded ring to the group case:

Theorem 6.31. *Let G be a groupoid, let $G' := \{g \in G : S_{s(g)} \neq \{0\}$ and $S_{r(g)} \neq \{0\}\}$ and let S be a nearly epsilon-strongly ring graded by G. The following statements are equivalent:*

(i) *S is prime;*

(ii) *$\oplus_{e\in G_0} S_e$ is G-prime, and for every $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ is prime;*

(iii) *$\oplus_{e\in G_0} S_e$ is G-prime, and for some $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ is prime;*

(iv) *S is graded prime, and for every $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ is prime;*

(v) *S is graded prime, and for some $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ is prime;*

(vi) *For every $e \in G_0'$, e is a support-hub and $\oplus_{g\in G_e^e} S_g$ is prime;*

(vii) *For some $e \in G_0'$, e is a support-hub and $\oplus_{g\in G_e^e} S_g$ is prime.*

Where G_e^e denotes the isotropy group of an element $e \in G_0'$. For more details about the statements in the theorem above see Definition 6.16 and Definition 6.19.

Furthermore, we show in Theorem 7.14 an equivalence for primeness of a partial skew groupoid ring associated to a groupoid partial action of group-type, see [5] and [2]. Since every global action of connected groupoids are group-type, we get an equivalence for primeness of skew groupoid rings of connected groupoids in Theorem 7.24.

Let A be a unital ring and let G be a group. In [18, Theorem 8], Connell proved that the group ring $A[G]$ is prime if and only if A is prime and G has no nontrivial finite normal subgroup. This result was generalized for A a s-unital ring in [40, Theorem 12.4]. We prove in Theorem 7.29 necessary and sufficient conditions for primeness of groupoid rings.

When it comes to torsion-free groups, the following theorem was proved in [40], generalizing [52, Corollary 4.6].

Theorem ([40, Theorem 1.4])**.** *Suppose that G is a torsion-free group and S is nearly epsilon-strongly G-graded. Then S is prime if and only if S_e is G-prime.*

We generalize [40, Theorem 1.4] for nearly epsilon-strongly groupoid graded rings in Theorem 6.35, and we describe this theorem in specific applications in Theorem 7.17 and Corollary 7.27.

Next, we outline how this part is divided. In Chapter 1, inspired by [46] and [58], we recall some basic definitions and properties about groupoids, groupoid graded rings and s-unital rings that will follow us throughout.

In Chapter 6, we show basic properties of nearly epsilon-strongly graded rings. Inspired by [40], if S is such a ring we show the relationship between the G-invariant ideals of $\oplus_{e\in G_0} S_e$ and the G-graded ideals of S. Moreover, we show necessary conditions for graded primeness of S and we finish proving an equivalence for primeness of nearly epsilon-strongly graded rings. Finally, in Chapter 7, we apply our results to partial skew groupoid rings, skew groupoid rings and groupoid rings.

1 PRELIMINARIES

In this section, we recall some notions and basic notation regarding groupoids, groupoid graded rings and groupoid partial actions on sets and rings. We refer the reader to [4] for more details.

1.1 GROUPOIDS

By a *groupoid* we shall mean a small category G in which every morphism is invertible. Each object of G will be identified with its corresponding identity morphism, allowing us to view G_0, the set of objects of G, as a subset of the set of morphisms of G. The set of morphisms of G will simply be denoted by G. This means that $G_0 := \{gg^{-1} : g \in G\} \subseteq G$.

The *range* and *source* maps $r,s : G \to G_0$, indicate the range (codomain) respectively source (domain) of each morphism of G. By abuse of notation, the set of *composable pairs* of G is denoted by $G^2 := \{(g,h) \in G \times G : s(g) = r(h)\}$. For each $e \in G_0$, we denote the corresponding *isotropy group* by $G_e^e := \{g \in G : s(g) = r(g) = e\}$.

Definition 1.1. *Let G be a groupoid. A non-empty subset H of G is said to be a* subgroupoid of *G, if $H^{-1} \subseteq H$ and $gh \in H$ whenever $g,h \in H$ and $(g,h) \in G^2$.*

1.2 GROUPOID GRADED RINGS

Definition 1.2. *Let G be a groupoid. A ring S is said to be G-*graded *if there is a family of additive subgroups $\{S_g\}_{g\in G}$ of S such that $S = \oplus_{g\in G} S_g$, and $S_g S_h \subseteq S_{gh}$, if $(g,h) \in G^2$, and $S_g S_h = \{0\}$, otherwise.*

Remark 1.3. *Suppose that G is a groupoid and that S is a G-graded ring.*

(a) If H is a subgroupoid of G, then $S_H := \oplus_{h\in H} S_h$ is an H-graded subring of S.

(b) For any element $c = \sum_{g\in G} c_g \in S$, with $c_g \in S_g$, we define

$$\mathrm{Supp}(c) := \{g \in G \mid c_g \neq 0\}.$$

(c) An ideal I of S is said to be a graded ideal *(or G-*graded ideal*) if $I = \oplus_{g\in G}(I \cap S_g)$.*

1.3 s-UNITAL RINGS

We briefly recall the definitions of s-unital modules and rings as well as some key properties.

Definition 1.4 ([46, cf. Definition 4])**.** *Let R be an associative ring and let M be a left (resp. right) R-module. We say that M is s-*unital *if $m \in Rm$ (resp. $m \in mR$) for every*

$m \in M$. If M is an R-bimodule, then we say that M is s-unital *if it is s-unital both as a left R-module and as a right R-module. The ring R is said to be* left s-unital *(resp.* right s-unital*) if it is left (resp. right) s-unital as a left (resp. right) module over itself. The ring R is said to be* s-unital *if it is s-unital as a bimodule over itself.*

The following results are due to Tominaga [58]. For the proofs, we refer the reader to [46, Proposition 2.8, Proposition 2.10].

Proposition 1.5. *Let R be a ring and let M be a left (resp. right) R-module. Then M is left (resp. right) s-unital if, and only if, for all $n \in \mathbb{N}$ and all $m_1, \ldots, m_n \in M$ there is $a \in R$ such that $am_i = m_i$ (resp. $m_i a = m_i$) for all $i \in \{1, \ldots, n\}$.*

Proposition 1.6. *Let R be a ring and let M be an R-bimodule. Then M is s-unital if, and only if, for all $n \in \mathbb{N}$ and all $m_1, \ldots, m_n \in M$ there is $a \in R$ such that $am_i = m_i a = m_i$ for all $i \in \{1, \ldots, n\}$.*

Remark 1.7. *The element a, in Proposition 1.6, is commonly referred to as an* s-unit for the set $\{m_1, \ldots, m_n\}$.

1.4 PARTIAL ACTIONS OF GROUPOIDS

Partial actions of (ordered) groupoids on sets and rings were first introduced in [28] and [3], respectively. Below we recall those definitions.

Definition 1.8. *A* partial action of a groupoid G on a set X *is a family of pairs $\alpha = (X_g, \alpha_g)_{g \in G}$, where X_g is a subset of X and $\alpha_g : X_{g^{-1}} \to X_g$ is a bijection, for all $g \in G$, that satisfy the following:*

(i) $X_g \subseteq X_{r(g)}$, *for all* $g \in G$,

(ii) $\alpha_e = \mathrm{id}_{X_e}$, *for all* $e \in G_0$,

(iii) $\alpha_h^{-1}(X_{g^{-1}} \cap X_h) \subseteq X_{(gh)^{-1}}$, *whenever* $(g,h) \in G^2$,

(iv) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, *for all* $x \in \alpha_h^{-1}(X_{g^{-1}} \cap X_h)$ *and* $(g,h) \in G^2$.

We say that α is *global* if $\alpha_g \alpha_h = \alpha_{gh}$ for all $(g,h) \in G^2$. Similarly to [4, Lemma 1.1 (i)], α is global if and only if $X_g = X_{r(g)}$ for all $g \in G$.

Definition 1.9. *A* topological partial action of a groupoid G on a topological space X *is a partial action $\alpha = (X_g, \alpha_g)_{g \in G}$ of G on the set X, such that X_g is an open subset of X and $\alpha_g : X_{g^{-1}} \to X_g$ is a homeomorphism, for all $g \in G$.*

Definition 1.10. *Let $\alpha = (X_g, \alpha_g)_{g \in G}$ be a partial action of a groupoid G on a set X. A subset M of X is said to be G*-invariant *if $\alpha_g(X_{g^{-1}} \cap M) \subseteq M$, for all $g \in G$.*

In the literature, there is a more restrictive notion of global groupoid action in which the groupoid acts on a fibred set (see e.g. [42]). We recall the definition.

Definition 1.11. *A* groupoid fibred action *is a* 4*-tuple* (G,ρ,θ,X) *consisting of a groupoid* G, *a set* X, *a surjective map* $\rho : X \to G_0$ *(called the* anchor *or* moment map*) and a map* $\theta : G \ltimes X \to X$, $(g,x) \longmapsto \theta_g(x) := g \cdot_\theta x$, *where* $G \ltimes X = \{(g,x) : s(g) = \rho(x)\}$ *is the pullback, that satisfy the following axioms:*

(i) $\rho(x) \cdot_\theta x = x$, *for all* $x \in X$,

(ii) if $(g,h) \in G^2$ *and* $(h,x) \in G \ltimes X$, *then* $(g, h \cdot_\theta x) \in G \ltimes X$,

(iii) $(gh) \cdot_\theta x = g \cdot_\theta (h \cdot_\theta x)$, *for all* $(g,h) \in G^2$ *and* $(h,x) \in G \ltimes X$.

An action of a topological groupoid on a topological space is an action (G,ρ,θ,X), *where* G *is a topological groupoid,* X *is a Hausdorff space, and the maps* ρ,θ *are continuous.*

For the convenience of the reader, we present a proof from [28, Proposition 4.1], which relates global actions to fibred actions.

Proposition 1.12. *Let* G *be a groupoid and let* X *be a set. The fibred actions of* G *on* X *are in one-to-one correspondence with the global actions of* G *on* X *satisfying* $X = \bigsqcup_{e\in G_0} X_e$.

Proof. Let (G,ρ,θ,X) be a groupoid fibred action. For each $g \in G$, define $X_g := \rho^{-1}(r(g))$ and $\alpha_g : X_{g^{-1}} \to X_g$ by $\alpha_g(x) := g \cdot_\theta x$, for all $x \in X_{g^{-1}}$. Notice that $x \in X_{g^{-1}}$ implies that $\rho(x) = s(g)$ and thus $(g,x) \in G \ltimes X$. Hence, α_g is well-defined. We also have that

$$X = \rho^{-1}(G_0) = \bigsqcup_{e\in G_0} \rho^{-1}(\{e\}) = \bigsqcup_{e\in G_0} X_e.$$

It is straightforward to verify that $\alpha = (X_g,\alpha_g)_{g\in G}$ is a global action of G on X.

Now, let $\alpha = (X_g,\alpha_g)_{g\in G}$ be a global action of G on X satisfying $X = \bigsqcup_{e\in G_0} X_e$. Define the anchor map $\rho : X \to G_0$ by $\rho(x) := e$ whenever $x \in X_e$. Since $\rho(X_e) = e$ for all $e \in G_0$, it follows that ρ is surjective. Define $\theta : G \ltimes X \to X$ by $g \cdot_\theta x := \alpha_g(x)$, for all $(g,x) \in G \ltimes X$. It is straightforward to verify that (G,ρ,θ,X) is a fibred action.

Clearly, the two procedures outlined above are mutually inverse. □

In [27], topological dynamics of (global) actions of topological groupoids on topological spaces was studied. In this paper, we aim to study topological dynamics of partial actions of discrete groupoids on topological spaces.

Definition 1.13. *A* partial action of a groupoid G on a ring R *is a partial action* $\alpha = (D_g,\alpha_g)_{g\in G}$ *of* G *on the set* R, *such that* $D_{r(g)}$ *is an ideal of* R, D_g *is an ideal of* $D_{r(g)}$, *and* $\alpha_g : D_{g^{-1}} \to D_g$ *is a ring isomorphism, for all* $g \in G$. *Given a partial action* α *of a groupoid* G *on a ring* R *one may define the* partial skew groupoid ring $R \rtimes_\alpha G$ *as the set of all formal sums* $\sum_{g\in G} a_g\delta_g$, *where* $a_g \in D_g$ *is zero for all but finitely many* $g \in G$

and δ_g is a symbol. Addition on $R \rtimes_\alpha G$ is defined in the natural way and multiplication is given by the rule

$$a_g\delta_g \cdot b_h\delta_h := \begin{cases} \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}, & \textit{if } (g,h) \in G^2, \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 1.14. *Let $\alpha = (D_g,\alpha_g)_{g\in G}$ be a partial action of a groupoid G on a ring R.*

(a) In general, the partial skew groupoid ring $R \rtimes_\alpha G$ need not be associative (not even for partial actions of groups). By [3, Proposition 3.1], if D_g is $(\mathcal{L},\mathcal{R})$-associative for every $g \in G$, then $R \rtimes_\alpha G$ is associative. If D_g is idempotent, then [21, Proposition 2.5] implies that D_g is $(\mathcal{L},\mathcal{R})$-associative (see [21, Definition 2.4]). In particular, if D_g is s-unital for every $g \in G$, then D_g is $(\mathcal{L},\mathcal{R})$-associative and consequently $R \rtimes_\alpha G$ is associative.

(b) Suppose that D_e is a unital ring, for all $e \in G_0$. Then, there exists a central idempotent $1_e \in R$ such that $D_e = 1_eR$. Moreover, if G_0 is finite, then [3, Proposition 3.3] implies that $R \rtimes_\alpha G$ is unital with multiplicative identity element $\sum_{e\in G_0} 1_e\delta_e$.

1.4.1 The G-grading on $R \rtimes_\alpha G$

Any partial skew groupoid ring $S := R \rtimes_\alpha G$ is, in a natural way, graded by the groupoid G. Indeed, we may endow it with a G-grading by putting $S_g := D_g\delta_g$, for every $g \in G$. Notice that an ideal I of the G-graded ring $R \rtimes_\alpha G$ is a *graded ideal* (or *G-graded ideal*) if $I = \bigoplus_{g\in G}(I \cap D_g\delta_g)$.

Let $s = \sum_{g\in G} a_g\delta_g \in R \rtimes_\alpha G$. Observe that $\mathrm{Supp}(s) = \{g \in G : a_g \neq 0\}$ and that $s = \sum_{g\in \mathrm{Supp}(s)} a_g\delta_g$.

1.4.2 Example: translation rings as skew groupoid rings

In [44] was introduced the notion of translation rings for arbitrary groups and in [44, Proposition 2.14] it was shown that every translation ring can be viewed as a skew group ring. We will generalize the definition of translation ring and prove that this generalization is isomorphic to a skew groupoid ring.

Given a groupoid G, $g \in G$ and $K \subset G$, consider $Kg = \{hg \ : \ h \in K \ \textit{and} \ (h,g) \in G^2\}$. For a ring R we define the *translation ring* $T(G,R)$ as the ring of all $G \times G$ matrices $M = (m_{(g,h)})$ with entries in R for which there is a *finite* subset K (which depends of M) of G such that $m_{(g,h)} = 0$ whenever $h \notin Kg$. Notice that if $M = (m_{(g,h)}) \in T(G,R)$ satisfies $m_{(g,g)} \neq 0$ for $g \in G$, then $r(g) \in K$. Hence, $T(G,R)$ is unital if, and only if, R is unital and G_0 is finite.

For each $e \in G_0$, let $G^e := \{g \in G : r(g) = e\}$, $D_e = \prod_{G^e} R$ and $A = \bigoplus_{e\in G_0} D_e$. We define a global action of G on A in the following way. For each $g \in G$, consider the isomorphism $\sigma_g : D_{s(g)} \longrightarrow D_{r(g)}$ defined by $\sigma_g(f)(h) = f(g^{-1}h)$, for all $f \in D_{s(g)}$ and

$h \in G^{r(g)}$. Observe that for $(g,h) \in G^2$, $f \in D_{s(gh)} = D_{s(h)}$ and $l \in G^{r(gh)} = G^{r(g)}$ we have that

$$\sigma_g(\sigma_h(f))(l) = \sigma_h(f)(g^{-1}l) = f(h^{-1}g^{-1}l) = \sigma_{gh}(f)(l).$$

Since $\sigma_e = \mathrm{id}_{D_e}$ for every $e \in G_0$, it follows that $\sigma = (D_{r(g)},\sigma_g)_{g\in G}$ is a global action of G on A and we can consider the skew groupoid ring $A \rtimes_\sigma G$.

Proposition 1.15. *Let G be a groupoid, R a ring and σ and A as above. Then $A \rtimes_\sigma G \simeq T(G,R)$ as rings.*

Proof. Consider the group homomorphism $\varphi : A \rtimes_\sigma G \longrightarrow T(G,R)$ defined by

$$\varphi(f_g\delta_g)_{(u,v)} = \begin{cases} f_g(u), & \text{if } u \in G^{r(g)} \text{ and } v = g^{-1}u \\ 0, & \text{otherwise,} \end{cases}$$

where $g \in G$, $f_g \in D_{r(g)}$ and $(u,v) \in G \times G$. Notice that $\varphi(f_g\delta_g) \in T(G,R)$. In fact, for $K = \{g^{-1}\}$ we have that $\varphi(f_g\delta_g)(u,v) = f_g(u)$ if $v \in Ku$ and $\varphi(f_g\delta_g)(u,v) = 0$ if $v \notin Ku$. It is straightforward to check that φ is a ring homomorphism. In order to prove that φ is surjective, given $M \in T(G,R)$, we denote the finite subset K of G associated to M by $K = \{g_1^{-1},...,g_n^{-1}\}$. Then, for $u \in G$ we have that $Ku = \{g_i^{-1}u \,:\, g_i^{-1} \in K \text{ and } u \in G^{r(g_i)}\}$. For each $i \in \{1,...,n\}$, let $f_i \in D_{r(g_i)}$ given by $f_i(u) = m_{(u,g_i^{-1}u)} \in R$, for all $u \in G^{r(g_i)}$. Thus, for $(u,v) \in G \times G$,

$$\begin{aligned}
\varphi\Big(\sum_{i=1}^n f_i\delta_{g_i}\Big)_{(u,v)} &= \begin{cases} f_i(u), & \text{if } u \in G^{r(g_i)} \text{ and } v = g_i^{-1}u \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} m_{(u,g_i^{-1}u)}, & \text{if } u \in G^{r(g_i)} \text{ and } v = g_i^{-1}u \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} m_{(u,v)}, & \text{if } v \in Ku \\ 0, & \text{otherwise,} \end{cases} \\
&= m_{(u,v)},
\end{aligned}$$

and consequently φ is surjective. Let $a = \sum_{h\in F} f_h\delta_h \in A \rtimes_\sigma G$ be such that $\varphi(a) = 0$, with F a finite subset of G. Then, for $h \in F$ and $u \in G^{r(h)}$, we have that $f_h(u) = \varphi(a)_{(u,h^{-1}u)} = 0$. Hence $a = 0$ and φ is injective. $\square$

PART 1

2 THE IDEAL STRUCTURE OF PARTIAL SKEW GROUPOID RINGS

Throughout this section, unless stated otherwise, R will denote an arbitrary associative ring, G will denote an arbitrary groupoid and $\alpha = (D_g, \alpha_g)_{g \in G}$ will be an arbitrary partial action of G on R, such that D_g is s-unital for every $g \in G$. Moreover, we will assume that $R = \bigoplus_{e \in G_0} D_e$.

Remark 2.1. *Note that, by our assumptions, R is clearly s-unital. Using that fact, it easily follows that the partial skew groupoid ring $R \rtimes_\alpha G$ is s-unital.*

We begin by establishing a couple of auxiliary results, which will be useful later on.

Lemma 2.2. *D_g is an ideal of R, for all $g \in G$.*

Proof. Take $g \in G$, $x \in D_g$ and $y \in R$. Choose an s-unit $u \in D_{r(g)}$ for x. Using that D_g is an ideal of $D_{r(g)}$, and that $D_{r(g)}$ is an ideal of R, we note that $xy = (xu)y = x(uy) \in D_g D_{r(g)} \subseteq D_g$. Similarly, $yx \in D_g$. □

Let $A := \bigoplus_{e \in G_0} D_e \delta_e \subseteq R \rtimes_\alpha G$. We now define a "projection" $P_0 : R \rtimes_\alpha G \to R$ by

$$P_0\left(\sum_{g \in G} a_g \delta_g\right) := \sum_{e \in G_0} a_e, \tag{2.1}$$

and an inclusion $\psi : R \to A$ by

$$\psi\left(\sum_{e \in G_0} a_e\right) := \sum_{e \in G_0} a_e \delta_e. \tag{2.2}$$

Remark 2.3. *Note that, in general P_0 is not a ring homomorphism.*

Lemma 2.4. *Let A be as above, and let I be an ideal of $R \rtimes_\alpha G$. The following assertions hold:*

(i) A is a subring of $R \rtimes_\alpha G$ and the map $\psi : R \to A$ is a ring isomorphism. In particular, R is commutative if, and only if, A is commutative.

(ii) If $\sum_{e \in G_0} a_e \delta_e \in I \cap A$, then $a_f \delta_f \in I \cap A$, for all $f \in G_0$.

(iii) If $a \in A$ and $b \in R \rtimes_\alpha G$, then $P_0(ab) = P_0(a)P_0(b)$ and $P_0(ba) = P_0(b)P_0(a)$. In particular, $P_0|_A$: $A \to R$ is a ring isomorphism whose inverse is ψ, defined in (2.2)*.*

(iv) The map $E : R \rtimes_\alpha G \to A$, defined by $E := \psi \circ P_0$, is an A–A-bimodule map, in the sense that it is additive and satisfies $E(a) = a$, $E(ab) = aE(b)$ and $E(ba) = E(b)a$, for all $a \in A$ and $b \in R \rtimes_\alpha G$.

(v) $P_0(I) = P_0\Big(I \cap \bigoplus_{e\in G_0}(D_e \rtimes_{\alpha^e} G_e^e)\Big)$*, where* $\alpha^e := (D_h, \alpha_h)_{h\in G_e^e}$ *is the partial action of the isotropy group* G_e^e *on* D_e*.*

Proof. The proof of (i) is straightforward. For (ii), suppose that $\sum_{e\in G_0} a_e\delta_e \in I\cap A$. Take $f \in G_0$ and choose $u_f \in D_f$ to be an s-unit for $\alpha_f(a_f)$. Notice that

$$a_f\delta_f = \alpha_f(\alpha_f(a_f)u_f)\delta_f = \left(\sum_{e\in G_0} a_e\delta_e\right) u_f\delta_f \in I\cap A.$$

For (iii), take $a = \sum_{e\in G_0} a_e\delta_e \in A$ and $b = \sum_{g\in G} b_g\delta_g \in R \rtimes_\alpha G$. Observe that

$$P_0(ab) = P_0\left(\sum_{g\in G} \alpha_{r(g)}(\alpha_{r(g)}(a_{r(g)})b_g)\delta_{r(g)g}\right) = \sum_{e\in G_0} a_e b_e = P_0(a)P_0(b).$$

Similarly, $P_0(ba) = P_0(b)P_0(a)$. The second part of (iii) is straightforward. The proof of (iv) is straightforward, using (iii) and the definitions of P_0 and ψ. Finally, for (v), we take $a = \sum_{g\in G} a_g\delta_g \in I$ and put $v := P_0(a)$. Take $e \in G_0$, put $F_e := G_e^e \cap \operatorname{Supp}(a)$ and $F_0 := G_0 \cap \operatorname{Supp}(a)$, and let $u_e \in D_e$ be an s-unit for a_g and $\alpha_{g^{-1}}(a_g)$, for all $g \in F_e$. Define $y_e := (u_e\delta_e)a(u_e\delta_e)$ and notice that

$$\begin{aligned}
y_e &= (u_e\delta_e)\left(\sum_{\substack{g\in G\\ s(g)=e}} \alpha_g\left(\alpha_{g^{-1}}(a_g)\,u_e\right)\delta_g\right)\\
&= (u_e\delta_e)\left(\sum_{g\in G_e^e} a_g\delta_g + \sum_{\substack{g\in G\\ s(g)=e\\ r(g)\neq e}} \alpha_g\left(\alpha_{g^{-1}}(a_g)\,u_e\right)\delta_g\right)\\
&= \sum_{g\in F_e} u_e a_g\delta_g + 0 = \sum_{g\in F_e} a_g\delta_g \in I\cap(D_e \rtimes_{\alpha^e} G_e^e).
\end{aligned}$$

Hence $y := \sum_{e\in F_0} y_e \in I \cap \bigoplus_{e\in G_0}(D_e \rtimes_{\alpha^e} G_e^e)$ satisfies $P_0(y) = \sum_{e\in F_0} a_e = P_0(a) = v$. Consequently, $v \in P_0\Big(I \cap \bigoplus_{e\in G_0}(D_e \rtimes_{\alpha^e} G_e^e)\Big)$. The reverse inclusion is trivial. □

2.1 G-INVARIANT IDEALS OF R VERSUS G-GRADED IDEALS OF $R \rtimes_\alpha G$

In this section, we describe a correspondence between G-invariant ideals of the ring R and graded ideals of $R \rtimes_\alpha G$. The proof of the next result is partially inspired by the proof of [31, Theorem 2.3].

Lemma 2.5. *If I is an ideal of $R \rtimes_\alpha G$, then $P_0(I)$ is a G-invariant ideal of R.*

Proof. Suppose that I is an ideal of $R \rtimes_\alpha G$. Take $a \in I$, $r \in R$, and consider $c := P_0(a) \in P_0(I)$. Observe that $\psi(r)a \in I$ and $a\psi(r) \in I$, where ψ is the map defined in (2.2). By Lemma 2.4 (iii), we have that $cr = P_0(a)P_0(\psi(r)) = P_0(a\psi(r)) \in P_0(I)$. Similarly, $rc \in P_0(I)$. Thus, $P_0(I)$ is an ideal of R.

Now, take $t \in G$ and $d_t \in P_0(I) \cap D_t$. Then, there is an element $b := \sum_{g \in G} b_g \delta_g \in I$ such that $P_0(b) = d_t \in D_t \subseteq D_{r(t)}$ and $b = d_t \delta_{r(t)} + \sum_{g \notin G_0} b_g \delta_g$. Let $u \in D_t$ be an s-unit for d_t. Consider the element $y := \left(\alpha_{t^{-1}}(u)\,\delta_{t^{-1}}\right) b\,(u\delta_t) \in I$. Notice that if $g \in \text{Supp}(b)$, then $t^{-1}gt \in G_0$ if, and only if, $g = tt^{-1} = r(t)$. Therefore,

$$\alpha_{t^{-1}}(u)\delta_{t^{-1}} d_t \delta_{r(t)} u\delta_t = \alpha_{t^{-1}}(u)\,\delta_{t^{-1}} d_t \delta_t = \alpha_{t^{-1}}\left(\alpha_t\left(\alpha_{t^{-1}}(u)\right) d_t\right) \delta_{t^{-1}t} = \alpha_{t^{-1}}(d_t)\,\delta_{s(t)}. \tag{2.3}$$

We now conclude that $\alpha_{t^{-1}}(d_t) = P_0(\alpha_{t^{-1}}(u)\delta_{t^{-1}}\left(d_t \delta_{r(t)}\right) u\delta_t) = P_0(y) \in P_0(I)$ and hence $P_0(I)$ is G-invariant. □

Lemma 2.6. *If I is an ideal of $R \rtimes_\alpha G$, then $P_0(I \cap \bigoplus_{e \in G_0} D_e \delta_e)$ is a G-invariant ideal of R.*

Proof. Suppose that I is an ideal of $R \rtimes_\alpha G$. Using the same arguments as in the proof of Lemma 2.5, it is easy to verify that $P_0(I \cap \bigoplus_{e \in G_0} D_e \delta_e)$ is an ideal of R. Take $t \in G$ and $d_t \in P_0(I \cap \bigoplus_{e \in G_0} D_e \delta_e) \cap D_t$. Then, there is an element $b \in I \cap \bigoplus_{e \in G_0} D_e \delta_e$ such that $P_0(b) = d_t \in D_t \subseteq D_{r(t)}$ and $b = d_t \delta_{r(t)}$. Let $u \in D_t$ be an s-unit for d_t. Consider the element $y := \alpha_{t^{-1}}(u)\delta_{t^{-1}} b u \delta_t \in I$. Similarly to (2.3), $y = \alpha_{t^{-1}}(d_t)\,\delta_{s(t)} \in \bigoplus_{e \in G_0} D_e \delta_e$. Thus, $\alpha_{t^{-1}}(d_t) = P_0(y) \in P_0\left(I \cap \bigoplus_{e \in G_0} D_e \delta_e\right)$ and hence $P_0\left(I \cap \bigoplus_{e \in G_0} D_e \delta_e\right)$ is G-invariant. □

We define the map Φ : {Ideals of $R \rtimes_\alpha G$} $\to$ {G-invariant ideals of R} by

$$\Phi(I) := P_0\left(I \cap \bigoplus_{e \in G_0} D_e \delta_e\right). \tag{2.4}$$

Observe that, by Lemma 2.6, Φ is well-defined.

Remark 2.7. *Let J be a G-invariant ideal of R. One can show that $\alpha_g(D_{g^{-1}} \cap J) = D_g \cap J$, for all $g \in G$. Thus, we may define a partial action $\alpha^J := (J \cap D_g, \alpha_g^J)_{g \in G}$ of G on J, where $\alpha_g^J : J \cap D_{g^{-1}} \to J \cap D_g$, for all $g \in G$. Note that there is a natural injection of rings $\imath : J \rtimes_{\alpha^J} G \to R \rtimes_\alpha G$. We will often suppress the use of $\imath$ and simply identify $\imath(J \rtimes_{\alpha^J} G)$ with $J \rtimes_{\alpha^J} G$.*

Proposition 2.8. *The following assertions hold:*

(i) If J is a G-invariant ideal of R, then $J \rtimes_{\alpha^J} G$ is a G-graded ideal of $R \rtimes_\alpha G$.

(ii) If I is a G-graded ideal of $R \rtimes_\alpha G$, then $P_0\left(I \cap \bigoplus_{e \in G_0} D_e \delta_e\right) = P_0(I)$.

Proof. Suppose that J is a G-invariant ideal of R and that I is a G-graded ideal of $R \rtimes_\alpha G$.

(i) By definition, $J \rtimes_{\alpha^J} G = \bigoplus_{g\in G}(J \cap D_g)\delta_g$. It is not difficult to verify that $J \rtimes_{\alpha^J} G$ is an ideal of $R \rtimes_\alpha G$, and clearly $(J \cap D_g)\delta_g = (J \rtimes_\alpha G) \cap D_g\delta_g$, for all $g \in G$. Hence, $J \rtimes_{\alpha^J} G$ is a G-graded ideal of $R \rtimes_\alpha G$.

(ii) Take $a \in P_0(I)$ and choose $x := \sum_{t\in G} x_t\delta_t \in I$ such that $P_0(x) = \sum_{e\in G_0} x_e = a$. Note that $\sum_{e\in G_0} x_e\delta_e \in I \cap \bigoplus_{e\in G_0} D_e\delta_e$, because I is G-graded. Therefore, $a = P_0\left(\sum_{e\in G_0} x_e\delta_e\right) \in P_0\left(I \cap \bigoplus_{e\in G_0} D_e\delta_e\right)$. The other inclusion is trivial. □

Proposition 2.9. *Let I be an ideal of $S := R \rtimes_\alpha G$, and write $J := P_0(I)$, $I_0 := I \cap \bigoplus_{e\in G_0} D_e\delta_e$ and $J_0 := P_0(I_0)$. The following assertions hold:*

(i) $I \subseteq J \rtimes_{\alpha^J} G$.

(ii) *If $b_t\delta_t \in J_0 \rtimes_{\alpha^{J_0}} G$ for some $t \in G$ and $b_t \in D_t \cap J_0$, then $b_t\delta_{r(t)} \in I_0$.*

(iii) $J_0 \rtimes_{\alpha^{J_0}} G \subseteq I$.

(iv) $J_0 \rtimes_{\alpha^{J_0}} G = SI_0S$.

Proof. Notice that, by Lemma 2.5 and Lemma 2.6, both J and J_0 are G-invariant ideals of R. Thus, $J \rtimes_{\alpha^J} G$ and $J_0 \rtimes_{\alpha^{J_0}} G$ are well-defined.

(i) Let $a = \sum_{g\in G} a_g\delta_g \in I$. Take $t \in G$ and let $u_t \in D_t$ be an s-unit for a_t. Consider the element $y := a(\alpha_{t^{-1}}(u_t)\delta_{t^{-1}}) \in I$, and observe that

$$y := \sum_{\substack{g\in G\\ s(g)=s(t)}} \alpha_g(\alpha_{g^{-1}}(a_g)\alpha_{t^{-1}}(u_t))\delta_{gt^{-1}}.$$

Thus, $P_0(y) = \alpha_t(\alpha_{t^{-1}}(a_t)\alpha_{t^{-1}}(u_t)) = \alpha_t(\alpha_{t^{-1}}(a_tu_t)) = a_t \in J \cap D_t$. We conclude that $a \in \bigoplus_{g\in G}(J \cap D_t)\delta_t = J \rtimes_{\alpha^J} G$.

(ii) Suppose that $b_t\delta_t \in J_0 \rtimes_{\alpha^{J_0}} G$. Then, $b_t \in D_t \cap J_0 = D_t \cap P_0(I_0) \subseteq D_{r(t)} \cap P_0(I_0)$. Therefore, there is an element $a = \sum_{e\in G_0} a_e\delta_e \in I_0$ such that $P_0(a) = \sum_{e\in G_0} a_e = b_t \in D_{r(t)}$. Notice that $b_t - a_{r(t)} = \sum_{\substack{e\in G_0\\ e\neq r(t)}} a_e \in D_{r(t)} \cap \sum_{\substack{e\in G_0\\ e\neq r(t)}} D_e = \{0\}$. Thus, $a = b_t\delta_{r(t)}$.

(iii) By Proposition 2.8 (i), $J_0 \rtimes_{\alpha^{J_0}} G$ is a G-graded ideal of $R \rtimes_\alpha G$. Take $t \in G$ and let $b_t\delta_t \in J_0 \rtimes_{\alpha^{J_0}} G$. By (ii) we get that, $b_t\delta_{r(t)} \in I_0 \subseteq I$. Let $v \in D_t$ be an s-unit for b_t. Then, $b_t\delta_t = (b_t\delta_{r(t)})(v\delta_t) \in I$.

(iv) Take $u := \sum_{g\in G} a_g\delta_g \in S$, $v := \sum_{h\in G} c_h\delta_h \in S$ and $x := \sum_{e\in G_0} b_e\delta_e \in I_0$. Since I is an ideal of S, it follows from Lemma 2.4 (ii) that $b_f \in J_0$, for all $f \in G_0$. Notice that

$$uxv = \left(\sum_{g\in G} a_g\delta_g\right)\left(\sum_{e\in G_0} b_e\delta_e\right)\left(\sum_{h\in G} c_h\delta_h\right) = \sum_{\substack{e\in G_0\\ s(g)=r(h)=e}} \alpha_g(\alpha_{g^{-1}}(a_g)b_{r(h)}c_h)\delta_{gh}.$$

Using that J_0 is a G-invariant ideal of R, we get that $uxv \in \bigoplus_{g\in G}\left(J_0 \cap D_g\right)\delta_g = J_0 \rtimes_{\alpha^{J_0}} G$. This shows that $SI_0S \subseteq J_0 \rtimes_{\alpha^{J_0}} G$.

Conversely, by Lemma 2.6 and Proposition 2.8 (i), $J_0 \rtimes_{\alpha^{J_0}} G$ is a G-graded ideal of S. Let $d_g\delta_g \in J_0 \rtimes_{\alpha^{J_0}} G$. Then, by (ii), $d_g\delta_{r(g)} \in I_0$. Let $u_g \in D_g$ be an s-unit for d_g. A short calculation now reveals that $d_g\delta_g = (u_g\delta_{r(g)})(d_g\delta_{r(g)})(u_g\delta_g) \in SI_0S$. $\square$

Now, consider the map $\Psi : \{G\text{-invariant ideals of } R\} \to \{G\text{-graded ideals of } R \rtimes_\alpha G\}$ defined by $\Psi(J) := J \rtimes_{\alpha^J} G$. By Proposition 2.8 (i), Ψ is well-defined. Also, consider the restriction $\Phi_{gr} : \{G\text{-graded ideals of } R \rtimes_\alpha G\} \to \{G\text{-invariant ideals of } R\}$ of the map Φ defined in (2.4). Notice that, by Proposition 2.8 (ii), $\Phi_{gr}(I) = P_0(I)$.

We will now show that the above map Ψ is in fact a bijection. The corresponding result for partial actions of groups was proved in [40, Theorem 4.7, Proposition 12.2].

Theorem 2.10. *Let R be a ring, let G be a groupoid and let $\alpha = (\alpha_g, D_g)_{g\in G}$ be a partial action of G on R, such that D_g is s-unital for every $g \in G$, and $R = \bigoplus_{e\in G_0} D_e$. Then Ψ is a bijection whose inverse is Φ_{gr} defined above. In particular, there is a one-to-one correspondence between G-graded ideals of $R \rtimes_\alpha G$ and G-invariant ideals of R.*

Proof. Let J be a G-invariant ideal of R. By Proposition 2.8 (i) and (ii),

$$J = P_0\left((J \rtimes_{\alpha^J} G) \cap \bigoplus_{e\in G_0} D_e\delta_e\right) = P_0(J \rtimes_{\alpha^J} G) = P_0(\Psi(J)) = \Phi_{gr}(\Psi(J)).$$

Now, let I be a G-graded ideal of $R \rtimes_\alpha G$, and write $J := P_0(I)$, $I_0 := I \cap \bigoplus_{e\in G_0} D_e\delta_e$ and $J_0 := P_0(I_0)$. Notice that, by Proposition 2.8 (ii), $J = J_0$. By Proposition 2.9 (i) and (iii), we get that $I = J \rtimes_{\alpha^J} G = \Psi(P_0(I)) = \Psi(\Phi_{gr}(I))$. This shows that Ψ and Φ_{gr} are mutual inverses. $\square$

2.2 THE RESIDUAL INTERSECTION PROPERTY

In order to characterize graded ideals in partial skew groupoid rings, we introduce the notion of a residual intersection property. To that end, we first need to define partial actions on quotient rings.

Remark 2.11. *Suppose that J is a G-invariant ideal of R. Notice that J is not necessarily contained in D_g for any $g \in G$. On the other hand, for each $g \in G$, $D_g \cap J$ is an ideal of D_g, and by the second isomorphism theorem, $\frac{D_g}{D_g\cap J}$ is isomorphic to $\frac{D_g+J}{J}$.*

Guided by the above remark, we make the following definition.

Definition 2.12. *Let $R \rtimes_\alpha G$ be a partial skew groupoid ring. Suppose that J is a G-invariant ideal of R. The* quotient partial action of G on R/J*, denoted by $\overline{\alpha} := (\overline{D}_g, \overline{\alpha}_g)_{g\in G}$, is defined by:*

$$\overline{D}_g := \frac{D_g + J}{J}, \quad \overline{\alpha}_g : \overline{D}_{g^{-1}} \to \overline{D}_g, \quad x + J \mapsto \alpha_g(x) + J, \quad \textit{for all } x + J \in \overline{D}_{g^{-1}}. \tag{2.5}$$

We will sometimes refer to $\overline{\alpha} = (\overline{D}_g, \overline{\alpha}_g)_{g \in G}$ *as the* quotient partial action (of α) with respect to J.

Notice that, since J is a G-invariant ideal of R, $\overline{\alpha}_g$ is well-defined, for all $g \in G$. Also observe that $\overline{\alpha}_g$ is a ring isomorphism with inverse $\overline{\alpha}_{g^{-1}}$.

Lemma 2.13. *Let R be a ring and let I, J, K be ideals of R. If I or K is s-unital, then $(I + J) \cap (K + J) = (I \cap K) + J$.*

Proof. Let $x \in (I + J) \cap (K + J)$. There exist $i \in I$, $k \in K$ and $j, j' \in J$ such that $x = i + j = k + j'$. Without loss of generality, suppose that K is s-unital. Let $u_k \in K$ be an s-unit for k, and notice that

$$x = k + j' = u_k k + j' = u_k(i + j - j') + j' = u_k i + u_k(j - j') + j' \in (K \cap I) + J.$$

The reverse inclusion is trivial. □

We will now show that the partial actions that we consider in this section (defined on s-unital ideals) will always have well-defined associated quotient partial actions.

Proposition 2.14. *Let J be a G-invariant ideal of R, and let $\overline{\alpha} = (\overline{D}_g, \overline{\alpha}_g)_{g \in G}$ be defined as in Definition 2.12. Then $\overline{\alpha}$ is a partial action of G on R/J.*

Proof. Using that D_g is s-unital, for every $g \in G$, by Lemma 2.13, we get that $(D_h + J) \cap (D_{g^{-1}} + J) = (D_h \cap D_{g^{-1}}) + J$, for all $g,h \in G$. Since α is a partial action of G on R, it follows that $\overline{D}_{r(g)}$ is an ideal of R/J, $\overline{D}_g$ is an ideal of $\overline{D}_{r(g)}$, for all $g \in G$, and $\overline{\alpha}_e = \mathrm{id}_{\overline{D}_e}$, for all $e \in G_0$. Notice that $\overline{D}_{g^{-1}} \cap \overline{D}_h = ((D_{g^{-1}} + J) \cap (D_h + J))/J$. By assumption, $((D_{g^{-1}} + J) \cap (D_h + J))/J = ((D_{g^{-1}} \cap D_h) + J)/J$. Let $(g,h) \in G^2$ and $x \in \overline{D}_{g^{-1}} \cap \overline{D}_h = ((D_{g^{-1}} \cap D_h) + J)/J$. There is some $z \in D_{g^{-1}} \cap D_h$ such that $x = z + J$. Therefore, $\alpha_{h^{-1}}(z) \in \alpha_{h^{-1}}(D_{g^{-1}} \cap D_h) \subseteq D_{h^{-1}g^{-1}}$. Consequently, $\overline{\alpha}_{h^{-1}}(x) = \alpha_{h^{-1}}(z) + J \in \overline{D}_{h^{-1}g^{-1}}$. It is clear that $\overline{\alpha}_{gh}(y) = \overline{\alpha}_g(\overline{\alpha}_h(y))$, for all $y \in \overline{\alpha}_{h^{-1}}(\overline{D}_{g^{-1}} \cap \overline{D}_h)$. □

Remark 2.15. *Let R be a ring and let I, J be ideals of R. If I is s-unital, then $(I + J)/J$ is s-unital. Indeed, consider $i + J$ for $i \in I$. Let $u \in I$ be an s-unit for i. We get that $(u + J)(i + J) = i + J$ and $(i + J)(u + J) = i + J$.*

Proposition 2.16. *Let J be a G-invariant ideal of R, and let $\overline{\alpha} = (\overline{D}_g, \overline{\alpha}_g)_{g \in G}$ be defined as in Definition 2.12. Then $\overline{D}_g$ is s-unital for every $g \in G$, the partial skew groupoid ring $R/J \rtimes_{\overline{\alpha}} G$ is associative, and $R/J = \bigoplus_{e \in G_0} \overline{D}_e$.*

Proof. For each $g \in G$, the ideal D_g of R is s-unital and hence, by Remark 2.15, the ideal $\overline{D}_g$ is s-unital. Thus, Remark 1.14 (a) implies that $R/J \rtimes_{\overline{\alpha}} G$ is associative. For the last part, if there is $e \in G_0$ such that $x \in \overline{D}_e \cap \sum_{f \in G_0 \setminus \{e\}} \overline{D}_f$, then $x = b_e + J \in \overline{D}_e$ and $x = \sum_{f \in G_0 \setminus \{e\}} (b_f + J) \in \sum_{f \in G_0 \setminus \{e\}} \overline{D}_f$. Let $u_e + J \in \overline{D}_e$ be an s-unit for $b_e + J$. Then, $x = (b_e + J)(u_e + J) = \sum_{f \in G_0 \setminus \{e\}} (b_f + J)(u_e + J) = 0 + J$. For any $a = \sum_{e \in G_0} a_e \in R$, we have $a + J = (\sum_{e \in G_0} a_e) + J = \sum_{e \in G_0} (a_e + J) \in \bigoplus_{e \in G_0} \overline{D}_e$, because $R = \bigoplus_{e \in G_0} D_e$. □

The following definition is inspired by [51] and [29].

Definition 2.17. *Let $R \rtimes_\alpha G$ be a partial skew groupoid ring.*

(a) The partial action α is said to have the intersection property *if $I \cap \bigoplus_{e \in G_0} D_e\delta_e \neq \{0\}$, for every nonzero ideal I of $R \rtimes_\alpha G$.*

(b) The partial action α is said to have the residual intersection property *if, for every G-invariant ideal J of R, the quotient partial action $\overline{\alpha}$ with respect to J has the intersection property.*

Remark 2.18. *By considering $J = \{0\}$, it immediately becomes clear that the residual intersection property implies the intersection property. In Example 4.19, however, we will see that the converse does not hold in general. That is, the residual intersection property is strictly stronger than the intersection property.*

Remark 2.19. *Let J be a G-invariant ideal of R. Then, we have a natural injection of rings $\iota : J \rtimes_{\alpha^J} G \to R \rtimes_\alpha G$ and a natural surjection of rings $\pi : R \rtimes_\alpha G \to R/J \rtimes_{\overline{\alpha}} G$. Moreover, it is easy to check that*

$$0 \longrightarrow J \rtimes_{\alpha^J} G \xrightarrow{\iota} R \rtimes_\alpha G \xrightarrow{\pi} R/J \rtimes_{\overline{\alpha}} G \longrightarrow 0 \tag{2.6}$$

is a short exact sequence of rings.

Proposition 2.20. *The partial action α has the residual intersection property if, and only if, every ideal of $R \rtimes_\alpha G$ is G-graded.*

Proof. We first show the "only if" statement. Suppose that α has the residual intersection property. Let I be an ideal of $R \rtimes_\alpha G$. By Lemma 2.6, $J_0 := P_0\left(I \cap \bigoplus_{e \in G_0} D_e\delta_e\right)$ is a G-invariant ideal of R. Hence, it follows from Proposition 2.8 (i) that $J_0 \rtimes_{\alpha^{J_0}} G$ is a G-graded ideal of $R \rtimes_\alpha G$. We claim that $I = J_0 \rtimes_{\alpha^{J_0}} G$. The inclusion $\supseteq$ follows from Proposition 2.9 (iii). For the other inclusion, observe that, since $\pi : R \rtimes_\alpha G \to R/J_0 \rtimes_{\overline{\alpha}} G$ is surjective, $\pi(I)$ is an ideal of $R/J_0 \rtimes_{\overline{\alpha}} G$.

Claim $\pi(I) \cap \bigoplus_{e \in G_0} \overline{D}_e\delta_e = \{0\}$, where $\overline{D}_e := \frac{D_e + J_0}{J_0}$.

Take $x \in \pi(I) \cap \bigoplus_{e \in G_0} \overline{D}_e\delta_e$ and $y = \sum_{t \in F} a_t\delta_t \in I$, such that $F = \mathrm{Supp}(y)$ and $\pi(y) = x$. Let $F_0 \subseteq G_0$ be finite and such that $x = \sum_{e \in F_0}(a_e + J_0)\delta_e$. Take $t \in F$. Let $u \in D_t$ be an s-unit for a_t and $\alpha_{t^{-1}}(a_t)$. Notice that $a_t\delta_{r(t)} = u\delta_{r(t)} y u\delta_{t^{-1}} \in I \cap \bigoplus_{e \in G_0} D_e\delta_e$. It now follows that $a_t\delta_t = (a_t\delta_{r(t)})(u\delta_t) \in I$. We get that $\sum_{e \in F_0} a_e\delta_e = y - \sum_{t \in F \setminus F_0} a_t\delta_t \in I \cap \bigoplus_{e \in G_0} D_e\delta_e$. Let $f \in F_0$ and choose an s-unit $u_f \in D_f$ for a_f. Observe that $a_f\delta_f = (u_f\delta_f)\left(\sum_{e \in F_0} a_e\delta_e\right) \in I \cap \bigoplus_{e \in G_0} D_e\delta_e$. Then, $a_f \in P_0(I \cap \bigoplus_{e \in G_0} D_e\delta_e) = J_0$ and hence $a_f + J_0 = 0 + J_0$, for all $f \in F_0$. Thus, $x = 0$.

Since α has the residual intersection property it follows from the claim above that $\pi(I) = \{0\}$. Consequently, $I \subseteq \ker \pi = \iota(J_0 \rtimes_{\alpha^{J_0}} G) = J_0 \rtimes_{\alpha^{J_0}} G$, where ι and π are the ring homomorphisms given in (2.6). Thus, $J_0 \rtimes_{\alpha^{J_0}} G = I$ and we have that I is G-graded.

Now we show the "if" statement. Suppose that every ideal of $R \rtimes_\alpha G$ is G-graded. Let J be a G-invariant ideal of R and suppose that I is an ideal of $R/J \rtimes_{\overline{\alpha}} G$ such that $I \cap \left(\bigoplus_{e \in G_0} \overline{D}_e \delta_e\right) = \{0\}$, where $\overline{D}_e := (D_e + J)/J$. We want to show that $I = \{0\}$. By assumption, $\pi^{-1}(I)$ is a G-graded ideal of $R \rtimes_\alpha G$ and thus $\pi^{-1}(I) = \bigoplus_{t \in G} \pi^{-1}(I) \cap D_t \delta_t$. As π is surjective, we have

$$\pi(\pi^{-1}(I) \cap \bigoplus_{e \in G_0} D_e \delta_e) \subseteq \pi(\pi^{-1}(I)) \cap \pi\left(\bigoplus_{e \in G_0} D_e \delta_e\right) = I \cap \bigoplus_{e \in G_0} \overline{D}_e \delta_e = \{0\}.$$

Therefore, $J_0 := \pi^{-1}(I) \cap \bigoplus_{e \in G_0} D_e \delta_e \subseteq \bigoplus_{e \in G_0} (J \cap D_e)\delta_e$ and consequently we obtain that $L := P_0(J_0) \subseteq \bigoplus_{e \in G_0} (J \cap D_e) = J$, because $R = \bigoplus_{e \in G_0} D_e$. Thus, $L \rtimes_{\alpha^L} G \subseteq J \rtimes_{\alpha^J} G$. Next we show that $\pi^{-1}(I) \subseteq L \rtimes_{\alpha^L} G$. Take $t \in G$ and $b_t \delta_t \in \pi^{-1}(I)$. Let $w \in D_{t^{-1}}$ be an s-unit for $\alpha_{t^{-1}}(b_t)$ and note that $b_t \delta_{r(t)} = (b_t \delta_t)(w \delta_{t^{-1}}) \in \pi^{-1}(I)$. Hence $b_t \delta_{r(t)} \in \pi^{-1}(I) \cap D_{r(t)} \delta_{r(t)}$, which implies that $b_t \in P_0(\pi^{-1}(I) \cap D_{r(t)} \delta_{r(t)}) \subseteq L$. Thus, $b_t \delta_t \in L \rtimes_{\alpha^L} G \subseteq J \rtimes_{\alpha^J} G = \iota(J \rtimes_{\alpha^J} G) = \ker \pi$. Consequently, $I = \pi(\pi^{-1}(I)) = \{0\}$. $\square$

The following theorem is similar to [29, Theorem 3.2] which was proved in the context of partial actions of groups on C^*-algebras.

Theorem 2.21. *Let R be a ring, let G be a groupoid and let $\alpha = (\alpha_g, D_g)_{g \in G}$ be a partial action of G on R, such that D_g is s-unital for every $g \in G$ and $R = \bigoplus_{e \in G_0} D_e$. The following statements are equivalent:*

(i) There is a one-to-one correspondence between the ideals of $R \rtimes_\alpha G$ and the G-invariant ideals of R (given by (2.4)*);*

(ii) Every ideal of $R \rtimes_\alpha G$ is G-graded;

(iii) α has the residual intersection property.

Proof. Notice that (iii)⇔(ii)⇒(i) follows from Proposition 2.20 and Theorem 2.10. In order to prove that (i)⇒(ii), suppose that Φ is bijective. Let I be an ideal of $R \rtimes_\alpha G$. By Lemma 2.6, $\Phi(I) = P_0(I \cap \bigoplus_{e \in G_0} D_e \delta_e)$ is a G-invariant ideal of R. Then, by Proposition 2.8 (i), $J := \Phi(I) \rtimes_{\alpha^{\varphi(I)}} G$ is a G-graded ideal of $R \rtimes_\alpha G$. Note that $\Phi(J) = P_0\Big(\Big(P_0\Big(I \cap \bigoplus_{e \in G_0} D_e \delta_e\Big) \rtimes_{\alpha^{\varphi(I)}} G\Big) \cap \bigoplus_{e \in G_0} D_e \delta_e\Big) = P_0\Big(I \cap \bigoplus_{e \in G_0} D_e \delta_e\Big) = \Phi(I)$. Since Φ is injective, $I = J$ and I is G-graded. $\square$

Let T be a ring and let S be a nonempty subset of T. Recall that the centralizer of S in T is the subring $C_T(S) := \{t \in T \,:\, ts = st \text{ for all } s \in S\}$ of T. A subring S of T is said to be a maximal commutative subring of T if $C_T(S) = S$.

We define $\tau : R \rtimes_\alpha G \to R$ by $\tau\left(\sum_{g \in G} a_g \delta_g\right) := \sum_{g \in G} a_g$, and note that it is additive. The following result partially generalizes [51, Theorem 3] and [51, Theorem 4] to possibly non-unital rings. It also generalizes [31, Theorem 2.1] from partial skew group rings to partial skew groupoid rings.

Proposition 2.22. *Let $R \rtimes_\alpha G$ be a partial skew groupoid ring, and suppose that R is commutative. The following statements are equivalent:*

(i) $\bigoplus_{e\in G_0} D_e\delta_e$ *is a maximal commutative subring of* $R \rtimes_\alpha G$;

(ii) α *has the intersection property.*

Proof. Write $S := R \rtimes_\alpha G$ and $A := \bigoplus_{e\in G_0} D_e\delta_e$.

(i)⇒(ii) Suppose that A is maximal commutative in S. Let I be a nonzero ideal of S. Choose $x = \sum_{t\in F} x_t\delta_t \in I$ to be a nonzero element of minimal support amongst all nonzero elements of I. Pick $g \in F$ such that $x_g \neq 0$. Let $u \in D_{g^{-1}}$ be an s-unit for $\alpha_{g^{-1}}(x_g)$ and define $y := xu\delta_{g^{-1}} \in I$. Notice that

$$y = x_g\delta_g u\delta_{g^{-1}} + \sum_{t\in F\setminus\{g\}} x_t\delta_t u\delta_{g^{-1}} = x_g\delta_{r(g)} + \sum_{\substack{t\in F\setminus\{g\}\\ s(t)=s(g)}} \alpha_t(\alpha_{t^{-1}}(x_t)u)\delta_{tg^{-1}}.$$

Hence, $|\operatorname{Supp}(y)| \leq |\operatorname{Supp}(x)|$ and $y \neq 0$. Let $b = \sum_{e\in G_0} b_e\delta_e \in A$ and define $z := by - yb \in I$. Observe that $|\operatorname{Supp}(z)| < |\operatorname{Supp}(y)| \leq |\operatorname{Supp}(x)|$, because $bx_g\delta_{r(g)} - x_g\delta_{r(g)}b = (b_{r(g)}x_g - x_g b_{r(g)})\delta_{r(g)} = 0$. By minimality of the support of x, we get that $z = 0$. Thus, $by = yb$ for all $b \in A$. By assumption, $y \in A$ and hence $I \cap A \neq \{0\}$.

(ii)⇒(i) We show the contrapositive statement. Suppose that A is not maximal commutative in S. There is a nonzero element $a = \sum_{t\in G} a_t\delta_t \in C_S(A) \setminus A$. In particular, there exist some $e \in G_0$ and $g \in \operatorname{Supp}(a) \setminus G_0$, with $s(g) = r(g) = e$, such that $a_g\delta_g \in C_S(A)$. Let L be the ideal of S generated by $a_g\delta_e - a_g\delta_g$. Using that S is s-unital, by Remark 2.1, it is clear that L is nonzero.

We claim that $L \subseteq \ker(\tau)$. If we assume that the claim holds, then since $\tau|_A = P_0|_A$ is a ring isomorphism, by Lemma 2.4, we get that $A \cap L \cong \tau|_A(A \cap L) \subseteq \tau(L) = \{0\}$. This shows that α does not have the intersection property. Now we show the claim.

By the definition of L it follows that it is enough to show that τ maps elements of the form $b_t\delta_t(a_g\delta_e - a_g\delta_g)c_k\delta_k$ to zero, where $t,k \in G$ and $b_t \in D_t$, $c_k \in D_k$ satisfy $s(t) = e = r(k)$.

Let $u \in D_e$ be an s-unit for $\{c_k, \alpha_{g^{-1}}(c_k a_g)\}$. Using that $a_g\delta_g \in C_S(A)$, and that R is commutative, we get that

$$\begin{aligned}
b_t\delta_t(a_g\delta_e - a_g\delta_g)c_k\delta_k &= b_t\delta_t(a_g c_k\delta_k - a_g\delta_g c_k\delta_e u\delta_k)\\
&= b_t\delta_t(a_g c_k\delta_k - c_k\delta_e a_g\delta_g u\delta_k)\\
&= b_t\delta_t(a_g c_k\delta_k - c_k a_g\delta_g u\delta_k)\\
&= b_t\delta_t(a_g c_k\delta_k - \alpha_g(\alpha_{g^{-1}}(c_k a_g)u)\delta_{gk})\\
&= b_t\delta_t(a_g c_k\delta_k - a_g c_k\delta_{gk})\\
&= \alpha_t(\alpha_{t^{-1}}(b_t)a_g c_k)\delta_{tk} - \alpha_t(\alpha_{t^{-1}}(b_t)a_g c_k)\delta_{tgk}.
\end{aligned}$$

Clearly, $\tau(b_t\delta_t(a_g\delta_e - a_g\delta_g)c_k\delta_k) = 0$. This proves the claim. □

Corollary 2.23. *Let J be a G-invariant ideal of R. Suppose that R is commutative. The following statements are equivalent:*

(i) $\bigoplus_{e\in G_0} \overline{D}_e\delta_e$ *is a maximal commutative subring of* $R/J \rtimes_{\overline{\alpha}} G$;

(ii) $\overline{\alpha}$ *has the intersection property.*

Proof. Notice that R/J is commutative. The proof follows from Proposition 2.22. □

We finish this section by summarizing our results in the case of partial actions on commutative rings.

Theorem 2.24. *Let R be a ring, let G be a groupoid and let $\alpha = (\alpha_g, D_g)_{g\in G}$ be a partial action of G on R, such that D_g is s-unital for every $g \in G$ and $R = \bigoplus_{e\in G_0} D_e$. If R is commutative, then the following statements are equivalent:*

(i) $\bigoplus_{e\in G_0} \overline{D}_e\delta_e$ *is a maximal commutative subring of* $R/J \rtimes_{\overline{\alpha}} G$ *for every G-invariant ideal J of R;*

(ii) α *has the residual intersection property;*

(iii) Every ideal of $R \rtimes_\alpha G$ is G-graded.

Proof. The proof follows from Proposition 2.20 and Corollary 2.23. □

2.3 SIMPLICITY AND PRIMENESS OF $R \rtimes_\alpha G$

In this section, we study simplicity and primeness of partial skew groupoid rings. To that end, we recall the following definitions.

Definition 2.25. *Let $R \rtimes_\alpha G$ be a partial skew groupoid ring.*

*(a) R is called G-*prime *if $IJ = \{0\}$, for G-invariant ideals I,J of R, implies $I = \{0\}$ or $J = \{0\}$.*

*(b) R is called G-*simple *if the only G-invariant ideals of R are $\{0\}$ and R.*

(c) $R \rtimes_\alpha G$ is said to be graded prime *if there are no nonzero graded ideals I,J of $R \rtimes_\alpha G$ such that $IJ = \{0\}$.*

(d) $R \rtimes_\alpha G$ is said to be graded simple *if there is no graded ideal of $R \rtimes_\alpha G$ other than $\{0\}$ and $R \rtimes_\alpha G$.*

Proposition 2.26. *R is G-simple if, and only if, $R \rtimes_\alpha G$ is graded simple.*

Proof. We first show the "only if" statement by contrapositivity. Suppose that $R \rtimes_\alpha G$ is not graded simple. There is a nonzero proper graded ideal I of $R \rtimes_\alpha G$. Write $J := P_0(I)$ and $J_0 := P_0(I \cap \bigoplus_{e\in G_0} D_e\delta_e)$. By Proposition 2.8 (ii) and Proposition 2.9 (iii), $J \rtimes_{\alpha^J} G = J_0 \rtimes_{\alpha^{J_0}} G \subseteq I \subsetneq R \rtimes_\alpha G$. Hence, $J = P_0(I)$ is a proper nonzero G-invariant ideal of R. Thus, R is not G-simple.

Now we show the "if" statement. Suppose that $R \rtimes_\alpha G$ is graded simple. Let J be a nonzero G-invariant ideal of R. By Proposition 2.8 (i), $J \rtimes_{\alpha^J} G$ is a nonzero graded ideal of $R \rtimes_\alpha G$. Hence, $R \rtimes_\alpha G = J \rtimes_{\alpha^J} G$, and, by Theorem 2.10, $J = P_0(J \rtimes_{\alpha^J} G) = P_0(R \rtimes_\alpha G) = R$. Thus, R is G-simple. $\square$

Proposition 2.27. *R is G-prime if, and only if, $R \rtimes_\alpha G$ is graded prime.*

Proof. We first show the "only if" statement by contrapositivity. Suppose that $R \rtimes_\alpha G$ is not graded prime. There are nonzero graded ideals I_1, I_2 of $R \rtimes_\alpha G$ such that $I_1 I_2 = \{0\}$. By Lemma 2.6, $P_0(I_1 \cap \bigoplus_{e \in G_0} D_e\delta_e)$ and $P_0(I_2 \cap \bigoplus_{e \in G_0} D_e\delta_e)$ are nonzero G-invariant ideals of R. We conclude that R is not G-prime, because Lemma 2.4 (iii) yields

$$\begin{aligned} P_0(I_1 \cap \bigoplus_{e \in G_0} D_e\delta_e)P_0(I_2 \cap \bigoplus_{e \in G_0} D_e\delta_e) &= P_0((I_1 \cap \bigoplus_{e \in G_0} D_e\delta_e)(I_2 \cap \bigoplus_{e \in G_0} D_e\delta_e)) \\ &\subseteq P_0(I_1 I_2) = \{0\}. \end{aligned}$$

Now we show the "if" statement. Suppose that $R \rtimes_\alpha G$ is graded prime. Let J_1 and J_2 be G-invariant ideals of R such that $J_1 J_2 = \{0\}$. By Proposition 2.8 (i), $J_1 \rtimes_{\alpha^{J_1}} G$ and $J_2 \rtimes_{\alpha^{J_2}} G$ are graded ideals of $R \rtimes_\alpha G$. Moreover, using that $J_1 J_2 = \{0\}$ and that J_1 is G-invariant, it is clear that $(J_1 \rtimes_{\alpha^{J_1}} G) \cdot (J_2 \rtimes_{\alpha^{J_2}} G) = \{0\}$. Hence, by assumption, $J_1 \rtimes_{\alpha^{J_1}} G = \{0\}$ or $J_2 \rtimes_{\alpha^{J_2}} G = \{0\}$, i.e. $J_1 = \{0\}$ or $J_2 = \{0\}$. This shows that R is G-prime. $\square$

We point out that part (i) of the theorem below was proved in [40, Corollary 5.8] for non-degenerately group graded rings, and part (iii) generalizes [31, Theorem 2.3].

Theorem 2.28. *Let R be a ring, let G be a groupoid, and let $\alpha = (\alpha_g, D_g)_{g \in G}$ be a partial action of G on R, such that D_g is s-unital for every $g \in G$ and $R = \bigoplus_{e \in G_0} D_e$. The following assertions hold:*

(i) If $R \rtimes_\alpha G$ is prime, then R is G-prime.

(ii) Suppose that α has the intersection property. Then R is G-prime if, and only if, $R \rtimes_\alpha G$ is prime.

(iii) R is G-simple and α has the intersection property if, and only if, $R \rtimes_\alpha G$ is simple.

Proof. (i) This follows from Proposition 2.27.
(ii) The "if" statement follows from (i). We show the "only if" statement by contrapositivity. Suppose that $R \rtimes_\alpha G$ is not prime. Let I, J be nonzero ideals of $R \rtimes_\alpha G$ such that $IJ = \{0\}$. Since α has the intersection property, we have that $I_0 := P_0\left(I \cap \bigoplus_{e \in G_0} D_e\delta_e\right) \neq \{0\}$ and $J_0 := P_0\left(J \cap \bigoplus_{e \in G_0} D_e\delta_e\right) \neq \{0\}$. By Proposition 2.8 (i), $I_0 \rtimes_{\alpha^{I_0}} G$ and $J_0 \rtimes_{\alpha^{J_0}} G$ are nonzero graded ideals of $R \rtimes_\alpha G$. Moreover, by Proposition 2.9 (iii), $(I_0 \rtimes_{\alpha^{I_0}} G) \cdot (J_0 \rtimes_{\alpha^{J_0}} G) \subseteq IJ = \{0\}$. This shows that $R \rtimes_\alpha G$ is not graded prime, and hence, by Proposition 2.27, R is not G-prime.

(iii) We first show the "only if" statement. Suppose that R is G-simple and that α has the intersection property. Let I be a nonzero ideal of $R \rtimes_\alpha G$. By Lemma 2.6, $P_0\Big(I \cap \bigoplus_{e \in G_0} D_e\delta_e\Big)$ is a nonzero G-invariant ideal of R. Thus, $P_0\Big(I \cap \bigoplus_{e \in G_0} D_e\delta_e\Big) = R$. By Lemma 2.4, we get that $I \cap \bigoplus_{e \in G_0} D_e\delta_e = \bigoplus_{e \in G_0} D_e\delta_e$. Take $t \in G$ and $b_t\delta_t \in R \rtimes_\alpha G$. Let $u \in D_t$ be an s-unit for b_t. Using that $u\delta_{r(t)} \in \bigoplus_{e \in G_0} D_e\delta_e \subseteq I$, it follows that $b_t\delta_t = (u\delta_{r(t)})(b_t\delta_t) \in I$. Thus, $R \rtimes_\alpha G = I$ and hence $R \rtimes_\alpha G$ is simple.

We now show the "if" statement. Suppose that $R \rtimes_\alpha G$ is simple. Clearly, α has the intersection property and, by Proposition 2.26, R is G-simple. □

Remark 2.29. *By [18, Theorem 8], the group ring of a group H over a ring R, denoted by* $R[H]$, *is prime if, and only if, R is prime and H has no nontrivial finite normal subgroup. In particular, notice that the converse of Theorem 2.28 (i) does not hold.*

Corollary 2.30. *Suppose that R is commutative and that* $\bigoplus_{e \in G_0} D_e\delta_e$ *is a maximal commutative subring of* $R \rtimes_\alpha G$. *Then R is G-prime if, and only if,* $R \rtimes_\alpha G$ *is prime.*

Proof. It follows from Proposition 2.22 and Theorem 2.28 (ii). □

Corollary 2.31. *Suppose that R is commutative. Then R is G-simple and* $\bigoplus_{e \in G_0} D_e\delta_e$ *is a maximal commutative subring of* $R \rtimes_\alpha G$ *if, and only if,* $R \rtimes_\alpha G$ *is simple.*

Proof. It follows from Theorem 2.28 (iii) and Proposition 2.22. □

In Example 2.32, we will see that neither maximal commutativity of $\bigoplus_{e \in G_0} D_e\delta_e$ in $R \rtimes_\alpha G$ nor the intersection property of α are necessary conditions for primeness of $R \rtimes_\alpha G$.

Example 2.32. *Let* $q \in \mathbb{C}$ *be a root of unity, that is* $q^k = 1$ *for some* $k \in \mathbb{Z}$. *For each* $n \in \mathbb{Z}$, *define the ring isomorphism* $\sigma_n : \mathbb{C}[x] \to \mathbb{C}[x]$, *by* $\sigma_n(f)(x) := f(q^n x)$. *Notice that* $\sigma_{m+n} = \sigma_m \circ \sigma_n$, *for all* $m,n \in \mathbb{Z}$. *Hence* σ *defines a global action of the additive group* $\mathbb{Z}$ *on the complex polynomial ring* $\mathbb{C}[x]$. *Consider the corresponding skew group ring* $\mathbb{C}[x] \rtimes_\sigma \mathbb{Z}$. *By [40, Theorem 13.5],* $\mathbb{C}[x] \rtimes_\sigma \mathbb{Z}$ *is prime. Nevertheless,* $\mathbb{C}[x]\delta_0$ *is not maximal commutative in* $\mathbb{C}[x] \rtimes_\sigma \mathbb{Z}$. *In particular,* σ *does not have the intersection property.*

3 PARTIAL ACTIONS OF GROUPOIDS ON TORSION-FREE COMMUTATIVE ALGEBRAS GENERATED BY IDEMPOTENTS

This chapter is inspired by [8, Examples 7.8–7.9] and serves as a preparation for Chapter 4. Let T be a commutative unital ring. The set of idempotents of T, denoted by $E(T)$, is a Boolean algebra with operations given by $e \vee f := e+f-ef$, $e \wedge f := ef$ and $\neg e := 1-e$, for all $e, f \in E(T)$. Recall that if $\mathcal{B}$ is a Boolean algebra and P is a subset of $\mathcal{B}$, then $\uparrow P := \bigcup_{z \in P} \uparrow z = \{y \in \mathcal{B} : z = z \wedge y \text{ for some } z \in P\}$.

Throughout this section, let G be a groupoid, let $\mathcal{R}$ be a commutative unital ring and suppose that R is a commutative, torsion-free, unital, $\mathcal{R}$-algebra generated by its idempotents $E(R)$. By torsion-free, we mean that if a is a nonzero element of $\mathcal{R}$ and e is a nonzero element of $E(R)$, then $ae \neq 0$.

As we will see in this section, every partial action of a groupoid G on $\mathcal{R}$ corresponds to a partial action of G on the commutative $\mathcal{R}$-algebra $\mathcal{L}_c(X(R),\mathcal{R})$ of all locally constant functions $f : X(R) \to \mathcal{R}$ with compact support, where $X(R)$ is the Stone dual space of the Boolean algebra $E(R)$. Before we define the partial action on $\mathcal{L}_c(X(R),\mathcal{R})$, we need to develop a few concepts.

Recall that $X(R)$ is the Hausdorff space comprised of all ultrafilters on $E(R)$. The basis of the topology on $X(R)$ is given by the compact open subsets $Z_e := \{\mathcal{F} \in X(R) : e \in \mathcal{F}\}$, for all $e \in E(R)$. By [39, Theorem 1], $R \simeq \mathcal{L}_c(X(R),\mathcal{R})$ as $\mathcal{R}$-algebras, via an isomorphism that identifies e with 1_{Z_e}, for all $e \in E(R)$. As usual, addition and multiplication in $\mathcal{L}_c(X(R),\mathcal{R})$ are defined pointwise.

Next, we will see how a homomorphism between algebras induces a continuous function between the Stone duals of their respective set of idempotents. Let R and S be commutative, unital, torsion-free, $\mathcal{R}$-algebras generated by their respective set of idempotents and $\rho : R \to S$ a unital $\mathcal{R}$-algebra homomorphism. Then, the restriction of ρ to $E(R)$, $\rho|_{E(R)}: E(R) \to E(S)$, is a Boolean algebra homomorphism. If $X(R)$ and $X(S)$ are the Stone duals of $E(R)$ and $E(S)$, respectively, then we define $\hat{\rho} : X(S) \to X(R)$ by

$$\hat{\rho}(\mathcal{F}) := \rho^{-1}(\mathcal{F}), \quad \text{for all } \mathcal{F} \in X(S). \tag{3.1}$$

In a Boolean algebra the notions of prime filter and ultrafilter agree, and hence it is easy to prove that the inverse image of an ultrafilter by a Boolean algebra homomorphism is again an ultrafilter. We will now show that $\hat{\rho}$ is continuous. Let $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be a net converging to $\mathcal{F} \in X(S)$. Suppose that $x \in E(R)$ is such that $x \in \hat{\rho}(\mathcal{F}) = \rho^{-1}(\mathcal{F})$. Then, $\rho(x) \in \mathcal{F}$. Since the net $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ converges to $\mathcal{F} \in X(S)$, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $\rho(x) \in \mathcal{F}_\lambda$. Thus, $x \in \rho^{-1}(\mathcal{F}_\lambda) = \hat{\rho}(\mathcal{F}_\lambda)$ for all $\lambda \geq \lambda_0$. The case where $x \in E(R)$ is such that $x \notin \hat{\rho}(\mathcal{F})$ is proved similary. We conclude that $\hat{\rho}$ is continuous.

The final ingredient that we need in order to define a partial action on $\mathcal{L}_c(X(R),\mathcal{R})$ is to extend the aforementioned isomorphism $R \cong \mathcal{L}_c(X(R),\mathcal{R})$ to ideals of R. Let I be a unital ideal of R and let u be the multiplicative identity element of I. Then, $u \in E(R)$ and we get that $I \cong \mathcal{L}_c(Z_u, \mathcal{R})$ via the above isomorphism.

Let $\tau = (D_g,\tau_g)_{g\in G}$ be a partial action of G on R, and suppose that D_g is unital with a multiplicative identity element u_g, for every $g \in G$, and that $R = \bigoplus_{e\in G_0} D_e$. By Lemma 2.2, D_g is an ideal of R, for every $g \in G$. Therefore, $D_g \cong \mathcal{L}_c(Z_{u_g},\mathcal{R})$.

We will define a topological partial action of G on $X(R)$. Notice that, for all $g \in G$, the restriction $\tau_g|_{E(D_{g^{-1}})}: E(D_{g^{-1}}) \to E(D_g)$ is a Boolean algebra homomorphism. Therefore, the corresponding map $\hat{\tau}_g : X(D_g) \to X(D_{g^{-1}})$, defined by (3.1), is continuous.

For each $g \in G$, there is a homeomorphism between $X(D_g)$, the Stone dual of $E(D_g)$, and Z_{u_g}, the set of ultrafilters on $E(R)$ containing u_g. This correspondence is given by

$$\zeta_g : X(D_g) \to Z_{u_g},\ \mathcal{G} \mapsto \uparrow \mathcal{G}, \qquad \zeta_g^{-1} : Z_{u_g} \to X(D_g),\ \mathcal{F} \mapsto \{u_g x : x \in \mathcal{F}\}.$$

Define $\theta := (U_g,\theta_g)_{g\in G}$, where, for all $g \in G$, $U_g := Z_{u_g}$ and

$$\theta_g : U_{g^{-1}} \to U_g, \quad \mathcal{F} \mapsto \uparrow_{E(R)} \hat{\tau}_{g^{-1}}(\{u_{g^{-1}} x : x \in \mathcal{F}\}). \tag{3.2}$$

Notice that $\theta_g = \zeta_g \circ \hat{\tau}_{g^{-1}} \circ \zeta_{g^{-1}}^{-1}$ and we have the following commutative diagram:

$$\begin{tikzcd} X(D_{g^{-1}}) \arrow[r, "\hat{\tau}_{g^{-1}}"] \arrow[d, "\zeta_{g^{-1}}"'] & X(D_g) \arrow[d, "\zeta_g"'] \\ U_{g^{-1}} \arrow[r, "\theta_g"'] & U_g \end{tikzcd}$$

Lemma 3.1. *$\theta = (U_g,\theta_g)_{g\in G}$, as defined in* (3.2)*, is a topological partial action of G on $X(R)$.*

Proof. Notice the following:

(a) For all $g \in G$, U_g is an element in the basis of the topology on $X(R)$. We claim that $U_g \subseteq U_{r(g)}$. Take $\mathcal{F} \in U_g$ and notice that $u_g \in \mathcal{F}$. From $D_g \subseteq D_{r(g)}$, we get that $u_g = u_{r(g)} \wedge u_g \leqslant u_{r(g)}$. Hence, $u_{r(g)} \in \mathcal{F}$ and $\mathcal{F} \in U_{r(g)}$.

(b) For all $g \in G$, $\hat{\tau}_g$ and ζ_g are homeomorphisms, and hence $\theta_g : U_{g^{-1}} \to U_g$ is also a homeomorphism that satisfies:

 (i) $\theta_e = \mathrm{id}_{U_e}$, for all $e \in G_0$, because $\tau_e = \mathrm{id}_{D_e}$;

 (ii) $\theta_{h^{-1}}(U_{g^{-1}} \cap U_h) \subseteq U_{(gh)^{-1}}$, whenever $(g,h) \in G^2$. Indeed, if $\mathcal{F} \in U_{g^{-1}} \cap U_h$, then $u_{g^{-1}} u_h = u_{g^{-1}} \wedge u_h \in \mathcal{F}$. Since $\tau_{h^{-1}}(D_{g^{-1}} \cap D_h) \subseteq D_{(gh)^{-1}}$, we have that $\tau_{h^{-1}}(u_{g^{-1}} u_h) = \tau_{h^{-1}}(u_{g^{-1}} u_h) u_{(gh)^{-1}} \leqslant u_{(gh)^{-1}}$. Therefore, $u_{(gh)^{-1}} \in \theta_{h^{-1}}(\mathcal{F})$ and $\theta_{h^{-1}}(U_{g^{-1}} \cap U_h) \subseteq U_{(gh)^{-1}}$;

 (iii) $\theta_g(\theta_h(x)) = \theta_{gh}(x)$, for all $x \in \theta_{h^{-1}}(U_{g^{-1}} \cap U_h)$ and $(g,h) \in G^2$. Indeed, let $\mathcal{F} \in \theta_{h^{-1}}(U_{g^{-1}} \cap U_h)$ and $z \in \theta_{gh}(\mathcal{F})$. Then, there exists $x \in \mathcal{F}$ such that

$$z \geqslant \tau_{gh}(xu_{h^{-1}g^{-1}}) \geqslant \tau_{gh}(xu_{h^{-1}g^{-1}})u_g = \tau_g(\tau_h(xu_{h^{-1}})u_{g^{-1}}).$$

Therefore, $z \in \uparrow_{E(R)} \tau_g(\{u_{g^{-1}}y : y \in \uparrow_{E(R)} \tau_h\{xu_{h^{-1}} : x \in \mathcal{F}\}) = \theta_g(\theta_h(\mathcal{F}))$. □

Let $\theta = (U_g, \theta_g)_{g \in G}$ be the partial action of the groupoid G on $X(R)$ defined above. For each $g \in G$, we consider the ring isomorphism

$$\rho_g : \mathcal{L}_c(U_{g^{-1}}, \mathcal{R}) \longrightarrow \mathcal{L}_c(U_g, \mathcal{R}), \quad f \longmapsto f \circ \theta_{g^{-1}}.$$

Then, $\rho = (\mathcal{L}_c(U_g, \mathcal{R}), \rho_g)_{g \in G}$ is a partial action of G on $\mathcal{L}_c(X(R), \mathcal{R})$.

Proposition 3.2. *The partial actions $\tau = (D_g, \tau_g)_{g \in G}$ and $\rho = (\mathcal{L}_c(U_g, \mathcal{R}), \rho_g)_{g \in G}$, as defined above, are G-equivariant.*

Proof. Take $g \in G$. Recall that there is an isomorphism $\varphi_g : \mathcal{L}_c(U_g, \mathcal{R}) \to D_g$ given by $\varphi_g(1_{U_{eu_g}}) := eu_g$, for every $e \in E(R)$. We claim that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_c(U_{g^{-1}}, \mathcal{R}) & \xrightarrow{\rho_g} & \mathcal{L}_c(U_g, \mathcal{R}) \\ \downarrow{\scriptstyle \varphi_{g^{-1}}} & & \downarrow{\scriptstyle \varphi_g} \\ D_{g^{-1}} & \xrightarrow[\tau_g]{} & D_g \end{array}$$

Let $e \in E(R)$ and $\mathcal{F} \in U_g$. Notice that

$$\varphi_g^{-1} \circ \tau_g \circ \varphi_{g^{-1}}(1_{U_{eu_{g^{-1}}}})(\mathcal{F}) = \varphi_g^{-1}(\tau_g(eu_{g^{-1}}))(\mathcal{F}) = 1_{U_{\tau_g(eu_{g^{-1}})}}(\mathcal{F})$$

and $\rho_g(1_{U_{eu_{g^{-1}}}})(\mathcal{F}) = 1_{U_{eu_{g^{-1}}}} \circ \theta_{g^{-1}}(\mathcal{F}) = 1_{U_{eu_{g^{-1}}}}(\uparrow_{E(R)} \tau_{g^{-1}}(\{u_g x : x \in \mathcal{F}\})$. We claim that $1_{U_{\tau_g(eu_{g^{-1}})}}(\mathcal{F}) = 1_{U_{eu_{g^{-1}}}}(\uparrow_{E(R)} \tau_{g^{-1}}(\{u_g x : x \in \mathcal{F}\})$. Indeed, if $1_{U_{\tau_g(eu_{g^{-1}})}}(\mathcal{F}) = 1_{\mathcal{R}}$, then $\tau_g(eu_{g^{-1}}) \in \mathcal{F}$. Hence, $eu_{g^{-1}} = \tau_{g^{-1}}(\tau_g(eu_{g^{-1}})u_g) \in \uparrow_{E(R)} \tau_{g^{-1}}(\{u_g x : x \in \mathcal{F}\}$. Therefore, $1_{U_{eu_{g^{-1}}}}(\uparrow_{E(R)} \tau_{g^{-1}}(\{u_g x : x \in \mathcal{F}\}) = 1_{\mathcal{R}}$.

On the other hand, if $1_{U_{eu_{g^{-1}}}}(\uparrow_{E(R)} \tau_{g^{-1}}(\{u_g x : x \in \mathcal{F}\}) = 1_{\mathcal{R}}$, then there exists $x \in \mathcal{F}$ such that $eu_{g^{-1}} \geqslant \tau_{g^{-1}}(u_g x)$. Since $u_g x \in \mathcal{F}$, we have that $eu_{g^{-1}} \in \tau_{g^{-1}}(\mathcal{F})$. Thus, $\tau_g(eu_{g^{-1}}) \in \mathcal{F}$ and $1_{U_{\tau_g(eu_{g^{-1}})}}(\mathcal{F}) = 1_{\mathcal{R}}$. □

4 TOPOLOGICAL PARTIAL ACTIONS OF GROUPOIDS, THEIR PARTIAL SKEW GROUPOID RINGS, AND STEINBERG ALGEBRAS

In Section 3, we saw that any partial action of a groupoid on the ring of locally constant functions with compact support over a Stone space, actually corresponds to a partial action $\tau = (D_g, \tau_g)_{g\in G}$ of the same groupoid on a torsion-free, unital, commutative algebra generated by its idempotents, such that D_g is a unital ideal of the commutative algebra, for every $g \in G$ (see Proposition 3.2). This motivates us to investigate algebraic partial actions induced by topological partial actions as follows.

Throughout this section, let G be a groupoid, let $\mathbb{K}$ be a field, let $\mathcal{R}$ be a commutative unital ring, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g,\theta_g)_{g\in G}$ be a topological partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e\in G_0} X_e$. Let $\mathcal{L}_c(X,\mathcal{R})$ denote the commutative $\mathcal{R}$-algebra of all locally constant functions $f : X \to \mathcal{R}$ with compact support, and with addition and multiplication defined pointwise. The *support of* $f \in \mathcal{L}_c(X,\mathcal{R})$ is defined as the set $\mathrm{Supp}(f) := \{x \in X : f(x) \neq 0\}$, and by [6, Proposition 1.3.4, Example 1.5.28] it is always clopen.

Remark 4.1. *The topological partial action $\theta = (X_g,\theta_g)_{g\in G}$ induces a partial action $\alpha := (D_g,\alpha_g)_{g\in G}$ of G on the ring $\mathcal{L}_c(X,\mathcal{R})$. Indeed, for each $g \in G$, we put $D_g := \{f \in \mathcal{L}_c(X,\mathcal{R}) : \mathrm{Supp}(f) \subseteq X_g\}$, and define the ring isomorphism $\alpha_g : D_{g^{-1}} \to D_g$ by*

$$\alpha_g(f)(x) := \begin{cases} f \circ \theta_{g^{-1}}(x), & \text{if } x \in X_g \\ 0, & \text{otherwise}, \end{cases}$$

for $f \in D_{g^{-1}}$, Furthermore, using that $X = \bigsqcup_{e\in G_0} X_e$ one gets that $\mathcal{L}_c(X,\mathcal{R}) = \bigoplus_{e\in G_0} D_e$.

In Section 4.1, we will show equivalences between, on the one hand topological properties of θ, and on the other hand algebraic properties of the induced partial action α of G on $\mathcal{L}_c(X,\mathbb{K})$, and of its associated partial skew groupoid ring $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$.

In Section 4.2, inspired by [7], we define the transformation groupoid $G \rtimes_\theta X$ induced by the topological partial action θ, and notice that $\mathcal{L}_c(X,\mathcal{R}) \rtimes_\alpha G$ and the Steinberg algebra of the transformation groupoid, $A_{\mathcal{R}}(G \rtimes_\theta X)$, are isomorphic as $\mathcal{R}$-algebras. We also point out correspondences between properties of θ and properties of $G \rtimes_\theta X$.

In Section 4.3, we apply Theorems 2.24 and 2.28 together with results from Section 4.1, to characterize primeness, simplicity etc. of the partial skew groupoid ring $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$. Using that $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G \cong A_{\mathbb{K}}(G \rtimes_\theta X)$, we are able to recover some known results.

4.1 CONNECTIONS BETWEEN TOPOLOGICAL AND ALGEBRAIC ACTIONS

The goal of this section is to identify correspondences between topological properties of θ and algebraic properties of the induced partial action α of G on $\mathcal{L}_c(X,\mathbb{K})$. We define a map $\mathcal{I}$: {Open subsets of X} $\to$ {Ideals of $\mathcal{L}_c(X,\mathbb{K})$} by

$$\mathcal{I}(U) := \{f \in \mathcal{L}_c(X,\mathbb{K}) : \mathrm{Supp}(f) \subseteq U\}, \tag{4.1}$$

for every open subset U of X. We also define a map $\mathcal{S}$: {Ideals of $\mathcal{L}_c(X,\mathbb{K})$} $\to$ {Open subsets of X} by

$$\mathcal{S}(J) := \{x \in X : \text{there is } f \in J \text{ such that } f(x) \neq 0\} = \bigcup_{f \in J} \mathrm{Supp}(f), \tag{4.2}$$

for every ideal J of $\mathcal{L}_c(X,\mathbb{K})$. It is not difficult to verify that $\mathcal{I}$ and $\mathcal{S}$ are both well-defined. Furthermore, we notice that $\mathcal{I}(\mathcal{S}(J)) = J$, for every ideal J of $\mathcal{L}_c(X,\mathbb{K})$, and that $\mathcal{S}(\mathcal{I}(U)) = U$, for every open subset U of X. For more details, see e.g. [6, Lemma 3.2.1, Remark 3.2.2].

Remark 4.2. *(a) Analogously to [6, pp. 116–117], if U is an open G-invariant subset of X, then $\mathcal{I}(U)$ is a G-invariant ideal of $\mathcal{L}_c(X,\mathbb{K})$. Conversely, if J is a G-invariant ideal of $\mathcal{L}_c(X,\mathbb{K})$, then $\mathcal{S}(J)$ is an open G-invariant subset of X. Thus, by restricting $\mathcal{I}$ we get a bijection between*

{Open G-invariant subsets of X} and {G-invariant ideals of $\mathcal{L}_c(X,\mathbb{K})$}.

(b) Clearly, $F \subseteq X$ is closed if, and only if, $X \setminus F$ is open. Moreover, it is not difficult to see that F is G-invariant if, and only if, $X \setminus F$ is G-invariant. Hence, the assignment $F \mapsto X \setminus F$ yields a bijection between

{Closed G-invariant subsets of X} and {Open G-invariant subsets of X}.

(c) For any closed subset F of X, we define $\mathcal{J}(F) := \mathcal{I}(X \setminus F)$ and this yields a bijection $\mathcal{J}$: {Closed subsets of X} $\to$ {Ideals of $\mathcal{L}_c(X,\mathbb{K})$}. Note that

$$\mathcal{J}(F) = \{f \in \mathcal{L}_c(X,\mathbb{K}) : f(x) = 0, \text{ for all } x \in F\}.$$

We now summarize our findings.

Proposition 4.3. *There are one-to-one correspondences between the set of G-invariant ideals of $\mathcal{L}_c(X,\mathbb{K})$ and any of the following:*

(i) The set of open G-invariant subsets of X, given by $\mathcal{I}$;

(ii) The set of closed G-invariant subsets of X, given by $\mathcal{J}$;

(iii) The set of G-graded ideals of $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$, given by Φ_{gr}.

Next, we record some topological-dynamical notions that are used in this section.

Definition 4.4. *Let θ be a topological partial action of G on X.*

(a) θ *is said to be* minimal*, if there is no proper open G-invariant subset of X*;

(b) θ *is said to be* topologically transitive*, if for all nonempty open subsets U, V of X there exists $g \in G$ such that $\theta_g(U \cap X_{g^{-1}}) \cap V \neq \emptyset$*;

(c) θ *is said to be* topologically free*, if for all $u \in G_0$ and $t \in G_u^u \setminus \{u\}$*, $\mathrm{Int}\{x \in X_{t^{-1}} : \theta_t(x) = x\} = \emptyset$;

(d) θ *is said to be* topologically free on the subset F *of X, if for all $u \in G_0$ and $t \in G_u^u \setminus \{u\}$*, $\mathrm{Int}_F\{x \in X_{t^{-1}} \cap F : \theta_t(x) = x\} = \emptyset$.

The following result follows immediately from Proposition 4.3.

Corollary 4.5. *θ is minimal if, and only if, $\mathcal{L}_c(X,\mathbb{K})$ is G-simple.*

Theorem 4.6. *Let $\mathbb{K}$ be a field, let G be a groupoid, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g,\theta_g)_{g\in G}$ be a topological partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e\in G_0} X_e$. Furthermore, let α be the induced partial action of G on $\mathcal{L}_c(X,\mathbb{K})$ (see Remark 4.1). Then θ is topologically transitive if, and only if, $\mathcal{L}_c(X,\mathbb{K})$ is G-prime.*

Proof. We first show the "only if" statement by contrapositivity. Suppose that $\mathcal{L}_c(X,\mathbb{K})$ is not G-prime. There are nonzero G-invariant ideals I_1,I_2 of $\mathcal{L}_c(X,\mathbb{K})$ such that $I_1 I_2 = \{0\}$. By Remark 4.2 and the fact that $\mathbb{K}$ is a field, there are nonempty open G-invariant subsets U_1,U_2 of X such that $I_1 = \mathcal{I}(U_1)$, $I_2 = \mathcal{I}(U_2)$, and $U_1 \cap U_2 = \emptyset$. Hence, $\theta_g(U_1 \cap X_{g^{-1}}) \cap U_2 \subseteq U_1 \cap U_2 = \emptyset$, for every $g \in G$. This shows that θ is not topologically transitive.

Now we show the "if" statement. Suppose that $\mathcal{L}_c(X,\mathbb{K})$ is G-prime. Let U,V be nonempty open subsets of X. Define $W := \bigcup_{g\in G}\theta_g(U \cap X_{g^{-1}})$ and $Z := \bigcup_{g\in G}\theta_g(V \cap X_{g^{-1}})$. Clearly, W and Z are open, and they are also G-invariant. Indeed, for any $h \in G$ we have

$$\theta_h(W \cap X_{h^{-1}}) = \theta_h\left(\left(\bigcup_{g\in G}\theta_g\left(U \cap X_{g^{-1}}\right)\right) \cap X_{h^{-1}}\right) = \bigcup_{g\in G}\theta_h\left(\theta_g\left(U \cap X_{g^{-1}}\right) \cap X_{h^{-1}}\right).$$

Note that $\theta_g(U \cap X_{g^{-1}}) \cap X_{h^{-1}} \subseteq X_g \cap X_{h^{-1}} \subseteq X_{r(g)} \cap X_{s(h)} = \emptyset$ if $s(h) \neq r(g)$. Therefore, take $g \in G$ such that $s(h) = r(g)$, and notice that

$$\begin{aligned}\theta_h\left(\theta_g\left(U \cap X_{g^{-1}}\right) \cap X_{h^{-1}}\right) &\subseteq \theta_h\left(\theta_g\left(\theta_{g^{-1}}\left(X_{h^{-1}} \cap X_g\right) \cap U\right)\right)\\ &= \theta_{hg}\left(U \cap \theta_{g^{-1}}\left(X_{h^{-1}} \cap X_g\right)\right)\\ &\subseteq \theta_{hg}\left(U \cap X_{g^{-1}h^{-1}}\right) \subseteq W.\end{aligned}$$

This shows that W is G-invariant, and similarly one can show that Z is G-invariant. By Remark 4.2, $I_1 := \mathcal{I}(W)$ and $I_2 := \mathcal{I}(Z)$ are nonzero G-invariant ideals of $\mathcal{L}_c(X,\mathbb{K})$. Hence, by G-primeness of $\mathcal{L}_c(X,\mathbb{K})$, we get that $I_1 I_2 \neq \{0\}$ and thus $W \cap Z \neq \emptyset$.

Choose some $z \in W \cap Z$. There are $g,h \in G$ such that $z \in \theta_g(U \cap X_{g^{-1}}) \cap \theta_h(V \cap X_{h^{-1}})$. Since $X_g \cap X_h \subseteq X_{r(g)} \cap X_{s(h^{-1})}$, we have that $s(h^{-1}) = r(g)$. By Definition 1.9,

$$\begin{aligned}\theta_{h^{-1}}(z) \in \theta_{h^{-1}}(\theta_g(U \cap X_{g^{-1}}) \cap X_h) \cap V &\subseteq \theta_{h^{-1}}\left(\theta_g\left(\theta_{g^{-1}}\left(X_h \cap X_g\right) \cap U\right)\right) \cap V \\ = \theta_{h^{-1}g}\left(U \cap \theta_{g^{-1}}\left(X_h \cap X_g\right)\right) \cap V &\subseteq \theta_{h^{-1}g}\left(U \cap X_{g^{-1}h}\right) \cap V,\end{aligned}$$

and this shows that θ is topologically transitive. □

Next, we characterize topological freeness of θ in terms of properties of the associated partial skew groupoid ring. Note that if $F \subseteq X$ is closed and G-invariant, then by Remark 4.2, $\mathcal{J}(F)$ is a G-invariant ideal of $\mathcal{L}_c(X,\mathbb{K})$. Thus, $\frac{\mathcal{L}_c(X,\mathbb{K})}{\mathcal{J}(F)} \rtimes_{\overline{\alpha}} G$ is well-defined (cf. Proposition 2.16).

Theorem 4.7. *Let $\mathbb{K}$ be a field, let G be a groupoid, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g,\theta_g)_{g\in G}$ be a topological partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e\in G_0} X_e$. Furthermore, let α be the induced partial action of G on $\mathcal{L}_c(X,\mathbb{K})$ (see Remark 4.1). Suppose that $F \subseteq X$ is closed and G-invariant. Then, θ is topologically free on F if, and only if, $\bigoplus_{e\in G_0} \overline{D}_e\delta_e$ is a maximal commutative subring of $\frac{\mathcal{L}_c(X,\mathbb{K})}{\mathcal{J}(F)} \rtimes_{\overline{\alpha}} G$.*

Proof. We first show the "only if" statement. Suppose that θ is topologically free on F. Let $a = \sum_{g\in H}(a_g + \mathcal{J}(F))\delta_g \in \frac{\mathcal{L}_c(X,\mathbb{K})}{\mathcal{J}(F)} \rtimes_{\overline{\alpha}} G$ be such that $ad = da$, for all $d \in \bigoplus_{e\in G_0} \overline{D}_e\delta_e$, and $a_g \notin \mathcal{J}(F)$ for all $g \in H$. Seeking a contradiction, suppose that there is some $t \in H\backslash G_0$. For any compact open subset K of $X_{r(t)}$, using that $a(1_K+\mathcal{J}(F))\delta_{r(t)} = (1_K + \mathcal{J}(F))\delta_{r(t)}a$, we get that

$$\sum_{\substack{g\in H \\ s(g)=r(t)}} \overline{\alpha}_g(\overline{\alpha}_{g^{-1}}(a_g + \mathcal{J}(F))(1_K + \mathcal{J}(F)))\delta_g = \sum_{\substack{g\in H \\ r(g)=r(t)}} (1_K + \mathcal{J}(F))(a_g + \mathcal{J}(F))\delta_g. \quad (4.3)$$

We also notice that if $f + \mathcal{J}(F) = g + \mathcal{J}(F)$, then $(g-f)|_F = 0$ and $g|_F = f|_F$.

Case 1 ($s(t) \neq r(t)$): Let $x_0 \in F \cap X_t$ be such that $a_t(x_0) \neq 0$. Using that X is locally compact, Hausdorff and zero-dimensional, there exists a compact open set K such that $x_0 \in K \subseteq X_t \subseteq X_{r(t)}$. By comparing the left-hand side with the right-hand side of (4.3), since $s(t) \neq r(t)$, we get that $0 + \mathcal{J}(F) = a_t 1_K + \mathcal{J}(F)$. Thus, $a_t 1_K \in \mathcal{J}(F)$ and $a_t 1_K|_F = 0$, which is a contradiction, since $x_0 \in F$ and $a_t(x_0)1_K(x_0) = a_t(x_0) \neq 0$.

Case 2 ($s(t) = r(t)$): Let $z \in F \cap X_t$ be such that $a_t(z) \neq 0$. Then, there is an open set A such that $z \in A \subseteq X_t$ and $a_t(A) = \{a_t(z)\}$. We have that $A \cap F \subseteq X_t \cap F$ is open. Notice that $t \neq r(t)$. Hence, by assumption, there is $y \in A \cap F$ such that $\theta_{t^{-1}}(y) \neq y$. Using that X is Hausdorff, there exist disjoint open sets V, W such that

$y \in V$ and $\theta_{t^{-1}}(y) \in W$. There is also a compact open set K such that $y \in K \subseteq V \cap X_t \subseteq V \cap X_{r(t)}$. By comparing the left-hand side with the right-hand side of (4.3), we get that $\alpha_t(\alpha_{t^{-1}}(a_t)1_K) + \mathcal{J}(F) = a_t 1_K + \mathcal{J}(F)$ which yields $a_t(1_K \circ \theta_{t^{-1}}) + \mathcal{J}(F) = a_t 1_K + \mathcal{J}(F)$. Finally, $a_t(1_K \circ \theta_{t^{-1}} - 1_K) \in \mathcal{J}(F)$ and $(a_t(1_K \circ \theta_{t^{-1}} - 1_K))|_F = 0$. This is a contradiction, since $y \in A \cap F$ and

$$a_t(1_K \circ \theta_{t^{-1}} - 1_K)(y) = a_t(y)(1_K \circ \theta_{t^{-1}}(y) - 1_K(y)) = a_t(y)(0-1) = -a_t(y) = -a_t(z) \neq 0.$$

Now we show the "if" statement. Suppose that θ is not topologically free on F. Then, there exist $u_0 \in G_0$, $t_0 \in G_{u_0}^{u_0} \setminus \{u_0\}$ and an open subset $V \subseteq X$ such that $V \cap F \subseteq X_{t_0^{-1}} \cap F$ and $\theta_{t_0}(x) = x$, for all $x \in V \cap F$. Using that X is locally compact, Hausdorff and zero-dimensional, there exists a compact open subset K such that $x \in K \subseteq V \subseteq X_{t_0^{-1}}$. Therefore, $1_K \in D_{t_0^{-1}}$.

Then, we show that $(1_K + \mathcal{J}(F))\delta_{t_0^{-1}}$ commutes with every element of $\bigoplus_{e\in G_0} \overline{D}_e\delta_e$. Let $b = \sum_{e\in G_0}(f_e + \mathcal{J}(F))\delta_e \in \bigoplus_{e\in G_0} \overline{D}_e\delta_e$. We get that

$$\begin{aligned}
&((1_K + \mathcal{J}(F))\delta_{t_0^{-1}})b - b(1_K + \mathcal{J}(F))\delta_{t_0^{-1}} = \\
&(\alpha_{t_0^{-1}}(\alpha_{t_0}(1_K)f_{r(t_0)}) + \mathcal{J}(F))\delta_{t_0^{-1}r(t_0)} - (f_{s(t_0)}1_K + \mathcal{J}(F))\delta_{s(t_0)t_0^{-1}} = \\
&(\alpha_{t_0^{-1}}(\alpha_{t_0}(1_K)f_{u_0}) + \mathcal{J}(F))\delta_{t_0^{-1}} - (f_{u_0}1_K + \mathcal{J}(F))\delta_{t_0^{-1}} = \\
&(\alpha_{t_0^{-1}}((1_K \circ \theta_{t_0^{-1}})f_{u_0}) + \mathcal{J}(F))\delta_{t_0^{-1}} - (f_{u_0}1_K + \mathcal{J}(F))\delta_{t_0^{-1}} = \\
&(1_K((f_{u_0} \circ \theta_{t_0}) - f_{u_0}) + \mathcal{J}(F))\delta_{t_0^{-1}}.
\end{aligned}$$

Let $x \in F$. If $x \in V \cap F$, then $\theta_{t_0}(x) = x$ and $((f_{u_0} \circ \theta_{t_0}) - f_{u_0})(x) = 0$. If $x \in F \setminus V$, then $1_K(x) = 0$, because $K \subseteq V$. Hence, $(1_K + \mathcal{J}(F))\delta_{t_0^{-1}}b - b(1_K + \mathcal{J}(F))\delta_{t_0^{-1}} = 0 + \mathcal{J}(F)$. □

By Theorem 2.24 and Theorem 4.7 we get the following.

Corollary 4.8. *θ is topologically free on every closed G-invariant subset of X if, and only if, the induced partial action α of G on $\mathcal{L}_c(X,\mathbb{K})$ has the residual intersection property.*

4.2 STEINBERG ALGEBRAS AND PARTIAL SKEW GROUPOID RINGS

In this section, we point out that the partial skew groupoid ring associated with θ is isomorphic to the Steinberg algebra of the induced transformation groupoid. We also identify correspondences between topological properties of θ and properties of the associated transformation groupoid.

The *transformation groupoid associated with* θ is defined as the set

$$G \rtimes_\theta X := \{(t,x) : t \in G \text{ and } x \in X_{t^{-1}}\},$$

and $(G \rtimes_\theta X)^2 := \{((v,y),(t,x)) : (v,y),(t,x) \in G \rtimes_\theta X,\ (v,t) \in G^2 \text{ and } \theta_t(x) = y\}$. The product in $G \rtimes_\theta X$ is given by $(v,y)(t,x) := (vt,x)$ for all $((v,y),(t,x)) \in (G \rtimes_\theta X)^2$. For $(t,x) \in G \rtimes_\theta X$, we define $(t,x)^{-1} := (t^{-1},\theta_t(x))$.

We equip $G \ltimes_\theta X$ with the topology induced by the product topology on $G \times X$. Since G is discrete and X is Hausdorff, $G \ltimes_\theta X$ is Hausdorff. Notice that the inversion and product operations are continuous, because θ_t is continuous for every $t \in G$, and that $(G \ltimes_\theta X)_0 = \{(e,x) : e \in G_0 \text{ and } x \in X_e\}$. Using that $X = \bigsqcup_{e \in G_0} X_e$, there is a homeomorphism

$$\rho : X \ni x \longmapsto (e, x) \in (G \ltimes_\theta X)_0. \tag{4.4}$$

Hence, $(G \ltimes_\theta X)_0$ is locally compact, Hausdorff and zero-dimensional.

For each $(g,x) \in G \ltimes_\theta X$, the source in given by $d(g,x) := (s(g),x)$, the range is given by $t(g,x) := (r(g),\theta_g(x))$ and its restriction $\{g\} \times X_{g^{-1}} \to \{r(g)\} \times X_g$ is a homeomorphism. In particular, $G \ltimes_\theta X$ is étale. Therefore, $G \ltimes_\theta X$ is ample and Hausdorff. Recall that the Steinberg algebra, $A_{\mathcal{R}}(G \ltimes_\theta X)$, is the set of all locally constant functions $f : G \ltimes_\theta X \to \mathcal{R}$ with compact support (see [7] for more details).

Theorem 4.9. *Let $\mathcal{R}$ be a commutative unital ring, let G be a groupoid, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g, \theta_g)_{g \in G}$ be a partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e \in G_0} X_e$. Furthermore, let α be the induced partial action of G on $\mathcal{L}_c(X,\mathcal{R})$ (see Remark 4.1). Then, $\mathcal{L}_c(X,\mathcal{R}) \ltimes_\alpha G$ and $A_{\mathcal{R}}(G \ltimes_\theta X)$ are isomorphic as $\mathcal{R}$-algebras.*

Proof. It is analogous to the proof of [7, Theorem 3.2]. □

In the next definition, we recall some standard notions regarding topological groupoids. Throughout the remainder of this section, let $\mathcal{G}$ be a topological groupoid, let $\mathcal{G}_0$ denote the unit space of $\mathcal{G}$, let s denote the *source map* and let r denote the *range map*. A subset D of $\mathcal{G}_0$ is *invariant* if $r(g) \in D$ whenever $g \in \mathcal{G}$ and $s(g) \in D$. If D is an invariant subset of $\mathcal{G}_0$, then we define $\mathcal{G}_D := \{g \in \mathcal{G} : s(g) \in D, r(g) \in D\}$. Notice that $\mathcal{G}_D$ is a subgroupoid and that $(\mathcal{G}_D)_0 = D$.

Definition 4.10. *Let $\mathcal{G}$ be a topological groupoid and write* $\mathrm{Iso}(\mathcal{G}) := \{g \in \mathcal{G} : s(g) = r(g)\}$.

(a) $\mathcal{G}$ is said to be effective*, if $\mathcal{G}_0 = \mathrm{Int}(\mathrm{Iso}(\mathcal{G}))$;*

(b) $\mathcal{G}$ is said to be strongly effective*, if for every nonempty invariant subset D of $\mathcal{G}_0$, the groupoid $\mathcal{G}_D$ is effective;*

(c) $\mathcal{G}$ is said to be minimal*, if $\mathcal{G}_0$ has no nontrivial open invariant subsets;*

(d) An étale groupoid $\mathcal{G}$ is said to be topologically transitive*, if $s^{-1}(U) \cap r^{-1}(V) \neq \emptyset$ whenever U, V are nonempty open subsets of $\mathcal{G}_0$.*

The homeomorphism ρ, defined in (4.4), also appears in the context of transformation groupoids induced by partial actions of groups (see e.g. [6, Section 2.1]).

Proposition 4.11. *The homeomorphism* ρ, *defined in* (4.4)*, yields a bijection between the set of open (resp. closed) G-invariant subsets of X and the set of open (resp. closed) invariant subsets of* $(G \rtimes_\theta X)_0$.

Proof. Let F be a G-invariant subset of X. We claim that $\rho(F)$ is an invariant subset of $(G \rtimes_\theta X)_0$. Take $(g,x) \in G \rtimes_\theta X$ such that $d(g,x) = (s(g),x) \in \rho(F)$. Notice that $t(g,x) = (r(g),\theta_g(x)) \in \rho(F)$, because $\theta_g(x) \in F \cap X_g \subseteq F \cap X_{r(g)}$, and $\rho(\theta_g(x)) = (r(g),\theta_g(x))$.

Let D be an invariant subset of $(G \rtimes_\theta X)_0$. Take $g \in G$. Let $x \in \rho^{-1}(D) \cap X_{g^{-1}} \subseteq \rho^{-1}(D) \cap X_{s(g)}$. Then, $\rho(x) = (s(g),x) \in D$. Using that D is invariant, $\rho(\theta_g(x)) = (r(g), \theta_g(x)) \in D$. Thus, $\theta_g(x) \in \rho^{-1}(D)$. This shows that $\theta_g(\rho^{-1}(D) \cap X_{g^{-1}}) \subseteq \rho^{-1}(D)$. □

Proposition 4.12. *The following assertions hold:*

(i) θ *is minimal if, and only if,* $G \rtimes_\theta X$ *is minimal.*

(ii) θ *is topologically transitive if, and only if,* $G \rtimes_\theta X$ *is topologically transitive.*

(iii) Let F be a closed G-invariant subset of X. Then, θ *is topologically free on F if, and only if,* $(G \rtimes_\theta X)_{\rho(F)}$ *is effective.*

Proof. (i) It follows from Proposition 4.11.

(ii) We first show the “if” statement. Suppose that $G \rtimes_\theta X$ is topologically transitive. Let U, V be nonempty open subsets of X. Then, $\rho(U)$ and $\rho(V)$ are nonempty open subsets of $(G \rtimes_\theta X)_0$. By assumption, there exists $(g,x) \in d^{-1}(\rho(U)) \cap t^{-1}(\rho(V))$. Hence, $x \in X_{g^{-1}}$ and $d(g,x) = (s(g),x) = \rho(x) \in \rho(U)$. Furthermore, $t(g,x) = (r(g),\theta_g(x)) = \rho(\theta_g(x)) \in \rho(V)$. Thus, $\theta_g(x) \in \theta_g(U \cap X_{g^{-1}}) \cap V \neq \emptyset$.

Now we show the “only if” statement. Suppose that θ is topologically transitive. Let A,B be nonempty open subsets of $(G \rtimes_\theta X)_0$. Then, $\rho^{-1}(A)$ and $\rho^{-1}(B)$ are nonempty open subsets of X. By assumption, there exists $g \in G$ such that $\theta_g(\rho^{-1}(A) \cap X_{g^{-1}}) \cap \rho^{-1}(B) \neq \emptyset$. Let $y \in \theta_g(\rho^{-1}(A) \cap X_{g^{-1}}) \cap \rho^{-1}(B)$. Then, $\theta_{g^{-1}}(y) \in \rho^{-1}(A) \cap X_{g^{-1}}$ and $d(g, \theta_{g^{-1}}(y)) = \rho(\theta_{g^{-1}}(y)) \in A$. Furhermore, $t(g, \theta_{g^{-1}}(y)) = \rho(y) \in B$. Hence, $(g, \theta_{g^{-1}}(y)) \in d^{-1}(A) \cap t^{-1}(B)$.

(iii) Suppose that F is nonempty, for otherwise there is nothing to prove.

We first show the “only if” statement. Suppose that θ is topologically free on F. We will show that $\mathrm{Int}(\mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)}) = \rho(F)$.

($\subseteq$) Let $(g,x) \in \mathrm{Int}(\mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)})$. Then, there is an open set U of $X_{g^{-1}}$ such that $(g,x) \in (\{g\} \times U) \cap (G \rtimes_\theta X)_{\rho(F)} \subseteq \mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)})$. Let $(g,y) \in (\{g\} \times U) \cap (G \rtimes_\theta X)_{\rho(F)}$. Then, $d(g,y) = t(g,y) \in \rho(F)$, that is, $s(g) = r(g)$ and $y = \theta_g(y) \in F$. In particular, $x \in U \cap F \subseteq \{y \in X_{g^{-1}} \cap F : \theta_g(y) = y\}$ and $x \in \mathrm{Int}_F\{y \in X_{g^{-1}} \cap F : \theta_g(y) = y\}$. Using that θ is topologically free on F, we have that $g \in G_0$ and $(g, x) \in \rho(F)$.

($\supseteq$) Let $(e,x) \in \rho(F)$. Notice that $(e,x) \in (\{e\} \times \{X_e\}) \cap (G \rtimes_\theta X)_{\rho(F)} \subseteq \mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)})$. Therefore, $\rho(F) \subseteq \mathrm{Int}(\mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)}))$ and we conclude that $(G \rtimes_\theta X)_{\rho(F)}$ is effective.

Now we show the "if" statement by contrapositivity. Suppose that θ is not topologically free on F. Then, there exist $g \in G \backslash G_0$ and some $x \in \mathrm{Int}_F\{y \in X_{g^{-1}} \cap F : \theta_g(y) = y\}$. Thus, there is an open set $U \subseteq X_{g^{-1}}$ such that $x \in U \cap F \subseteq \{y \in X_{g^{-1}} \cap F : \theta_g(y) = y\}$. Let $z \in U \cap F$. Then, $\theta_g(z) = z$, and $z \in (X_{r(g)} \cap X_{s(g)}) \cap F$. Therefore, $s(g) = r(g)$ and $r(g,z) = d(g,z) = \rho(z) \in \rho(F)$. We notice that $(g,z) \in \mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)})$. Using that F is G-invariant, $(g,x) \in \{g\} \times (U \cap F) = (\{g\} \times U) \cap (G \rtimes_\theta X)_{\rho(F)} \subseteq \mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)})$ and we have that $(g,x) \in \mathrm{Int}(\mathrm{Iso}((G \rtimes_\theta X)_{\rho(F)}))$, but $(g,x) \notin \rho(F)$, because $g \notin G_0$. Hence, $(G \rtimes_\theta X)_{\rho(F)}$ is not effective. □

Remark 4.13. *Note that by [56, Corollary 5.5] and [55, Proposition 3.3] the statements (i) and (ii) in Proposition 4.12 are already known for transformation groupoids induced by global actions of groups. Moreover, Corollary 4.14 below was proved for transformation groupoids induced by partial actions of groups in [6, Proposition 4.2.1] and [6, Example 4.1.12].*

Corollary 4.14. *θ is topologically free if, and only if, $G \rtimes_\theta X$ is effective.*

Corollary 4.15. *θ is topologically free on every closed G-invariant subset of X if, and only if, $G \rtimes_\theta X$ is strongly effective.*

4.3 APPLICATIONS TO PARTIAL SKEW GROUPOID RINGS INDUCED BY TOPOLOGICAL PARTIAL ACTIONS

In this section, we let $\alpha = (D_g, \alpha_g)_{g \in G}$ be the partial action of G on $\mathcal{L}_c(X,\mathbb{K})$ induced by θ (see Remark 4.1). We aim to characterize primeness, simplicity, and when every ideal of the skew groupoid ring $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ is G-graded using the topological properties of the partial action θ. Using that $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ and the Steinberg algebra of the transformation groupoid, $A_{\mathbb{K}}(G \rtimes_\theta X)$, are isomorphic (see Theorem 4.9), we are able to recover some known results.

Theorem 4.16. *Let $\mathbb{K}$ be a field, let G be a groupoid, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g, \theta_g)_{g \in G}$ be a topological partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e \in G_0} X_e$. Furthermore, let α be the induced partial action of G on $\mathcal{L}_c(X,\mathbb{K})$ (see Remark 4.1). The following statements are equivalent:*

(i) *θ is topologically free on every open G-invariant subset of X;*

(ii) *θ is topologically free on every closed G-invariant subset of X;*

(iii) *$\bigoplus_{e \in G_0} \overline{D}_e \delta_e$ is a maximal commutative subring of $\frac{\mathcal{L}_c(X,\mathbb{K})}{\mathcal{J}(F)} \rtimes_{\overline{\alpha}} G$ for every closed G-invariant subset F of X;*

(iv) *α has the residual intersection property;*

(v) *Every ideal of $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ is G-graded;*

(vi) The transformation groupoid $G \rtimes_\theta X$ is strongly effective.

Proof. The equivalence between (i) and (ii) is straightforward. The other equivalences are established through Theorem 4.7, Corollary 4.8, Theorem 2.24, and Corollary 4.15 □

By combining Theorem 4.16, Proposition 4.3 and Theorem 2.21, we get the following result which partially generalizes [17, Corollary 3.7].

Corollary 4.17. *θ is topologically free on every closed G-invariant subset of X if, and only if, there is a one-to-one correspondence (given by $\Gamma := \mathcal{I}^{-1} \circ \Phi$) between the ideals of $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ and the open G-invariant subsets of X.*

A result similar to Theorem 4.18 below was proved in [45, Theorem 1.3] for a groupoid dynamical system in the context of global actions.

Theorem 4.18. *Let $\mathbb{K}$ be a field, let G be a groupoid, let X be a locally compact, Hausdorff, zero-dimensional space, and let $\theta = (X_g,\theta_g)_{g\in G}$ be a topological partial action of G on X, such that X_g is clopen, for every $g \in G$, and $X = \bigsqcup_{e\in G_0} X_e$. Furthermore, let α be the induced partial action of G on $\mathcal{L}_c(X,\mathbb{K})$ (see Remark 4.1). The following statements are equivalent:*

(i) θ is topologically free;

(ii) $\bigoplus_{e\in G_0} D_e\delta_e$ is a maximal commutative subring of $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$;

(iii) α has the intersection property;

(iv) The transformation groupoid $G \rtimes_\theta X$ is effective.

Proof. The equivalence between (i) and (ii) follows from Theorem 4.7 by considering $F = X$. The equivalence between (ii) and (iii) follows from Proposition 2.22, and Corollary 4.14 implies the equivalence between (i) and (iv). □

The following example shows that the condition of being topologically free on every closed G-invariant subset is stronger than topological freeness. In particular, the residual intersection property is a stronger condition than the intersection property.

Example 4.19 ([53]). *Let θ be an action of $\mathbb{Z}$ on the Alexandroff compactification $\mathbb{Z}\cup\{\infty\}$ such that, for all $n \in \mathbb{Z}$, $\theta_n : \mathbb{Z} \cup \{\infty\} \ni x \longmapsto x + n \in \mathbb{Z} \cup \{\infty\}$. Notice that $\mathrm{Fix}(\theta_n) = \{\infty\}$, for all $n \in \mathbb{Z}$. Hence, θ is topologically free, because $\mathrm{Int}\{x \in \mathbb{Z} \cup \{\infty\} : \theta_n(x) = x\} = \mathrm{Int}\{\infty\} = \emptyset$, for all $n \in \mathbb{Z}$. Nevertheless, $\{\infty\}$ is a closed $\mathbb{Z}$-invariant subset of $\mathbb{Z} \cup \{\infty\}$ such that $\mathrm{Int}_{\{\infty\}}\{x \in \{\infty\} : \theta_n(x) = x\} = \mathrm{Int}_{\{\infty\}}\{\infty\} = \{\infty\}$. Thus, θ in not topologically free on $\{\infty\}$.*

Proposition 4.20. *The following assertions hold:*

(i) $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ is graded simple if, and only if, θ is minimal;

(ii) $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ *is graded prime if, and only if,* θ *is topologically transitive.*

Proof. (i) It follows from Proposition 2.26 and Corollary 4.5.

(ii) It follows from Proposition 2.27 and Theorem 4.6. □

In the next theorem, we will recover particular cases of [55, Theorem 4.3, Theorem 4.5] and [10, Theorem 4.1].

Theorem 4.21. *Let* $\mathbb{K}$ *be a field, let* G *be a groupoid, let* X *be a locally compact, Hausdorff, zero-dimensional space, and let* $\theta = (X_g, \theta_g)_{g\in G}$ *be a topological partial action of* G *on* X*, such that* X_g *is clopen, for every* $g \in G$*, and* $X = \bigsqcup_{e\in G_0} X_e$*. Furthermore, let* α *be the induced partial action of* G *on* $\mathcal{L}_c(X,\mathbb{K})$ *(see Remark 4.1). The following assertions hold.*

(i) If $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ *is prime, then* θ *is topologically transitive.*

(ii) If θ *is topologically free and topologically transitive, then* $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ *is prime.*

(iii) θ *is minimal and topologically free if, and only if,* $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha G$ *is simple.*

Proof. The proof follows from Corollary 4.5, Theorem 4.6 and Theorem 2.28. □

In [48, Theorem 1.3], a result similar to Theorem 4.21 (iii) was proved for groupoid dynamical systems. We finish this section by specializing to the case when G is not just a groupoid, but in fact a torsion-free group.

Proposition 4.22. *Let* H *be a torsion-free group. Then,* $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha H$ *is prime if, and only if,* θ *is topologically transitive.*

Proof. By [40, Theorem 13.5], if H is torsion-free, then $\mathcal{L}_c(X,\mathbb{K}) \rtimes_\alpha H$ is prime if, and only if, $\mathcal{L}_c(X,\mathbb{K})$ is H-prime. The desired conclusion now follows from Theorem 4.6. □

5 APPLICATIONS TO ULTRAGRAPHS VIA LABELLED SPACES

Throughout this section, $\mathcal{G}$ denotes an ultragraph. In [14, Definition 2.7], a normal labelled space $(\mathcal{E}_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ associated with $\mathcal{G}$ was defined and, in [15, Section 3], a topological partial action φ of the free group generated by the labels of the edges on the tight spectrum T of a labelled space was defined. The description of φ when the labelled space is associated with an ultragraph was given in [14, Section 4.1], in which it was also shown that the associated partial skew group ring coincides with the ultragraph Leavitt path algebra of $\mathcal{G}$, and with the partial skew group ring associated with certain non-topological partial actions [34].

The purpose of this section is to prove that φ is topologically free on every closed $\mathbb{F}$-invariant subset of T if, and only if, the ultragraph $\mathcal{G}$ satisfies Condition (K). By letting α be the algebraic partial action induced by φ, we will show that every ideal of $\mathcal{L}_c(\mathsf{T}, \mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{F}$-graded if, and only if, $\mathcal{G}$ satisfies Condition (K), a result that is also new in the context of Leavitt path algebras of graphs. A version of this theorem was proved in the context of C^*-algebras of Boolean dynamical systems. See [12, Theorem 6.3] and [16, Proposition 6.1].

5.1 PRELIMINARIES

Following [9], [14], [15] and, mainly [8], we repeat some basic definitions and properties concerning labelled spaces, ultragraphs via labelled spaces and the aforementioned topological partial action φ.

5.1.1 Labelled spaces

A (directed) graph is a quadruple $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$, where $\mathcal{E}^0$ and $\mathcal{E}^1$ are nonempty sets and r,$s : \mathcal{E}^1 \to \mathcal{E}^0$ are called *range* and *source* maps. The elements of $\mathcal{E}^0$ are denominated *vertices* and the elements of $\mathcal{E}^1$ are denominated *edges*. If $\mathcal{E}^0$ and $\mathcal{E}^1$ are countable, then we say that the graph $\mathcal{E}$ is *countable*.

A sequence of edges $\lambda_1\lambda_2\ldots\lambda_n$ such that $r(\lambda_i) = s(\lambda_{i+1})$ for all $i \in \{1, \ldots, n-1\}$ is called a *path* of length n. We denote by $\mathcal{E}^n$ the set of paths of length n and define $\mathcal{E}^* := \cup_{n\geq 0}\mathcal{E}^n$. An infinite sequence of edges $\lambda_1\lambda_2\ldots$ such that $r(\lambda_i) = s(\lambda_{i+1})$ for all $i \geq 1$ is called an *infinite path*. We denote by $\mathcal{E}^\infty$ the set of infinite paths. We define vertices to be paths of length 0.

Let $\mathcal{E}$ be a graph and let $\mathcal{A}$ be an *alphabet*, that is a nonempty set whose elements are called *letters*. A *labelled graph* $(\mathcal{E}, \mathcal{L})$ consists of a graph $\mathcal{E}$ and a surjective *labelling map* $\mathcal{L} : \mathcal{E}^1 \to \mathcal{A}$.

$\mathcal{A}^*$ denotes the set of all finite *words* over $\mathcal{A}$, including the *empty word* ω, and $\mathcal{A}^\infty$ denotes the set of all infinite words over $\mathcal{A}$. We consider $\mathcal{A}^*$ as a monoid whose binary operation is given by concatenation.

We may extend the labelling map $\mathcal{L}$ to $\mathcal{L} : \mathcal{E}^n \to \mathcal{A}^*$ and $\mathcal{L} : \mathcal{E}^\infty \to \mathcal{A}^\infty$. Denote the set of *labelled paths* α *of length* $|\alpha| = n$ by $\mathcal{L}^n := \mathcal{L}(\mathcal{E}^n)$ and the set of *infinite labelled paths* by $\mathcal{L}^\infty := \mathcal{L}(\mathcal{E}^\infty)$. Recall that ω is considered a labelled path with $|\omega| = 0$ and we define $\mathcal{L}^{\geq 1} := \cup_{n\geq 1}\mathcal{L}^n$, $\mathcal{L}^* := \{\omega\} \cup \mathcal{L}^{\geq 1}$ and $\mathcal{L}^{\leqslant\infty} := \mathcal{L}^* \cup \mathcal{L}^\infty$.

If α, β are labelled paths such that $\beta = \alpha\beta'$ for some labelled path β', then we say that α is a *beginning* of β. Denote the set of all infinite words all of whose beginnings appear as finite labelled paths by $\overline{\mathcal{L}^\infty} := \{\alpha \in \mathcal{A}^\infty : \alpha_{1,n} \in \mathcal{L}^*, \text{ for every } n \in \mathbb{N}\}$ and define $\overline{\mathcal{L}^{\leqslant\infty}} := \mathcal{L}^* \cup \overline{\mathcal{L}^\infty}$. We let $\mathbb{P}(X)$ denote the power set of X.

Definition 5.1 ([9, Definition 2.8])**.** *For* $\alpha \in \mathcal{L}^*$ *and* $A \in \mathbb{P}(\mathcal{E}^0)$*, the* relative range of α with respect to A*, denoted by* $r(A,\alpha)$*, is the set*

$$r(A,\alpha) := \{r(\lambda) : \lambda \in \mathcal{E}^*, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\},$$

if $\alpha \in \mathcal{L}^{\geq 1}$*, and* $r(A,\omega) := A$*, if* $\alpha = \omega$*. The* range of α*, denoted by* $r(\alpha)$*, is the set* $r(\alpha) := r(\mathcal{E}^0,\alpha)$.

Note that $r(\omega) = \mathcal{E}^0$ and, if $\alpha \in \mathcal{L}^{\geq 1}$, then $r(\alpha) = \{r(\lambda) \in \mathcal{E}^0 \ : \ \mathcal{L}(\lambda) = \alpha\}$.

Definition 5.2 ([9, Definition 2.9])**.** *Let* $(\mathcal{E}, \mathcal{L})$ *be a labelled graph and* $\mathcal{B} \subseteq \mathbb{P}(\mathcal{E}^0)$*. We say that* $\mathcal{B}$ *is* closed under relative ranges*, if* $r(A,\alpha) \in \mathcal{B}$*, for all* $A \in \mathcal{B}$ *and* $\alpha \in \mathcal{L}^{\geq 1}$*. If additionally* $\mathcal{B}$ *is closed under finite intersections and finite unions, and contains* $r(\alpha)$*, for every* $\alpha \in \mathcal{L}^{\geq 1}$*, then we say that* $\mathcal{B}$ *is* accommodating *for* $(\mathcal{E}, \mathcal{L})$*, and in that case* $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *is called a* labelled space*.*

A labelled space $(\mathcal{E}, \mathcal{L},\mathcal{B})$ is *weakly left-resolving* if $r(A \cap B,\alpha) = r(A,\alpha) \cap r(B,\alpha)$, for all $A,B \in \mathcal{B}$ and $\alpha \in \mathcal{L}^{\geq 1}$. We say that a weakly left-resolving labelled space $(\mathcal{E}, \mathcal{L},\mathcal{B})$ is *normal*, if $\mathcal{B}$ is closed under relative complements.

For $\alpha \in \mathcal{L}^*$, define $\mathcal{B}_\alpha := \mathcal{B} \cap \mathbb{P}(r(\alpha)) = \{A \in \mathcal{B} : A \subseteq r(\alpha)\}$. In the case of a normal labelled space, $\mathcal{B}_\alpha$ is a Boolean algebra, for each $\alpha \in \mathcal{L}^*$.

By [9, Proposition 3.4], there is an inverse semigroup S, with zero element 0, associated with the normal labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$. This inverse semigroup is defined by the set

$$S := \{(\alpha, A, \beta) : \alpha, \beta \in \mathcal{L}^* \text{ and } A \in \mathcal{B}_\alpha \cap \mathcal{B}_\beta \text{ with } A \neq \emptyset\} \cup \{0\}$$

with a binary operation as follows. Let $s = (\alpha, A, \beta)$ and $t = (\gamma, B, \delta)$ be nonzero elements of S. Then,

$$s \cdot t := \begin{cases} (\alpha\gamma', r(A, \gamma') \cap B, \delta), & \text{if } \gamma = \beta\gamma' \text{ and } r(A, \gamma') \cap B \neq \emptyset, \\ (\alpha, A \cap r(B, \beta'), \delta\beta'), & \text{if } \beta = \gamma\beta' \text{ and } A \cap r(B, \beta') \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

$E(S)$ denotes the semilattice of idempotents of S. To be precise

$$E(S) = \{(\alpha, A, \alpha) \ : \ \alpha \in \mathcal{L}^* \text{ and } A \in \mathcal{B}_\alpha\} \cup \{0\}.$$

The set of all tight filters in $E(S)$ is denoted by T, which is called the *tight spectrum* (see [25, Section 12]). For each $e \in E(S)$, define $V_e := \{\xi \in \mathsf{T} : e \in \xi\}$. The subsets

$$V_{e:e_1,\dots,e_n} := V_e \cap V_{e_1}^c \cap \cdots \cap V_{e_n}^c = \{\xi \in \mathsf{T} : e \in \xi, e_1 \notin \xi, \dots, e_n \notin \xi\},$$

with $\{e_1, \dots, e_n\}$ a finite (possibly empty) subset of $E(S)$, form a basis of compact-open sets for a Hausdorff topology on T (see [43] and [15, Corollary 3.3]). If $E(S)$ is countable, then T is second-countable and therefore metrizable. See [8, Sections 2.4-2.5], for more details.

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a weakly-left resolving labelled space. Let $\alpha \in \overline{\mathcal{L}^{\leqslant\infty}}$ and let $\{\mathcal{F}_n\}_{0\leq n\leq|\alpha|}$ (where $0 \leq n \leq |\alpha|$ means that $0 \leq n < \infty$ for $\alpha \in \overline{\mathcal{L}^\infty}$) be a family such that $\mathcal{F}_n$ is a filter in $\mathcal{B}_{\alpha_{1,n}}$ for every $n > 0$, and $\mathcal{F}_0$ is either a filter in $\mathcal{B}$ or $\mathcal{F}_0 = \emptyset$. The family $\{\mathcal{F}_n\}_{0\leq n\leq|\alpha|}$ is a *complete family for* α, if $\mathcal{F}_n = \{A \in \mathcal{B}_{\alpha_{1,n}} \ : \ r(A, \alpha_{n+1}) \in \mathcal{F}_{n+1}\}$ whenever $0 \leq n < |\alpha|$.

Theorem 5.3 ([9, Theorem 4.13])**.** *Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space and S its associated inverse semigroup. There is a bijection between filters in $E(S)$ and pairs $(\alpha, \{\mathcal{F}_n\}_{0\leq n\leq|\alpha|})$, where $\alpha \in \overline{\mathcal{L}^{\leqslant\infty}}$ and $\{\mathcal{F}_n\}_{0\leq n\leq|\alpha|}$ is a complete family for α.*

Remark 5.4. *Theorem 5.3 was originally proved, but misstated, in [9, Theorem 4.13]. A comment about this later appeared in [15, Theorem 2.4].*

We say that a filter ξ in $E(S)$ is of *finite type*, if it is associated with a pair $(\alpha, \{\mathcal{F}_n\}_{0\leq n\leq|\alpha|})$ where $|\alpha| < \infty$, and of *infinite type* otherwise. The filter ξ is denoted by ξ^α and the filters in the complete family will be denoted by ξ_n^α (or just ξ_n). To be precise $\xi_n^\alpha = \{A \in \mathcal{B} \ : \ (\alpha_{1,n}, A, \alpha_{1,n}) \in \xi^\alpha\}$.

Remark 5.5 ([15, Remark 2.6])**.** *For a filter ξ^α in $E(S)$ and an element $(\beta, A, \beta) \in E(S)$, we have that $(\beta, A, \beta) \in \xi^\alpha$ if, and only if, β is a beginning of α and $A \in \xi^\alpha_{|\beta|}$.*

Let $\beta \in \overline{\mathcal{L}^{\leqslant\infty}}$. Denote by T_β the set of all tight filters in $E(S)$ associated with the word β. For labelled paths $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \overline{\mathcal{L}^{\leqslant\infty}}$, consider $\mathsf{T}_{(\alpha)\beta} := \{\xi \in \mathsf{T}_\beta : r(\alpha) \in \xi_0\}$.

Remark 5.6. *Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \overline{\mathcal{L}^{\leqslant\infty}}$. By [15, Remark 2.10], if $\xi^\beta \in T_{(\alpha)\beta}$, then $\alpha\beta \in \overline{\mathcal{L}^{\leqslant\infty}}$.*

In particular, for $\alpha \in \mathcal{L}^*$, denote the set of all filters in $E(S)$ whose associated labelled path β can be “glued” to α by the disjoint union:

$$\bigsqcup_\beta \mathsf{T}_{(\alpha)\beta} := \bigcup \{\mathsf{T}_{(\alpha)\beta} : \beta \in \overline{\mathcal{L}^{\leqslant\infty}} \text{ and } \alpha\beta \in \overline{\mathcal{L}^{\leqslant\infty}}\}.$$

Furthermore, denote the set of all filters in $E(S)$ whose associated word begins with α by the disjoint union:

$$\bigsqcup_{\beta} \mathsf{T}_{\alpha\beta} := \bigcup\{\mathsf{T}_{\alpha\beta} : \beta \in \overline{\mathcal{L}^{\leqslant\infty}} \text{ and } \alpha\beta \in \overline{\mathcal{L}^{\leqslant\infty}}\}.$$

Lemma 5.7 ([15, Lemma 3.4]). *Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a weakly-left resolving labelled space. Fix $\alpha \in \mathcal{L}^{\geq 1}$. Then, $V_{(\alpha,r(\alpha),\alpha)} = \bigsqcup_\beta T_{\alpha\beta}$, and $V_{(\omega,r(\alpha),\omega)} = \bigsqcup_\beta T_{(\alpha)\beta}$.*

5.1.2 Ultragraphs via labelled spaces

Ultragraph C^*-algebras were defined in [57]. We now recall some definitions and results about ultragraphs and labelled spaces associated with ultragraphs.

Definition 5.8 ([30, Definition 2.1]). *An* ultragraph *is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consisting of two countable sets $G^0, \mathcal{G}^1$, a map $s : \mathcal{G}^1 \to G^0$, and a map $r : \mathcal{G}^1 \to \mathbb{P}(G^0) \setminus \{\emptyset\}$, where $\mathbb{P}(G^0)$ is the power set of G^0.*

Definition 5.9 ([30, Definition 2.3],[14, Definition 2.6]). *Let $\mathcal{G}$ be an ultragraph. Define $\mathcal{G}^0$ to be the smallest subset of $\mathbb{P}(G^0)$ that contains $\{v\}$, for all $v \in G^0$, contains $r(e)$, for all $e \in \mathcal{G}^1$, contains $\emptyset$, and is closed under finite unions and finite intersections. Elements of $\mathcal{G}^0$ are called* generalized vertices.

Definition 5.10 ([14, Definition 2.6]). *The* accommodating family $\mathcal{B}$ associated with $\mathcal{G}$ *is the smallest family of subsets of G^0 that contains $\mathcal{G}^0$ and is closed under relative complements, finite unions and finite intersections.*

An element of $\mathcal{G}^0$ or a sequence of edges $e_1 e_2 \ldots e_n \in \mathcal{G}^1$ such that $s(e_{i+1}) \in r(e_i)$ for all $1 \leqslant i \leq n$ are called *finite paths* in $\mathcal{G}$. We denote by $\mathcal{G}^*$ the set of finite paths in $\mathcal{G}$. An infinite sequence of edges $e_1 e_2 \ldots$ such that $s(e_{i+1}) \in r(e_i)$ for all $i \geq 1$ is called an *infinite path* in $\mathcal{G}$.

A finite path $\alpha \in \mathcal{G}^*$ with $|\alpha| > 0$ is called a *loop*, if $s(\alpha) \in r(\alpha)$. A loop α is said to be *based at $A \in \mathcal{G}^0$*, if $s(\alpha) \in A$. A loop $\alpha = \alpha_1 \ldots \alpha_n$ is a *simple loop*, if $s(\alpha_i) \neq s(\alpha_1)$ for $i \neq 1$.

Next, we recall how ultragraphs are realized as labelled graphs.

Definition 5.11 ([14, Definition 2.7]). *Fix an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$. Let $\mathcal{E}_{\mathcal{G}} = (\mathcal{E}^0_{\mathcal{G}}, \mathcal{E}^1_{\mathcal{G}}, r', s')$, where $\mathcal{E}^0_{\mathcal{G}} = G^0$, $\mathcal{E}^1_{\mathcal{G}} = \{(e, \upsilon) : e \in \mathcal{G}^1, \upsilon \in r(e)\}$ and define $r'(e,\upsilon) := \upsilon$ and $s'(e,\upsilon) := s(e)$. Set $\mathcal{A} = \mathcal{E}^1_{\mathcal{G}}$, let $\mathcal{B}$ be the accommodating family of $\mathcal{G}$, and define $\mathcal{L}_{\mathcal{G}} : \mathcal{E}^1_{\mathcal{G}} \to \mathcal{A}$ by $\mathcal{L}_{\mathcal{G}}(e,\upsilon) := e$. Then, $(\mathcal{E}_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ is the normal labelled space associated with $\mathcal{G}$.*

Notice that labelled paths correspond to paths on the ultragraph and that $\mathcal{L}^{\infty}_{\mathcal{G}} = \overline{\mathcal{L}^{\infty}_{\mathcal{G}}}$.

Remark 5.12 ([14, Remark 3.1]). *Notice that, for all $A \subseteq G^0$ and $e \in \mathcal{G}^1$, we have that $r(A,e) = r(e)$, if $s(e) \in A$, and $r(A,e) = \emptyset$ otherwise. In particular, $r(\{s(e)\},e) = r(e)$.*

Lemma 5.13 ([14, Lemma 3.2]). *Let α be a path in $\mathcal{G}$ such that $|\alpha| \geq 1$, and let $\{\mathcal{F}_n\}_{n=0}^{|\alpha|}$ be a complete family of filters for α. If $0 \leq n < |\alpha|$, then $\mathcal{F}_n = \uparrow_{\mathcal{B}_{\alpha_n}} \{s(\alpha_{n+1})\}$.*

Lemma 5.14 ([14, Lemma 3.4]). *Let α be an infinite path in $\mathcal{G}$. Then $\{\mathcal{F}_n\}_{n=0}^{\infty} := \{\uparrow_{\mathcal{B}_{\alpha_n}} \{s(\alpha_{n+1})\}\}_{n=0}^{\infty}$ is the only complete family of filters (in particular, ultrafilters) for α.*

Remark 5.15. *By [14, Proposition 3.5, Proposition 3.6], for each infinite path α in $\mathcal{G}$, there is a unique element $\xi^{\alpha} \in \mathsf{T}$ whose associated word is α. And, a filter of infinite type is completely described by the infinite path associated to it.*

5.1.3 The topological partial action on T

A partial action of the free group generated by the alphabet $\mathcal{A}$ on the tight spectrum T of a labelled space was defined in [15, Section 3]. A description of that partial action, when the labelled space is associated with an ultragraph, was given in [14, Section 4.1]. For the convenience of the reader, we recall the characterization from [14, Section 4.1] in the following two paragraphs.

Fix a weakly-left resolving labelled space $(\mathcal{E},\mathcal{L},\mathcal{B})$ and let $\mathbb{F}$ be the free group generated by $\mathcal{A}$ (identifying the identity of $\mathbb{F}$ with ω). Then, by [15, Proposition 3.12], for every $t \in \mathbb{F}$ there is a compact open set $V_t \subseteq \mathsf{T}$ and a homeomorphism $\varphi_t : V_{t^{-1}} \to V_t$ such that

$$\varphi = (\{V_t\}_{t\in\mathbb{F}},\{\varphi_t\}_{t\in\mathbb{F}}) \tag{5.1}$$

is a topological partial action of $\mathbb{F}$ on T. In particular, $V_{\omega} := \mathsf{T}$ and if $\alpha,\beta \in \mathcal{L}^*$, then $V_{\alpha} := V_{(\alpha,r(\alpha),\alpha)}$, $V_{\alpha^{-1}} := V_{(\omega,r(\alpha),\omega)}$, and $V_{(\alpha\beta^{-1})^{-1}} = \varphi_{\beta^{-1}}^{-1}(V_{\alpha^{-1}})$, with $V_{(\alpha\beta^{-1})^{-1}} \neq \emptyset$ if, and only if, $r(\alpha) \cap r(\beta) \neq \emptyset$ [15, Lemma 3.10].

For the labelled space associated with an ultragraph, we can intuitively describe the map $\varphi_{\alpha\beta^{-1}}$, for $\alpha,\beta \in \mathcal{L}^*$, as cutting β from the beginning of an element $\xi \in V_{\alpha^{-1}\beta}$ and then gluing α in front of it. For filters of finite type we just have to take into account the last filter of the corresponding complete family. In most cases, the last filter is kept fixed, unless the empty word is involved. If β is the labelled path associated with an element $\xi \in V_{\alpha^{-1}\beta}$ then, by cutting β, we get the filter associated with the pair $(\omega,\uparrow_{\mathcal{B}}\xi_{|\beta|})$ and, by gluing α afterwards, we get the filter associated with the pair $(\alpha,\mathcal{F})$, where $\mathcal{F} = \{A \cap r(\alpha) : A \in \uparrow_{\mathcal{B}}\xi_{|\beta|}\}$.

Let $\mathbb{F} \rtimes_{\varphi} \mathsf{T} = \{(t,\xi) \in \mathbb{F}\times\mathsf{T} : \xi \in V_t\}$ denote the transformation groupoid associated with the partial action φ, as defined in [1].

The following results were proven in [15] and [8] for the partial action of the free group generated by the alphabet $\mathcal{A}$ on the tight spectrum T of a normal labelled space (not necessarily associated with an ultragraph).

Remark 5.16. *Let $t \in \mathbb{F}$ be in reduced form. By [15, Lemma 3.11], if $t \notin \{\omega\} \cup \{\alpha : \alpha \in \mathcal{L}^{\geq 1}\} \cup \{\alpha^{-1} : \alpha \in \mathcal{L}^{\geq 1}\} \cup \{\alpha\beta^{-1} : \alpha, \beta \in \mathcal{L}^{\geq 1}, r(\alpha) \cap r(\beta) \neq \emptyset\}$, then $V_t = V_{t^{-1}} = \emptyset$.*

Remark 5.17. *By [15, Remark 6.11], if $\xi^{\alpha} \in \mathrm{Fix}(t)$ for some $t \in \mathbb{F} \setminus \{\omega\}$, then there must exist $\beta \in \mathcal{L}^*$ and $\gamma \in \mathcal{L}^{\geq 1}$ such that the associated labelled path of ξ^{α} is $\alpha = \beta\gamma^{\infty}$ and t is either $\beta\gamma\beta^{-1}$ or $\beta\gamma^{-1}\beta^{-1}$.*

Lemma 5.18 ([8, Lemma 8.9])**.** *Let $(t,\xi) \in \mathrm{Iso}(\mathbb{F} \rtimes_{\varphi} T)$ with $t = \beta\gamma\beta^{-1}$ or $t = \beta\gamma^{-1}\beta^{-1}$, where $\beta\gamma^{\infty}$ is the labelled path associated with ξ. Then, for every neighborhood $U \subseteq T$ of ξ, there exists $k > 1$ and $B \in \xi_{|\beta\gamma^k|}$ such that $\xi \in V_{(\beta\gamma^k, B, \beta\gamma^k)} \subseteq U$.*

5.2 APPLICATIONS TO ULTRAGRAPHS, THEIR ASSOCIATED PARTIAL ACTIONS AND PARTIAL SKEW GROUPOID RINGS

Let $\mathcal{G}$ be an ultragraph, let $(\mathcal{E}_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ be the normal labelled space associated with $\mathcal{G}$, and let φ be the topological partial action of the free group $\mathbb{F}$ generated by the labels of the ultragraph on the tight spectrum T as defined above. We aim to prove that φ is topologically free on every closed $\mathbb{F}$-invariant subset of T if, and only if, $\mathcal{G}$ satisfies Condition (K).

In the context of ultragraphs, Condition (K) was first defined in [38, Definition 2.4]. We present an equivalent definition below.

Definition 5.19 ([13, Section 4])**.** *An ultragraph $\mathcal{G}$ satisfies* Condition (K)*, if for every $v \in G^0$, there is either no simple loop based at v or at least two simple loops based at v.*

The following definition is based on [26, Definition 37.20].

Definition 5.20. *Let γ be a loop in $\mathcal{G}$.*

(a) γ is said to be recurrent*, if there exists another loop ρ in $\mathcal{G}$ such that $s(\rho) = s(\gamma)$ and $\gamma\rho\gamma^{\infty} \neq \gamma^{\infty}$.*

(b) γ is said to be transitory*, if it is not recurrent.*

Proposition 5.21. *Every loop in $\mathcal{G}$ is recurrent if, and only if, $\mathcal{G}$ satisfies Condition (K).*

Proof. We first show the "only if" statement. Suppose that every loop in $\mathcal{G}$ is recurrent. Let $v \in G^0$ be a vertex and let α be a simple loop based at v. In particular, $s(\alpha) = v$. By assumption, there is a loop β such that $\alpha^{\infty} \neq \alpha\beta\alpha^{\infty}$. Notice that $\beta = \gamma^{(1)} \ldots \gamma^{(n)}$ is such that $\gamma^{(j)}$ is a simple loop based at $s(\alpha)$, for every $j \in \{1, \ldots, n\}$. Since $\alpha^{\infty} \neq \alpha\beta\alpha^{\infty}$, there exists $k \in \{1, \ldots, n\}$ such that $\gamma^{(k)}$ is distinct from α. Thus, $\mathcal{G}$ satisfies Condition (K).

Now we show the "if" statement. Suppose that $\mathcal{G}$ satisfies Condition (K). Let α be a loop in $\mathcal{G}$ and notice that $\alpha = \gamma^{(1)} \ldots \gamma^{(n)}$ is such that $\gamma^{(j)}$ is a simple loop based at $s(\alpha)$, for every $j \in \{1, \ldots, n\}$. If there is $k \in \{1, \ldots, n\}$ such that $\gamma^{(k)} \neq \gamma^{(1)}$, then let

$\beta := \gamma^{(k)}$. Note that, $\alpha\beta\alpha^\infty \neq \alpha^\infty$. If $\alpha = \gamma^{(1)}\gamma^{(1)}\ldots\gamma^{(1)}$, then by Condition (K), there exists a simple loop δ based at $s(\alpha)$ and distinct from $\gamma^{(1)}$. Therefore, $\alpha\delta\alpha^\infty \neq \alpha^\infty$, and hence α is recurrent. □

Lemma 5.22. *Let $\alpha \in \mathcal{L}_{\mathcal{G}}^{\geq 1}$ and $\beta \in \mathcal{L}_{\mathcal{G}}^{\infty}$ be such that $\alpha\beta \in \mathcal{L}_{\mathcal{G}}^{\infty}$ and $\xi^\beta \in T_\beta$. Then $\xi^\beta \in T_{(\alpha)\beta}$.*

Proof. By the definition of $T_{(\alpha)\beta}$, we only need to prove that $r(\alpha) \in \xi_0^\beta$. Since $\beta \in \mathcal{L}_{\mathcal{G}}^{\infty}$, Lemma 5.14 implies that $\xi_0^\beta = \uparrow_{\mathcal{B}_\omega} \{s(\beta_1)\} = \uparrow_{\mathcal{B}} \{s(\beta_1)\}$. By assumption, $\alpha\beta \in \mathcal{L}_{\mathcal{G}}^{\infty}$. Therefore, $s(\beta_1) \in r(\alpha)$ and $r(\alpha) \in \uparrow_{\mathcal{B}} \{s(\beta_1)\}$. □

Proposition 5.23 and Proposition 5.25 below are inspired by techniques from [26, Proposition 37.21] and [24, Chapter 12], and will be used to prove the main result of this section.

Proposition 5.23. *Let $t \in \mathbb{F} \setminus \{\omega\}$ and $\xi^\alpha \in \mathrm{Fix}(t)$ be such that $\alpha = \beta\gamma^\infty$ and that t is either $\beta\gamma\beta^{-1}$ or $\beta\gamma^{-1}\beta^{-1}$ (see Remark 5.17). Then, ξ^α is an isolated point in $\mathrm{Orb}(\xi^\alpha)$ if, and only if, γ is a transitory loop.*

Proof. We first show the "if" statement. Suppose that γ is a transitory loop and let U be an open set such that $\xi^\alpha \in U$. Since $\xi^\alpha \in \mathrm{Fix}(t)$, we have that $(t, \xi^\alpha) \in \mathrm{Iso}(\mathbb{F} \rtimes_\varphi \mathrm{T})$. By Lemma 5.18, there exists $k > 1$ and $B \in \xi^\alpha_{|\beta\gamma^k|}$ such that $\xi^\alpha \in V_{(\beta\gamma^k, B, \beta\gamma^k)} \subseteq U$.

We claim that $V_{(\beta\gamma^k, B, \beta\gamma^k)} \cap \mathrm{Orb}(\xi^\alpha) = \{\xi^\alpha\}$. Let $\eta^\delta \in V_{(\beta\gamma^k, B, \beta\gamma^k)} \cap \mathrm{Orb}(\xi^\alpha)$. Using that $\eta^\delta \in \mathrm{Orb}(\xi^\alpha)$, there exists $s \in \mathbb{F}$ such that $\xi^\alpha \in V_{s^{-1}}$ and $\eta^\delta = \varphi_s(\xi^\alpha)$. Since $V_{s^{-1}} \neq \emptyset$, Remark 5.16 implies that $s \in \{\omega\} \cup \{\alpha' : \alpha' \in \mathcal{L}_{\mathcal{G}}^{\geq 1}\} \cup \{\alpha'^{-1} : \alpha' \in \mathcal{L}_{\mathcal{G}}^{\geq 1}\} \cup \{\alpha'\beta'^{-1} : \alpha', \beta' \in \mathcal{L}_{\mathcal{G}}^{\geq 1}, r(\alpha') \cap r(\beta') \neq \emptyset\}$. Since $\alpha = \beta\gamma^\infty$ and by the definition of the partial action φ, the labelled path δ associated to η^δ is eventually periodic with period component γ.

By assumption $\eta^\delta \in V_{(\beta\gamma^k, B, \beta\gamma^k)}$, and we have that $(\beta\gamma^k, B, \beta\gamma^k) \in \eta^\delta$. By Remark 5.5, $\beta\gamma^k$ is a beginning of δ. Using that δ eventually returns to γ, and that γ is not recurrent, we have that $\delta = \beta\gamma^\infty = \alpha$. Therefore, η is associated with the word α, and hence, by Remark 5.15, $\eta^\alpha = \xi^\alpha$.

Now we show the "only if" statement. Suppose that ξ^α is an isolated point in $\mathrm{Orb}(\xi^\alpha)$. There exists an open set $U \subseteq \mathrm{T}$ such that $U \cap \mathrm{Orb}(\xi^\alpha) = \{\xi^\alpha\}$. By Lemma 5.18, there exist $k > 1$ and $B \in \xi^\alpha_{|\beta\gamma^k|}$ such that $\xi^\alpha \in V_{(\beta\gamma^k, B, \beta\gamma^k)} \subseteq U$. Hence, $V_{(\beta\gamma^k, B, \beta\gamma^k)} \cap \mathrm{Orb}(\xi^\alpha) = \{\xi^\alpha\}$.

Seeking a contradiction, suppose that γ is a recurrent loop. Then, there exists a loop ρ such that $s(\rho) = s(\gamma)$ and $\gamma\rho\gamma^\infty \neq \gamma^\infty$. Define $\delta := \beta\gamma^k\rho\gamma^\infty$. By Remark 5.15, there is a unique element η^δ associated with δ. Notice that, since $\beta\gamma^k$ is a beginning of δ and $s(\rho) = s(\gamma)$, by Lemma 5.14, the elements of the complete family of ultrafilters associated with α and δ satisfy $\xi_n^\alpha = \eta_n^\delta$, for all $n \in \{1, \ldots, |\beta\gamma^k|\}$. In particular, since $B \in \xi^\alpha_{|\beta\gamma^k|} = \eta^\delta_{|\beta\gamma^k|}$, by Remark 5.5, $(\beta\gamma^k, B, \beta\gamma^k) \in \eta^\delta$. Hence, $\eta^\delta \in V_{(\beta\gamma^k, B, \beta\gamma^k)}$.

We claim that $\eta^\delta \in \mathrm{Orb}(\xi^\alpha)$. Let $s = \beta\gamma^k\rho\beta^{-1}$. Observe that $\xi^\alpha \in V_{s^{-1}}$, because $V_{s^{-1}} = \varphi_{\beta^{-1}}^{-1}(V_{(\beta\gamma^k\rho)^{-1}})$ and $\varphi_{\beta^{-1}}(\xi^\alpha) = \mu^{\gamma^\infty} \in \mathrm{T}_{\gamma^\infty}$. Since $s(\gamma) = s(\rho) \in r(\rho)$, then $\beta\gamma^k\rho\gamma^\infty \in \mathcal{L}_{\mathcal{G}}^\infty$ and Lemma 5.22 implies that $\mu^{\gamma^\infty} \in \mathrm{T}_{(\beta\gamma^k\rho)\gamma^\infty}$. Thus, by Lemma 5.22, $\mu^{\gamma^\infty} \in \mathrm{T}_{(\beta\gamma^k\rho)\gamma^\infty} \subseteq V_{(\beta\gamma^k\rho)^{-1}}$ and $\xi^\alpha = \varphi_{\beta^{-1}}^{-1}(\mu^{\gamma^\infty}) \in V_{s^{-1}}$. By Remark 5.15, there is a unique element associated with the word δ. Hence, $\varphi_s(\xi^\alpha) = \eta^\delta$ and we conclude that $\eta^\delta \in V_{(\beta\gamma^k,B,\beta\gamma^k)} \cap \mathrm{Orb}(\xi^\alpha)$. This is a contradiction. □

Corollary 5.24. *Let $t \in \mathbb{F}\setminus\{\omega\}$ and $\xi^\alpha \in \mathrm{Fix}(t)$ be such that $\alpha = \beta\gamma^\infty$ and that t is either $\beta\gamma\beta^{-1}$ or $\beta\gamma^{-1}\beta^{-1}$ (see Remark 5.17). Then, ξ^α is not an isolated point in $\mathrm{Orb}(\xi^\alpha)$ if, and only if, γ is a recurrent loop.*

Proposition 5.25. *The partial action φ of $\mathbb{F}$ on T is topologically free on every closed $\mathbb{F}$-invariant subset of T if, and only if, for every $t \in \mathbb{F}\setminus\{\omega\}$ and $\xi \in \mathrm{Fix}(t)$, ξ is not an isolated point in $\mathrm{Orb}(\xi)$.*

Proof. We first show the "if" statement by contrapositivity. Suppose that φ is not topologically free on every closed $\mathbb{F}$-invariant subset of T. Then, there exist a closed $\mathbb{F}$-invariant subset $C \subseteq \mathrm{T}$ and an element $t \in \mathbb{F}\setminus\{\omega\}$ such that $\mathrm{Int}_C(\mathrm{Fix}(t) \cap C) \neq \emptyset$.

Let $\xi^\alpha \in \mathrm{Int}_C(\mathrm{Fix}(t) \cap C)$. Since $\xi^\alpha \in \mathrm{Fix}(t)$, by Remark 5.17, there exist $\beta \in \mathcal{L}^*$ and $\gamma \in \mathcal{L}^{\geq 1}$, such that the associated labelled path of ξ^α is $\alpha = \beta\gamma^\infty$, and t is either $\beta\gamma\beta^{-1}$ or $\beta\gamma^{-1}\beta^{-1}$. By Remark 5.15, since α is an infinite path in $\mathcal{G}$, ξ^α is the unique element in T whose associated word is α. Therefore, $\mathrm{Int}_C(\mathrm{Fix}(t) \cap C) = \{\xi^\alpha\}$. Notice that

$$\xi^\alpha \in \mathrm{Orb}(\xi^\alpha) = \bigcup_{t\in\mathbb{F}} \varphi_t(\{\xi^\alpha\} \cap V_{t^{-1}}) \subseteq \bigcup_{t\in\mathbb{F}} \varphi_t(C \cap V_{t^{-1}}) \subseteq C.$$

Thus, ξ^α is an isolated point in $\mathrm{Orb}(\xi^\alpha)$.

Now we show the "only if" statement by contrapositivity. Suppose that there exist $t \in \mathbb{F}\setminus\{\omega\}$ and $\xi \in \mathrm{Fix}(t)$, such that ξ is an isolated point in $\mathrm{Orb}(\xi)$. Then, there exist an open subset V of T such that $\mathrm{Orb}(\xi) \cap V = \{\xi\}$. We claim that $\overline{\mathrm{Orb}(\xi)} \cap V = \{\xi\}$. Seeking a contradiction, suppose that there is $\eta \in \overline{\mathrm{Orb}(\xi)} \cap V$ such that $\eta \neq \xi$. Then $\eta \in V\setminus\{\xi\}$, the latter being an open set. But, $V\setminus\{\xi\}\cap\mathrm{Orb}(\xi) = \emptyset$ and this is a contradiction, since $\eta \in \overline{\mathrm{Orb}(\xi)}$. Therefore, $C = \overline{\mathrm{Orb}(\xi)}$ is a closed $\mathbb{F}$-invariant subset of T such that $\xi \in \mathrm{Int}_C(\mathrm{Fix}(t) \cap C)$. Hence, φ is not topologically free on C. □

Proposition 5.26. *The partial action φ of $\mathbb{F}$ on T is topologically free on every closed $\mathbb{F}$-invariant subset of T if, and only if, every loop in $\mathcal{G}$ is recurrent.*

Proof. We first show the "if" statement by contrapositivity. Suppose that there is a closed $\mathbb{F}$-invariant subset on which φ is not topologically free. By Proposition 5.25, there exists $t \in \mathbb{F}\setminus\{\omega\}$ such that $\xi^\alpha \in \mathrm{Fix}(t)$ is a isolated point in $\mathrm{Orb}(\xi^\alpha)$. By Remark 5.17, there exist $\beta \in \mathcal{L}^*$, $\gamma \in \mathcal{L}^{\geq 1}$ such that the associated labelled path of ξ^α is $\alpha = \beta\gamma^\infty$ and t is either $\beta\gamma\beta^{-1}$ or $\beta\gamma^{-1}\beta^{-1}$. By Proposition 5.23, γ is a transitory loop.

Now we show the "only if" statement by contrapositivity. Suppose that $\mathcal{G}$ has a transitory loop γ. Set $\alpha := \gamma^\infty$ and notice that $\xi^\alpha \in \mathrm{Fix}(\gamma)$. By Proposition 5.23, ξ^α is an isolated point in $\mathrm{Orb}(\xi^\alpha)$, and, by Proposition 5.25, φ is not topologically free on every closed $\mathbb{F}$-invariant subset of T. □

Let $\mathbb{K}$ be a field and let α be the partial action of $\mathbb{F}$ on $\mathcal{L}_c(\mathrm{T},\mathbb{K})$ induced by the topological partial action φ. The natural $\mathbb{F}$-grading on the partial skew group ring $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is given by $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F} = \bigoplus_{t\in\mathbb{F}} \mathcal{L}_c(V_t,\mathbb{K})\delta_t$. Using the map $\alpha\beta^{-1} \mapsto |\alpha| - |\beta|$, for $\alpha\beta^{-1} \in \mathbb{F}$ in reduced from, and the fact that the partial action φ is semi-saturated [15, Proposition 3.12], it follows that $\mathcal{L}_c(\mathrm{T},R) \rtimes_\alpha \mathbb{F}$ has a $\mathbb{Z}$-grading with homogeneous component of degree $n \in \mathbb{Z}$ given by $D_n := \mathrm{span}_{\mathbb{K}}\{f_{\alpha\beta^{-1}}\delta_{\alpha\beta^{-1}} : \alpha,\beta \in \mathcal{L}^* \text{ and } |\alpha|-|\beta| = n\}$. See [8, Section 4], for more details.

Theorem 5.27. *Let $\mathbb{K}$ be a field and let $\mathcal{G}$ be an ultragraph. Then, every ideal of $\mathcal{L}_c(T,\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{F}$-graded if, and only if, every ideal of $\mathcal{L}_c(T,\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{Z}$-graded.*

Proof. By [8, Theorem 5.6], there is a $\mathbb{Z}$-graded isomorphism between $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F}$ and the Leavitt path algebra, $L_{\mathbb{K}}(\mathcal{E}_{\mathcal{G}},\mathcal{L}_{\mathcal{G}},\mathcal{B})$, of the labelled space $(\mathcal{E}_{\mathcal{G}},\mathcal{L}_{\mathcal{G}},\mathcal{B})$. Moreover, $L_{\mathbb{K}}(\mathcal{E}_{\mathcal{G}},\mathcal{L}_{\mathcal{G}},\mathcal{B})$ is isomorphic to the ultragraph Leavitt path algebra $L_{\mathbb{K}}(\mathcal{G})$, see [8, Example 7.2]. Finally, by [37, Theorem 4.3], since $\mathbb{K}$ is a field, every ideal of $L_{\mathbb{K}}(\mathcal{G})$ is $\mathbb{Z}$-graded if, and only if, the ultragraph $\mathcal{G}$ satisfies Condition(K). Thus, every ideal of $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{Z}$-graded if, and only if, $\mathcal{G}$ satisfies Condition (K). Proposition 5.21 and Proposition 5.26 imply that $\mathcal{G}$ satisfies Condition (K) if, and only if, φ is topologically free on every closed $\mathbb{F}$-invariant subset of T. By Theorem 4.16, this is equivalent to every ideal of $\mathcal{L}_c(\mathrm{T},\mathbb{K}) \rtimes_\alpha \mathbb{F}$ being $\mathbb{F}$-graded. □

Now we state the main result of this section, partially generalizing [8, Proposition 9.1].

Theorem 5.28. *Let $\mathbb{K}$ be a field, let $\mathcal{G}$ be an ultragraph, let $(\mathcal{E}_{\mathcal{G}},\mathcal{L}_{\mathcal{G}},\mathcal{B})$ be the normal labelled space associated with $\mathcal{G}$, let φ be the topological partial action of the free group $\mathbb{F}$ generated by the labels of the ultragraph on the tight spectrum T, and let α be the partial action of $\mathbb{F}$ on $\mathcal{L}_c(T,\mathbb{K})$ induced by the topological partial action φ. The following statements are equivalent:*

(i) $\overline{D}_0\delta_0$ is a maximal commutative subring of $\frac{\mathcal{L}_c(T,\mathbb{K})}{\mathcal{J}(F)} \rtimes_{\overline{\alpha}} \mathbb{F}$, for every closed $\mathbb{F}$-invariant subset F of T;

(ii) α has the residual intersection property;

(iii) Every ideal of $\mathcal{L}_c(T,\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{F}$-graded;

(iv) Every ideal of $\mathcal{L}_c(T,\mathbb{K}) \rtimes_\alpha \mathbb{F}$ is $\mathbb{Z}$-graded;

(v) φ is topologically free on every closed $\mathbb{F}$-invariant subset of T;

(vi) φ is topologically free on every open $\mathbb{F}$-invariant subset of T;

(vii) *$\mathbb{F} \rtimes_\varphi T$ is strongly effective;*

(viii) *Every loop in $\mathcal{G}$ is recurrent;*

(ix) *$\mathcal{G}$ satisfies Condition (K).*

Proof. The equivalences between (i), (ii), (iii), (v), (vi), and (vii) follow from Theorem 4.16. The other equivalences follow from Proposition 5.21, Proposition 5.26 and Theorem 5.27. $\square$

Remark 5.29. *We point out that the equivalences between (iv), (vii), and (viii) in the above theorem are already known in the context of Leavitt path algebras of ultragraphs (see [37, Theorem 4.3]), but the other equivalences are new even in the context of graphs.*

PART 2

6 PRIMENESS OF GROUPOID GRADED RINGS

Throughout this chapter, unless stated otherwise, G denotes an arbitrary groupoid.

6.1 NEARLY EPSILON-STRONGLY GROUPOID GRADED RINGS

In this section, we will introduce the notion of a nearly epsilon-strongly graded ring and show some of its basic properties.

Definition 6.1 ([41, Definition 4.6])**.** *Let S be a G-graded ring. We say that S is* nearly epsilon-strongly G-graded*, if for each $g \in G$, $S_g S_{g^{-1}}$ is an s-unital ring and $S_g S_{g^{-1}} S_g = S_g$.*

Remark 6.2. *The above definition simultaneously generalize [47, Definition 3.3] and [49, Definition 34].*

The following characterization is inspired by [47, Proposition 11].

Proposition 6.3. *Let S be a G-graded ring. The following statements are equivalent:*

(i) S is nearly epsilon-strongly graded;

(ii) For each $g \in G$ and each $d \in S_g$, there exist $\varepsilon_g(d) \in S_g S_{g^{-1}}$ and $\varepsilon'_g(d) \in S_{g^{-1}} S_g$ such that $\varepsilon_g(d) d = d \varepsilon'_g(d) = d$.

Proof. Suppose that (i) holds. Let $g \in G$ and $d \in S_g$. We may write $d = \sum_{i=1}^{n} a_i b_i c_i$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n, c_1, \ldots, c_n \in S_g$ and $b_1, \ldots, b_n \in S_{g^{-1}}$. Notice that $a_i b_i \in S_g S_{g^{-1}}$ and $b_i c_i \in S_{g^{-1}} S_g$ for every $i \in \{1, \ldots, n\}$. By assumption, $S_g S_{g^{-1}}$ and $S_{g^{-1}} S_g$ are s-unital and hence, by Proposition 1.5, there exist $\varepsilon_g(d) \in S_g S_{g^{-1}}$ and $\varepsilon'_g(d) \in S_{g^{-1}} S_g$ such that $\varepsilon_g(d) a_i b_i = a_i b_i$ and $b_i c_i \varepsilon'_g(d) = b_i c_i$ for every $i \in \{1, \ldots, n\}$. Thus, $\varepsilon_g(d) d = d \varepsilon'_g(d) = d$. This shows that (ii) holds.

Conversely, suppose that (ii) holds. Let $g \in G$. Note that, by assumption, S_g is s-unital as a left $S_g S_{g^{-1}}$-module and $S_{g^{-1}}$ is s-unital as a right $S_g S_{g^{-1}}$-module. Let $m \in S_g S_{g^{-1}}$. We may write $m = \sum_{i=1}^{n} a_i b_i$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in S_g$ and $b_1, \ldots, b_n \in S_{g^{-1}}$. By s-unitality of the left $S_g S_{g^{-1}}$-module S_g, and Proposition 1.5, there is some $u \in S_g S_{g^{-1}}$ such that $u a_i = a_i$ for every $i \in \{1, \ldots, n\}$. Similarly, there is some $u' \in S_g S_{g^{-1}}$ such that $b_i u' = b_i$ for every $i \in \{1, \ldots, n\}$. Hence, $um = m$ and $mu' = m$. This shows that $S_g S_{g^{-1}}$ is s-unital. Note that $S_g S_{g^{-1}} S_g \subseteq S_{g g^{-1} g} = S_g$. Using that S_g is s-unital as a left $S_g S_{g^{-1}}$-module we get that $S_g \subseteq (S_g S_{g^{-1}}) S_g$. Thus, S is symmetrically G-graded. This shows that (i) holds. □

The following result generalizes [40, Proposition 2.13].

Proposition 6.4. *Let S be a nearly epsilon-strongly G-graded ring. The following assertions hold:*

(i) S_e *is an s-unital ring, for every* $e \in G_0$.

(ii) $d \in d(\oplus_{e\in G_0} S_e) \cap (\oplus_{e\in G_0} S_e)d$*, for every* $d \in S$.

(iii) S *is s-unital and* $\oplus_{e\in G_0} S_e$ *is an s-unital subring of* S.

(iv) Suppose that H *is a subgroupoid of* G*. Then* $\oplus_{h\in H} S_h$ *is a nearly epsilon-strongly* H*-graded ring.*

(v) $\oplus_{g\in G_e^e} S_g$ *is an s-unital ring, for every* $e \in G_0$.

(vi) The set

$$G' := \{g \in G : S_{s(g)} \neq \{0\} \text{ and } S_{r(g)} \neq \{0\}\} \tag{6.1}$$

is a subgroupoid of G.

(vii) $S = \oplus_{g\in G'} S_g$.

Proof. (i) Take $e \in G_0$. By assumption, $S_e = S_e S_e S_e$ and $S_e S_e$ is an s-unital ring. Hence, $S_e = (S_e S_e)S_e \subseteq S_e S_e \subseteq S_{ee} = S_e$. Thus, $S_e = S_e S_e$. We conclude that S_e is an s-unital ring.

(ii) Let $d = \sum_{t\in G} d_t \in S$, with $d_t \in S_t$. Take $g \in \mathrm{Supp}(d)$. By Proposition 6.3, there exist $u_g \in S_g S_{g^{-1}} \subseteq S_{r(g)}$ and $u'_{g^{-1}} \in S_{g^{-1}} S_g \subseteq S_{s(g)}$ such that $u_g d_g = d_g u'_{g^{-1}} = d_g$. The set $B := \{r(t) : t \in \mathrm{Supp}(d)\}$ is finite, because $\mathrm{Supp}(d)$ is finite. For every $f \in B$, by (i), S_f is s-unital, and we let $v_f \in S_f$ be an s-unit for the finite set $\{u_t : t \in \mathrm{Supp}(d) \text{ and } r(t) = f\} \subseteq S_f$. Define $a := \sum_{f\in B} v_f \in \oplus_{e\in G_0} S_e$. We get that

$$ad = \sum_{f\in B} v_f \sum_{t\in G} d_t = \sum_{t\in G} v_{r(t)} d_t = \sum_{t\in G} v_{r(t)}(u_t d_t) = \sum_{t\in G}(v_{r(t)} u_t) d_t = \sum_{t\in G} u_t d_t = \sum_{t\in G} d_t = d.$$

Similarly, we define the finite set $D := \{s(t) : t \in \mathrm{Supp}(d)\}$. For every $f \in D$, we let $v'_f \in S_f$ be an s-unit for the finite set $\{u'_{t^{-1}} : t \in \mathrm{Supp}(d) \text{ and } s(t) = f\} \subseteq S_f$. Define $a' := \sum_{f\in D} v'_f \in \oplus_{e\in G_0} S_e$. We get that

$$\begin{aligned} da' &= \sum_{t\in G} d_t \sum_{f\in D} v'_f = \sum_{t\in G} d_t v'_{s(t)} = \sum_{t\in G} (d_t u'_{t^{-1}}) v'_{s(t)} \\ &= \sum_{t\in G} d_t (u'_{t^{-1}} v'_{s(t)}) = \sum_{t\in G} d_t u'_{t^{-1}} = \sum_{t\in G} d_t = d. \end{aligned}$$

(iii) It follows immediately from (ii).

(iv) It follows immediately from Remark 1.3 and the fact that $H \subseteq G$.

(v) Take $e \in G_0$. Clearly, the isotropy group G_e^e is a subgroupoid of G. By (iv) and (iii) we get that $\oplus_{g\in G_e^e} S_g$ is s-unital.

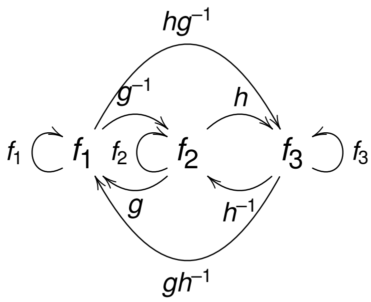
(vi) Clearly, $g^{-1} \in G'$ whenever $g \in G'$. Suppose that $g,h \in G'$ and $(g,h) \in G^2$. Then $S_{s(gh)} = S_{s(h)} \neq \{0\}$ and $S_{r(gh)} = S_{r(g)} \neq \{0\}$. Therefore, $gh \in G'$. This shows that G' is a subgroupoid of G.

(vii) Take $g \in G$ such that $S_g \neq \{0\}$. We claim that $g \in G'$. If we assume that the claim holds, then clearly $S = \oplus_{g\in G'} S_g$. Now we show the claim. Let $d \in S_g$ be nonzero.

By Proposition 6.3, there are $\varepsilon_g(d) \in S_g S_{g^{-1}} \subseteq S_{r(g)}$ and $\varepsilon'_g(d) \in S_{g^{-1}} S_g \subseteq S_{s(g)}$ such that $\varepsilon_g(d)d = d\varepsilon'_g(d) = d$. In particular, $S_{r(g)} \neq \{0\}$ and $S_{s(g)} \neq \{0\}$. $\square$

Remark 6.5. *Suppose that S is a nearly epsilon-strongly G-graded ring. By Proposition 6.4 (vii), $g \in G'$ whenever $S_g \neq \{0\}$. The converse, however, need not hold, see Example 7.22.*

Example 6.6. *Let $G := \{f_1, f_2, f_3, g, h, g^{-1}, h^{-1}, hg^{-1}, gh^{-1}\}$ be a groupoid with $G_0 = \{f_1, f_2, f_3\}$ and depicted as follows:*



Let $S := \mathcal{M}_3(\mathbb{Z})$, be the ring of the 3×3 matrices over $\mathbb{Z}$, and let $\{e_{ij}\}_{i,j}$, denote the standard matrix units. We define:

$$\begin{aligned}
S_g &:= \{\lambda e_{12} : \lambda \in \mathbb{Z}\}, & S_{g^{-1}} &:= \{\lambda e_{21} : \lambda \in \mathbb{Z}\}, & S_h &:= \{\lambda e_{32} : \lambda \in \mathbb{Z}\},\\
S_{h^{-1}} &:= \{\lambda e_{23} : \lambda \in \mathbb{Z}\}, & S_{gh^{-1}} &:= \{\lambda e_{13} : \lambda \in \mathbb{Z}\}, & S_{hg^{-1}} &:= \{\lambda e_{31} : \lambda \in \mathbb{Z}\},\\
S_{f_1} &:= \{\lambda e_{11} : \lambda \in \mathbb{Z}\}, & S_{f_2} &:= \{\lambda e_{22} : \lambda \in \mathbb{Z}\}, & S_{f_3} &:= \{\lambda e_{33} : \lambda \in \mathbb{Z}\}.
\end{aligned}$$

Notice that $S = \oplus_{l \in G} S_l$. Therefore, the ring $\mathcal{M}_3(\mathbb{Z})$ is graded by the groupoid G.

6.2 INVARIANCE IN GROUPOID GRADED RINGS

Inspired by [40, Sections 3–4], we shall now examine the relationship between graded ideals of a G-graded ring S and G-invariant ideals of the subring $\oplus_{e \in G_0} S_e$.

Definition 6.7 ([40, Definition 3.1, Definition 3.3])**.** *Let S be a G-graded ring.*

(i) For any $g \in G$ and any subset I of S, we write $I^g := S_{g^{-1}} I S_g$.

*(ii) Let H be a subgroupoid of G and let I be a subset of S. Then, I is called H-*invariant *if $I^g \subseteq I$ for every $g \in H$.*

Remark 6.8. *Let S be a G-graded ring. Note that if $g \in G$, and $I \subseteq S$, then*

$$I^g = \left\{ \sum_{k=1}^{n} a_k x_k b_k : n \in \mathbb{N}, a_k \in S_{g^{-1}}, x_k \in I \text{ and } b_k \in S_g \text{ for each } k \in \{1, \ldots, n\} \right\}.$$

In the upcoming five results we are assuming that S is a G-graded ring.

Lemma 6.9. *If $g \in G$ and J is an ideal of $\oplus_{e \in G_0} S_e$, then J^g is an ideal of $\oplus_{e \in G_0} S_e$.*

Proof. Let J be an ideal of $\oplus_{e\in G_0} S_e$ and $g \in G$. Notice that J^g is an additive subgroup of $\oplus_{e\in G_0} S_e$. Moreover, $(\oplus_{e\in G_0} S_e)J^g = (\oplus_{e\in G_0} S_e)(S_{g^{-1}} J S_g) = S_{s(g)} S_{g^{-1}} J S_g \subseteq S_{g^{-1}} J S_g = J^g$. Analogously,
$J^g(\oplus_{e\in G_0} S_e) \subseteq J^g$. $\square$

Proposition 6.10. *If I is an ideal of S, then $I \cap \oplus_{e\in G_0} S_e$ is an ideal of $\oplus_{e\in G_0} S_e$.*

Proof. It is easy to prove that $I \cap \oplus_{e\in G_0} S_e$ is an additive subgroup of $\oplus_{e\in G_0} S_e$. Using that I is an ideal of S and that $\oplus_{e\in G_0} S_e \subseteq S$, we get that $(\oplus_{e\in G_0} S_e)(I \cap \oplus_{e\in G_0} S_e) \subseteq I \cap \oplus_{e\in G_0} S_e$. Similarly, $(I \cap \oplus_{e\in G_0} S_e)(\oplus_{e\in G_0} S_e) \subseteq (I \cap \oplus_{e\in G_0} S_e)$. $\square$

Proposition 6.11. *Suppose that J is an ideal of $\oplus_{e\in G_0} S_e$. Then SJS is a G-graded ideal of S.*

Proof. It is clear that SJS is an ideal of S and that $\oplus_{g\in G}((SJS) \cap S_g) \subseteq SJS$. Now we show the reversed inclusion. Take $g,h \in G$, $a_g \in S_g$, $c_h \in S_h$ and $b = \sum_{e\in G_0} b_e \in J$. If $s(g) \neq r(h)$, then $a_g b c_h = 0 \in (SJS) \cap S_{gh}$. Otherwise, $s(g) = r(h)$, and then $a_g b c_h = a_g b_{s(g)} c_h \in (SJS) \cap S_{gh}$. Thus, $SJS = \oplus_{g\in G}((SJS) \cap S_g)$. $\square$

Lemma 6.12. *Suppose that $\oplus_{e\in G_0} S_e$ is s-unital and that J is an ideal of $\oplus_{e\in G_0} S_e$. Then J is G-invariant if, and only if, $(SJS) \cap \oplus_{e\in G_0} S_e = J$.*

Proof. We first show the “only if” statement. Suppose that J is G-invariant. For each $e \in G_0$, we have

$$(SJS) \cap S_e \subseteq \left(\sum_{\substack{g\in G \\ s(g)=e}} (S_{g^{-1}} J S_g) \right) \cap S_e \subseteq \left(\sum_{\substack{g\in G \\ s(g)=e}} J^g \right) \cap S_e \subseteq J.$$

Let $a = \sum_{f\in G_0} a_f \in (SJS) \cap \oplus_{e\in G_0} S_e$. By Proposition 6.11, SJS is G-graded and we notice that $a_f \in (SJS) \cap S_f \subseteq J$. Thus, $(SJS) \cap \oplus_{e\in G_0} S_e \subseteq J$. By assumption, $\oplus_{e\in G_0} S_e$ is s-unital and $J \subseteq \oplus_{e\in G_0} S_e$. Thus,

$$J \subseteq (\oplus_{e\in G_0} S_e) J (\oplus_{e\in G_0} S_e) \subseteq (SJS) \cap \oplus_{e\in G_0} S_e.$$

Now we show the “if” statement. Suppose that $(SJS) \cap \oplus_{e\in G_0} S_e = J$. Take $g \in G$ and notice that $J^g = S_{g^{-1}} J S_g \subseteq (SJS) \cap S_{s(g)} \subseteq (SJS) \cap \oplus_{e\in G_0} S_e = J$. Thus, J is G-invariant. $\square$

Lemma 6.13. *If I is a G-graded ideal of S, then $I \cap \oplus_{e\in G_0} S_e$ is a G-invariant ideal of $\oplus_{e\in G_0} S_e$.*

Proof. Let I be a G-graded ideal of S and $g \in G$. Notice that $S_{g^{-1}}(I \cap S_{r(g)})S_g \subseteq I \cap S_{s(g)}$. Furthermore, if $e \in G_0 \setminus \{r(g)\}$, then $S_{g^{-1}}(I \cap S_e)S_g = \{0\}$. Therefore, $(I \cap \oplus_{e\in G_0} S_e)^g = S_{g^{-1}}(I \cap \oplus_{e\in G_0} S_e)S_g \subseteq I \cap \oplus_{e\in G_0} S_e$. $\square$

Lemma 6.14. *Let S be a nearly epsilon-strongly G-graded ring. If I is a G-graded ideal of S, then $I = (I \cap \oplus_{e\in G_0} S_e)S = S(I \cap \oplus_{e\in G_0} S_e) = S(I \cap \oplus_{e\in G_0} S_e)S$.*

Proof. Let I be a G-graded ideal of S. By Proposition 6.4, S is s-unital and hence $(I \cap \oplus_{e\in G_0} S_e) \subseteq (I \cap \oplus_{e\in G_0} S_e)S$. Thus, $S(I \cap \oplus_{e\in G_0} S_e) \subseteq S(I \cap \oplus_{e\in G_0} S_e)S \subseteq I$. Analogously, $(I \cap \oplus_{e\in G_0} S_e)S \subseteq S(I \cap \oplus_{e\in G_0} S_e)S \subseteq I$.

We claim that $I \subseteq S(I \cap \oplus_{e\in G_0} S_e)$. Since I is G-graded, consider $a_g \in I \cap S_g$. By Proposition 6.3, there is $u_g \in S_g S_{g-1}$ such that $a_g = u_g a_g$. Then, $u_g = \sum_j b_j c_j$ such that for each j, $b_j \in S_g$ and $c_j \in S_{g^{-1}}$. Notice that $c_j a_g \in (S_{g^{-1}} S_g) \cap I \subseteq S_{s(g)} \cap I$ for every j. Hence, $a_g = u_g a_g = \sum_j b_j c_j a_g \in S_g(S_{s(g)} \cap I) \subseteq S(\oplus_{e\in G_0} S_e \cap I)$. Similarly, $I \subseteq (\oplus_{e\in G_0} S_e \cap I)S$. Thus, $I = S(\oplus_{e\in G_0} S_e \cap I) = (\oplus_{e\in G_0} S_e \cap I)S$ and $I = S(\oplus_{e\in G_0} S_e \cap I) \subseteq S(\oplus_{e\in G_0} S_e \cap I)S \subseteq I$. □

Corollary 6.15. *Let S be a nearly epsilon-strongly G-graded ring. If J is a G-invariant ideal of $\oplus_{e\in G_0} S_e$, then $SJS = JS = SJ$.*

Proof. Let J be a G-invariant ideal of $\oplus_{e\in G_0} S_e$. By Proposition 6.11, SJS is a G-graded ideal of S, and, by Lemma 6.14, $SJS = ((SJS) \cap \oplus_{e\in G_0} S_e)S = S \cap ((SJS) \cap \oplus_{e\in G_0} S_e)$. Thus, by Proposition 6.4 (iii) and Lemma 6.12, $SJS = JS = SJ$. □

6.3 GRADED PRIMENESS OF GROUPOID GRADED RINGS

In this section we will characterize graded primeness for groupoid graded rings. We will generalize Proposition 2.27 and we will show some necessary conditions for graded primeness of S.

Definition 6.16. *Let G be a groupoid and let S be a G-graded ring.*

*(i) The ring $\oplus_{e\in G_0} S_e$ is said to be G-*prime *if there are no nonzero G-invariant ideals I,J of $\oplus_{e\in G_0} S_e$ such that $IJ = \{0\}$.*

(ii) The ring S is said to be graded prime *if there are no nonzero graded ideals I,J of S such that $IJ = \{0\}$.*

The following theorem generalizes Proposition 2.27.

Theorem 6.17. *Let S be a nearly epsilon-strongly G-graded ring. Then S is graded prime if, and only if, $\oplus_{e\in G_0} S_e$ is G-prime.*

Proof. We first show the "if" statement. Suppose that $\oplus_{e\in G_0} S_e$ is G-prime and let I_1, I_2 be nonzero graded ideals of S. By Lemma 6.13, $(I_1 \cap \oplus_{e\in G_0} S_e)$ and $(I_2 \cap \oplus_{e\in G_0} S_e)$ are nonzero G-invariant ideals of $\oplus_{e\in G_0} S_e$. Then $\{0\} \neq (I_1 \cap \oplus_{e\in G_0} S_e) \cdot (I_2 \cap \oplus_{e\in G_0} S_e) \subseteq I_1 I_2$.

Now we show the "only if" statement. Suppose that S is graded prime and let J_1, J_2 be G-invariant ideals of $\oplus_{e\in G_0} S_e$. By Proposition 6.4 (iii) and Proposition 6.11, SJ_1S and SJ_2S are nonzero graded ideals of S. By Corollary 6.15 and our assumption, $SJ_1 \cdot J_2S = SJ_1S \cdot SJ_2S \neq \{0\}$. Thus $J_1 \cdot J_2 \neq \{0\}$. □

Now, we show some necessary conditions for graded primeness of groupoid graded rings.

Lemma 6.18. *Let S be a G-graded ring which is s-unital. Suppose that S is graded prime. Let $a_g \in S_g$ and $c_h \in S_h$ be nonzero elements, for some $g, h \in G$. Then there is some $t \in G$ and $b_t \in S_t$ such that $a_g b_t c_h$ is nonzero.*

Proof. We prove the contrapositive statement. Suppose that $a_g b_t c_h = 0$ for every element $b_t \in S_t$ with $t \in G$. Consider the sets $A := Sa_gS$ and $C := Sc_hS$. By s-unitality of S it is clear that both A and C are graded nonzero ideals of S. Moreover, by assumption we have $AC = Sa_gSc_hS = 0$. This shows that S is not graded prime. □

Definition 6.19. *Let S be a G-graded ring. An element $e \in G_0'$ (see* (6.1)*) is said to be a* support-hub *if for every nonzero $a_g \in S_g$, with $g \in G$, there are $h, k \in G$ such that $s(h) = e$, $r(k) = e$ and a_gS_h and S_ka_g are both nonzero.*

Remark 6.20. *(a) Suppose that $e \in G_0'$ is a support-hub and that $a_g \in S_g$ is nonzero, for some $g \in G$. Notice that there are $h, k \in G$ according to the following diagram.*

$$\begin{tikzcd} & e \arrow[dl, "h"'] \\ d(g) \arrow[r, "g"'] & c(g) \arrow[u, "k"'] \end{tikzcd}$$

(b) Notice that, if S is a ring which is nearly epsilon-strongly graded by a group G, then the identity element e of G is always a support-hub.

Proposition 6.21. *Let S be a G-graded ring which is s-unital. If S is graded prime, then every $e \in G_0'$ is a support-hub.*

Proof. We prove the contrapositive statement. Suppose that there is some $e \in G_0'$ which is not a support-hub. Then there are $g \in G$ and a nonzero element $a_g \in S_g$, such that for every $h \in G$ such that $s(h) = e$, we have that $a_gS_h = \{0\}$ or for every $k \in G$ such that $r(k) = e$, we have that $S_ka_g = \{0\}$. Let a_e be a nonzero element of S_e. Since S is s-unital, $A = Sa_eS$ and $B = Sa_gS$ are nonzero graded ideals of S.

Notice that if for every $h \in G$ such that $s(h) = e$, we have that $a_gS_h = \{0\}$, then $BA = Sa_gSa_eS = \{0\}$. Moreover, if for every $k \in G$ such that $r(k) = e$, we have that $S_ka_g = \{0\}$, then $AB = Sa_eSa_gS = \{0\}$. Therefore, S is not graded prime. □

Definition 6.22. *Let G be a groupoid. G is said to be* connected*, if for all $e, f \in G_0$, there exists $g \in G$ such that $s(g) = e$ and $r(g) = f$.*

Proposition 6.23. *Let S be a G-graded ring which is s-unital. The following assertions hold:*

(i) If G is a connected groupoid, then G' is a connected subgroupoid of G.

(ii) If there is a support-hub in G'_0, then G' is a connected subgroupoid of G.

(iii) If S is graded prime, then G' is a connected subgroupoid of G.

Proof. (i) Suppose that G is connected. Take $e,f \in G'_0$. By assumption, there is $g \in G$ such that $s(g) = e$ and $r(g) = f$. Since S_e and S_f are nonzero, we must have $g \in G'$, and hence G' is connected.
(ii) Suppose that $e \in G'_0$ is a support-hub. Let $f_1,f_2 \in G'_0$. By the definition of G', there are nonzero elements $a_{f_1} \in S_{f_1}$ and $b_{f_2} \in S_{f_2}$. Since e is a support-hub, there is $k_1 \in G$ such that $r(k) = e$ and $S_k a_{f_1} \neq \{0\}$. In particular, $s(k) = f_1$. Using again that e is a support-hub, there is $h \in G$ such that $s(h) = e$ and $b_{f_2} S_h \neq \{0\}$. Then, $r(h) = f_2$. Define $t := hk$ and note that $s(t) = f_1$ and $r(t) = f_2$.
(iii) It follows from Proposition 6.21 and (ii). □

6.4 PRIMENESS OF GROUPOID GRADED RINGS

In this section, we will provide necessary and sufficient conditions for primeness of a nearly epsilon-strongly G-graded ring S. Furthermore, we will generalize [40, Theorem 1.3] to the context of groupoid graded rings.

Proposition 6.24. *Let S be a nearly epsilon-strongly G-graded ring. If S is prime, then $\oplus_{g\in G_e^e} S_g$ is prime for every $e \in G_0$.*

Proof. We prove the contrapositive statement. Let $e \in G_0$. Suppose that I,J are nonzero ideals of $\oplus_{g\in G_e^e} S_g$ such that $IJ = \{0\}$. By Proposition 6.4, S is s-unital and hence $A' := SIS$ and $B' := SJS$ are nonzero ideals of S. Clearly, $A'B' = SISSJS = SISJS$ and we claim that $ISJ \subseteq IJ$. If we assume that the claim holds, then it follows that $A'B' = \{0\}$, and we are done. Now we show the claim. Take $g \in G$, $c_g \in S_g$, $a = \sum_{k\in G_e^e} a_k \in I$, and $b = \sum_{t\in G_e^e} b_t \in J$. Let $k,t \in G_e^e$. If $e = s(k) \neq r(g)$ or $s(g) \neq r(t) = e$, then $a_k c_g b_t = 0 \in IJ$. Otherwise, $g \in G_e^e$, and then, since I and J are ideals of $\oplus_{g\in G_e^e} S_g$, we get that $ac_g b \in IJ$. Therefore, $ISJ \subseteq IJ$ and this proves the claim. □

Remark 6.25. *Let S be a nearly epsilon-strongly ring graded by G. By Proposition 6.24 and Proposition 6.23 (iii), if S is prime, then $\oplus_{g\in G_e^e} S_g$ is prime for every $e \in G_0$ and G' is connected. However, the converse is not true as shown in Example 7.22.*

The following lemma generalizes [50, Lemma 2.4]. For the convenience of the reader we include the proof.

Lemma 6.26. *Let G be a groupoid and let S be a G-graded ring. Suppose that H is a subgroupoid of G. Define $\pi_H : S \to S_H$ by $\pi_H\left(\sum_{g\in G} c_g\right) := \sum_{h\in H} c_h$. The following assertions hold:*

(i) The map $\pi_H : S \to S_H$ is additive.

(ii) If $a \in S$ and $b \in S_H$, then $\pi_H(ab) = \pi_H(a)b$ and $\pi_H(ba) = b\pi_H(a)$.

Proof. (i) This is clear.

(ii) Take $a \in S$ and $b \in S_H$. Put $a' := a - \pi_H(a)$. Clearly, $a = a' + \pi_H(a)$ and $\mathrm{Supp}(a') \subseteq G \setminus H$. If $g \in G \setminus H$ and $h \in H$, then either the composition gh does not exist or it belongs to $G \setminus H$. Thus, $\mathrm{Supp}(a'b) \subseteq \emptyset \cup G \setminus H$. Hence, $\pi_H(ab) = \pi_H((a' + \pi_H(a))b) = \pi_H(a'b) + \pi_H(\pi_H(a)b) = 0 + \pi_H(a)b$. Analogously, one may show that $\pi_H(ba) = b\pi_H(a)$. □

The next result partially generalizes [40, Lemma 2.19].

Lemma 6.27. *Let S be a nearly epsilon-strongly G-graded ring and let I be a nonzero ideal of S. If there is some $e \in G'_0$ which is a support-hub, then $\pi_{G_e^e}(I)$ is a nonzero ideal of $\oplus_{g \in G_e^e} S_g$.*

Proof. By Lemma 6.26, $\pi_{G_e^e}(I)$ is an ideal of $\oplus_{g \in G_e^e} S_g$. We claim that $\pi_{G_e^e}(I) \neq \{0\}$. Let $d = d_{g_1} + d_{g_2} + \ldots + d_{g_n} \in I$ be an element where all the homogeneous coefficients are nonzero and the $g_i's$ are distinct. By Proposition 6.3, there is some nonzero $c_{g_1^{-1}} \in S_{g_1^{-1}}$ such that $d_{g_1} c_{g_1^{-1}}$ is nonzero and contained in $S_{r(g_1)}$.

Notice that $dc_{g_1^{-1}}$ is nonzero and contained in I. Thus, without loss of generality, we may assume that $g_1 \in G'_0$. Since e is a support-hub, there is an element $k \in G'$ such that $r(k) = e$ and $S_k d_{g_1}$ is nonzero. In particular, there is an element $b_k \in S_k$ such that $b_k s_{g_1}$ is nonzero. Using again that e is a support-hub, there is an element $l \in G'$ such that $s(l) = e$ and $(b_k d_{g_1}) S_l$ is nonzero. Therefore, there is an element $b_l \in S_l$ such that $(b_k d_{g_1}) b_l$ is nonzero. Thus, $b_k d b_l \in I$ and

$$\pi_{G_e^e}(b_k d b_l) = \sum_{\substack{s(k)=r(g_i)\\ s(g_i)=r(l)}} b_k d_{g_i} b_l = \sum_{\substack{g_1=r(g_i)\\ s(g_i)=g_1}} b_k d_{g_i} b_l.$$

Notice that $b_k d_{g_i} b_l \in S_{kl}$ if and only if $g_i = g_1$. Thus, $0 \neq \pi_{G_e^e}(b_k d b_l) \in \pi_{G_e^e}(I)$. □

Theorem 6.28. *Let S be a nearly epsilon-strongly G-graded ring. If there is some $e \in G'_0$ such that e is a support-hub and $\oplus_{g \in G_e^e} S_g$ is prime, then S is prime.*

Proof. Let I and J be nonzero ideals of S. By Lemma 6.27, $\pi_{G_e^e}(I)$ and $\pi_{G_e^e}(J)$ are nonzero ideals of $\oplus_{g \in G_e^e} S_g$. Since $\oplus_{g \in G_e^e} S_g$ is prime, $\pi_{G_e^e}(I)\pi_{G_e^e}(J) \neq \{0\}$.

We claim that $\pi_{G_e^e}(I) \subseteq I$. Let $b = \sum_{t \in G} b_t \in I$ and consider the finite set $F := \{t \in \mathrm{Supp}(b) : r(t) = e\}$. Since $S_t = S_t S_{t^{-1}} S_t$ for all $t \in F$, there are $a_i^t \in S_t S_{t^{-1}} \subseteq S_{r(t)}$ and $b_i^t \in S_t$ such that $b_t = \sum_i a_i^t b_i^t$. Since $S_{r(t)} = S_e$ is s-unital, there is $u_e \in S_{r(t)}$ such that $u_e a_i^t = a_i^t$ for all i and for all $t \in F$. Thus, $b_t = u_e b_t$ for every $t \in F$. Therefore, $u_e \sum_{t \in G} b_t = \sum_{e = r(t)} u_e b_t = \sum_{e = r(t)} b_t$.

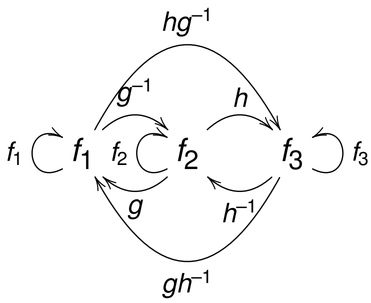
Now, since $S_t = S_t S_{t^{-1}} S_t$ for all $t \in G_e^e$, there are $a_j^t \in S_{t^{-1}} S_t \subseteq S_{r(t^{-1})}$ and $b_j^t \in S_t$ such that $b_t = \sum_j b_j^t a_j^t$. Since $S_{s(t)} = S_e$ is s-unital, there is $v_e \in S_{s(t)}$ such that $a_j^t v_e = a_j^t$ for all j and for all $t \in G_e^e$. Thus, $b_t = b_t v_e$ for every $t \in G_e^e$. Therefore,

$$I \ni \left(u_e \sum_{t \in G} b_t \right) v_e = \left(\sum_{e = r(t)} b_t \right) v_e = \sum_{t \in G_e^e} b_t v_e = \sum_{t \in G_e^e} b_t = \pi_{G_e^e}(b).$$

Analogously, $\pi_{G_e^e}(J) \subseteq J$. Thus, $\{0\} \neq \pi_{G_e^e}(I) \pi_{G_e^e}(J) \subseteq IJ$ and S is prime. $\square$

Remark 6.29. *The assumption on the existence of a support-hub in Theorem 6.28 cannot be dropped. Indeed, consider the groupoid $G = \{e, f\} = G_0$ and the groupoid ring $S := \mathbb{C}[G]$. Then $G_e^e = \{e\}$ and $G_f^f = \{f\}$. Furthermore, $S_e \cong \mathbb{C}$ and $S_f \cong \mathbb{C}$ are both prime. Nevertheless, S is not prime.*

Example 6.30. *Let $G = \{f_1, f_2, f_3, g, h, g^{-1}, h^{-1}, hg^{-1}, gh^{-1}\}$ be a groupoid such that $G_0 = \{f_1, f_2, f_3\}$ as follows:*



Define S as the ring of the matrices over $\mathbb{Z}$ of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

Denote by $\{e_{ij}\}_{i,j}$, the canonical basis of S and define:

$S_g := \{\lambda e_{12} : \lambda \in \mathbb{Z}\}$, $S_{g^{-1}} := \{\lambda e_{21} : \lambda \in \mathbb{Z}\}$, $S_h := \{\lambda e_{43} : \lambda \in \mathbb{Z}\}$,
$S_{h^{-1}} := \{\lambda e_{34} : \lambda \in \mathbb{Z}\}$, $S_{f_1} := \{\lambda e_{11} : \lambda \in \mathbb{Z}\}$, $S_{f_3} := \{\lambda e_{44} : \lambda \in \mathbb{Z}\}$,
$S_{f_2} := \{\lambda_1 e_{22} + \lambda_2 e_{33} : \lambda_1, \lambda_2 \in \mathbb{Z}\}$,

and $S_l := \{0\}$, otherwise. Notice that $S = \oplus_{l \in G} S_l$ and that $f_2 \in G_0'$ is a support-hub. Although, notice that $e_{12} \in S_g$, $e_{43} \in S_h$ are nonzero elements and there is no element a_l in S_l such that $l \in G'$ and $e_{12} a_l e_{43} \neq 0$. Therefore, by Lemma 6.18, S is not graded prime.

Theorem 6.31. *Let S be a nearly epsilon-strongly ring graded by G. The following statements are equivalent:*

(i) *S is prime;*

(ii) $\oplus_{e\in G_0} S_e$ *is G-prime, and for every* $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ *is prime;*

(iii) $\oplus_{e\in G_0} S_e$ *is G-prime, and for some* $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ *is prime;*

(iv) *S is graded prime, and for every* $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ *is prime;*

(v) *S is graded prime, and for some* $e \in G_0'$, $\oplus_{g\in G_e^e} S_g$ *is prime;*

(vi) *For every* $e \in G_0'$, *e is a support-hub and* $\oplus_{g\in G_e^e} S_g$ *is prime;*

(vii) *For some* $e \in G_0'$, *e is a support-hub and* $\oplus_{g\in G_e^e} S_g$ *is prime.*

Proof. It follows from Proposition 6.24 and by the definition of primeness that (i) $\Rightarrow$ (iv) $\Rightarrow$ (v). By Proposition 6.21, (v) $\Rightarrow$ (vii) and (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii). By Theorem 6.28, (vii) $\Rightarrow$ (i). Therefore, (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (i). And, (i) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i). Finally, note that by Theorem 6.17, (ii) is equivalent to (iv) and (iii) is equivalent to (v). $\square$

Corollary 6.32. *Let G be a groupoid and let S be a nearly epsilon-strongly G-graded ring. Then S is not prime if, and only if, there is some* $e \in G_0'$ *such that* $\oplus_{g\in G_e^e} S_g$ *is not prime or all of the following assertions hold:*

(i) $\oplus_{e\in G_0} S_e$ *is not G-prime;*

(ii) *S is not graded prime;*

(iii) *There is some* $e \in G_0'$ *which is not a support-hub.*

Proof. We first show the "if" statement. Suppose that there exists $e \in G_0'$ such that $\oplus_{g\in G_e^e} S_g$ is not prime or that (i), (ii) and (iii) hold. By Theorem 6.31, S is not prime.

Now we show the "only if" statement. Suppose that S is not prime. If there is some $e \in G_0'$ such that $\oplus_{g\in G_e^e} S_g$ is not prime, then the proof is done. Otherwise, $\oplus_{g\in G_e^e} S_g$ is prime for every $e \in G_0'$, and by Theorem 6.31 (ii), we get that (i) holds. Thus, by Theorem 6.17, (ii) also holds. Finally, Theorem 6.31 (vi) implies that there is some $e \in G_0'$ such that e is not a support-hub. $\square$

We recall that Passman proved in [52] an equivalent condition for the case that a unital strongly group graded ring is not prime. This result was generalized in [40, Theorem 1.3] for nearly epsilon-strongly group graded ring.

Theorem 6.33 ([40, Theorem 1.3])**.** *Let G be a group and S be a nearly epsilon-strongly G-graded ring. The following statements are equivalent:*

(a) *S is not prime;*

(b) *There exist:*

(i) *subgroups* $N \lhd H \subseteq G$,

(ii) *an H-invariant ideal I of* S_e *such that* $I^g I = \{0\}$ *for all* $g \in G \setminus H$, *and*

(iii) *nonzero ideals* $\tilde{A}, \tilde{B}$ *of* S_N *such that* $\tilde{A}, \tilde{B} \subseteq IS_N$ *and* $\tilde{A} S_H \tilde{B} = \{0\}$.

(c) There exist:

(i) subgroups $N \lhd H \subseteq G$ with N finite,

(ii) an H-invariant ideal I of S_e such that $I^g I = \{0\}$ for all $g \in G \setminus H$, and

(iii) nonzero ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A} S_H \tilde{B} = \{0\}$.

(d) There exist:

(i) subgroups $N \lhd H \subseteq G$ with N finite,

(ii) an H-invariant ideal I of S_e such that $I^g I = \{0\}$ for all $g \in G \setminus H$, and

(iii) nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N with $\tilde{A}, \tilde{B} \subseteq IS_N$ such that $\tilde{A} S_H \tilde{B} = \{0\}$.

(e) There exist:

(i) subgroups $N \lhd H \subseteq G$ with N finite,

(ii) an H-invariant ideal I of S_e such that $I^g I = \{0\}$ for all $g \in G \setminus H$, and

(iii) nonzero H/N-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Remark 6.34. *Note that, in Corollary 6.32, $\oplus_{g \in G_e^e} S_g$ is nearly epsilon-strongly graded by the group G_e^e. Hence, one can apply Theorem 6.33 to determine whether $\oplus_{g \in G_e^e} S_g$ is prime.*

The following Theorem generalizes [40, Theorem 1.4].

Theorem 6.35. *Let S be a nearly epsilon-strongly G-graded ring such that $\oplus_{e \in G_0} S_e$ is G-prime. Suppose that there is $e \in G_0'$ such that G_e^e is torsion-free. Then S is prime if and only if S_e is G_e^e-prime.*

Proof. It follows from Theorem 6.31 and [40, Theorem 1.4]. □

7 APPLICATIONS TO PARTIAL SKEW GROUPOID RINGS

In this section, we will apply our main results on primeness for nearly epsilon-strongly graded rings to the context of partial skew groupoid rings, (global) skew groupoid rings and groupoid rings. In particular, we will describe primeness of a partial skew groupoid ring induced by a partial action of group-type (cf. [5]). Furthermore, we will generalize [40, Theorem 12.4] and [40, Theorem 13.7].

7.1 PARTIAL SKEW GROUPOID RINGS

Throughout this section, let A be an arbitrary associative ring, let G be an arbitrary groupoid, and let $\sigma = (A_g, \sigma_g)_{g\in G}$ be a partial action of G on A. Denote by $A \rtimes_\sigma G$ its associated partial skew groupoid ring (see Definition 1.11 and Definition 1.13).

Remark 7.1. *(a) We will assume that A_g is an s-unital ring, for every $g \in G$, and that $A = \oplus_{e\in G_0} A_e$. As a consequence, $A \rtimes_\sigma G$ will always be an associative ring (see Remark 1.14 (a)).*

(b) Under the above assumptions, by Lemma 2.2, A_g is an ideal of A for all $g \in G$.

(c) Recall that the partial skew groupoid ring $S := A \rtimes_\sigma G$ carries a natural G-grading defined by letting $S_g := A_g\delta_g$, for every $g \in G$.

The following result generalizes [40, Proposition 13.1].

Proposition 7.2. *The partial skew groupoid ring $A \rtimes_\sigma G$ is a nearly epsilon-strongly G-graded ring.*

Proof. Let $g \in G$. Using that $A_{g^{-1}}$ is s-unital, and hence idempotent, we get that

$$(A_g\delta_g)(A_{g^{-1}}\delta_{g^{-1}}) = \sigma_g(\sigma_{g^{-1}}(A_g)A_{g^{-1}})\delta_{r(g)} = \sigma_g(A_{g^{-1}}A_{g^{-1}})\delta_{r(g)} = \sigma_g(A_{g^{-1}})\delta_{r(g)} = A_g\delta_{r(g)}.$$

Now, using that A_g is s-unital we get that $A_g\delta_{r(g)}$ is s-unital, and that A_g is idempotent. Hence,

$$(A_g\delta_g)(A_{g^{-1}}\delta_{g^{-1}})(A_g\delta_g) = A_g\delta_{r(g)}A_g\delta_g = A_g^2\delta_g = A_g\delta_g.$$

This shows that $A \rtimes_\sigma G$ is nearly epsilon-strongly G-graded. □

Remark 7.3. *Notice that Proposition 6.4 and Proposition 7.2 imply that the partial skew groupoid ring $A \rtimes_\sigma G$ is s-unital.*

Remark 7.4. *Since $A = \oplus_{e\in G_0} A_e$, there is a ring isomorphism $\psi : A \longrightarrow \oplus_{e\in G_0} A_e\delta_e$ defined by*

$$\psi\left(\sum_{e\in G_0} a_e\right) := \sum_{e\in G_0} a_e\delta_e. \tag{7.1}$$

Propositions 7.5 and 7.6 generalize [40, Remark 13.4].

Proposition 7.5. *I is a G-invariant ideal of A in the sense of Definition 1.10 if and only if $\psi(I)$ is a G-invariant ideal of $\oplus_{e\in G_0} A_e\delta_e$ in the sense of Definition 6.7.*

Proof. Let $g \in G$. By Remark 7.1 (b) and s-unitality of A_g, we get that $A_g I = I \cap A_g = IA_g$ and $IA_g = A_g IA_g = A_g I$. Furthermore, $\psi(I) = \oplus_{e\in G_0}(IA_e)\delta_e$. Notice that $A_g = A_{r(g)}A_g$. We get that

$$\begin{aligned}\psi(I)^g &= A_{g^{-1}}\delta_{g^{-1}} \cdot \psi(I) \cdot A_g\delta_g = A_{g^{-1}}\delta_{g^{-1}} \cdot (IA_{r(g)})A_g\delta_g = A_{g^{-1}}\delta_{g^{-1}} \cdot IA_g\delta_g \\ &= \sigma_{g^{-1}}(\sigma_g(A_{g^{-1}})IA_g)\delta_{s(g)} = \sigma_{g^{-1}}(A_g IA_g)\delta_{s(g)}.\end{aligned}$$

Therefore,

$$\psi(I)^g \subseteq \psi(I) \Longleftrightarrow \sigma_{g^{-1}}(A_g IA_g)\delta_{s(g)} \subseteq \psi(I) \Longleftrightarrow \sigma_{g^{-1}}(A_g IA_g) \subseteq I \Longleftrightarrow \sigma_{g^{-1}}(I \cap A_g) \subseteq I.$$

□

Proposition 7.6. *A is G-prime if and only if $\oplus_{e\in G_0} A_e\delta_e$ is G-prime.*

Proof. It follows from Proposition 7.5. □

Remark 7.7. (a) *Recall that, with the natural G-grading on $A \rtimes_\sigma G$, an element $e \in G_0'$ is a* support-hub *if for every nonzero element $a_g\delta_g$, with $g \in G$, there are $h,k \in G$ such that $s(h) = e$, $r(k) = e$ and $a_g\delta_g A_h\delta_h$ and $A_k\delta_k a_g\delta_g$ are nonzero.*

(b) *For $e \in G_0$, denote by $\sigma^e := (A_h, \sigma_h)_{h\in G_e^e}$ the partial action of the isotropy group G_e^e on the ring A_e, obtained by restricting σ. The associated partial skew group ring is denoted by $A_e \rtimes_{\sigma^e} G_e^e$.*

Theorem 7.8. *Let $\sigma = (A_g, \sigma_g)_{g\in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$ and $A = \oplus_{e\in G_0} A_e$. Then, the following statements are equivalent:*

(i) *The partial skew groupoid ring $A \rtimes_\sigma G$ is prime.*

(ii) *A is G-prime and, for every $e \in G_0'$, $A_e \rtimes_{\sigma^e} G_e^e$ is prime;*

(iii) *A is G-prime and, for some $e \in G_0'$, $A_e \rtimes_{\sigma^e} G_e^e$ is prime;*

(iv) *$A \rtimes_\sigma G$ is graded prime and, for every $e \in G_0'$, $A_e \rtimes_{\sigma^e} G_e^e$ is prime;*

(v) *$A \rtimes_\sigma G$ is graded prime and, for some $e \in G_0'$, $A_e \rtimes_{\sigma^e} G_e^e$ is prime;*

(vi) *For every $e \in G_0'$, e is a support-hub and $A_e \rtimes_{\sigma^e} G_e^e$ is prime;*

(vii) *For some $e \in G_0'$, e is a support-hub and $A_e \rtimes_{\sigma^e} G_e^e$ is prime.*

Proof. It follows from Proposition 7.2, Theorem 6.31 and Proposition 7.6. □

In Propositions 7.16 and 7.14, we will show sufficient conditions for (vii) in Theorem 7.8. In particular, we find sufficient conditions for primeness of a partial skew groupoid ring.

Definition 7.9 ([5, Remark 3.4]). *A partial action $\sigma = (A_g, \sigma_g)_{g\in G}$ of a connected groupoid G on A is said to be of* group-type *if there exist an element $e \in G_0$ and a family of morphisms $\{h_f\}_{f\in G_0}$ in G such that $h_f : e \to f$, $h_e = e$, $A_{h_f^{-1}} = A_e$ and $A_{h_f} = A_f$, for every $f \in G_0$.*

Remark 7.10. *(a) If a partial action σ is of group type, then every element of G_0 can take the role of e in the above definition (see [5, Remark 3.4]).*

(b) By [5, Lemma 3.1], every global groupoid action is of group-type. The converse does not hold. For an example of a non-global partial action of group type we refer the reader to [5, Example 3.5].

Lemma 7.11. *Let $\sigma = (A_g, \sigma_g)_{g\in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$ and $A = \oplus_{e\in G_0} A_e$. Consider the following statements:*

(i) σ is of group-type;

(ii) There is some $e \in G_0'$ such that for every nonzero element $a_g\delta_g \in A \rtimes_\sigma G$ there is some $k \in G$ such that $s(k) = r(g)$, $r(k) = e$ and $a_g \in A_{k^{-1}}$;

(iii) There is some $e \in G_0'$ such that for every nonzero element $a_g\delta_g \in A \rtimes_\sigma G$ there is some $k \in G$ such that $s(k) = r(g)$, $r(k) = e$ and $A_{k^{-1}} a_g \neq \{0\}$;

(iv) There is some $e \in G_0'$ which is a support-hub.

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. (i)$\Rightarrow$(ii) Suppose that (i) holds. Let $e \in G_0'$. Let $g \in G$ such that $a_g\delta_g \neq 0$. Since σ is of group-type, there is a morphism $h_{r(g)} : e \to r(g)$ such that $A_{h_{r(g)}^{-1}} = A_e$ and $A_{h_{r(g)}} = A_{r(g)}$. Note that $a_g \in A_g \subseteq A_{r(g)} = A_{h_{r(g)}}$. Define $k := h_{r(g)}^{-1}$ and the proof is done.

(ii)$\Rightarrow$(iii) Suppose that (ii) holds. Let $e \in G_0'$ and $a_g\delta_g \in A \rtimes_\sigma G$. By assumption, there is some $k \in G$ such that $s(k) = r(g)$, $r(k) = e$ and $a_g \in A_{k^{-1}}$. Since $A_{k^{-1}}$ is s-unital, we get $A_{k^{-1}} a_g \neq \{0\}$.

(iii)$\Rightarrow$(iv) Suppose that (iii) holds. Let $e \in G_0'$ be as in (iii). Let $g \in G$ such that $a_g\delta_g \neq 0$. By assumption, there is some $k \in G$ such that $s(k) = r(g)$ and $r(k) = e$ and $A_{k^{-1}} a_g \neq \{0\}$. Then, there is $d_{k^{-1}} \in A_{k^{-1}}$ such that $0 \neq d_{k^{-1}} a_g \in A_{k^{-1}} \cap A_g$. Let $u \in A_{g^{-1}}$ be an s-unit for $\sigma_{g^{-1}}(d_{k^{-1}} a_g)$ and let $w \in A_{k^{-1}}$ be an s-unit for $d_{k^{-1}} a_g$. Note that

$$b := ((\sigma_k(w)\delta_k)(d_{k^{-1}}\delta_{s(k)}))(a_g\delta_g)((u\delta_{g^{-1}})(w\delta_{k^{-1}})) \in A_k\delta_k(a_g\delta_g)A_{g^{-1}k^{-1}}\delta_{g^{-1}k^{-1}}.$$

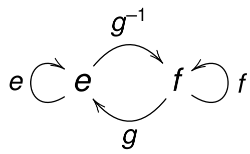
And,

$$\begin{aligned} b &= (\sigma_k(w)\delta_k)(d_{k^{-1}} a_g\delta_g)(u\delta_{g^{-1}})(w\delta_{k^{-1}}) = (\sigma_k(w)\delta_k)(\sigma_g(\sigma_{g^{-1}}(d_{k^{-1}} a_g)u)\delta_{gg^{-1}})(w\delta_{k^{-1}}) \\ &= (\sigma_k(w)\delta_k)(d_{k^{-1}} a_g\delta_{r(g)})(w\delta_{k^{-1}}) = (\sigma_k(w)\delta_k)(d_{k^{-1}} a_g\delta_{k^{-1}}) = \sigma_k(\sigma_{k^{-1}}(\sigma_k(w))d_{k^{-1}} a_g)\delta_{kk^{-1}} \\ &= \sigma_k(d_{k^{-1}} a_g)\delta_e \neq 0. \end{aligned}$$

(iv)⇒(iii) Suppose that (iv) holds. Let $e \in G_0'$ be as in (iv) and let $a_g\delta_g$, with $g \in G$, be a nonzero element. By assumption, there is some $k \in G$ such that $r(k) = e$ and $A_k\delta_k a_g\delta_g$ is nonzero. Note that $A_k\delta_k a_g\delta_g = \sigma_k(A_{k^{-1}} a_g)\delta_{kg}$. Therefore, $A_{k^{-1}} a_g \neq \{0\}$. □

Remark 7.12. *In general, statement (i) in Lemma 7.11 is stronger than statements (iii) and (iv). This is shown by the following example.*

Example 7.13. *We will construct a partial action of a groupoid on a ring, which is not of group-type, and such that the corresponding partial skew groupoid ring is prime. Let $G := \{e,f,g,g^{-1}\}$ be a groupoid such that $G_0 = \{e,f\}$, $s(g) = f$ and $r(g) = e$ as follows:*



Let $A := \mathbb{Z} \oplus \mathbb{Z}$, and define the partial action $\sigma := (A_g,\sigma_g)_{g\in G}$ of G on A, by

- $A_e := \mathbb{Z} \oplus \{0\}$ *and* $A_f := \{0\} \oplus \mathbb{Z}$;
- $A_g := 2\mathbb{Z} \oplus \{0\}$ *and* $A_{g^{-1}} := \{0\} \oplus 2\mathbb{Z}$;
- $\sigma_e := \mathrm{id}_{\mathbb{Z}\oplus\{0\}}$ *and* $\sigma_f := \mathrm{id}_{\{0\}\oplus\mathbb{Z}}$;
- $\sigma_g(0,x) := (x-2,0)$ *and* $\sigma_{g^{-1}}(x,0) := (0,x+2)$.

Notice that σ is not of group type, and that $A_e \rtimes_{\sigma^e} G_e^e = A_e\delta_e \cong \mathbb{Z}$ and $A_f \rtimes_{\sigma^f} G_f^f = A_f\delta_f \cong \mathbb{Z}$ are prime rings. Moreover, $G = G'$ and G is connected.

We claim that A is G-prime. Let I and J be nonzero G-invariant ideals of A. Then, there are nonzero elements $x = (x_1,x_2) \in I$ and $y = (y_1,y_2) \in J$. Without loss of generality, we may assume that $x_2 \neq 0$. If $y_2 \neq 0$, then $0 \neq xy \in IJ$. Otherwise, $y_2 = 0$ in which case $\sigma_{g^{-1}}(y) = \sigma_{g^{-1}}(y_1,0) = (0,y_1+2) \in J$ and $\sigma_{g^{-1}}(2y) = \sigma_{g^{-1}}(2y_1,0) = (0,2y_1+2) \in J$. Therefore, $0 \neq x\sigma_{g^{-1}}(y) \in IJ$ or $0 \neq x\sigma_{g^{-1}}(2y) \in IJ$. By Theorem 7.8, $A \rtimes_\sigma G$ is prime. In particular, by Theorem 7.8, e is a support-hub.

Theorem 7.14. *Let $\sigma = (A_g,\sigma_g)_{g\in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$, $A = \oplus_{e\in G_0} A_e$ and σ is of group-type. Then the partial skew groupoid ring $A \rtimes_\sigma G$ is prime if, and only if, there is some $e \in G_0'$ such that $A_e \rtimes_{\sigma^e} G_e^e$ is prime.*

Proof. It follows from Lemma 7.11 and Theorem 7.8. □

We will now make use of the example from [5, Example 3.5] of a non-global partial action of a groupoid on a ring, not of group-type, and apply Theorem 7.14 to it.

Example 7.15. *Let $G = \{e,f,g,h,l,m,l^{-1},m^{-1}\}$ be the groupoid with $G_0 = \{e,f\}$ and the following composition rules:*

$$g^2 = e, \quad h^2 = f, \quad lg = m = hl, \quad g \in G_e^e, \quad h \in G_f^f \quad \text{and} \quad l,m : e \to f.$$

We present in the following diagram the structure of G:

$$\begin{tikzcd} e \arrow[r, "l"] & f \arrow[d, "h"] \\ e \arrow[u, "g"] \arrow[r, "m"'] & f \end{tikzcd}$$

Let $\mathbb{C}$ be the field of complex numbers and let $A = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$, where $e_ie_j = \delta_{i,j}e_i$ and $e_1 + \ldots + e_4 = 1$. We define the partial action $(A_t, \sigma_t)_{t\in G}$ of G on A as follows:

$$A_e = \mathbb{C}e_1 \oplus \mathbb{C}e_2 = A_{l^{-1}}, \quad A_f = \mathbb{C}e_3 \oplus \mathbb{C}e_4 = A_l,$$
$$A_g = \mathbb{C}e_1 = A_{g^{-1}} = A_{m^{-1}}, \quad A_m = A_h = \mathbb{C}e_3 = A_{h^{-1}},$$

and

$$\sigma_e = \mathrm{id}_{A_e}, \quad \sigma_f = \mathrm{id}_{A_f}, \quad \sigma_g : ae_1 \mapsto \overline{a}e_1, \quad \sigma_h : ae_3 \mapsto \overline{a}e_3, \quad \sigma_m : ae_1 \mapsto \overline{a}e_3,$$
$$\sigma_{m^{-1}} : ae_3 \mapsto \overline{a}e_1, \quad \sigma_l : ae_1 + be_2 \mapsto ae_3 + be_4, \quad \sigma_{l^{-1}} : ae_3 + be_4 \mapsto ae_1 + be_2,$$

where $\overline{a}$ denotes the complex conjugate of a, for all $a \in \mathbb{C}$. Let $e \in G_0$ and denote $h_e = e$ and $h_f = l$. By definition, the partial action σ is of group-type.

Now, we characterize the group partial action $\sigma^e = (A_t, \sigma_t)_{t\in G_e^e}$ of G_e^e on A_e. Note that

$$G_e^e = \{e, g\}, \quad \sigma_e = \mathrm{id}_{A_e} \quad \textit{and} \quad \sigma_g : ae_1 \mapsto \overline{a}e_1.$$

We claim that A_e is not G_e^e-prime. Let $I = \mathbb{C}e_1 \oplus \{0\}e_2$ and $J = \{0\}e_1 \oplus \mathbb{C}e_2$. Note that I and J are nonzero G_e^e-invariant ideals of A_e and $IJ = \{0\}$. By Theorem 7.8, $A_e \rtimes_{\sigma^e} G_e^e$ is not prime and Theorem 7.14 implies that $A \rtimes_\sigma G$ is not prime.

Proposition 7.16. *Let $\sigma = (A_g, \sigma_g)_{g\in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$ and $A = \oplus_{e\in G_0} A_e$. Suppose that there exists $e \in G_0'$ such that for every nonzero element $a_g\delta_g \in A \rtimes_\sigma G$ there is some $k \in G$ such that $s(k) = r(g)$, $r(k) = e$ and $a_g \in A_{k^{-1}}$. Then the partial skew groupoid ring $A \rtimes_\sigma G$ is prime if, and only if, $A_e \rtimes_{\sigma^e} G_e^e$ is prime.*

Proof. It follows from Lemma 7.11 and Theorem 7.8. □

The following result generalizes [40, Theorem 13.5].

Theorem 7.17. *Let $\sigma = (A_g, \sigma_g)_{g\in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$, and $A = \oplus_{e\in G_0} A_e$. Suppose that A is G-prime or that σ is of group-type. Furthermore, suppose that there is some $e \in G_0'$ such that G_e^e is torsion-free. Then $A \rtimes_\sigma G$ is prime if, and only if, A_e is G_e^e-prime.*

Proof. It follows from Theorem 7.8, Proposition 7.14 and [40, Theorem 13.5]. □

Proposition 7.18. *Let $\sigma = (A_g, \sigma_g)_{g \in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$ and $A = \oplus_{e \in G_0} A_e$. If σ is of group-type or A is G-prime and at least one of the following conditions holds, then the partial skew groupoid ring $A \rtimes_\sigma G$ is prime.*

(i) There is some $e \in G_0'$ such that G_e^e contains no subgroup H which contains a non-trivial finite normal subgroup and A_e is prime.

(ii) There is some $e \in G_0'$ such that $\sigma^e := (A_h, \sigma_h)_{h \in G_e^e}$, the partial action of the isotropy group G_e^e on A_e, has the ideal intersection property and A_e is G_e^e-prime.

(iii) A is commutative, there is some $e \in G_0'$ such that $A_e \delta_e$ is maximal commutative in $A_e \rtimes' G_e^e$ and A_e is G_e^e-prime.

Proof. (i) Let H is a subgroup of G_e^e. Then the only finite normal subgroup of H is $N = \{e\}$. Since A_e is prime, for every nonzero $H/\{e\}$-invariant ideals $\tilde{A}, \tilde{B}$ of $A_e \rtimes_\sigma \{e\}$, we have that $\tilde{A}\tilde{B} \neq 0$. Hence, the third condition of Theorem 6.33 (e) does not hold and $A_e \rtimes'_\sigma G_e^e$ is prime.

And the result follows from from Theorem 7.8, Proposition 7.14 and Proposition 2.28 and Corollary 2.30. □

Proposition 7.19. *Let $\sigma = (A_g, \sigma_g)_{g \in G}$ be a partial action of G on A such that A_g is s-unital for every $g \in G$, and $A = \oplus_{e \in G_0} A_e$. The partial skew groupoid ring $A \rtimes_\sigma G$ is not prime if, and only if, there is some $e \in G_0'$ such that $A_e \rtimes_{\sigma^e} G_e^e$ is not prime or all of the following assertions hold:*

(a) A is not G-prime;

(b) $A \rtimes_\sigma G$ is not graded prime;

(c) There exists $e \in G_0'$ such that e is not a support-hub.

Proof. It follows from Proposition 7.2 and Theorem 6.32. □

Theorem 7.20 ([40, Theorem 13.7])**.** *Let G be a group and let $A \rtimes_\sigma G$ be a s-unital partial skew group ring. $A \rtimes_\sigma G$ is not prime if and only if there are:*

(i) subgroups $N \lhd H \subseteq G$ with N finite,

(ii) an ideal I of A such that

- $\sigma_h(IA_{h^{-1}}) = IA_h$ *for every* $h \in H$,
- $IA_g \cdot \sigma_g(IA_{g^{-1}}) = \{0\}$ *for every* $g \in G \setminus H$, *and*

(iii) nonzero ideals $\tilde{B}, \tilde{D}$ of $A \rtimes_\sigma N$ such that $\tilde{B}, \tilde{D} \subseteq I\delta_e(A \rtimes_\sigma N)$ and $\tilde{B} \cdot A_h\delta_h \cdot \tilde{D} = \{0\}$ for every $h \in H$.

Remark 7.21. *Note that, in Proposition 7.19, $A_e \rtimes_{\sigma^e} G_e^e$ is a s-unital partial skew group ring. Thus, one can apply Theorem 7.20 to determine whether $A_e \rtimes_{\sigma^e} G_e^e$ is prime.*

Example 7.22. *Let $G = \{e,f,g,g^{-1}\}$ be a groupoid such that $G_0 = \{e,f\}$, $s(g) = f$ and $r(g) = e$ as follows:*

$$\begin{tikzcd} e \arrow[loop left, "e"] \arrow[r, bend left, "g^{-1}"] & f \arrow[loop right, "f"] \arrow[l, bend left, "g"] \end{tikzcd}$$

Consider $A = \mathbb{Z} \oplus \mathbb{Z}$. We define the partial action $\sigma = (A_g,\sigma_g)_{g\in G}$ of G on A. Let:

- $A_e = \mathbb{Z} \oplus \{0\}$ *and* $A_f = \{0\} \oplus \mathbb{Z}$;
- $A_g = A_{g^{-1}} = \{0\}$;
- $\sigma_e = \mathrm{id}_{\mathbb{Z}\oplus\{0\}}$, $\sigma_f = \mathrm{id}_{\{0\}\oplus\mathbb{Z}}$ *and* $\sigma_g = \sigma_{g^{-1}} = \mathrm{id}_{\{0\}}$.

Notice that $A_e \rtimes_{\sigma^e} G_e^e = A_e\delta_e \cong \mathbb{Z}$ and $A_f \rtimes_{\sigma^f} G_f^f = A_f\delta_f \cong \mathbb{Z}$ are prime rings. Observe that $G = G'$ and G is connected. Although, there is no element $c_t\delta_t$ in $A \rtimes_\sigma G$ such that $t \in G$ and $\delta_e(c_t\delta_t)\delta_f \neq 0$ and, by Lemma 6.18, $A \rtimes_\sigma G$ is not graded prime.

7.2 SKEW GROUPOID RINGS

Let G be a groupoid, let A be a ring and let $\{A_e\}_{e\in G_0}$ be a collection of s-unital ideals of A. Suppose that, for each $g \in G$, there is a ring isomorphism $\sigma_g : A_{s(g)} \to A_{r(g)}$. Moreover, $\sigma_g \circ \sigma_h = \sigma_{gh}$ whenever $(g,h) \in G^{(2)}$. We say that σ is a global action of G on A and we will use the notation $\sigma = (A_{r(g)},\sigma_g)_{g\in G}$.

Remark 7.23. *Notice that this is a special case of groupoid partial actions. In fact, a groupoid partial action $\sigma = (A_g,\sigma_g)_{g\in G}$ of a groupoid G on a ring A is global, if and only if, $A_g = A_{r(g)}$ for every $g \in G$, see [4, Lema 1.1].*

The *skew groupoid ring $A \rtimes_\sigma G$* is defined as the set of all formal sums of the form $\sum_{g\in G} a_g\delta_g$ where $a_g \in A_{r(g)}$ for all $g \in G$. Multiplication is defined by the rule

$$(a\delta_g)(b\delta_h) = \begin{cases} a\sigma_g(b)\delta_{gh} & \text{if } (g,h) \in G^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

for $g,h \in G$ and $a \in A_{r(g)}$, $b \in A_{r(h)}$. Throughout this section, we assume $A := \oplus_{e\in G_0} A_e$.

Theorem 7.24. *Let G be a connected groupoid, let $\{A_e\}_{e\in G_0}$ be a collection of s-unital rings, $A := \oplus_{e\in G_0} A_e$ and let $\sigma = (A_{r(g)},\sigma_g)_{g\in G}$ be a groupoid global action. Then the skew groupoid ring $A \rtimes_\sigma G$ is prime if, and only if, there is some $e \in G_0'$ such that $A_e \rtimes_{\sigma^e} G_e^e$ is prime.*

Proof. It follows from Remark 7.10 (b) and Proposition 7.14. □

Proposition 7.25. *Let $\{A_e\}_{e\in G_0}$ be a collection of s-unital rings, $A := \oplus_{e\in G_0} A_e$ and let $\sigma = (A_{r(g)},\sigma_g)_{g\in G}$ be a groupoid global action. The following statements are equivalent:*

(i) G' is connected;

(ii) For every $e \in G'_0$, e is a support-hub;

(iii) For some $e \in G'_0$, e is a support-hub.

Proof. (i)⇒(ii). Let $e \in G'_0$ and let $a_g\delta_g \in A \rtimes_\sigma G$ be a nonzero element. Then $g \in G'$ and, by hypothesis, there is $k \in G'$ such that $s(k) = r(g)$ and $c(k) = e$. Notice that $a_g \in A_{r(g)} = A_{s(k)}$. By Lemma 7.11 (ii) ⇒ (iv), e is a support-hub. It is immediate that (ii) implies (iii). And, (iii) implies (i) follows from Proposition 6.23 (ii). □

Below, we combine the necessary and sufficient conditions of primeness of a skew groupoid ring.

Theorem 7.26. *Let G be a groupoid and let $\{A_e\}_{e\in G_0}$ be a collection of s-unital rings such that $A := \oplus_{e\in G_0} A_e$. Then, the following assertions are equivalent.*

(i) The skew groupoid ring $A \rtimes_\sigma G$ is prime;

(ii) A is G-prime, and for every $e \in G'_0$, $A_e \rtimes_{\sigma^e} G^e_e$ is prime;

(iii) A is G-prime, and for some $e \in G'_0$, $A_e \rtimes_{\sigma^e} G^e_e$ is prime;

(iv) $A \rtimes_\sigma G$ is graded prime and for every $e \in G'_0$, $A_e \rtimes_{\sigma^e} G^e_e$ is prime;

(v) $A \rtimes_\sigma G$ is graded prime, and for some $e \in G'_0$, $A_e \rtimes_{\sigma^e} G^e_e$ is prime;

(vi) G' is connected, and for every $e \in G'_0$, $A_e \rtimes_{\sigma^e} G^e_e$ is prime;

(vii) G' is connected, and for some $e \in G'_0$ such that $A_e \rtimes_{\sigma^e} G^e_e$ is prime.

Proof. It follows from Theorem 7.8 and Proposition 7.25. □

Corollary 7.27. *Let $A \rtimes_\sigma G$ be an s-unital skew groupoid ring such that A is G-prime or G is connected. Suppose that there is $e \in G'_0$ such that G^e_e is torsion-free. Then $A \rtimes_\sigma G$ is prime if, and only if, A_e is G^e_e-prime.*

Proof. It follows from Remark 7.10 (b) and Theorem 7.17. □

7.3 GROUPOID RINGS

Let R be an s-unital ring and let G be a groupoid. The groupoid ring $R[G]$ consists of elements of the form $\sum_{g\in G} a_g\delta_g$ where $a_g \in R$ for every $g \in G$. For $g,h \in G$ and $a,b \in R$, the multiplication in $R[G]$ is defined as $a\delta_g \cdot b\delta_h := ab\delta_{gh}$, if g,h are composable, and 0 otherwise.

Remark 7.28. *Let R be an s-unital ring and let G be a groupoid. Consider the global action $\sigma = (A_g, \sigma_g)_{g\in G}$ of G on A, defined by letting $A_g := R$ and $\sigma_g := \mathrm{id}_R$ for every $g \in G$, and $A := \oplus_{e\in G_0} A_e = \oplus_{e\in G_0} R$. Notice that the corresponding skew groupoid ring $A \rtimes_\sigma G$ is isomorphic to the groupoid ring $R[G]$.*

The following theorem generalizes [40, Theorem 12.4].

Theorem 7.29. *Let R be a nonzero s-unital ring and let G be a groupoid. The following assertions are equivalent.*

(i) The groupoid ring $R[G]$ is prime;

(ii) G is connected and there is some $e \in G_0$ such that $R[G_e^e]$ is prime;

(iii) G is connected and, for every $e \in G_0$, $R[G_e^e]$ is prime;

(iv) G is connected, R is prime and there is some $e \in G_0$ such that G_e^e has no non-trivial finite normal subgroup;

(v) G is connected, R is prime and, for every $e \in G_0$, G_e^e has no non-trivial finite normal subgroup.

Proof. Notice that $G' = G$. The proof follows from Theorem 7.26 and [40, Theorem 12.4]. $\square$

A subset $X \subseteq G_0$ is said to be *R-dense* if for every nonzero $a = \sum_{g \in F} a_g \delta_g \in R[G]$ there is $g \in F$ such that $a_g \neq 0$ and $s(g) \in X$. In particular, we have that:

Proposition 7.30. *Let G be a groupoid and let R be a unital commutative ring. Then G is connected if, and only if, there is some $e \in G_0$ such that $\mathcal{O}_e := \{f \in G_0 : \exists g \in G, s(g) = e, r(g) = f\}$ is R-dense.*

Proof. Suppose that there is $e \in G_0$ such that $\mathcal{O}_e$ is R-dense. Let $f, h \in G_0$. Notice that $1\delta_f$ and $1\delta_h \in R[G]$. Since $\mathcal{O}_e$ is R-dense, there is $g, k \in G$ such that $s(g) = s(k) = e$, $r(g) = s(f)$ and $r(k) = s(h)$. Notice that $hkg^{-1} \in G$, $s(hkg^{-1}) = s(g^{-1}) = r(g) = s(f)$ and $r(hkg^{-1}) = r(h)$. Therefore, G is connected. For the converse, let $e \in G_0$ and $0 \neq a = \sum_{g \in F} a_g \delta_g \in R[G]$. Then, there is $g \in F$ such that $a_g \neq 0$. Since G is connected, there is $h \in G$ such that $s(h) = e$ and $r(h) = s(g)$. Therefore, $s(g) \in \mathcal{O}_e$. $\square$

A commutative ring with unit is prime if and only if it is an integral domain. Therefore, in the case where A is a commutative and unital ring, an analogous result was proved in the context of Steinberg algebras of discrete groupoids, see [54, Remark 4.10]. We recover [55, Proposition 4.4] and [55, Theorem 4.9] in the specific case of groupoid rings in the next theorem.

Theorem 7.31. *Let G be a groupoid and let R be a unital commutative ring. The following statements are equivalent:*

(i) $R[G]$ is prime;

(ii) There is some $e \in G_0$ such that $\mathcal{O}_e$ is R-dense and $R[G_e^e]$ is prime;

(iii) There is some $e \in G_0$ such that $\mathcal{O}_e$ is R-dense, R is an integral domain and G_e^e has no finite non-trivial normal subgroups.

Proof. It follows from Proposition 7.30 and Theorem 7.29. □

BIBLIOGRAPHY

[1] F. Abadie. “On partial actions and groupoids”. In: *Proc. Amer. Math. Soc.* 132.4 (2004), pp. 1037–1047.

[2] D. Bagio, Sant’Ana A., and Tamusiunas T. “Galois correspondence for group-type partial actions of groupoids.” In: *Bull. Belg. Math. Soc. Simon Stevin.* 28.5 (2022), pp. 745–767.

[3] D. Bagio, D. Flores, and A. Paques. “Partial actions of ordered groupoids on rings.” In: *J. Algebra Appl.* 9.3 (2010), pp. 501–517.

[4] D. Bagio and A. Paques. “Partial groupoid actions: globalization, Morita theory, and Galois theory.” In: *J. Comm. Algebra* 40.10 (2012), pp. 3658–3678.

[5] D. Bagio, A. Paques, and H. Pinedo. “On partial skew groupoid rings.” In: *International Journal of Algebra and Computation* 31.1 (2021), pp. 1–7.

[6] V. Beuter. “Partial actions of inverse semigroups and their associated algebras”. PhD thesis. Brazil: Universidade Federal de Santa Catarina, 2018.

[7] V. Beuter and D. Gonçalves. “The interplay between Steinberg algebras and skew rings.” In: *J. Algebra* 497 (2018), pp. 337–362.

[8] G. Boava, G. de Castro, D. Goonçalves, and D. van Wyk. “Leavitt path algebras of labelled graphs”. arXiv preprint arXiv:2106.06036. 2021.

[9] G. Boava, G. de Castro, and F. Mortari. “Inverse semigroups associated with labelled spaces and their tight spectra.” In: *Semigroup Forum* 94.3 (2017), pp. 582–609.

[10] J. Brown, L. Clark, C. Farthing, and A. Sims. “Simplicity of algebras associated to étale groupoids”. In: *Semigroup Forum* 88.2 (2014), pp. 433–452.

[11] A. Buss and R. Exel. “Inverse semigroup expansions and their actions on C^*-algebras”. In: *Illinois J. Math.* 56.4 (2012), pp. 1185–1212.

[12] T. M. Carlsen and E. J. Kang. “Condition (K) for Boolean dynamical systems”. In: *J. Aust. Math. Soc.* 112.2 (2022), pp. 145–169.

[13] G. de Castro, D. Gonçalves, and D. van Wyk. “Topological full groups of ultragraph groupoids as an isomorphism invariant.” In: *Münster J. Math.* 14.1 (2021), pp. 165–189.

[14] G. de Castro, D. Gonçalves, and D. W. van Wyk. “Ultragraph algebras via labelled graph groupoids, with applications to generalized uniqueness theorems”. In: *J. Algebra* 579 (2021), pp. 456–495.

[15] G. de Castro and D. van Wyk. “Labelled space C^*-algebras as partial crossed products and a simplicity characterization”. In: *J. Math. Anal. Appl.* 491.1 (2020), p. 124290.

[16] G. G. de Castro and E. J. Kang. “C^*-algebras of generalized Boolean dynamical systems as partial crossed products”. arXiv preprint arXiv:2202.02008. 2022.

[17] L. Clark, C. Edie-Michell, A. Huef, and A. Sims. “Ideals of Steinberg algebras of strongly effective groupoids, with applications to Leavitt path algebras”. In: *Trans. Amer. Math. Soc.* 371.8 (2019), pp. 5461–5486.

[18] I. Connell. “On the group ring”. In: *Canadian J. Math.* 15 (1963), pp. 650–685.

[19] L. Cordeiro, D. Gonçalves, and R. Hazrat. “The talented monoid of a directed graph with applications to graph algebras”. In: *Rev. Mat. Iberoam.* 38.1 (2022), pp. 223–256.

[20] M. Dokuchaev. “Recent developments around partial actions”. In: *São Paulo J. Math. Sci.* 13.1 (2019), pp. 195–247.

[21] M. Dokuchaev and R. Exel. “Associativity of crossed products by partial actions, enveloping actions and partial representations”. In: *Trans. Amer. Math. Soc.* 357.5 (2005), pp. 1931–1952.

[22] M. Dokuchaev and M. Khrypchenko. “Partial cohomology of groups”. In: *J. Algebra* 427 (2015), pp. 142–182.

[23] T. Duyen, D. Gonçalves, and T. Nam. “On the ideals of ultragraph Leavitt path algebras”. 2021.

[24] R. Exel. “Cuntz-Krieger algebras for infinite matrices”. In: *J. Reine Angew. Math.* 512 (1999), pp. 119–172.

[25] R. Exel. “Inverse semigroups and combinatorial C^*-algebras”. In: *Bull. Braz. Math. Soc. (N.S.)* 39 (2008), pp. 191–313.

[26] R. Exel. *Partial dynamical systems, Fell bundles and applications*. Vol. 224. American Mathematical Society., 2017.

[27] F. Flores and M. Mantoiu. “Topological Dynamics of Groupoid Actions”. 2020.

[28] N. Gilbert. “Actions and expansions of ordered groupoids”. In: *J. Pure Appl. Algebra* 198.1-3 (2005), pp. 175–195.

[29] T. Giordano and A. Sierakowski. “Purely infinite partial crossed products”. In: *J. Funct. Anal.* 266.9 (2014), pp. 5733–5764.

[30] D. Goçalves and D. Royer. “Ultragraphs and shift spaces over infinite alphabets.” In: *Bull. Sci. Math.* 141.1 (2017), pp. 25–45.

[31] D. Gonçalves, J. Öinert, and D. Royer. “Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics”. In: *J. Algebra* 420 (2014), pp. 201–216.

[32] D. Gonçalves and D. Royer. “Leavitt path algebras as partial skew group rings”. In: *Comm. Algebra* 42.8 (2014), pp. 3578–3592.

[33] D. Gonçalves and G. Yoneda. “Free path groupoid grading on Leavitt path algebras”. In: *Internat. J. Algebra Comput.* 26.6 (2016), pp. 1217–1235.

[34] Daniel Gonçalves and Danilo Royer. “Simplicity and chain conditions for ultragraph Leavitt path algebras via partial skew group ring theory”. In: *J. Aust. Math. Soc.* 109.3 (2020), pp. 299–319.

[35] R. Hazrat. “The dynamics of Leavitt path algebras”. In: *J. Algebra* 384 (2013), pp. 242–266.

[36] R. Hazrat. “The graded Grothendieck group and the classification of Leavitt path algebras”. In: *Math. Ann.* 355.1 (2013), pp. 273–325.

[37] M. Imanfar, A. Pourabbas, and H. Larki. “The Leavitt Path Algebras of Ultragraphs”. In: *Kyungpook Math. J.* 60.1 (2020), pp. 21–43.

[38] T. Katsura, Muhly P., A. Sims, and M. Tomforde. “Ultragraph C^*-algebras via topological quivers”. In: *Studia Math.* 187.2 (2008), pp. 137–155.

[39] K. Keimel. “Algèbres commutatives engendrées par leurs éléments idempotents”. In: *Canadian J. Math.* 22.5 (1970), pp. 1071–1078.

[40] D. Lännström, P. Lundström, J. Öinert, and S. Wagner. “Prime group graded rings with applications to partial crossed products and Leavitt path algebras”. 2021.

[41] D. Lännström and J. Öinert. “Graded von Neumann regularity of rings graded by semigroups”. 2022.

[42] M. Lawson. *Inverse semigroups: the theory of partial symmetries*. World Scientific Publishing Co., River Edge, NJ, 1998.

[43] M. Lawson. "Non-commutative Stone duality: inverse semigroups, topological groupoids and C^*-algebras". In: *International Journal of Algebra and Computation* 22.6 (2012), p. 1250058.

[44] K. Lorensen and J. Öinert. "Generating numbers of rings graded by amenable groups". 2022.

[45] P. Lundström and J. Öinert. "Skew category algebras associated with partially defined dynamical systems". In: *Internat. J. Math.* 23.4 (2012), p. 1250040.

[46] P. Nystedt. "A survey of s-unital and locally unital rings". In: *Rev. Integr. Temas Mat.* 37.2 (2019), pp. 251–260.

[47] P. Nystedt and J. Öinert. "Group gradations on Leavitt path algebras". In: *Journal of Algebra and its Applications* 19.9 (2020), p. 2050165.

[48] P. Nystedt and J. Öinert. "Simple skew category algebras associated with minimal partially defined dynamical systems". In: *Discrete Contin. Dyn. Syst.* 33.9 (2013), pp. 4157–4171.

[49] P. Nystedt, J. Öinert, and H. Pinedo. "Epsilon-strongly groupoid-graded rings, the Picard inverse category and cohomology". In: *Glasg. Math. J.* 62.1 (2020), pp. 233–259.

[50] J. Öinert. "Units, zero-divisors and idempotents in rings graded by torsion-free groups". 2019.

[51] J. Öinert and P. Lundström. "The ideal intersection property for groupoid graded rings". In: *Comm. Algebra* 40.5 (2012), pp. 1860–1871.

[52] D. Passman. "Infinite crossed products and group-graded rings." In: *Trans. Amer. Math. Soc.* 284.2 (1984), pp. 707–727.

[53] A. Sims, G. Szabó, and D. Williams. *Operator algebras and dynamics: groupoids, crossed products, and Rokhlin dimension*. Springer, 2020.

[54] B. Steinberg. "A groupoid approach to discrete inverse semigroup algebras." In: *Adv. Math.* 223.2 (2010), pp. 689–727.

[55] B. Steinberg. "Prime étale groupoid algebras with applications to inverse semigroup and Leavitt path algebras". In: *J. Pure and Appl. Algebra* 223.6 (2019), pp. 2474–2488.

[56] B. Steinberg. “Simplicity, primitivity and semiprimitivity of étale groupoid algebras with applications to inverse semigroup algebras”. In: *J. Pure Appl. Algebra* 220.3 (2016), pp. 1035–1054.

[57] M. Tomforde. “A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs”. In: *J. Operator Theory* 50.2 (2003), pp. 345–368.

[58] H. Tominaga. “On *s*-unital rings.” In: *Math. J. Okayama univ.* 18 (1976), pp. 117–134.

[59] L. Vas. “Every graded ideal of a Leavitt path algebra is graded isomorphic to a Leavitt path algebra”. In: *Bull. Aust. Math. Soc.* 105.2 (2022), pp. 248–256.

[60] L. Vas. “Graded irreducible representations of Leavitt path algebras: A new type and complete classification”. In: *J. Pure Appl. Algebra* 227.3 (2023), p. 107213.