2017 - "Dig where you stand" 4. Proceedings of the Fourth International Conference on the History of Mathematics Education


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Proceedings of the Fourth International Conference on the History of Mathematics Education September 23-26, 2015, at University of Turin, Italy

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Edizioni Nuova Cultura

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In the front cover: A classroom of a primary school (rural area of Northern Italy in the 1940s)
In the back cover: View of Turin, venue of the conference

These proceedings are published with the contribution of INdAM
(Istituto Nazionale di Alta Matematica)

ISBN: 9788868128647
DOI: 10.4458/8647

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Contents

Introduction

From 23 to 26 September 2015 the fourth International Conference on the History of Mathematics Education (ICHME-4) was held at Academy of Sciences and University of Turin, Italy. The local organizers were Livia Giacardi and Erika Luciano. The Scientific Program Committee was composed by Kristín Bjarnadóttir (University of Iceland), Fulvia Furinghetti (University of Genoa, Italy), Livia Giacardi (University of Turin, Italy), Erika Luciano (University of Turin, Italy), Johan Prytz (Uppsala University), Gert Schubring (Universität Bielefeld, Germany/Universidade Federal do Rio de Janeiro, Brazil), with the scientific support of Ferdinando Arzarello (University of Turin, Italy), president of ICMI.

Altogether there were 51 participants from 16 countries, 44 contributions (research reports and posters) were presented. After processing by peer reviews, 28 papers are published in these Proceedings. They may be categorized according to the following thematic dimensions:

Ideas, people and movements
Kristín Bjarnadóttir; Kajsa Bråting and Tove Österman; Livia Giacardi and Margherita Raspitzu; Erika Luciano; Gert Schubring; Harm Jan Smid.

Transmission of ideas
Fulvia Furinghetti; Jan Guichelaar; Alexander Karp; Jenneke Krüger; Antonio M. Oller-Marcén, Chiara Pizzarelli.

Teacher education
Rui Candeias; Marta Menghini.

Geometry and textbooks
Evelyne Barbin; Gregg De Young; Guillaume Moussard; Johanna Pejlare.

Textbooks – changes and origins
Andreas Christiansen; Veronica Gavagna; Miguel Picado, Luis Rico, and Bernardo Gómez; Johan Prytz.

Curriculum and reforms
Jeremy Kilpatrick; Leo Rogers.

Teaching in special institutions
Rita Binaghi; Elisa Patergnani.

Teaching of geometry
Geert Vanpaemel and Dirk De Bock; João Pedro Xavier and Eliana Manuel Pinho.

To emphasize the continuity of the project behind the conference held in Turin the volume containing the proceedings keeps the original title of the first conference, i.e. "*Dig where you stand*" (followed by *4*, which is the number of the conference). This sentence, which is the English title of the book *Gräv där du står* (1978) by the Swedish author Sven Lindqvist, stresses the importance of knowing the historical path that brought us to the present. As we explained in the Introduction of the Proceedings of ICHME-3 we deem that "Dig where you stand" may be a suitable motto for those (historians, educators, teachers, educationalists) who wish to sensitively and deeply understand the teaching and learning of mathematics.

The editors:
Kristín Bjarnadóttir, Fulvia Furinghetti, Marta Menghini,
Johan Prytz, Gert Schubring

Preface

The volume of the Fourth International Conference on the History of Mathematics Education (ICHME4) shows the international and scientific relevance that this topic has reached in the community of mathematics educators.

Just over a decade has passed since July 2004, when in Copenhagen, at ICME 10, the Topic Study Group 29 about *The History of Learning and Teaching Mathematics* proved that the history OF mathematics education had become a well-established area of research.

The interest for history IN mathematics education had been always present in most of conferences and in many researches within mathematics education: in 1976 *The international study group on the relations between the history and pedagogy of mathematics* (HPM) was founded as an affiliated study group of ICMI.

However the change from IN to OF in the title stresses a relevant shift, according to which the historical data are looked at and investigated. It also realizes a change of paradigm in this topic.

Two issues seem to me featuring this new approach.

First the narrative within which the story telling of the researches is formulated has reversed the standpoint: historical data are not any longer framed according to a logic that obeys to pedagogical or strictly didactical goals. On the contrary the paradigm is typically declined according to an historical point of view, where what we could call outer dimensions (e.g. the institutional ones, in a wide sense of the word) are more present.

The second is a consequence of the first: the shift of paradigm, organized according to the new basis, allows to put forward fresh "problématiques", not so present in the previous studies of the history in mathematics education.

This constitutes an enlargement and a richness that this type of researches offers to the whole community of researchers and practitioners, but it could also be useful to policy makers so that they can take their decisions with more information about the whole frame of mathematics education.

For example, a type of problem that as ICMI community we face every day concerns the way developing countries approach and design their school policy and particularly Science and Mathematics education for their students as a tool of sustaining their economical and social progress. For them it is essential equipping the students with those mathematical skills that will enable them to compete effectively in the global market. But this will be possible only avoiding the dangers of the alienation generated by the loss of the variety of cultural richness existing in the different regions of the world.

For these reasons I like very much the motto that features the ICHME conferences: "*Dig where you stand*".

The motto is particularly interesting for ICMI, since ICMI stands everywhere (it has 93 countries as members or associate members): looking at the motto from an ICMI perspective puts forward its dramatic meaning.

It is only digging where we stand that we can grasp the network of different forces, instances, traditions, which are behind and within the educational designs all over the world: we need the contributions of all scholars for this, each digging where she/he stands.

However this research must be common and shared: in the era of globalisation each one's digging is linked the others' digging and the connections and specificities of such investigations are crucial to grasp what is globally happening in the field of education.

The studies of ICHME can accomplish this double aim: from the one side, they bring the rigour of detailed researches on a specific subject, but from the other side they create the plowed ground where we possibly can find the rationale of more general events, which span over longer periods, and whose sense we can possibly glimpse through the lens of the findings at a more specific level. There is a dialectic that recalls me the distinction made by Fernand Braudel between the "eventful history of short, rapid, nervous oscillations" and the "almost immobile history of man's relations with the milieu surrounding him"[1].

In our times of fast changes the history is possibly not any longer so immobile as pointed out by Braudel more than sixty years ago, but the dialectic between the two aspects of historical phenomena and the consequent historical methodology of research is still alive. And we, as mathematics educators, are interested in both sides of the coin. They embody different levels and instances to which mathematics educators must be very attentive, particularly those working within an international context and attentive to the instances of renewing mathematics teaching and learning, for example pushing forward the necessity of new curricula.

In fact, in recent years, globalization of economy, universality of technological development and related needs for manpower skills play the role of strong historical motivations for a reform that should bring to unified standards for mathematics in school all over the world.

But from the other side, only a multiple cultural perspective allows to take into account:

- the existence of different epistemological and cultural positions concerning mathematics and its relevance in the culture;

[1] Braudel, Fernand (1949). *La Méditerranée et le monde méditerranéen a l'époque de Philippe II*, 3 Vols. Paris: A. Colin.

- the possible, cultural distance of proposed curricular reforms from the mathematical culture of the different countries;
- the relationships with the culture and the personal contributions brought by the students in the classroom, so relevant to avoid the students' alienation from their cultural environment and to allow students to engage in learning in a productive way.
- the importance of anchoring professional development in teachers' activities.

Many documents of UN and of UNESCO underline the necessity of developing new curricula that make it possible acquiring a common mathematical toolkit to deal with technology, quantitative and graphical information provided by media, problem solving and decision making in the workplace and in ordinary life.

But from the other side, they also pinpoint crucial issues to be dealt with in order to avoid the dangers both of cultural refusal of the innovation, and of cultural alienation and of loss of the cultural richness existing in the different regions of the world.

A volume like this one, containing the contributions of 34 scholars, whose papers focus on historical data concerning the teaching/learning of mathematics in different regions of the world and in different historical periods, constitutes an important contribution for understanding the links between the eventful and the long duration history in our field and this will certainly contribute at least to avoid mistakes in decisions concerning the new mathematics curricula.

For this I am honoured to write this short preface and express my deep gratitude to the Scientific Committee that made possible ICHM4 and to all people who contributed to the production of this wonderful volume.

Turin, December 31, 2016

Ferdinando Arzarello
(President of ICMI)

The role of Reye's *Geometrie der Lage* in the teaching of "modern geometry"

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Abstract

In 1866, nearly twenty years after the edition of the Geometrie der Lage by von Staudt, Theodor Reye published the first volume of a textbook with the same title, which was the course given by him at the Polytechnic school of Zurich. Our paper gives an outline of the pedagogical basis of Reye's textbook, specially the place of problems and the learning of visualization, then it shows the impact of this textbook in the teaching of "modern geometry" in secondary school in several countries, particularly in Italy, Germany and United States. Indeed, for teachers, authors or translators, it was an interesting geometry for beginners, because it leans on visualization and it permits an early presentation of duality in teaching.

Introduction: von Staudt's *Geometrie der Lage* and its readers

In his *Geometrie der Lage,* edited in 1847, Karl Christian von Staudt established the geometry of position as an autonomous science without measure. In the preface, he wrote that his project also concerns the teaching of geometry in general. Indeed, this teaching has to proceed from general considerations, which exercise intuition, like the principle of duality that does not need measure. He wrote: "Each teacher who draws the attention of his students on the principle of duality, makes the experience of the fact that the principle is, for a student who is sensible to geometry, more stimulating than any particular theorem" (von Staudt, 1847, p. iv). With his book, he hoped to encourage teachers to place the essentials of the geometry of position before the geometry of measure. His *Beiträge zur Geometrie der Lage* had been edited in three parts (1856, 1857, 1860).

His treatise did not meet success: there are two editions only, the last one in 1870 (Hartshorne, 2008). The translations of the *Geometrie der Lage* appeared later, as in Italian by Mario Pieri (1887). Why were there no new editions during thirty years? An answer given often concerns von Staudt's style, considered as arid because the text contains no figures and no comments. Hermann Hankel wrote: "Classical in its singularity, this work is a great attempt made with the aim to submit the nature, whose essence is to manifest itself under a thousand of various

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

forms, to the uniformity of an abstract and systematic schematization", but he mentioned the "excellent" Reye's book (Hankel, 1885, p. 237).

Why an edition in 1870? This date corresponds to works on the foundations of projective geometry by Hilbert, Pieri and others. Also, in this period, Felix Klein used the projective coordinates of the *Beiträge zur Geometrie der Lage* for non-Euclidean geometries (Klein, 1871) and Reye's tetrahedral complex (Rowe, p. 231). He also criticized von Staudt's proof of the fundamental theorem of projective geometry and this attracted mathematicians' attention to von Staudt's books. He wrote later about them: "These books contain an extraordinary wealth of ideas in a gapless form, rigidified almost to the point of lifelessness – a form corresponding to Staudt's thorough, systematic nature and to his age: he was already 63 when he finished the second work. I myself have always found his manner of exposition completely inaccessible" (Klein, 1928, p. 122).

Nevertheless, from the years 1860, von Staudt's books began to interest teachers of graphical methods. The first one was Karl Culmann, who used the works of Poncelet and von Staudt to apply the graphical methods to the problems treated by engineers. He wrote in the French edition of *Die Graphische Statik*: "When we have been appointed at the time of the creation of the Polytechnic school of Zurich in 1855, we have been obliged to introduce Poncelet's graphical methods […]. That's how, so to speak irresistibly, we have been led to replace algebra as much as possible by the geometry of position" (Culmann, 1880, p. x-xi). In 1863, Theodor Reye became the assistant of Culmann in Zurich, and that led him to write a new version of the geometry of position.

Reye's project

Theodor Reye was born in 1838 in Germany, he studied mechanics for the engineers in the Polytechnic school of Hanover, then in Zurich where Culmann was his professor. In 1861, he defended his thesis in Göttingen and then his *Habilitation* thesis in Zurich. He became lecturer of Culmann's in the Polytechnic school of Zurich in 1863, then professor in 1870. Reye published his *Die Geometrie der Lage: Vorträge* in two volumes in 1866 and 1868. He became professor of mathematics in the (German) University of Strasbourg in 1872 and taught here until 1918. He published numerous papers and his *Synthetische Geometrie der Kugeln und linearen Kugelsystem* in 1879. He died in 1919.

A pedagogical project

In the preface of his *Geometrie der Lage*, Reye explained the necessity to write a textbook intended for his students by these words:

> Von Staudt's own work, evidently not written for a beginner, embodies peculiarities which are praiseworthy enough in themselves, but which essentially increase the difficulties of the study. It is especially marked by a scantiness of expression, and a very condensed, almost laconic, form of statement; nothing is said except what is absolutely necessary, rarely is there a word of explanation given, and it is left to the student to form for himself suitable examples illustrative of the theorems, which are enunciated in their most general form. […] The presentation is so abstract that ordinarily the energies of a beginner are quickly exhausted by his study (Reye, 1898, p. xii-xiii, translation by Holgate).

But his textbook is not a simple adaptation of von Staudt's treatise to a special audience. Indeed, the first goal was "to supply the want of a textbook which offers to the student the necessary material in concise form" (Reye, 1866, p. xi). But there is another goal, linked with the strong interest of Reye for the geometry developed by von Staudt. For him, this geometry has the property to exercise the imagination in the student:

> One principal object of geometrical study appears to me to be the exercise and the development of the power of imagination in the student, and I believe that this object is best attained in the way in which von Staudt proceeds. That is to say, von Staudt excludes all calculations whether more or less complicated which make no demands upon the imagination, and to whose comprehension there is requisite only a certain mechanical skill having little to do with geometry in itself; and instead, arrives at the knowledge of the geometric truths upon which he bases the geometry of position by direct visualization. In order to make the accomplishment of this end easier for the student, I have added plates of diagrams to my lectures (Reye, 1898, p. xii).

The idea of Reye was to associate the geometry of von Staudt with that of Jacob Steiner. These two mathematicians had the same idea, that is to present the principle of duality in the beginning of geometry as a consequence of the relations between points, straight lines and planes. Reye stressed on the important advantage of von Staudt' s method from this point of view: "it turns to account most beautifully, in all its clearness and to its full compass, the important and fruitful principle of duality or reciprocity, by which the whole geometry of position is controlled" (Reye, 1898, p. xiv). But, for him, it was not a reason to abolish the measure: "Metric relations, I must add, especially those of the conic sections, have by no means been neglected in my lectures, on the contrary, I have throughout developed these relations to a greater extent than did either Steiner or von Staudt, wherever they could naturally present themselves as special cases of general theorems" (Reye, 1898, p. xiv). His textbook goes from general theorems inspired by von Staudt to applications in terms of metric relations. For his purpose, visualization of figures has an important role. But more, he introduced a new way to see the figures.

A new visualization of figures

In the introduction, Reye gave an inaugural problem to show that, when metric relations are not considered, then theorems and problems can be more general and comprehensive. The problem is to draw a third straight line through the inaccessible point of intersection of two straight lines *a* and *b*. The metric geometry solves it by using proportional segments but the geometry of position affords a simpler solution. If we take some point *P* outside the two given straight lines and pass through it transversals, then the points of intersection of the diagonals in each of the quadrangles formed by two of these transversals taken with the lines *a* and *b* lie upon one straight line which passes through the intersection of *a* and *b* (Fig. 1). A "very simple proof" of this result uses the harmonic property of a quadrangle: if we choose three points *A*, *B*, *C*, upon a straight line, and construct any quadrangle such that two opposite sides pass through *A*, one diagonal through *B*, and the other two opposite sides through *C*, then the second diagonal meets the line *ABC* in a perfectly definite fourth point *D* independently of the considered quadrangle (Fig. 2).

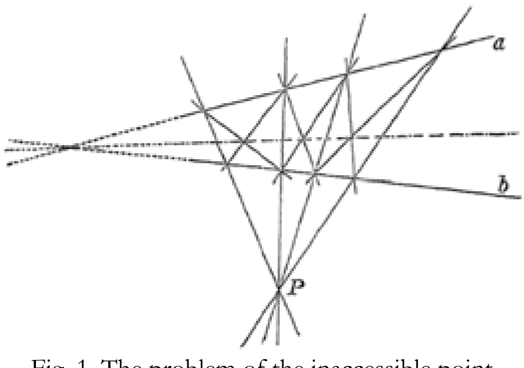

Fig. 1. The problem of the inaccessible point

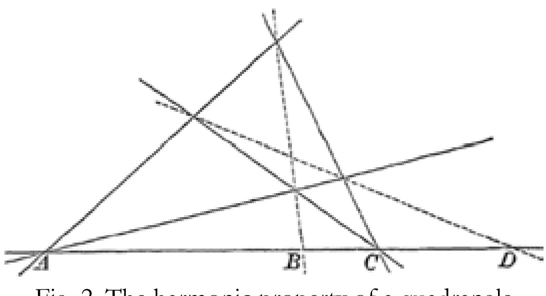

Fig. 2. The harmonic property of a quadrangle

Reye pointed out the new visualization involved in the geometry of position with the example of a "theorem on triangles". If two triangles ABC and $A_1B_1C_1$ are such that AA_1, BB_1, CC_1 intersect in S then the three points of intersection of AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 lie upon one straight line u (Fig. 3).

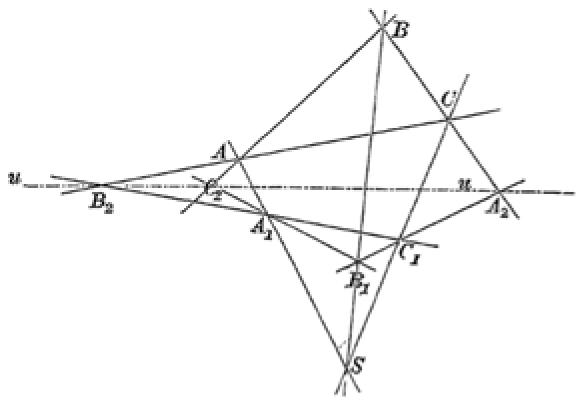

Fig. 3. A theorem on triangles

But further, Reye explained that it is not Desargues' theorem exactly. Indeed, in the figure considered by the geometry of position, we have not to see triangles, but only straight lines and their intersections. He wrote: "the diagram illustrating this theorem is worthy of notice as representing a class of remarkable configurations characterized by a certain regularity of form. It consists of ten points and ten straight lines; three of the ten points lie upon each of the straight lines, and three of the ten lines pass through each of the points" (Reye, 1898, p. 5). He introduced the word "configuration" to specify this kind of relation in the second edition in 1876.

Perspective or central projection plays an important part in the visualization because "we are able to discover, and at the same time can prove, important theorems through mere visualization, with the help of the methods of projection and section" (Reye, 1898, p. 10). All through the textbook, these methods give a general context to make comprehensive the notions and the proofs.

Reye's geometry of position

The textbook contains fifteen lectures. It begins with the method of projection and section and with the definition of the six primitive forms. The infinitely distant elements are introduced in the second lecture at the same time at the notion of correlative primitive forms. The principle of duality is introduced as soon as the third lecture. It is followed by the correlation between harmonic forms where metric relations appears in terms of proportions.

Correlation and duality

Point, straight line and plane are the simple elements of the geometry of position. Primitive forms of the first grade are "a range of points", "a sheaf of lines" and "a sheaf of planes". The first one is the totality of points lying upon one straight line (its basis), the second is the totality of rays lying in one plane and passing through one point (is basis) and the third is the totality of planes passing through one straight line (its basis). Primitive forms of the second grade are "a plane system" and "a bundle of rays" (translation by Holgate). The first one is the totality of points and lines contained in a plane and the second is the totality of rays and planes passing through any point in space. The primitive form of the third grade is "the space system", with all points, lines, and planes.

The geometry of position is based on the notion of correlation between primitive forms. Two forms are said correlated to each other if every element of the one is associated to an element of the other. Reye explained that two primitive forms are correlated to each other in the simplest and clearest manner by making the one a section or a projector of the other. The first example is a sheaf of rays *S* correlated with a range of points *u*. Here, the parallel *p* of *S* corresponds to the infinitely distant point of *u* (Fig. 4). Correlated primitive forms can be correlated as sections or projectors of the same third form. In a first example, Reye considered two sheaves of rays *S* and *S1* correlated as projectors of a range of points (Fig. 5). In a second example, two ranges of points *u* and u_1 are correlated as sections of a sheave of rays *S* and the infinitely distant point *P* corresponds to P_1 while the infinitely distant point Q_1 corresponds to *Q* (Fig. 6).

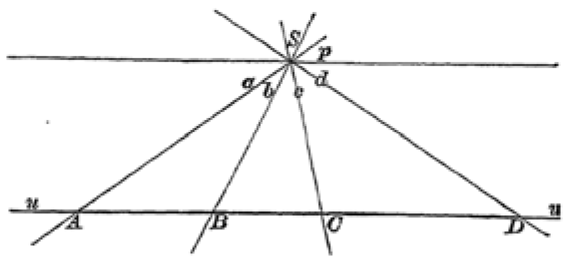

Fig. 4. A sheaf of rays *S* correlated with a range of points *u*

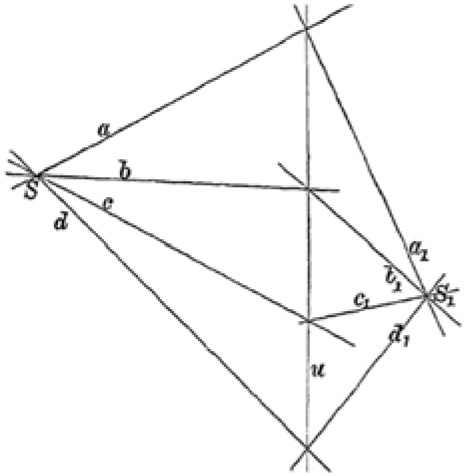

Fig. 5. Two correlated sheaves of rays S and S_1

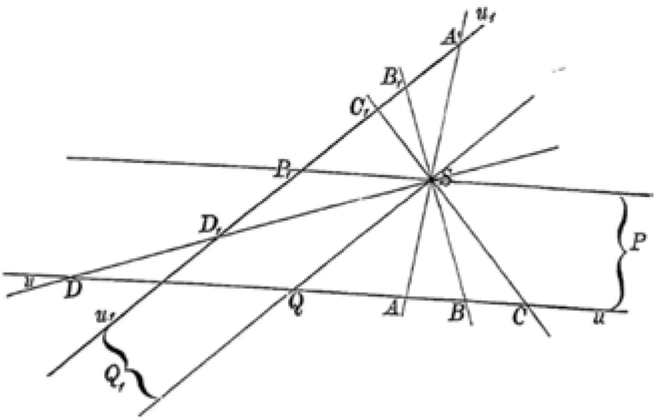

Fig. 6. Two correlated ranges of points u and u_1

Reye gave a commentary before introducing duality: "I must make mention of a geometrical principle which will occupy an important place in these lectures. This principle very greatly simplifies the study of the geometry of position, in that it divides the voluminous material of the subject into two parts, and sets these over against each other in such a way that the one part arises immediately out of the other" (Reye, 1898, p. 25). Next, he gave many examples of duality where he used the two columns introduced by Gergonne. In particular, duality permits us to obtain two ways to visualize a figure (Fig. 7):

A complete quadrangle *ABCD* has six sides; any two of these sides which do not pass through the same vertex are opposite sides of the quadrangle, so that in a quadrangle there are three pairs of opposite sides.

A complete quadrilateral *abcd* has six vertices; any two of these vertices which do not lie upon the same side are opposite vertices of the quadrilateral, so that in a quadrilateral there are three pairs of opposite vertices.

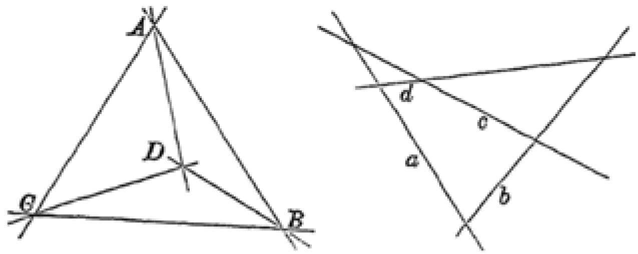

Fig. 7. Complete quadrangle and complete quadrilateral in duality

He used the figure of two complete quadrangles correlated to each other, to define four harmonic points *A, B, C, D*, and to show that three points *A, B, C* of a straight line and the order of their succession completely determine the fourth harmonic point *D* (Reye, 1898, pp. 37-38).

Projective properties of primitive forms

Reye listed four simple methods to correlate two primitive forms of the first grade. Firstly, when a sheaf of rays or planes and a range of points, or a sheaf of planes and a sheaf of rays are such that each element of the latter lies upon the corresponding element of the former. Secondly, when two ranges of points are sections of the same sheaf of rays. Thirdly, when two sheaves of rays are projectors of the same range of points, or sections of the same sheaf of planes, or both. Fourthly, when two sheaves of planes are projectors of the same sheaf of rays. In these cases, two primitive forms are said "perspective to each other". Two primitive forms are said to be "projective to each other" when to every set of four harmonic elements correspond four harmonic elements in the other.

The case of two superposed projective primitive forms (which have the same basis) is commented by these words: "the investigation as to how many self-corresponding elements may exist in two projective one-dimensional primitive forms which are superposed, that is, how many elements of one form coincide with their homologous elements in the other, is of great importance in all that follows" (Reye, 1898, p. 55). Thus, if two sheaves of rays S_1 and S_2 are projectors of the same range of points *u* and are intersected by a straight line *v*, then there are determined upon this straight line v two projective ranges of points u_1 and u_2, in which the intersections $P_1$and P_2 are self-corresponding points (Fig. 8).

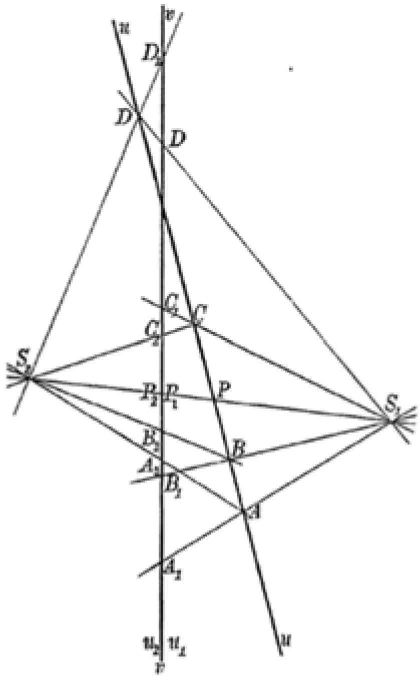

Fig. 8. Self-corresponding elements

This permits him to state the fundamental theorem of the projective geometry: "If two projective one-dimensional primitive forms have three self-corresponding elements *A*, *B*, *C*, then are all their elements self-corresponding and the forms are consequently identical" (Reye, 1898, p. 57). Reye gave a proof ad absurdum for the case of two ranges of points, where he used the motions of points. Then he extended to sheaves of rays and sheaves of planes.

Curves and cones of the second order are introduced in the sixth lecture. The study of these curves lies on the consideration of their tangents and on the duality between curves and sheaves of rays of second order. The study continues with Pascal and Brianchon's theorems, then with the notions of pole and polar. The diameters and axes of curves of second order are introduced in the ninth lecture before the algebraic equations of these curves. The involution is introduced in the twelfth lecture, and then its metric relations.

Three features characterize Reye's style: it contains many comments about ins and outs, the whys and the how, and it resorts to the consideration of many cases and figures, it furnishes many examples.

Reye's reception within teaching of geometry (1873-1900)

Reye's *Geometrie der Lage* received five editions until 1923: between 1877 and 1880 in two volumes, between 1886 and 1892 in three volumes, between 1899 and 1910 and between 1909 and 1923. It was translated into French, by Chemin in 1881, and into English, by Holgate in 1898.

Reception in Italy

In his *Lezioni di statica grafica nell'insegnamento tecnico superiore* (1873), Antonio Favaro referred to von Staudt's *Geometrie der Lage* and considered Reye as a simple popularizer of the first one. But, in the French translation of his book in 1879, he recognized that he followed Reye and he wrote about Von Staudt: "His work is marked out by a great research of expressions, by an excessive terseness and by a wanted deletion of figures. He contented himself to expose the more general propositions, leaving it to the reader to develop and to apply them" (Favaro, 1879, p. xvii). In the foreword of the French translation of his book, Reye wrote that Favaro's textbook is a translation of his one.

In 1866, Luigi Cremona became professor of geometry and graphical statics in the Polytechnical Institute of Milan. He published *Elementi di geometria projettiva* in 1873, where he referred to von Staudt and Reye, and *Elementi di calcolo grafico* in 1874, where there is no presentation of the geometry of position. The first book was translated some years later: *Éléments de géométrie projective* into French by Dewulf (1875), *Elemente der projectivischen Geometrie* in German by Trautvetter (1882), *Elements of projective geometry* in English by Leudesdorf (1885).

In the foreword of the French edition of 1875, there are four paragraphs, which are not in the edition of 1873. In them, Cremona explained that he had made lectures on the geometry of position as an indispensable preparation of his course on graphical statics in Milan and he added: "In Zurich Professor Reye has given a course on Geometrie der Lage to prepare the students to follow the lessons of the professor Culmann, the creator of the graphical statics" (Cremona, 1875, p. vi). He followed Reye about von Staudt's *Geometrie der Lage*:

> If this excellent book did not meet success, it is because of the absolute lack of figures and because of his remarkably arid and concise style. Guided by the same thought, other writers have directly established the notions of collineation and reciprocity after having established fundamental notions of space, surface, line, point, straight line, plane. Maybe, that it this thought which will give the solution to the problem of the elementary teaching of geometry; then, but only then, if I don't make a mistake, we will have a method which will be worthy to be substituted to the one of Euclid, method whose introduction had been the object of fierce attacks in our Lycées (Cremona, 1875, p. vii, translation by Barbin).

Cremona hesitated about the title of his textbook. He did not want to adopt "Higher Geometry", nor "Modern Geometry", nor "Geometry of position" because it excludes the metrical properties of figures. He concluded: "I chose the title of "Projective Geometry", as expressing the true nature of the methods, which are based essentially on central projection or perspective" (Cremona, 1873, p. viii-ix, translation by Leudesdorf). Another reason is that the "great Poncelet" gave the title of *Traité des propriétés projectives des figures* to his "immortal book". Behind this choice, there is also a choice for the teaching of geometry, which is decided with the importance of constructions: "It is, I think, desirable that theoretical instruction in geometry should have the help afforded it by the practical construction and drawing of figures. I have accordingly laid more stress on descriptive properties than on metrical ones; and have followed rather the methods of the *Geometrie der Lage* of Staudt than those of the *Géometrie supérieure* of Chasles. It has not however been my wish entirely to exclude metrical properties, for to do this would have been detrimental to other practical objects of teaching" (Cremona, 1873, p. x, translation by Leudesdorf). On this last point, Cremona quoted Reye's *Geometrie der Lage*.

The choice of the order of the chapters is prominent for the coherence of the graphical and metric viewpoints. Cremona introduced the principle of duality as soon as possible. He mentioned Reye: "Professor Reye remarks, with justice, in the preface to his book, that Geometry affords nothing so stirring to a beginner, nothing so likely to stimulate him to original work, as the principle of duality; and for this reason it is very important to make him acquainted with it as soon as possible, and to accustom him to employ it with confidence" (Cremona, 1873, p. xi). The order taken by Cremona for his 23 chapters follows closely the one of Reye, but with some gaps. Central projection and figures in perspective are presented as soon as the second chapter. The principle of duality is defined in the sixth chapter, by using the six primitive forms given in the fifth chapter. But pole and polar appear fifteen chapters later. Metric relations are introduced in the ninth chapter on anharmonic properties, where Cremona presented the two classes of geometrical propositions, metrical and descriptive. Self-corresponding elements are defined in the eighteenth chapter: that means that the fundamental theorem does not seem a priority in a textbook intended for beginners.

Reception in Germany

We mention three textbooks, which illustrate three manners to consider and to situate Reye's *Geometrie der Lage* in teaching at different levels. Wilhelm Fiedler was a German (then Swiss) mathematician who made his thesis on central projection in 1858 with Möbius in Leipzig. He became a teacher of descriptive geometry in the Technical University in Prague in 1864, then, thanks to Culmann, teacher in the Federal polytechnic school of Zurich in 1867, where he stayed until 1907. In

Die darstellende Geometrie in organischer Verbindung mit der Geometrie der Lage, edited in 1871, he mentioned Reye whose the textbook showed the interest of the geometry of position (Fiedler, 1871, p. xii). The geometry of position is introduced in the first lesson on method, with the notions of systems of figures, duality and corresponding elements. So, it is considered as a propaedeutics to the descriptive geometry.

Johannes Thomae was a mathematician, who became professor at the University of Jena in 1879. In 1873, he edited his *Ebene geometrische Gebilde vom Standpunkte der Geometrie der Lage*, a short elementary textbook of 44 pages. In his foreword, he referred to Von Staudt and Reye. Like Reye, he introduced the geometry of position straightaway in the context of projections, and he gave first examples in a rich configuration of straight lines and points (Figure 9) (Thomae, 1873, p. 2). In this context, duality appears as a natural notion as soon as the page three of the textbook, given with the disposition in columns. The duality between points of a circle and its tangents precedes the introduction of curves. It is remarkable that a greatest part of the textbook is written by presenting properties and theorems in duality with the two columns. Equality of segments and angles appeared later in the chapter on ellipse. Here, the geometry of position is a propaedeutics to synthetic geometry

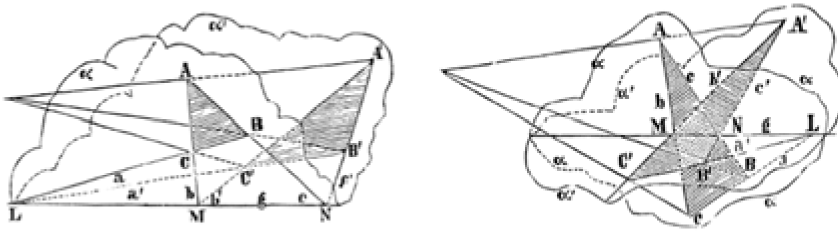

Figure 9. First figures in Thomae's textbook

In 1897, Rudolf Böger published his *Die Geometrie der Lage in der Schule* in the context of a teaching given in a Realgymnasium in Hamburg. In his foreword, he wrote that the essential books of Steiner and von Staudt are difficult to understand, but that the textbooks of Schröter and Reye had shown that the difficulties are not in the subject itself. He added:

> My textbook shows, as I hope, that the geometry of position […] can lead to theorems on conics, which are obtained in a loud and hard manner today, but nevertheless not satisfactory. The foundations of the geometry of position are so simple, they lead to all the conclusions by essential propositions with such clearness and obviousness, that the constructions are natural, the procedures of proof are so complete and coherent, their methods so fertile

> that this geometry is like the best way of mathematical training for the teaching of plane geometry (Böger, 1897, p. I, translation by Barbin and Xavier Lefort).

Die Geometrie der Lage in der Schule is a short booklet of 47 pages. It uses duality systematically with Gergonne's columns. The applications concern conics, Pascal and Brianchon's theorems, then poles and polars. It includes many problems to be solved by students at home and in classroom. Here, a new step is crossed: the geometry of position is presented as a way to teach conics in an elementary level.

"La géométrie de position" in France

The Reye's translation into French is titled *Leçons de géométrie de position*. It is presented in the *Bulletin des sciences mathématiques* in these terms: "Written with the clarity and the elegant style which distinguish M. Reye, it made accessible the theories of the Geometrie der Lage of the profound geometer of Erlangen, von Staudt. [...] It will find a warm welcome, not only among persons who would like to begin the study of the graphical statics of Culmann, but also among those who are interested in the geometry" (C.S, 1882, pp. 281-282). So, it is considered as useful for graphical statics, but it can interest the geometers "also"!

In France, the name "géométrie de position" was associated to Lazare Carnot's geometry. In *Les nouvelles bases de la géomètrie supérieure (Géométrie de position)* edited in 1892, A. Mouchot wrote about von Staudt:

> Maybe, it is in desperation and for not being able to decipher the enigmas of Algebra that Von Staudt proposed to liberate the Higher geometry of position from any idea of number. Nobody could deny that his method, in part founded on the parallels between plane figures and space ones, supplies precious ways of investigation. But, until now, nobody provided the proof that its is enough to dispel all the mysteries of the Geometry of position. So, the work of Von Staudt has not the importance that one would like to give to it (Mouchot, 1892, p. 174, translation Barbin).

This shows that the German geometry of position was not appreciated in the country of Descartes and Carnot. In his *Premiers principes de géométrie moderne*, used by many generations of students, Ernest Duporcq did not referred to von Staudt or Reye, but to Poncelet, Chasles, Laguerre. In his foreword, he stressed on the "highly analytic character" of modern geometry (Duporcq, 1899, p. vi).

The geometry of position was not mentioned in the famous *Traité de géométrie* of Rouché and Comberousse in 1866. But, it appeared in an eccentric manner in the seventh edition in 1900, in a note written by Eugène Rouché, who edited his *Éléments de statique graphique* in 1889. In this note concerning "linear and quadratic transformations, conics associated to a triangle and systems on three directly similar figures", Rouché defined the six primitive forms of the "géométrie de position" (Rouché & Comberousse, 1900, pp. 595-596). He did not refer to von Staudt and Reye but to Lemoine, Brocard, McCay, Neuberg, Tarry, etc., that means the authors of the geometry of the triangle, which is presented in the textbook. So, in France, the geometry of position became a theoretical framework of the geometry of the triangle.

Reception in the United States

Thomas F. Holgate translated and edited Reye's textbook in 1898, under the title *Lectures on the Geometry of Position*, he was professor of applied mathematics in the Northwestern University. He wrote in his foreword: "The increasing interest in this study during recent years has seemed to demand a text-book at once scientific and sufficiently comprehensive to give the student a fair view of the field of modern pure geometry, and also sufficiently suggestive to incite him to investigation. […] I trust that I have not altogether destroyed the charm of the original writing" (Reye, 1898, p. v). The reception of the translation is enthusiastic, as we can read in the reviews of American journals.

Charlotte Angas Scott, professor in Bryn Mawr College, was author of books on the geometry of position and on projective geometry. She wrote: "Cremona's Projective Geometry, in Leudesdorf's translation, is curiously uninteresting and unattractive, and does not seem to take the student sufficiently into the heart of the subject. […] Reye's sympathetic style is such as to commend the subject" (Angas Scott, 1899, pp. 175-176). John Henry Tanner, professor in Cornell University, also preferred Reye to Cremona: "Reye has beautifully shown that metric relations, especially those connected with the conic sections, present themselves very naturally as special cases of general non-metric theorems. […] Von Staudt is too brief to be easily read by a beginner, and Cremona, as translated by Leudesdorf, seems rather unattractive, and certainly lacks the charm of Reye's lucid style" (Tanner, 1899, pp. 577-578).

In *The Monist*, McCormack associated the synthetic geometry to the geometry of position and assigned two advantages to it. Firstly, the preliminary knowledge necessary for the prosecution of this geometry is not great and a profound knowledge of the old geometry is not required. Secondly, Reye's geometry of position produces mental images and increases imagination and abstraction both:

> The tendency of all recent educational methods lies in this direction: the utmost development of powers of sense and consequent imagination. Here lies the basis on which the abstract must build, and the more perfect that basis the more solid will be the superstructure. The most beautiful parts of the greatest of recent mathematicians have been made by "vizualisation";

> and it is to be hoped that the present beautiful volume will mark, in English-speaking countries also, a distinct age in the progress towards this goal (McCormack, 1899, p. 466).

Conclusion: Reye's style and the spread of the geometry of position

The reference to Reye's style in American reviews is interesting to examine. It permits us to understand a kind of enigma: how is it possible that van Staudt's geometry, considered as very difficult by mathematicians like Felix Klein, during twenty years and today too, had could be seen as a possible part of secondary teaching in geometry? It leads us to see in Reye's geometry an important historical step: it provides an initiation to von Staudt's text, but more, it proposed an original way to teach geometry to beginners. I had the opportunity to test the value of Reye's approach among students who had a very light knowledge of geometry. It is true that they entered well into the world of duality, amused and relaxed in the face of their difficulties with calculations on segments.

We proposed a short reading to Reye's textbook in this paper to try to understand what its style means and also because it seems a little forgotten in historiography today. Indeed, von Staudt and Reye's books were well referred in old classical books on history of mathematics. Eugene David Smith wrote that "Unfortunately von Staudt wrote in an unattractive style, and to Reye is due much of the popularity which now attends the subject" (Smith, 1906, p. 63). Klein named Reye in his historical book also (Klein, 1928). But it disappeared in recent classical history, like the Morris Kline's book (Kline, 1972). This forgetfulness can correspond to a minimization of the role of the teaching of mathematics into history of mathematics. In fact, we saw that a new writing of the geometry of position is closely linked with the teaching of graphical statics, which led authors to read Reye and to spread his ideas, like Cremona or Rouché. It is an example also of the role of the teaching for engineers in the implementation of modern mathematics in secondary education in 19th century.

Acknowledgment. I thank Leo Rogers very much for his help to write the paper in English.

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The teaching of mathematics, architecture and engineering in the *Ancien Régime* in Turin

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Abstract

In Turin, as elsewhere in Europe, the establishment of university courses devoted to pure mathematics, characterized by cutting-edge theoretical contents, can be traced back to no earlier than the middle of the 19th century, after Italy's Unification. The reason lies in a deliberate effort on the part of the State in the Ancien Régime to attach primacy in mathematics to its practical applications. A careful analysis of what is known about the teaching of mathematics in terms of contents and methodologies in the schools of Turin – based on the handwritten and printed documents that are conserved today in the Turin State Archives, in the University Archives and in various Italian libraries – has brought to light an unexpected scenario. According to this research, it appears that between the 17th and 18th centuries special emphasis was placed on the teaching of an extensive set of theoretical basic mathematics, in order to train not mathematicians as such, but rather professionals in civil and military architecture who could contribute to giving shape to a new political and social state.

Introduction

The aim of this paper is show that in Piedmont the teaching of mathematics, civil and military architecture and engineering were closely entwined. The demand for appropriate methods of teaching was dictated by the need to train skilled professionals in the sectors of construction and hydraulic engineering. This need heavily influenced the teaching of mathematics in military schools and universities for more than three centuries (from the mid-16th century to the mid-19th century). It reflected significantly and negatively on the inclusion in university curricula of "pure" mathematics and the establishment of a specific degree course, which would occur only after the mid-19th century, as evidenced by documents conserved in the Turin State Archives, in the University archives and in various Italian libraries.

Curiously, this aspect has been largely disregarded by historians of architecture and engineering in Piedmont and beyond. The most widely accepted opinion is that, in the centuries that preceded the establishment of polytechnic schools, the

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

training of these students was limited to working on actual building sites. In other words, there were no specific project methodologies and calculation systems and everything revolved around the experience acquired in the field. It is also widely believed, incorrectly, that architecture treatises – from Vitruvius onwards – were used as collections of models to be replicated solely with an eye to the ultimate formal and aesthetic result (Scotti Tosini, 2003; Curcio & Kieven, 2000).

In Turin, the Regia Scuola di Applicazione per gli Ingegneri (Royal School of Application for Engineers) – established by the Casati Act of 1859 and the precursor of the present Politecnico di Torino – is largely believed to have been established *ex novo* according to teaching methodologies that were imposed from the outside based on the French model (Redondi. 1980, pp. 766-767; Conte & Giacardi 1989, vol. II, pp. 281-296; Conte 1994, pp. 589-609; Marchis 1994, pp. 570-597). In actual fact, the Politecnico di Torino was the logical evolution of how mathematics and physics were taught at the university, at the Real Collegio di Savoia, also known as the Collegio dei Nobili, owned and operated by Jesuits for the education of the scions of Piedmontese aristocracy and at the Reale Accademia Sabauda (Royal Academy) in the Ancien Régime. The Collegio dei Nobili and the Reale Accademia Sabauda were founded by Carlo Emanuele II (1634-1675), but the former became operational in 1688 and the latter in 1678, both under the regency of Marie Jeanne Baptiste of Nemours (Binaghi, 2012, pp. 118-119); the University of Turin has been in existence since the 15th century.

The mathematics taught in these schools and at university was mostly of a practical nature, as witnessed by numerous civil and military writings that have been preserved to date (Giacardi & Roero, 1987; Carassi et alii (Eds), 2011). An example of the subjects that were taught in school are found in the works by 17th-century Jesuit mathematicians, one of which was particularly relevant in Turin: the *Cursus seu Mundus Mathematicus* by Claude François Milliet De Chales (Deschales)[1] (Chambéry 1621 – Turin 1678), who taught mathematics at the Jesuit Collegio dei Nobili in Turin in the later years of his life, where he earned a solid reputation as a teacher. The index of this work lists a number of subjects that range from Euclid to civil and military architecture, from hydraulic engineering to optics, from perspective to the construction of sundials to the art of navigation. Of particular interest, however, are the chapters devoted to stereotomy and the art of carpentry, which reveal a close connection to the laws of statics and provide an exhaustive addition to the information provided in the pages on civil and military architecture. Significantly, this work is of the greatest interest also for the training of architects and engineers.

[1] Milliet De Chales (1674). The copy consulted is conserved in Turin, Biblioteca Nazionale: Q-I-12/13); see also Milliet De Chales (1677). Both are dedicated to Carlo Emanuele II, duke of Savoy.

Outside of educational institutions, such subjects were also taught by private tutors (Binaghi, 2012, pp. 126-127 n. 76; Delpiano, 1997, p. 13 n. 40), many of whom came from military and architectural circles, from Jesuit Colleges or from schools run by other religious orders. A case in point is Guarino Guarini (Modena 1624 - Milan 1683), a Theatine and leading architect and mathematician present in Turin in the second half of the 17th century (Binaghi, 2012, pp. 113-114; Cecchini, 2002; Frank, 2015). Others came from schools of art such as the Accademia di San Luca in Rome. The demand was for "mixed" or applied mathematics, the only type of teaching that was suitable to solve the problems faced by architects and engineers in their professional work. It also served to provide an educational basis to young noblemen who wished to continue their education at university or pursue a career in the military. The practical-applied component was so relevant that the official title of "Court Mathematician" – initially appointed by the duke and later by the king – entailed tutoring members of the Court and their offspring and, most notably, the ability to address specific issues in the fields of architecture and hydraulic engineering (Binaghi, 2012, pp. 114-120). For example, the mathematician Milliet De Chales was known, in addition to his skills as a teacher, for his ability to build mechanical devices and tools. The documents available in Turin State Archive indicate that, while the teaching of mathematics had been part of the curricula of the Faculties of Arts (Ricuperati, 2000, pp. 3-30) since the Middle Ages, only a small number of students attending official university courses in mathematics would eventually earn a degree (in theology, law or medicine). Many were auditors or attended the first year course but did not pursue their education any further.

The 18th century reform

At the beginning of the 18th century, the University of Turin was in a state of decline and teaching had nearly come to a halt. This state of affairs was clearly inadequate to the new role that Vittorio Amedeo II (1666-1732) intended to attribute to the university as the only institution allowed to confer academic degrees. Thus, upon ascending to the throne in 1713, he launched a reform of the university charter and allocated large funds to increase the salary of university professors. This decision issued directly from Vittorio Amedeo II (Roggero, 1987, p. 39) – a higher salary reflected a special interest on the part of the sovereign in the subject to be taught and was a deliberate effort to attract more skilled professors.

The purpose was to assure the best training for the degrees in theology, law and medicine, with the addition of other subjects aimed to train higher education teachers and professionals who could be employed by the state, in particular: architects, engineers and land-surveyors. Access to these occupations did not entail the earning of a specific degree (*laurea*), but only a university certificate of approval issued by professors of mathematics in order to enter the profession. There was no specific indication as to when and how one was to prepare for this examination. From the documents that will be discussed below it can be inferred that students were expected, but not required, to attend the course of mathematics at the University. In 1730 this certificate was made compulsory through a state act[2], but by the end of the century no university course had been established. It was only in the second half of the 19th century that a degree course would be set up at the Regia Scuola di Applicazione per gli Ingegneri.

Around 1720 Vittorio Amedeo II had solicited numerous opinions to better understand which chairs to choose and which professor to appoint to each post. These opinions are still conserved today in the Turin State Archives[3]. As regards the teaching of mathematics, it would appear that the king largely accepted the interesting plan of studies, submitted by the military architect and engineer Giuseppe Francesco Ignazio Bertola (Turin 1676 - Turin 1755) that will be described below[4]. The fact that – in the very early days of the new, reformed 18th-century university, the syllabus of the mathematics course would be drawn up by an architect-engineer – who had himself taught and would continue to teach privately and publically[5] the subjects he had identified in his proposal – is a clear indication of the king's intentions. Furthermore, this fact provides a direct connection between the university and the Scuole Teoriche e Pratiche di Artiglieria that would be founded by Bertola himself in 1739, about which more will be said in what follows[6].

The teachings of mathematics at the University of Turin

The reform of the university undertaken by Vittorio Amedeo II provided for two chairs of mathematics. The main chair was assigned to the Abbot Ercole Corazzi (Bologna 1673 - Turin 1726)[7]. Corazzi was a professor of military architecture at

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[2] (Binaghi, 2001, p. 164); (Carpanetto, 2002, pp. 187-191). The registries of the examination are available at the Archivio Storico dell'Università di Torino (hereafter ASUT): X-D-1; X-D-2; X-D-3.

[3] Archivio di Stato di Torino, Corte, Pubblica Istruzione Regia Università (hereafter ASTo, Corte, P.I., R.U.), mz. 2 (1701-1724), n. 4; mz. I di Add.; See (Carpanetto, 1998, p. 109, p. 114 n. 67).

[4] (Naretto, 2002-2003); (Bianchi, 2002, pp. 159-162); (Ferraresi, 2004, p. 20 n. 9).

[5] He was a teacher at the "Accademia Reale"(Royal Academy). The manuscripts that contain his lessons are in Turin in the Biblioteca Nazionale (Q-II-28) and in Archivio di Stato, Biblioteca Antica, (Jb-VI-18). His father, the engineer Francesco Antonio Bertola (Muzzano 1647-1719), was also a tutor of arithmetic and fortification (Naretto, 2002-2003). The teaching of both Bertolas, father and son, has not yet been examined in any study.

[6] ASTo, Corte, Materie Militari, Intendenza Generale di Artiglieria, mz. 3 di Add., n. 5; (Binaghi, 2000c, pp. 268-270).

[7] ASTo, Sezioni Riunite, Controllo generale delle finanze (hereafter ASTo, S.R., C.G.F.), Patenti, mz. 2, c. 29v.

the Istituto delle Scienze in Bologna and also taught algebra at the university of the same city[8]. For the chair in Turin he was granted the remarkable annual salary of "2,500 Lire di Piemonte". The propaedeutic course (arithmetic and Euclid's elements) was entrusted instead to Carlo Tommaso Bocca, a lawyer, who received a salary of 800 Lire (Carpanetto, 1998, p. 165).

Corazzi's salary was soon raised to 3,000 Lire, a reflection of the expectations attached to this particular chair. Bertola's plan for the curriculum, mentioned above, is extremely rich and the subjects presented closely reflect those contained in the *Cursus* by Milliet De Chales: from mathematics (rectilinear trigonometry, use of the sine table, tangents and secants and logarithms) for the purposes of cartography, to geometry for topographical applications. These are followed by hydraulic engineering applied to the management of rivers and streams, mechanics and statics, military architecture for offence and defense, and civil architecture. The list of subjects is extensive and includes nautical science, astronomy and geography, spherical trigonometry, cylindrical sections, conic sections, numerical and literal algebra, music, optics, dioptrics, catoptrics, perspective, centrobarica, gnomonics, and pyrotechnics.

The most innovative aspects of Bertola's program are the presence of instructions for teaching (for example the explanation of the connections between mathematics and physics), and the hope that the two respective professors will collaborate. In particular, the text encourages the teaching of the basics of those subjects that will not be discussed in detail, so as to enable the students to continue their studies in the future, if need be, by providing to them a solid method of investigation and study[9].

However the most interesting information contained in the Bertola's plan is the reference to civil architecture and the need to go beyond the teaching of the five orders identified by Vitruvius (Tuscan, Doric, Ionic, Corinthian and Composite) and presented in the manuals, since the professionals are required to correctly design civil and religious spaces and to check the appropriateness of the building materials. As regards materials, the main interest at the time centered on the calculation of their structural resistance (Binaghi, 2001, p. 190 n. 150). Geometry served to assign adequate shapes and was key to planning spaces, creating bearing structures and verifying their structural coherence. To this end, project calculation was geometrical. It was clear that students needed to learn the basics of mathematics, as is noted in Corazzi's two manuscripts about the lessons[10]. But the teaching extended beyond "ex cathedra" lessons and also entailed practice on actual building sites. The field of architecture was one of the most popular.

[8] No modern biography of this interesting figure has been written so far. The most interesting insights come from Spallanzani (1993, pp. 120-141), who stressed the maieutic skills of Abbot Corazzi.

[9] ASTo, Corte, P.I., R. U., mz.2, n. 4.

[10] The copy consulted is conserved in Turin, Biblioteca Nazionale, K^3-IV-4 and 5.

The documents regarding methods and contents for the teaching of mathematics: inaugural addresses and year-end public demonstrations

The crucial relevance of civil and military architecture in the teaching of mathematics is testified by the fact that in November 1721 Corazzi, professor of mathematics, devoted the inaugural address for the opening of the academic year to the use of mathematics in civil and military architecture (*De uso mathéseos in civili et militari architectura*[11]), which was printed in a pamphlet and has been conserved. He speaks of the importance of geometry for architecture, citing the works of the royal architect Filippo Juvarra (Messina 1678 - Madrid 1736) as a case in point. Inaugural addresses at university served to present a course's syllabus, as shown in the calendar of the academic year 1724, where Corazzi announces that he will teach civil architecture according to a new teaching method[12].

During his years as a professor in Turin, Corazzi organized public "exercitationes" (practice exercises) for his students at the end of each academic year, although not all students participated. In fact, at the end of the first year (1720), he laments the refusal of some students, especially among the aristocracy and in spite of their skills, to participate in those public practice exercises. He lists only nine students[13], while documents attest that no fewer than 53[14] were actually enrolled, including auditors.

The public exercises to test the knowledge acquired by the students continued over the years. Undoubtedly they were deemed important and provide evidence of a deliberate interest in the practical application of knowledge. Unfortunately only the occurrence of the exercises was duly recorded, not their contents, with one notable exception. In the academic year 1725 a printed document written by Corazzi attests that a public exercise was held for Corazzi's students; it was dedicated to Vittorio Amedeo II and it took place on 6 May[15]. The short publication, only nine pages long, provides interesting information about the teaching methods adopted in those years. To date this is the only evidence that documents both the contents and the kind of activity that the exercises entailed for the students[16]. Unfortunately the drawings and maquettes mentioned therein have not been preserved.

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[11] The copy consulted is conserved in Rome, Biblioteca Nazionale Centrale: 8.E.11.6.

[12] ASTo, Corte, P.I., R.U., mz. 2 (1702-1724), n. 23.

[13] Ibid.

[14] Ibid.

[15] Hercules Corazzi (1725), *De architettura exercitatio academica in Regio Turinensi Archigymnasio invictissimo Regi Victorio Amadeo dicata*, Augusta Taurinorum: ex Typographia Jo. Baptistae Valettae S.S.R.M. impressor (the copy consulted is conserved in Rome, Biblioteca Nazionale Centrale: 34.9.G.19.1).

[16] One of the students is Bernardo Vittone (Turin 1704 - Turin 1770), the most important architect in Piedmont after Guarini and Juvarra (Binaghi (2016), pp. 79-92). Vittone would later teach mathematics at the Collegio delle Province in Turin (Binaghi, 2005, pp. 87-91).

In the introduction (*Dissertatio*) Corazzi writes about "how and why" architecture needed mathematics: to distinguish the architect from the mason. Whereas the mason can only repeat what he has learned in his work in the building yard, while the architect can invent new forms based on his scientific (mathematical) education (Binaghi, 2001, pp. 175-180). It should be noted that at the time architects and civil engineers unsettled the social and economic status quo among the crafts, in that their profession entailed an active decision-making and planning process and their reliance on theoretical thinking set them apart from master craftsmen and masons in particular. For the Abbot the best examples of unusual shapes in Turin are the church of San Lorenzo and the chapel of the Holy Shroud, both by the architect and mathematician Guarino Guarini. Corazzi's work continues with a narration of the method adopted for teaching: the students' work began with a visit to an active building site, that of the Certosa (Carthusian monastery) of Collegno, near Turin. Some students drew the Vitruvian orders, others surveyed the built portions already in place, while the most advanced proposed their own plans to complete the construction work. Other students created projects for military defense purposes. The guiding thread, that bound together all these efforts, was the need for the students to use geometry-based arguments both to ensure the mechanical resistance of each Order and to provide a foundation for their choices. The method hinged on the principle that creating new shapes and ensuring their solidity would be impossible without the mathematical notions acquired in the classroom – over the course of one or more years, as seen from the appearance of some students' names in several exercises.

After Corazzi's death in 1726, his chair was assigned to Giulio Accetta (Francavilla 1690 - Turin 1752) (Gliozzi, 1960). He started working in Turin only in 1730[17] and in those years of vacancy the years-end public demonstrations were named *Accademie* (Academies)[18], and was increasing their importance. Five Academies were set up at the University: three for humanities and two for sciences, namely mathematics and physics. The latter required the use of machines and tools and, consequently, large spaces, for which the aula magna and the anatomical theater were inadequate, in spite of various alternative options[19], no solution was found.

However, this did not prevent the continuation of science lessons and academies: according to available documents[20], many were held by Accetta's students and, start-

[17] ASTo, S.R., C.G.F., Patenti, mz. 7, cc. 145-145v.

[18] ASTo, Corte, P.I., R.U., mz.2 (1701-1724), nn. 18 and 24.

[19] For this reason the First Royal Architect Filippo Juvarra, before departing for Spain (1735) and probably not long after the arrival of Accetta (1730), designed a project to reorganize some areas near the Reale Accademia Sabauda and in front of the University building for the creation of an "Accademia di Architettura Civile e Militare" (Academy of Civil and Military Architecture). This project never came to be. See (Binaghi, 2001, pp. 178-180).

[20] ASTo, Corte, P.I., R.U., mz. 1 di Add. bis.

ing in 1731, also by those of Ansano Vaselli, the newly established chair of geometry[21]. Unfortunately no written record has survived about their contents, except for a partial description of a year-end public demonstration, held in the aula magna on 14 June 1735 by the students of mathematics[22] of Accetta and Vaselli alike. This information is contained in the introductory address (*orazione*) penned by Accetta, that appeared in print[23]. No description remains of the students' work. In spite of its celebratory tone, Accetta's speech – dedicated to Carlo Emanuele III and to the University "Regent" Ludovico Caissotti di Santa Vittoria – maintains from the very beginning that mechanical sciences depend on geometrical demonstrations, thus hinting at the kind of exercises that the students were to perform.

A similar testimony is found also a year later (1736), on a different occasion. In this case it was the celebration of the new institutional organization of the *Accademia di pittura, scultura ed architettura detta di San Luca* where architecture was also taught[24]. Gerolamo Tagliazucchi, professor of eloquence and a private tutor of mathematics, held an Academy "in laudem Picturae, Achitecturae et Sculpturae"[25] on 8 June 1736, of which the introductory address and some poems have survived[26]. Tagliazucchi's words make it clear that he regarded civil and military architecture and engineering as political tools, able to attest to the glory of the sovereign over the centuries, hence their relevance.

In those years the debate on the teaching of civil and military architecture was gaining momentum. A few months earlier, on 6 April 1636, the military engineer Giuseppe Ignazio Bertola – the same who had drawn up the plan to establish a chair of mathematics at the University in 1720 – submitted a proposal for a reform of the teaching methodology (Ferraresi, 2004, p. 27 n. 21, p. 29 n. 26). He suggested the creation of a new military institution, that would be implemented in 1739

[21] The chair of geometry, distinct from that of mathematics, was created in 1731 for Ansano Vaselli (ASTo, S.R., C.G.F., Patenti, mz. 9 c. 85). See (De Angelis, 1810, p. 29).

[22] Among the students was Filippo Antonio Revelli (Monastero di Lanzo 1716-Turin 1801), who would later hold the chair of geometry (1750) (Revelli, 1918).

[23] Accetta (1735), *All'il.mo ed eccell.mo signore il Conte Ludovico Caissotti di Santa Vittoria (…). Orazione recitata nella Scuola Magna della regia Università di Torino per l'accademia fatta dagli Scolari de professori delle matematiche sul terminare l'anno scolastico del 1735*, in *Torino: per Gio Battista Valetta stampatore di S.M. dé Regi Magistrati*, 1735. The copy consulted is conserved in Milan, Archivio Storico e Biblioteca Trivulziana: G 2173.

[24] In Turin there existed an Academy of Art (Accademia d'Arte), which is indicated in documents as "Accademia dei pittori, scultori ed architetti detta di San Luca", twinned in 1675 with the one in Rome. Starting in 1716 the Turinese Academy was housed in the same building as the University, thus recreating a situation that is similar to the one found in Bologna, where Corazzi had come from, where the Accademia Clementina (Academy of Art) and the Istituto delle Scienze (where mathematics was taught) were both housed in Palazzo Poggi (Binaghi, 2001, pp. 181-192).

[25] ASTo, Corte, P.I., R.U., mz.1 di add. bis

[26] Tagliazucchi (1736)*, Orazione e poesie per l'istituzione de l'Accademia del Disegno, della Dipintura, Scultura e Architettura Militare e Civile, dedicate a S.A.R. il signor Duca di Savoia*, *in Torino MDCCXXXVI, nella Regia Università, appresso Giambattista Chiais stampatore e libraio di S.R.M. con lic. dé superiori.* The copy consulted is conserved in Turin, Biblioteca della Provincia: P.g. 234/4.

as the Scuole Teoriche e Pratiche di Artiglieria (Theoretical and Practical Schools of Artillery) (Binaghi, 2007, p. 141 n. 45; p. 148 n. 78). Bertola was convinced that in some professions the practical training is essential, and that in such cases, teaching should not be confined to the university classroom, where theory was predominant. The practical section of the school was located near the river Po, but the classrooms were housed in a building in the vicinity of the University and the Reale Accademia Sabauda. A year earlier (1738) the Corpo degli Ingegneri Topografi (Corp of Topographer Engineers) had been established and the Ufficio Topografico (Office of Topography) (Binaghi, 2001, pp. 188-189) was housed in the same building that was home to the *Accademia dei pittori, scultori ed architetti detta di San Luca* (Binaghi, 2004, pp. 33-36; Binaghi, 2007, p. 150 n. 88).

All these institutions required the teaching of mathematics. Their physical proximity was intentional and aimed to facilitate exchanges between teachers and students (as noted, auditors were common and officially accepted in university classrooms). At the same time, mathematicians, physicists, botanists, physicians and astronomers required the collaboration of professional draughtsmen – that is to say scientific illustrators – who are mentioned in existing documents[27]. This spurred scientific interests among artists and influenced the teaching of scientific subjects, along the basic lines that Corazzi had imported from Bologna. Significantly, Turin maintained a close connection to Bologna throughout the century (Binaghi, 2001, p. 187).

In the mid-18th century (6 October 1750) Filippo Antonio Revelli (Monastero di Lanzo 1716 - Turin 1801) (Revelli, 1918), a pupil of Vaselli, was appointed to the chair of geometry[28]. Upon Accetta's death (25 September 1752), his chair was given to Francesco Domenico Michelotti (Cinzano 1710 - Turin 1778) (Roero, Luciano, 2010, pp. 249-251; Ferraresi, 2004, pp. 70). Michelotti earned Turin a reputation throughout Europe in the field of hydraulics (Redondi, 1980, pp. 770-773). Documents show that the two chairs continued to organize introductory addresses (*prelezioni*) and year-end exercises in collaboration, but do not provide insights as to their contents[29].

Joseph-Louis Lagrange

In the years between 1750 and 1752 the Piedmontese future mathematician Giuseppe Luigi Lagrange (Turin 1736 - Paris 1813) attended the University of Turin,

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[27] ASTU, Mandati, XII-C-I, c. 46, c. 64, c. 80, c. 231. See (Binaghi, 2000b, pp. 147-180; 2001, p. 188 n. 142; 2004, pp. 30-36).

[28] He published *Elementi dell'aritmetica universale e della geometria piana e solida*, vols. I-II. Torino: Gianmichele Briolo, 1778.

[29] AST, Corte, P.I., R.U., mz. 1 di Add. bis.

where he earned the diploma of *Magistero* (teaching) from the Faculty of the Arts (1752) (Pepe, 1993, p. XIV; Pepe, 2004, pp. 75-80; Giacardi, 2014). He interrupted his university studies and did not earn a degree, deciding instead to devote himself to the intensive private study of mathematics[30]. It should be noted that Bertola's plan contained a clear reference to the need to provide students with the tools – that is to say, an approach to research and learning – that would allow them to pursue their studies autonomously. This was the most significant lesson that the University of Turin and the local scientific community taught Lagrange, along with the need to associate any theoretical teaching to practical bases and particularly to practical applications. Thanks to a Royal Letters Patent (26 September 1755)[31] he became substitute professor of mathematics at the Scuole Teoriche e Pratiche di Artiglieria in Turin. He was nineteen years old: he had enrolled at university at the age of fourteen, after being tutored privately. The young man's talent originated partly from a natural gift, but, as noted, he also benefitted greatly from the inputs provided by the Turin scientific community – in and outside the university – with which Lagrange maintained a close connection through some of its leading figures, even after completing his education.

His fundamental skills, the development of hypothetical-deductive and abstract reasoning, without neglecting to maintain a direct relation to the practical application of every discovery, was what he sought to transmit to his students. This can also be gathered, indirectly, from his ability as a teacher at the military school, where he started working very early on and teaching according to the patterns he had become familiar with. As a teacher he collaborated with the students to prepare year-end public demonstrations which were also typical of military schools. For those students he wrote manuals (Borgato, Pepe 1987) on analytical geometry and differential calculus[32]. It is likely that his ability to explain and think clearly, a trait that is found in all his scientific publications, owes much to his teaching experience. The main scientific work carried out by Lagrange in Torino includes by his famous contribution on vibrating chords, published in 1759 in the first volume of the series edited by the Private Society founded by Cigna, Saluzzo and Lagrange himself in 1757, which later became the Accademia delle Scienze di Torino in 1783. Other significant contributions of this period spent in Torino regard differential equations, finite difference techniques and variational methods (Pepe, 2008, p. 37). The use of variational methods in mechanics, including the principle of the least action sent through Euler to the president of the Berlin Academy of Science Pierre Luis Maupertuis, permitted Lagrange to achieve prestige and fame on a Eu-

[30] Turin, Biblioteca Reale, Miscellanea patria, vol. 113, n. 35.

[31] ASTo, Sezioni Riunite, Ministero della guerra, Regia segreteria di guerra e marina (1717-1801), Patenti e commissioni, mz. 7, cc. 61-62.

[32] *Principi di Analisi Sublime (1755-1760)* (Turin, Biblioteca Reale, Ms. Saluzzo 736).

ropean level by age of twenty, when in September 1756 he became a foreign member of the Berlin Academy. (Roero, 2008, p. 36).

On 21 August 1767, Lagrange, who was then thirty one years old and culturally mature, left Turin never to return again and embarked on a shining career as a mathematician in Berlin and in Paris.

Meanwhile, in Turin the 1760s and 1770s were characterized by a debate about the training of architects, engineers and surveyors where mathematics was gaining growing relevance. Significantly, candidates wishing to take the examination that would earn them the Letters Patents to practice the profession were now required to attend courses in mathematics at the university. Between 1761 and 1772 a number of provisions[33] were issued to gradually make attendance of university courses compulsory: four years for architects and five for hydraulic engineers (Ferraresi, 2004). One year later (9 March 1762) the "Manifesto del Magistrato della Riforma" (Provisions by the Board of Public Instruction) laid down the rules and the subjects of the examinations to be held in front of the "Classe dei Matematici" (Class of Mathematicians) and illustrated the penalty for any abuse of the title.

The hydraulic and civil architect Francesco Benedetto Feroggio (Turin 1760 - Montpellier 1842) (Binaghi, 1996, pp. 369-372; 2001, pp. 194-196, pp. 218-220), upon earning certifications in both subjects, for his "aggregazione" (admittance) to the Class of Mathematicians (1788) proposed a study entitled "On the usefulness and applications of mathematics to civil architecture"[34]. The title of his work seems to point to the 1721 inaugural address by Corazzi, but there is a significant distance between the two works in terms of content, since Feroggio supports a mechanical approach to construction – an analytical instrument that, in his view, could provide a rule to apply to different cases.

In that same year (1788) and stemming from the same conceptual premises (analytic thinking: principle of virtual work), Lagrange published in Paris the work that earned him worldwide fame: *Méchanique Analytique* [35]. The building culture in Piedmont – which was shaped at the Reale Accademia Sabauda, the Scuole Teoriche e Pratiche di Artiglieria, the Accademia dei pittori, scultori ed architetti detta di San Luca, the University and, after the mid-18th century, the Accademia delle Scienze – laid the foundations in the Ancien Régime for a cultural substratum that would make it possible in the 19th century to profoundly change the theoretical "instruments" used by engineers in their profession. In particular, infinitesimal

[33] ASTo, Corte, P.I., R.U., mz.6, n.13; (Levra, 2000, pp. 31-98; Binaghi, 2000c, p. 274 n. 43; Ferraresi, 2004, pp. 62-63).

[34] Feroggio (1788), *Dell'utilità ed applicazione delle matematiche all'architettura civile*, Torino: Stamperia Reale. The copy consulted is conserved in Rome, Biblioteca Apostolica Vaticana: Cicognara III, 503.

[35] Lagrange (1788), *Mécanique Analitique par M. De La Grange, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin etc.*, Paris: chez La Veuve Desaint, Libraire rue du Foin S. Jacques. The copy consulted is conserved in Turin, Biblioteca Nazionale: Q-V-97.

analysis – which was discussed by Feroggio without an in-depth scientific study but which reached its highest peaks with Lagrange – paved the way for the passage, at the Regia Scuola di Applicazione per gli Ingegneri, from the drawn model – conceived exclusively in geometric-formal terms – to the structural project conceived in a modern sense, that is to say as a mathematical model that could anticipate reality and allow the static testing of the designed structure before actually building it on the construction site. Turin's contribution to the advancement of science and technology in the field of structural mechanics between the 18^{th} and 19^{th} centuries is very interesting (Secchi Landriani & Giorgilli, 2008; Chorino, 2014, pp. 61-84) and merits further in-depth studies. It was due not only to favorable historical circumstances, but to a well-organized educational system.

At the turn of the century (1799) the hydraulic engineer Ignazio Maria Michelotti (Turin 1764 - Turin 1846), son of the professor of geometry Francesco Domenico and since 1795 professor of mathematics, proposed a reformed curriculum of studies in which the foundational role of applied mathematics was evident, and whose methods and contents were deliberately aimed at the training of engineers (Ferraresi, 2004, pp. 62-63 n. 101). The reason is stated quite clearly in a later document (dated 1844) by Andrea Tecco, the official Architect of the University. Tecco presents a number of considerations to accompany the graphical proposals for the rethinking of the teaching spaces of the university building, of which he was in charge, with the aim to improve the premises that were available to the school of architecture and the school of mathematics (engineering)[36]. He remarks that the number of students attending those schools had significantly increased over the past twenty years. This was because a career in those fields – and particularly in engineering – opened up many opportunities to work for both private citizens and for the State on the design and construction of bridges, roads, railways or as skilled professionals in the cadaster. No relevant increases were noted or envisaged for the more traditional careers (in law, medicine and theology). Tecco affirms that since both the State and private citizens were investing large sums in construction projects and the creation of new infrastructures, they expected to rely on worthy professionals with a university education, who could ensure excellent results.

Conclusion

Based on the scenario that can be drawn according to available documentation (including reference documents studied for previous publications as listed in the bibliography and others only recently discovered by the author of this paper, which

[36] ASTo, Corte, Genio Civile, versamento 1936, mz. 23, n. 99.

have shed new light on the contents and the methods of the teaching of mathematics at the University of Turin), it can be asserted that pure mathematics was not cultivated in Turin's educational institutions in the period of the Ancien Régime and in the first half of the following century. This is further confirmed by the documents concerning the University of Turin in the first half of the 19th century studied by Alessandra Ferraresi (2004), where it is clear that the term "mathematics" referred in fact to engineering, leaving no room to misinterpretation. Pure mathematics was included in the curriculum of the Degree Course in Mathematical and Physical Sciences – which was already in place in 1848 – only after the Regia Scuola di Applicazione per gli Ingegneri, established in 1859, became fully operational. Its academic staff, teaching methods and even the equipment used would issue from the University, thus ensuring continuity with the past. It is this continuity with the past, that characterizes the Regia Scuola di Applicazione per gli Ingegneri, created to meet the needs for technical expertise, and which, in Savoy Piedmont, had found in the close connection between the teaching of mathematics and that of architecture and civil and military engineering its basic epistemology, which I have aimed to underscore.

Acknowledgments. I wish to thank Dr. Riccardo Mazza and Dr. Giuseppe D'Errico of the Biblioteca Nazionale Centrale in Rome and Dr. Luca De Vecchi of the Biblioteca Trivulziana in Milan for their kindness. In Turin I would like to express my thanks to the staff of the Archivio di Stato and to Dr. Paola Novaria of the Archivio Storico dell'Università.

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Recommendations of the Royaumont Seminar on primary school arithmetic. Influences in the Nordic countries

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Abstract

Following the seminar on new thinking in school mathematics, held in Royaumont, France, in 1959, the Nordic countries took up cooperation on analysing the situation in mathematics education, to work out curriculum plans and write experimental texts. A Danish author, A. Bundgaard, and her collaborator wrote a textbook series for primary level which was translated into Icelandic. The text is analysed with respect to presentations on arithmetic education at the seminar and compared to previous and later texts in use. The results show that the declared intention of the reform movement to emphasize the structure of the number system and build its presentation on set-theoretical concepts gradually faded out while the study of numbers built on primes and divisibility became a revived topic in Icelandic school mathematics. Furthermore new topics such as statistics and introduction to probability entered the curriculum.

Introduction

In 1959, a seminar for mathematicians, mathematics educators and mathematics teachers was organized by the OEEC at Royaumont, France, to discuss a reform of school mathematics. Radical reforms were proposed, for arithmetic and algebra teaching, even at primary level. We shall explore

- which ideas on arithmetic teaching, proposed at the Royaumont seminar, were implemented in a primary level textbook series, composed by the Nordic Commission for Modernizing Mathematics [*Nordiska kommittén för modernisering av matematikundervisningen*], abbreviated NKMM
- if the ideas were new in Iceland
- if they survived the first wave of enthusiasm for the New Math and became permanent contribution to school mathematics.

The NKMM primary level textbook series was translated into Icelandic. Their content will be analysed with respect to presentations and recommendations of the Royaumont Seminar, and compared to earlier and later textbook publications in order to clarify permanent influences of the seminar.

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The Royaumont Seminar

The Royaumont Seminar was a seminar on new thinking in school mathematics, held in Royaumont, France, in November 1959. It was arranged by the OEEC (later OECD) and attended by all member countries except Portugal, Spain, and Iceland (OEEC, 1961, pp. 213-220). At the seminar, the European proponents for reform met representatives of the New Math movement in the United States. For a more detailed account of the seminar, see De Bock and Vanpaemel (2015).

During the 1959s and 1960s, CIEAEM, the International Commission for the Study and Improvement of Mathematics Teaching, was an arena in Europe with the aim of finding new approaches to mathematics education suitable to the changed mathematical and social context. Among its founding members were the Swiss psychologist Jean Piaget (1896-1980), mathematicians Gustave Choquet (1915-2006) and Jean Dieudonné (1906-1992) from France. The main concern of the CIEAEM was a growing attention to the student and the process of teaching (CIEAEM website).

The CIEAEM had strong representation at the seminar, with Dieudonné and Choquet among the invited speakers. Choquet introduced the theme of new thinking in mathematics education, and proposed new approaches to arithmetic and algebra. According to the programme of the seminar, the Danish mathematician Svend Bundgaard spoke on teachers' mathematical competencies and their training on its second last day (Schubring, personal communication).

Dieudonné belonged to the Bourbaki group, a group of mathematicians (mainly French), who worked at a mathematical encyclopedia, where the borders between the different mathematical topics were abolished. The group's central concept was "structure". When describing the structures, the importance lay in the elements' relationships, determined by axioms.

Among speakers on behalf of the reform movement in the United States were Howard Fehr, head of department of teaching mathematics at Teacher College of Columbia University in New York, and Marshall Stone, chairman of ICMI, the Commission of Mathematics Instruction of the International Union of Mathematics. Stone was the chairman of the Royaumont seminar.

Stone emphasized the need for reform to give quite as much attention in schools to the technical preparation of skilled workers as to the training of future university students, engineers and scientists (OEEC, 1961, p. 19), thus promoting education for all. He called for a thorough up-to-date analysis of simple uses of the more elementary kinds of mathematics in the skilled occupations of modern industry and in the daily life of the average citizen called on to vote and pay taxes. Stone expressed his concern that elementary mathematics must not repel the child. It was all too evident that primary schools were failing to develop adequately the latent mathematical talents and interest of the average child. It was imperative to find remedies for these defects in elementary mathematical instruction. Fortunately, illuminating psychological investigations, particularly those of Piaget, were pointing the way to hitherto unrecognized pedagogical possibilities (OEEC, 1961, p. 22-23). However, Stone's plea for research and for reforming primary teaching, remained without success, and his programme of "mathematics for all" was not endorsed in the follow-up discussion nor the final report of the meeting (Schubring, 2014b).

During the last two days of the seminar, the participants developed jointly its conclusions. The conclusions, published in a report (OEEC, 1961) by Howard Fehr, are not identical to the conclusions preserved in the archives of OECD. However, the original conclusions of the seminar concerning arithmetic teaching (OEEC, 1961, pp. 108-111; Schubring, 2014a, pp. 93-94) do not differ considerably with respect to the topics discussed in this article. In the following, we shall explore the official report with respect to recommendations on arithmetic and algebra teaching and how they are reflected in a Nordic primary level textbook series.

Nordic cooperation

One of the final recommendations of the Royaumont Seminar was that each country could reform its mathematics teaching according to its own needs, but it was recommended to establish as much cooperation as possible. The Nordic participants in the Royaumont Seminar agreed upon cooperation on reform of mathematics teaching. The ideas about Nordic cooperation were presented to governmental bodies, and the issue was taken up in the Nordic Council, which decided to set up a committee under its Culture Commission. Each of four countries – Denmark, Finland, Norway and Sweden – appointed four persons to the Nordic Committee for Modernizing Mathematics Teaching, NKMM.

The committee was active from 1960 until 1967, when its report was ready in the autumn. The members of the committee were mathematicians and mathematics teachers, or they came from school administration. Their programme for reform was to

- analyse the situation in mathematics education
- work out preliminary and revised curriculum plans, and
- write experimental texts for courses at all school levels.

The committee appointed several teams of writers. Its focus was on the mathematical content, and the teaching of seventh to twelfth grades was its main object. Consequently its main contribution was in this field. However it was decided to handle mathematics courses throughout the school and the committee contacted for that purpose extra experts for the first to sixth grades.

Writing sessions were arranged in summer 1961. Some texts were ready that autumn, and the others were to be so successively until the beginning of 1966. Joint Nordic manuscripts were planned. Several persons from each country would translate and adapt the joint publications to each language. This committee dominated mathematics instruction in the Nordic countries for most of the 1960s (Gjone, 1983, II. pp. 78-80; Nordisk råd, 1967).

Denmark was one of the countries which went the furthest when it came to introducing the Bourbaki tradition into university programs, and eventually also to high school programs. Svend Bundgaard was a highly influential professor of mathematics at Aarhus University, and when he took up this professorship around 1954, after spending time abroad, he declared: "This New Math is something we must do in Denmark. We have to really revamp the entire program and modernize it." He was also one of those who arranged for Danish participation at the Royaumont seminar in France in 1959, and one of its guest speakers. That movement became very influential in Denmark (Karp, 2015).

Svend Bundgaard's sister, primary teacher Agnete Bundgaard, was a member of the writing team. She wrote a textbook series for the first two primary grades together with the Finnish Eeva Kyttä and alone for the remaining four primary grades. Iceland did not participate in the NKMM cooperation. Iceland was part of the Danish Realm since the Kalmar Union around year 1400 until 1944 and many students received their vocational or academic training there still in the 1960s. Icelanders were informed about the textbook series written by Agnete Bundgaard by Svend Bundgaard who had been a co-student of G. Arnlaugsson, an Icelandic mathematician. The series was translated into Icelandic, gradually as it was published in Denmark from 1966 (Bjarnadóttir, 2007, pp. 267-268).

Analysis of the primary level Bundgaard-textbook series

Three arithmetic textbook series were legalized in 1929 for use at primary level in Iceland. Among them were a series by Sigurbjörn Á. Gíslason (1911-1914), here called SÁG, and another by Elías Bjarnason (1927-1929), EB. The EB series was chosen for free distribution in 1939 and was thereafter the only textbook series in use until 1966.

Translation of the NKMM primary level mathematics textbook series for age 7-12 by Agnete Bundgaard and Eeva Kyttä (1967-1968; Bundgaard, 1969-1971) was pursued during 1966-1971. Jens Høyrup (1979) deemed that material as a most orthodox adjustment to the demands of the mathematicians at Royaumont. The series, here called Bundgaard series, caused some disturbance among teachers and parents.

A new series SFG, *Stærðfræði handa grunnskólum 1A … 6B* (Bjarnadóttir et al., 1971-1977), was composed quickly on behalf of the state monopolistic enterprise, State Textbook Publishing House (Ríkisútgáfa námsbóka, RN) and run as the main syllabus for about three decades. By the turn of the century, a new series, GP (Mogensen and Balzer Petersen, 1999-2001; Pálsdóttir et al., 2002-2004) was initiated on behalf of the National Centre for Educational Material (NCEM), the heir of RN, and was run as the main option until 2010.

We shall now compare the two older series, SG and EB, and two more recent series, SFG and GP, to the Bundgaard series with respect to the topics mentioned in Choquet's presentation and the recommendations of the Royaumont seminar. The analysis of the influences splits into three parts; the introduction of

- set theoretical concepts and notation
- structure of the number field, and
- study of numbers.

Proposals on arithmetic teaching realized in Bundgaard series

The second section of the seminar programme was on *new thinking in mathematics education*. The task of the section was to seek answers to what mathematics should be taught, to whom and how. Introducing these and other problems, Gustave Choquet considered the psychological implications of teaching mathematics as well as the presentation of the subject matter. The start of Choquet's address consisted of an exposition of the experiments carried out by Jean Piaget on the understanding of number and magnitude by children up to the age of seven years (OEEC, 1961, p. 62-63). Piaget said that the inclusion of a part in a whole implied a preliminary algebraic structure.

Choquet then spoke on tendencies in modern mathematics: to do away with boundaries between arithmetic, algebra, geometry and calculus, which could be done through the study of structures. The sets of N and Z were endowed with numerous structures, and the set Z constituted an excellent basis for study in that it might be regarded as taking concrete form in the child's mind very early. Its "discrete" character made it tangible so that it might be used for introducing and studying such concepts as one-to-one correspondence, function, conversion and equivalence (OEEC, pp. 63-64). These topics are reflected in the Bundgaard series; see Figures 1, 2 and 3.

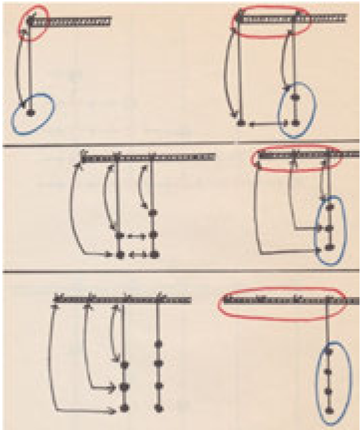


Figure 1. Building up the number concept by one-to-one correspondence to the number line, represented by knots. Age 7 (Bundgaard and Kyttä, 1967, p. 1:20).

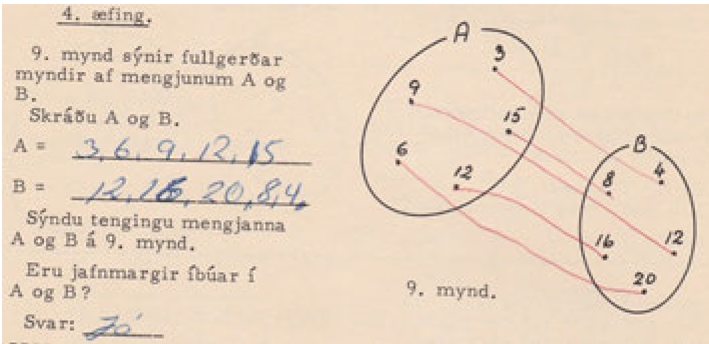


Figure 2. Establishing equivalence between two sets. Age 10 (Bundgaard, 1970, p. 4a:24).

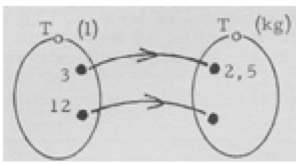


Figure 3. The function concept, relating liters of oil to mass, measured in kg. Age 11 (Bundgaard, 1971, p. 5b:13).

Concerning arithmetic in elementary schools, finite cardinal and ordinal number could be shown by Cuisenaire rods – and concomitant with this, the concepts of the subset of a set, of complementary set, union and intersection of two or three sets could be shown, according to Choquet. The concept of order could be studied from simple examples. See Figure 4 for the reflection in the Bundgaard material. Here, one may notice nurturing of Piaget's idea when he said that the inclusion of a part in a whole implied a preliminary algebraic structure.

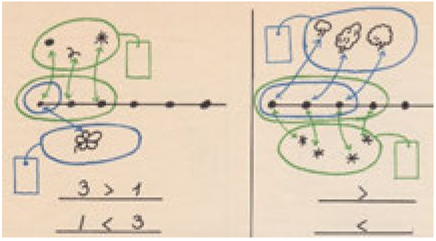


Figure 4. Subsets and ordering. Age 7 (Bundgaard and Kyttä, 1967, p. 1:29).

Addition and multiplication were to be introduced by the union of finite disjoint sets and the product of finite sets, respectively, see Figure 5 for addition and Figure 6 for multiplication in the Bundgaard series.

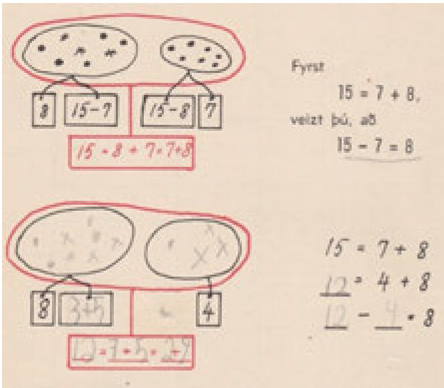


Figure 5. Addition and subtraction. The reader may notice that the pupil, working on the task, was somewhat confused when filling in the empty spaces. Age 8 (Bundgaard and Kyttä, 1968, p. 2a:20).

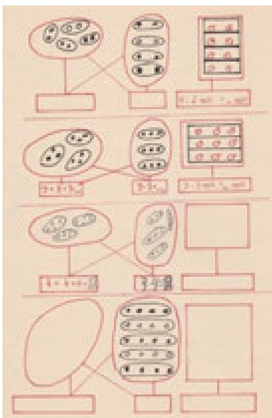


Figure 6. Multiplication, counting apples. The pupil knew too well that 3·4 = 4·3. Age 8 (Bundgaard and Kyttä, 1968, p. 2a:34).

Below, in Table 4, topics related to set theory are collected. These were completely new in the sense that they had not appeared in earlier textbooks. Remnants were seen in later texts, but not as systematically used to build up the number concept. In the following tables, the numbers indicate the age level when the topic in question was introduced. Age within parenthesis indicates that the topic was only marginally introduced and not worked on in what followed. Normally, the topics were readdressed regularly after they had been introduced.

Table 1. Set-theoretical concepts in five Icelandic textbook series for primary level.

Textbook / Topic	SÁG	EB	Bundgaard	SFG	GP
Years in use	1911-37	1927-80	**1966-80**	1971-00	1999-10
Age level	10-13	10-12	**7-12**	7-12	6-12
Sets			**7**	7	10
1 to 1 correspondence			**7**	(7)	
Subset			**7**	7	
Union			**7–10**		11
Intersection			**10**		11
Introduction to set algebra			**10**		
Set difference			**10**		
Complementary set			**10**		
Notation, mod. symbolism			**10**	(11)	
Cuisenaire rods				7	

According to Choquet, the introduction of positive and negative integers raised no difficulty, using translation operators to the right or left. This could be helped by games, e.g. of winning and losing. This would give a better understanding of zero and allow the introduction of simple algebra.

Decimal numeration must be introduced fairly early, confined to a useful system for the notation of large numbers and not for a study of properties of operations. Long multiplications and divisions were unnecessary burdens while children must know simple and rapid mental calculations, and exercise estimation of large numbers.

Fractions could not be avoided but at elementary stage one could not consider p/q as a number but as an operator, operating on magnitudes, e.g. finding 2/3 of a quantity. – Later, when the set of real numbers were introduced as an Archimedean ordered commutative field, the question of fractions would no longer arise since, by definition, p/q would be an element of the field (OEEC, pp. 64-66).

The axioms of the number field were introduced step by step in the Bundgaard series, beginning with the commutative law of addition in its first volume at age 7, see Figure 7, after having presented ordering, see Figure 4.

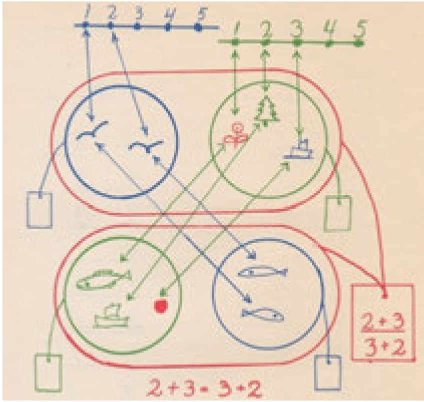


Figure 7. The commutative law. The first introduction of addition and the "+" symbol. Age 7 (Bundgaard and Kyttä, 1967, p. 1:32).

The commutative law was followed by the associative law for addition later in first grade, see Figure 8. Figure 9, demonstrating the distributive law, is the only illustration of people or daily life allowed in the textbook series. The author, Agnete Bundgaard, clearly stressed that she did not want the pupils' minds be distracted by illustrations unrelated to the topics presented. In a letter to Icelandic teachers she said: "Dear Colleagues. It is you who shall try to show the children that the subject <u>in itself</u> is fun and for that aim one can surely only use items that are <u>relevant for the subject</u>" (Bundgaard, 1968, a letter attached to a handbook for teachers). The textbooks for the 7- and 8-year olds were printed in colours, but later textbooks only in black printing.

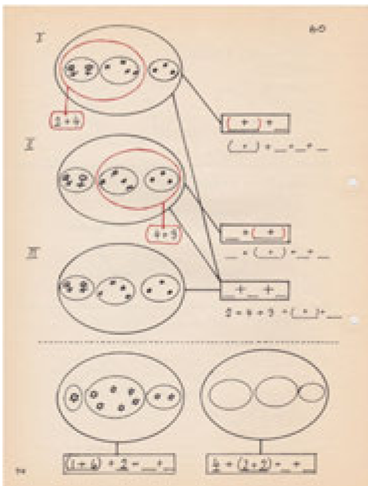
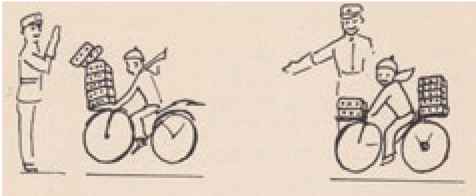
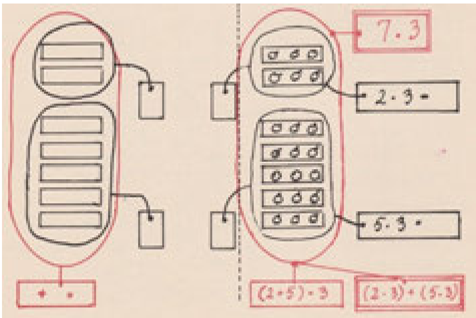


Figure 8. Associative law for addition. Age 7 (Bundgaard and Kyttä, 1967, p. 1:74).





Figures 9 and 10. Distributive law. Age 8 (Bundgaard and Kyttä, 1968, p. 2b:72).

The axioms of the number field were systematically introduced as indicated in Table 2. The commutative and associative laws for addition and multiplication were presented each after other at the age of 7, clearly expressing their names. The additive and multiplicative identities were presented at age 9. The multiplicative inverse was presented in connection with division of fractions at age 12. The additive inverse was not presented as negative numbers were not included in the series.

Table 2. Axioms of the number field in the Bundgaard series compared to other series.

Textbook Topic	SÁG	EB	Bundgaard	SFG	GP
Years in use	1911-37	1927-80	1966-80	1971-00	1999-10
Age level	10-13	10-12	7-12	7-12	6-12
The commutative law					
of addition			7	7	11
of multiplication	(10)		8	10	11
The distributive law			8	10	11
The associative law					
of addition			7		11
of multiplication			9	12	11
Identity - additive			9		12
- multiplicative			9		12
Inverse - additive				12	
- multiplicative			12		
0 in multiplication 0 in division	(11)		8 10		
Negative numbers				11	11
Inverse operations					
- addition –subtraction					11
- Multiplication – div.					11

In the section of the seminar report, Case for reform – A summary, psychological implications of learning procedures used in primary schools and the shift of school aims to developing concepts and modes of thinking were conceived to demand a corresponding change in arithmetic instruction, probably with the use of some kind of physical objects. Learning must be the result of understanding arising from guided experimentation and discovery. In this way, the child must be led to the abstraction of the quality of a set called its number. In getting to this abstraction it was necessary to use the ideas – but not necessarily the language – of sets, subsets, correspondence and order. A necessary part of the early instruction was the understanding and use of the decimal place number system of numeration. Brighter children could be introduced to the study of number relations involving odd and even numbers, primes, factorization, greatest common factor, least common multiple and place-numeration systems other than ten. There were also areas of disagreement such as the introduction at an early year to negative numbers, use of symbols such as 8 + 1, 7 + 2, … as another name for 9 rather than as operations (pp. 108–110).

One can hardly say that the Bundgaard series promoted guided experimentation or discovery. On the contrary, it emphasized the language of sets and set theory, while it also promoted carefully the study of number relations and the decimal place number system, see Table 3.

Table 3. Topics on numbers in the Bundgaard series compared to other series.

Textbook Topic	SÁG	EB	Bundgaard	SFG	GP
Years in use	1911-37	1927-80	1966-80	1971-00	1999-10
Age level	10-13	10-12	7-12	7-12	6-12
Number line			7	7	6
Number relations: even & odd numbers primes	 13 13	11	 8 9	 8 12	 8 12
Factorization, GCF, LCM	13	(12)	10	12	12
Divisibility – the placement system	13	(11)	9		12
Bases other than ten			9		
Modular systems			11		
Symbols as 7 + 2 for 9			8	7	
Variables, as placeholders - as quantities that vary				7	10 11
Equations				10	10
Probability				9	10
Statistics				8	10
Mental arithmetic	10		8	9	6
Approximation, estimation			11	8	9
Use of calculators					10

Table 3 shows that topics on numbers, such as primes and divisibility, were revived in the Bundgaard series from the early 1900s series SÁG. Bases other than ten and modular systems were novelties that awoke attention. They did not appear in later textbook series, whereas approximation and estimation did. The controversial use of symbols as 7 + 2 for 9, enjoyed some attention when used to promote mental arithmetic, such as 13 + 9 = 13 + (7 + 2) = (13 + 7) + 2 = 20 + 2 = 22.

The topic statistical averages was recommended in the original conclusions (Schubring, 2014a, p. 93) while otherwise probability and statistics, which appeared in later primary level textbooks, were recommended for secondary school level in the seminar report (OEEC, 1961, p. 106-107). Simple equations and variables also began to appear later.

Conclusions

The new topics in the Bundgaards series were primarily the use of set theoretical concepts and notation for building up the number concept and understanding of operations through repeated reference to the axioms of the number field, even if negative numbers were missing. The axioms were carefully introduced with respect to structure. One can therefore agree with Høyrup that the Bundgaard series went far towards meeting the mathematicians' demands. In later textbooks these concepts appeared more as aids to calculations than emphasizing structure.
Building up the system of natural numbers from primes and divisibility, and emphasis on mental arithmetic were revived topics that have survived in the school curriculum to this day. Approximation and estimation have also become permanent contribution to school mathematics in Iceland. What did not become permanent was replaced by introduction to statistics and probability, discussed at Royaumont but less recommended for primary level, and the use of variables and solving simple equations.

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John Dewey and mathematics education in Sweden

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Abstract

International comparisons such as TIMSS and PISA have shown that the mathematical skills of Swedish students have declined notably in the last fifteen years. The Swedish government has implemented disciplinary efforts with an increasing amount of national tests, and grades in early years have been suggested. Yet, the results of Swedish school students have not improved. We suggest that this negative trend is not only the result of a lack of discipline in the classrooms, but is also the effect of a more recent didactic turn in mathematics education: the emphasis on verbal and meta-mathematical knowledge at "the expense" of numeracy skills. In this paper we discuss the state of mathematics education in Sweden using a distinction made by John Dewey between the psychological and logical aspects of a subject. To exemplify our argument we consider curriculum documents, national tests and evaluation material from the last 50 years.

Introduction

The results of Swedish students in international comparisons such as TIMSS and PISA have significantly weakened in the last fifteen years (National Agency for Education, 2012, p. 108). The same tendency can be noted in national tests: the amount of students in secondary education who fail mathematics has increased considerably in the past fifteen years. Some argue that this is due to the progressive ideals that have dominated Swedish pedagogy since at least the 1970's, characterized by democratic classrooms, student participation and absence of homework and tests. As a reaction, the Swedish government has in the last decades implemented disciplinary efforts with an increasing amount of national tests. Grades in early years and grades for conduct have been suggested. Other efforts, such as financing research projects in mathematics education and special courses for mathematics teachers have also been arranged. Yet, the results of Swedish school students on TIMSS and PISA have not improved. We believe that this negative trend is not only the result of a lack of discipline in the classrooms, but perhaps also an effect of a more recent didactic turn in mathematics education: We suggest that during the past twenty years the emphasis on "verbal, meta-mathematical knowledge" has increased which may have affected the importance of numeracy skill in a negative way.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Here the expression "verbal, meta-mathematical knowledge" should be understood as the ability to describe mathematics and solutions to mathematical problems as well as to communicate mathematics with other people. A concrete example is when a student describes a strategy to solve a mathematical problem, but the actual solution is not included. The expression refers to an ability to talk *about* mathematics, that is a meta-mathematical knowledge. However, we would like to point out that our suggestion should be viewed as a tentative explanation regarding the negative trend of Swedish school students' results on TIMSS and PISA.

We will discuss the state of mathematics education in Sweden with the help of the terminology of John Dewey, an American philosopher who had a significant impact on the Swedish school system with its ideal of focusing on student interest and student activity. With tools provided by Dewey we aim to highlight the problem of separating verbal understanding and numeracy skills in mathematics education. We have compared curricula, national tests and evaluation material from the last 50 years in order to investigate the amount of mathematical knowledge as well as verbal, meta-mathematical knowledge. Here we will present two examples from our investigation, but first we will give a short introduction to Dewey's educational philosophy.

John Dewey's progressivism

John Dewey was an American philosopher (1859-1952), known as one of the main proponents of the American progressive movement in the beginning of the 1900's that tried to change the school from the highly authoritarian and disciplinary institution that it had been in most countries during the 1800's. This was a time when concepts such as student democracy, student motivation and activation were introduced. These ideas became influential in the US in the 30's, with the Lewin, Lippitt and White study, which concluded that groups led by democratic leaders were more productive, efficient and cooperative and less hostile and aggressive than groups with either autocratic leaders or totally permissive leaders. The influence of this study was enormous, in a time when people feared rising fascism and Nazism in Europe – it has been called one of the most educationally influential studies ever launched (Raywid, p. 253). This helped to boost the so called progressive ideals, and to spread the influence of Dewey in the US and in other countries, such as Sweden.

Let's have a look at Dewey's educational philosophy. He argued that the teacher shouldn't be a dictator but someone who can activate and motivate the students. The teacher should "psychologize" the subject, i.e., make it interesting for the students through relating it to their experiences (Dewey, 1959, p. 105). This is important because Dewey regards learning as an active undertaking, something that the students have to participate in if they are really to learn anything of value. If the motivation to learn is external to the subject (punishments and rewards, as well as grades), the students only learn to meet the requirements, to say and do what the teacher wants them to – or, rather, to *seem* to meet the requirements, because what they really learn is unimportant as long as it looks good in the eyes of the teacher (Dewey, 1966, p. 156). And since competition rather than cooperation is encouraged in this kind of environment, it fosters selfish and aggressive individuals who only care about their own accomplishments. Instead, Dewey thinks, the motivation should come from the task itself, from a problem or difficulty that the student wants to solve. This encourages real understanding and learning, as well as cooperation. And if we want to foster democratic citizens, we should teach children not only to memorize data and follow orders, but to think critically about the information given so that they can form their own opinions. At the bottom lies an ethical ideal of democracy and the open society (Dewey, 1966, pp. 301, 356).

Dewey in Sweden

The so called progressive ideals came to Sweden from different sources (for example via the German and Austrian Arbeitsschule). And even if Dewey had been translated into Swedish as early as 1902, his influence on the Swedish school dates most clearly to sometime after the Second World War. Here, too, the idea was to avoid the horrors of fascism and to instil democratic values in the students. A school commission was appointed by the government in 1946, with the purpose of investigating the possibility for a common, compulsory school for all children. Through one of the members, Alva Myrdal – who, with her husband Gunnar Myrdal, is known as one of the main driving forces in the creation of the Swedish welfare state – Dewey came to influence the formation of the Swedish school (Hartman, Lundgren & Hartman, p. 34). The Myrdals had recently stayed in the US and become very impressed with the progressive school movement there, and brought those ideas with them to Sweden. The recommendation that the commission wrote to the Swedish government stated: "In school, the individuality and personal capacities of each child must not only be paid attention to and respected, but be the actual starting point" (Myrdal, p. 115).

So the progressive, democratic ideals formed the modern Swedish school. But now the era of progressive ideals seems to be over, not only because of privatization of the schools and a bigger stress on individualism and market needs, but also because of bad Pisa results during many years. Many blame the progressive ideals of student democracy and participation, they argue that the teachers lack authority and that the impulses of the students set the agenda, rather than hard arguments and knowledge. And, allegedly, this is why Sweden is doing so poorly on international tests. In the last ten years or so the politicians have therefore been eager to increase disciplinary efforts, traditional teaching (with the teacher standing in front of the class talking rather than activating the students), more testing and earlier grading.

But these measures have not improved the results, on the contrary, they keep declining. And furthermore, the results on international mathematics tests didn't actually start to decline until quite recently, in the 1990's Sweden was still doing fairly well. So it seems that we can't blame Dewey's influence for the failure of school mathematics. And if we look at what Dewey actually said, it becomes clear that he did not undervalue knowledge in the way that is often thought. He thought of the "psychologization" of knowledge (relating the subject matter to the experiences of the students and making them interested) as a *starting point* of education, not the end point (Dewey, 1959, p. 99).

According to Dewey you can view a subject, like mathematics, from two view points, a psychological and a logical. To emphasise the psychological aspect is to stress student interest and experiences and to relate the subject matter to what the student is familiar with. To emphasise the logical aspect is instead to focus on the subject as it appears to the expert, as an abstract body of knowledge driven by its internal laws and rules. The teacher needs to be familiar with both of these aspects in order to be able to teach the students: The teacher of mathematics must master mathematics as an abstract body of knowledge, but also know how it can be made intelligible and interesting to the students. Dewey's point is that these aspects cannot be separated, they are both part of a well functioning education. He says that the child's interest (the psychological aspect) and the subject (the logical aspect) are two limits that define a single process: "Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction" (Dewey, 1959, p. 97).

This means that the teacher ideally starts from what interests the students, but that which gives direction to the instruction, the goal, is the "organized bodies of truth"– mathematics as abstract knowledge. Education, then, is what goes on in between these two defining points, the movement from student experience to abstract subject matter.

When the small child is asked to count three apples in a basket, and to count again after one apple has been removed, s/he is dealing with concrete, physical things. But at the same time the child is taught an abstract operation: 3-1. Here the instruction is psychologized, adapted to the child's level and to his/her interest in colourful, eatable things. But learning won't stop there, counting apples will make possible other, more abstract operations. During the years to come the instruction will become more and more abstract and move toward what Dewey calls the logical aspect of mathematics: toward mathematics as a goal in itself without a connection to practical interests. But it is important to keep in mind that these two aspects are interdependent. They are two sides of the same coin, impossible to categorically separate: the logical aspect, the part of mathematics that is driven forward by the internal development of the field rather than by the need of practical applications, would be unthinkable unless mathematics also played a practical role in human life. And the practical applications of mathematics (in for example technology) are dependent on the progress made in the theoretical field of mathematics research.

We think Dewey's psychological perspective, where student interest and practicality are central, can be likened to a didactical perspective on mathematics, whereas the logical perspective includes the more abstract mathematical practices that are part of higher mathematics, such as for example calculus, algorithms and proofs. Ironically, the recent turn away from the progressive ideals in mathematics education in Sweden hasn't resulted in the students acquiring "real knowledge" instead of "only what the students are interested in", as the proponents of these disciplinary measures claim, but rather, we argue, the opposite: As we will show there has recently been a tendency to downplay the importance of teaching students to calculate, and to think that they don't need to learn algorithms and rules.

This is due to a recent research trend in which verbal, meta-mathematical skills are emphasized at the expense of numeracy skills in mathematics education, or, with Dewey's terminology, to emphasise the psychological at the expense of the logical aspects of mathematics. This trend seems to be the result of a tendency to see numeracy as a potentially mechanical process, and therefore it is thought that the genuine mathematical understanding is best expressed verbally. We do not have the room to expand on the grounds for this view in the present paper, we can only note that the reasoning behind the calculus is viewed as separate from the calculus and therefore to be tested separately. This is why traditional math tests won't do. Swedish students used to be able to complete a math test and get full points for using the correct mathematical methods and arriving at the right answer. This is not the case anymore, as we will show in the next section.

Two concrete examples

In this section we will consider two examples from our investigation of Swedish curricula and evaluation material from national tests in school mathematics.

Swedish curricula

In 1962 a reform took place within the Swedish school system. The old school form was replaced by the primary school (grundskolan) which was implemented during a ten year period (Prytz, 2010, p. 310). The first curriculum was published in 1962 and has been replaced by five new curricula published in the following years: 1969, 1980, 1994 and 2011. In our investigation we have compared these five curricula.

In general one can deduce that the amount of "everyday mathematics" and practical mathematics have increased over time, while the pure mathematical content has decreased. The recent curriculum stresses that mathematics not only consists of calculations and "learning rules by heart", a large portion deals with the usage of mathematics as a tool, language and resource in order to solve practical problems related to private economy, social life, electronics, newspapers and medicine dosage (National Agency of Education, 2011).

We do not claim that it is wrong to focus on practical skills in mathematics, instead our point is that during the past fifty years the numeracy skill and practical skill have been separated and the latter seems to have been emphasized at "the expense" of the former. With Dewey's terminology one could perhaps argue that the goal of mathematics education today is the psychological aspect of mathematics rather than the logical. We will return to this issue in the next Section where we consider national tests.

A related tendency within Swedish school mathematics is to emphasise the ability of *communicating* mathematics rather than the ability to calculate. It seems that verbal understanding has gained importance compared to previous years. A typical example is that the Swedish curricula from 1962, 1969 and 1980 are based on the different topics of school mathematics, for instance arithmetic, geometry, algebra and probability theory. Meanwhile, the two most recent curricula, from 1994 and 2011, are both based on general and verbal, meta-mathematical *abilities* that are the same for all school levels and every different topic of school mathematics. The abilities in the current curriculum from 2011, *Läroplan för grundskolan, förskoleklassen och fritidshemmet (Lgr11)*, are the following:

1. Conceptual ability
2. Procedural ability
3. Problem solving ability
4. Modelling ability
5. Reasoning ability
6. Communication ability
7. Relevance ability

The commentary material to the 2011 curriculum of mathematics emphasises that students should learn to use "metacognitive reflections in order to think out loud, look for alternative solutions as well as discuss and evaluate solutions, methods, strategies and results" (National Agency for Education, 2011). This is related to "Bloom's taxonomy" from the 1950's where the aim was to achieve a system in order to categorize different levels of learning abstraction based on cognitive skills such as to describe, analyse, compare and evaluate (see Bloom et. al., 1956). (This will be further clarified in our next Section where we discuss national tests.)

The implementation of general and verbal, meta-mathematical abilities in Swedish curricula seems to be a result of an international trend within the research field of mathematics education. Over the past 20 years researchers have made efforts in order to understand what "mathematical skill" really means. One example is the American report "Adding it up" whose purpose was to describe mathematical knowledge by means of different *competencies* (Kilpatrick, et. al., 2001). A similar example is the Danish "KOM project" (Competencies and the Learning of Mathematics), initiated by the Danish Ministry of Education and led by Mogens Niss, whose aim was to describe mathematics curricula on the notion of mathematical *competencies* rather than on syllabi in the traditional sense of lists of topics (Niss, 1999; Niss & Höjgaard-Jensen, 2002). The competencies introduced in the "KOM project" are the following: Thinking mathematically, posing and solving mathematical problems, modelling mathematically, reasoning mathematically, representing mathematical entities, handling mathematical symbols and formalisms, communicating in, with and about mathematics, and finally making use of aids and tools (Niss & Höjgaard-Jensen, 2002).

As Helenius (2006) points out, our Swedish 1994 curriculum can be traced to the idea of competencies and we argue that the similarity becomes even clearer in the 2011 curriculum. Furthermore, we agree with Helenius that one of the advantages of using competencies is that it makes it easier to describe the progression in the curricula. But at the same time we believe that it is difficult to describe mathematical progression by means of general and verbal competencies (in a similar way as in Bloom's taxonomy) since mathematics has an "internal" progression *between* its different topics as well as *within* each topic. The former means that you for example need a certain amount of arithmetic before you can learn algebra, while the latter can mean that you must acquire basic algebraic skills before you can learn more complex algebraic structures such as "groups" or "rings".

We will now turn our attention to an additional problem; how do we measure general and verbal competencies in mathematics? We will investigate this issue by considering examples from evaluation materials of Swedish national tests.

Swedish national tests and their evaluation material

National tests were introduced in Sweden in the 1960s. During the years the tests have been given in various subjects and various grades. Today there are national tests in mathematics in grades 3, 6, 9 and at upper secondary school level. The tests are divided into different parts, for instance, during the last couple of years the national test in mathematics for grade 9 has consisted of four parts (A-D). Part A examines the student's ability to verbally express and follow mathematical reasoning and the ability to comment on other students' explanations and arguments. Part B consists of tasks that should be solved without digital tools or given formulas. Part C consists of a more comprehensive task of investigative nature where the solution should be clearly described. Finally, part D consists of tasks belonging to a certain theme (National Agency for Education, 2013).

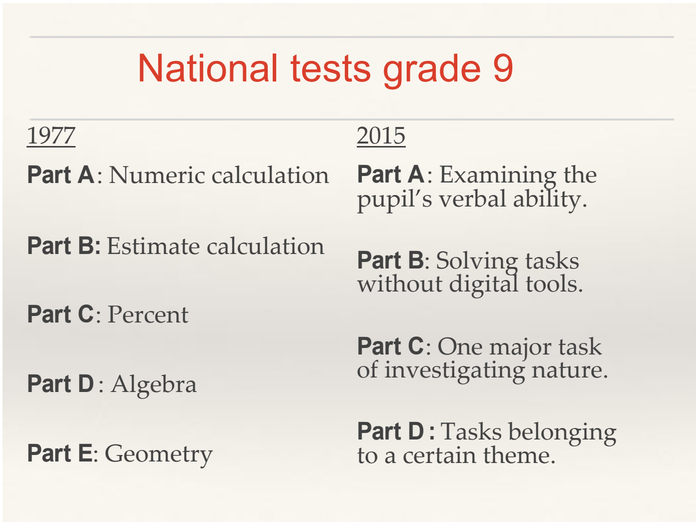

Fig. 1. The different parts of the national test in 1977 and 2015, respectively.

It is interesting to compare the current national tests with national tests from the 1970s, 1980s and the beginning of the 1990s. For instance, in 1977 the national test in mathematics for grade 9 consisted of five different parts; numeric calculation, estimate calculation, percentage, algebra and geometry (National test in mathematics, grade 9, 1977). If we compare these five parts with the four different parts of today's national test (mentioned above) there is an essential difference. In 1977 the different parts were classified on the basis of different topics of mathematics, meanwhile, today the classification is based on more general aspects such as verbal reasoning, access to tools, problem solving, investigation and different themes (see Figure 1).

Another difference between the national tests from the 1970's, 1980's and the beginning of the 1990's compared with recent national tests is the evaluation material. For instance, in 1977 the evaluation material consisted of an answer key of 1-2 pages for each part of the test. Meanwhile, today there are around 20 pages of evaluation material for each part of the test, that is, around 100 pages for the whole test. Clearly, the amount of evaluation material has increased over time, and we believe that the main reason for this is the difficulty of measuring general and verbal, meta-mathematical abilities compared to numeracy skills. In fact, today's national tests in mathematics are based on the general abilities from the 2011 mathematics curriculum (which were enumerated in our previous Section). In order to understand the complexity of evaluating today's national tests let us consider the evaluation material to the test given in grade 9.

In Sweden the grading scale is A-F where A is the highest grade, E is the lowest passing grade and F stands for “failed”. In the students’ test paper for grade 9 the maximal score at each assignment is denoted on the basis of the different grades, for instance, an assignment with the notation (3/2/1) means that the maximal score is 6 where 1 point is at the E-level, 2 points at the C-level and 1 point at the A-level (observe that this notation always refers to the A-, C-, and E-levels). In the teachers’ evaluation material the points are not only connected to different grades, they are also connected to the different abilities (which were enumerated in the previous Section). Here the abilities are denoted C (Concept), PR (Procedure), P (Problem solving), M (Modelling), R (Reasoning) and K (Communication). In the evaluation material the notations E_P and A_R should be interpreted as one “problem solving point at E-level” and one “reasoning point at A-level” respectively.

	Lower E	Middle C	Higher A
Problem solving and method (P)	Reads the graph or draws simple conclusions E_P	Interprets the relationship between the variables and describes how the graph shows change over time C_P	Draws correct conclusions from the graph A_P
Concepts (C)	Shows basic knowledge of the gradation of the axes in the diagram E_C	Expresses knowledge of how the inclination of the graph shows the change of the water level *or* that the structure of the graph is connected to the geometric form of the tank C_C	Reasons correctly about how the form of the tank *or* about the proportions between its parts *or* about the structure of the graph with respect to volume/ time. A_C
Reasoning 1 (R)	Shows simple reasoning skills with respect to the graph or the gradation of the axes E_R	Reasons about the structure of the graph and its connection to the form of the tank C_R	Shows well developed reasoning skills with respect to the form of the tank *or* the proportions of its parts *or* about the structure of the graph in relation to volume/ time A_R
Reasoning 2 (R)	Contributes with a question or comment that to some extent advances the reasoning of other students E_R	Contributes with ideas and explanations that advances the reasoning of other students C_R	Takes part in the argumentation of other students and develops and expands their reasoning A_R
Communication (K)	Expresses her/ himself simply and the train of thought is easy to follow E_K	Expresses her/ himself clearly with the suitable mathematical language C_K	Expresses her/ himself with certainty and consistently uses the relevant and correct mathematical language A_K

Fig. 2. Evaluation table for part A of the national test in mathematics, grade 9. (PRIM-gruppen).

In Figure 2 an example of an “evaluation table” for part A of the national test in mathematics for grade 9 is given. Part A is an oral exam carried out in groups of three to four students. The test consists of one assignment, in this case the problem deals with a water tank that is pumping out water and a graph that shows the change in water level over time. In Figure 2 the vertical axis consists of the abilities evaluated in this test; problem solving (P), concept (C), reasoning (R), and communication (K). On the horizontal axis the grades A, C and E are given. The maximal score of this assignment is 15 points distributed in the following way: a maximum of 3 points for problem solving ability (the upper row in Figure 2), 3 points for concept ability (the second row in Figure 2), 3+3 points for reasoning ability (rows three and four in Figure 2) and 3 points for ability to communicate (the bottom row in Figure 2).

Let us consider the reasoning ability in the evaluation table in Figure 2. In order to get one point at the E-level (that is E_R) the student must "contribute with a question or a comment" to the group's discussion. To achieve the higher grades in the reasoning ability category, C_R and A_R, the student must be able to "contribute with ideas and explanations that advance the reasoning of other students" and "develop and expand the reasoning of other students". Note that the mathematical content is not mentioned, neither is the volume, graph or the velocity of the water. Clearly, within the "reasoning category" the focus is on verbal understanding rather than any numeracy skill.

The same tendency appears in other categories as well, for instance, in order to get the grade E_K within the "communication category" the student must be able to "express her/himself simply and the train of thought should be easy to follow". Moreover, to get the highest grade A_K in the same category the student must be able to "express her/himself with certainty and consistently use the relevant and correct mathematical language". That is, to get the highest grade in mathematics in grade 9 one must not only be able to use a correct mathematical language, one is also required to express this language with certainty. Consequently, a student who is shy or introvert may find it difficult to get the highest grade in mathematics, regardless of her/his ability to solve mathematical problems.

Final remarks

On the basis of our two examples we can conclude that in mathematics education in Sweden today it is not sufficient to give the correct solution to the mathematical problems – the students must also be able to reflect, discuss and evaluate their solutions, methods and results. This would not necessarily have to be problematic, but taking the evaluation material into account, it seems that the verbal, metamathematical abilities are emphasized at "the expense" of the numeracy skills. One could perhaps say, with Dewey, that in Sweden the logical aspect of mathematics has been separated from the psychological. A typical example of this is the structure of the evaluation material discussed above; the problem solving ability (which can be viewed as a logical aspect of mathematics) and the ability to communicate (which can be viewed as a psychological aspect of mathematics) are measured separately.

A potential risk of this separation is that the requirement of verbal ability becomes an obstacle for those having numeracy skills but a weak self-confidence or verbal ability. Another potential problem with a tendency that emphasises verbal, meta-mathematical abilities at "the expense" of numeracy skills is that our future teachers prefer to discuss mathematics rather than solving mathematical problems, since they have not practiced enough numeracy during their teacher training. If the teachers have been taught mathematics from a meta-perspective without first having learned to calculate properly, it will be difficult to manage the declining TIMSS- and PISA results in Sweden.

In this paper we have focused on mathematics education in Sweden, but one should have in mind that Sweden is not the only country where verbal, meta-mathematical knowledge have gained much greater importance in the curriculum compared to numeracy skills over the last 20 years. An interesting next step within our project would be to consider the curricula and TIMSS results in Sweden's neighbouring countries Finland and Norway. The results of the latest TIMSS tests in Finland have been much better compared to Sweden's results. However, they have recently (two years ago) implemented a new curriculum which effects we cannot draw any conclusions of yet. Moreover, Norway has improved their TIMSS results both in 2003 and 2011 (see Yang Hansen, Ed., 2014). An interesting future project would be to investigate the design of the curricula in Finland and Norway with particular attention on verbal, meta-mathematical abilities.

Just like Dewey, we think that the logical and psychological aspects of mathematics go hand in hand: These two should not be separated in teaching or in testing mathematical understanding. If you think of the psychological aspect as freestanding and independent, you might very well think of the ability to reason about and to discuss mathematics from an external perspective as the important part, and the calculating, which can look mechanical, as unimportant. But then you end up with students who don't know how to solve mathematical problems correctly, which has been shown to be the case with Swedish students in international as well as historical comparisons. This prompts the question whether they really have the needed mathematical understanding.

We can conclude that the progressive ideals of John Dewey are not to blame for the declining mathematical skills of Swedish students. Rather, if understood correctly, his views could provide us with the solution.

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Mathematics in the initial pre-service education of primary school teachers in Portugal (1926-1974)

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Abstract

Aiming to know the governmentally issued order for the teaching of mathematics in the courses of schools in the initial pre-service education of primary school teachers during the Portuguese New State regime (1926-1974), this article analyses the mathematics curriculum of Primary Teachers Training Schools during the study period. The main sources used are the official Ministerial Diary (Diário do Governo) that published all legislative pieces. The study is located within the perspective of the history of mathematics teaching.

The military dictatorship established in 1926 and especially the so-called New State that followed, influenced the formation of teachers and schools where this training was made. The period 1926-1936 was marked by numerous legislative measures. In 1930, the Primary Normal Schools (Escolas Normais Primárias), where the training of future primary school teachers was performed, were renamed, becoming Primary Teachers Training Schools (Escolas do Magistério Primário), designation that will kept until their extinction.

The mathematical content of the syllabi for Primary Teachers Training Schools of 1943 was essentially the same as primary school mathematics syllabi which meant that what were essentially taught were the didactic and methodological dimensions of teaching those contents. They did not address mathematics scientific content.

The reshaped curriculum framework and syllabi of 1960 strengthened the discussion about teaching methodologies either by separating Special Didactics A, discussing teaching of humanities, from Special Didactics B, addressing natural sciences and mathematics teaching methodologies, or by allotting more class time for these disciplines. However, the syllabi remained identical to the 1943 syllabi.

Introduction

The schools that did the training of future primary school teachers played a central role in shaping pedagogical knowledge in Portugal (Nóvoa, 2003). Historical research has been addressing in detail schools and conceptual proposals (Chervel, 1990), valuing the autonomy of school subjects, proved productive, particularly with regard to the study of representations associated with the subjects called Education or Teaching allowing to map the initial development of pedagogical think-

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

ing in Portugal (Pintassilgo, 2012). This study aims to contribute to the understanding of mathematical knowledge and its teaching developed in schools for the formation of primary teachers between 1926 and 1974 and complements a previous work focused on the period 1772-1910 (Candeias & Matos, 2016). The school culture, as proposed by Julia (1995), distinguishes between norms and practices. This paper focuses on the first aspect, and it is a historical study based on legislation (laws, decrees, orders, circulars and instructions sent to schools). At this stage of the investigation we sought to collect data, organize and present it, without performing a contextualization of the development that mathematics teaching had in this kind of schools in other countries. The main sources were the legal documents, published in the official Ministerial Diary (Diário do Governo) between 1926 and 1974 and we followed a documentary historical study methodology (Mcculloch, 2004). The analysis of the documents followed a qualitative approach according to Creswell (2012). The first step was the preparation and organization of the collected documentation. Then we proceeded to a descriptive analysis of the general ideas contained in each document. The documents were analyzed independently of each other, trying to get a sense of the whole and finding them common features. At this stage we organized a first scheme with the themes to be analyzed, giving a name to each one. In the following phase was made a detailed analysis of each document with the division of the text into different themes, according to what was identified in the descriptive analysis. This process involved the partition of paragraphs or phrases into categories, giving them an evocative name. Then we proceeded to an interpretative analysis of the data, trying to establish a relationship between the various themes. At this stage we made a representation of the results through a narrative and tables. In the last phase was made an interpretation of the significance of the results. This phase led to the drafting of conclusions.

Thus, after a brief characterization of the evolution of teacher education for primary schools, the text details the presence of mathematics: 1) in the entrance examinations of schools for the initial pre-service education of primary teachers, 2) in the curriculum framework, the disciplines and the syllabi of these schools, and 3) in the qualifications of the teachers who taught in the schools. In this work curriculum framework is understood as the disciplines, there name and the number of hours they occupy in initial training courses for teachers. The syllabus is defined as the contents that are part of the disciplines.

The development of teacher education for primary schools until 1974

The second half of the eighteenth century was a defining moment for the beginning of the composition of the teaching profession in non-higher education in

Portugal. In the framework of an absolutist regime still centered on the royal power, the reform of Prime-Minister Pombal in 1772 regulated for the first time the professionalization of teachers in the areas of reading, writing, and arithmetic. The admission criteria, the rules for professional certification, the profession and their remuneration came out of the Church's domain and became a responsibility of the state (Nóvoa, 2003; Pintassilgo, 2012).

From the late eighteenth century on it was no longer allowed to teach without a license or government approval, which was granted after a test that could be claimed by individuals who met certain conditions, such as qualifications, age and moral behavior. This decision helped to define the professional field, becoming endowed to teachers the exclusive right to intervene in this area. However, a teacher training system was still very far from completion. The access to the profession through qualification exams would be valid until 1901.

From 1816 until 1860, the idea that to become a teacher it was necessary a long training held in institutions created specifically for this purpose consolidated. At this time the first attempts to promote the training of teachers appeared, for example the Normal School for Qualification of Teachers of the Military Regimental Schools (1816-1818) and the Normal School of Mutual Education and Normal School of Lisbon (1824-1835). These institutions were created on an *ad hoc* basis and without a national plan and remained until 1869.

In 1835 the creation of two normal schools in Lisbon and Porto was proposed, and in 1836 it was planned to build mutual schools in the capital of each administrative region that would function simultaneously as normal schools. Only some of these schools opened and its activity remained irregular. The reform of the minister Costa Cabral (1844) proposed a new primary teacher training system and in 1845 established a Normal Primary School in Lisbon. However, this school was never put into operation. (Pintassilgo & Mogarro, 2015).

The gradual homogenization of the school system that occurred throughout the nineteenth century in Portugal (Barroso, 2005), and formalized secondary education, also led to a structured teacher education for primary schools in Portugal. This occurred with the publication of the regulation for the Primary Normal School in Lisbon in 1860, followed in 1862 by the commissioning of the Primary Normal School of Marvila, Lisbon. This school, a boarding school supported by the state, intended for the training of primary male school teachers, was the first integrated official Primary Normal School in a wider plan of teacher training for primary education.

Other schools were gradually established (Candeias & Matos, 2016), but only from 1901 the qualification for the practice of primary school teaching began to depend on the mandatory approval in the course of the Primary Normal Schools ending the possibility of people without this course to gain access to the profession. In 1910, at the end of the monarchy in Portugal, there were six central Primary Normal Schools in Lisbon, Porto and Coimbra (one for each sex in each of these cities) and seventeen Habilitation Schools for Primary Teaching (Escolas de Habilitação para o Magistério Primário), that constituted a district network of primary teachers training schools, where teachers, by attending the course, obtained the license that enabled them to teach. This set of schools formed more teachers than what was deemed necessary, and many become unemployed or were working in activities unrelated to teaching. This would extend until 1921, when schools were closed and replaced by the new republican Primary Normal Schools. This republican reform intended to improve the quality of primary standard education, focusing the training of future teachers in specific issues of the teaching profession and in the republican values (Pintassilgo, 2012). This training model was characterized by a secular education, coeducation regime, higher requirement for access to training schools and fewer training schools. With regard to the curriculum framework and the syllabi, the main features were the importance given to scientific disciplines and educational sciences and the strong influence of the New Education (Pintassilgo, Mogarro, & Henriques, 2010).

In Portugal, the military dictatorship implanted in 1926 and especially the New State regime (Estado Novo) that followed aimed to change the formation of primary school teachers (Pintassilgo, 2012). In 1930, still in the transition from military dictatorship to the New State regime, the Normal Primary Schools (Escolas Normais Primárias) were replaced by Primary Teachers Training Schools (Escolas do Magistério Primário) involving a radical change in school organization, the curriculum framework and, in 1943, the syllabi. The implementation of the intended training models of the 1930 reform was not an easy task for the government. There were many opponents on the educational field who were supporters of the previous existing training model, very influenced by the republican regime (Baptista, 2004). This political conflict extended beyond the normal primary schools and encompassed all levels of education, leading to the imprisonment of prominent educators (Carvalho, 1996).

Claiming that there were an excessive number of graduates, the Government suspended enrollment in Primary Teachers Training Schools in 1936. As a consequence, in 1940, the lack of primary school teachers required a fast and emergency recruitment of "professionals" and individuals whose education was limited to 3rd grade of primary school were enrolled as primary teachers. In 1942, the government took the decision to reopen the Primary Teachers Training Schools, recognizing that this model was better for the formation of primary teachers and, in 1943, syllabi for the disciplines were published. The last reform under study occurred in 1960 when a restructuring of the course was adopted. The period under scrutiny ends in 1974 when a democratic revolution introduced new changes to these schools.

Admission to schools for primary teacher education

In the early years after the military coup of 1926, the conditions for access to Primary Normal Schools remained similar to the previous period. Candidates had to complete the middle secondary school (9 years of schooling) and perform entrance tests that included arithmetic and geometry.

With the institution of Primary Teachers Training Schools, in 1930, the minimum qualifications became very low (4th grade of elementary primary education) and students with the middle secondary school were exempt from entrance exams. Table 1 summarizes the admission conditions.

Table 1. Admission conditions for schools for primary teacher education. Source: Decrees (1928, 1929, 1930, 1931, 1932, 1936, 1942, 1960)

Year	Age	Qualifications	Entrance exams
1928	14 years old	Examination pass of the 2nd cycle of middle secondary school (6th grade).	- Written, practical and oral tests. - Written test: an arithmetic problem (1 hour), and perform a geometric drawing (2 hours). - Oral tests: arithmetic and geometry; drawing.
1929	14 years old	Middle secondary school (9th grade).	- Written, practical and oral tests. - Written tests: one arithmetic and one geometry problem (90 minutes); perform a geometric drawing (2 hours). - Oral tests: arithmetic, geometry, drawing
1930	16 years old	4th grade of primary education.	- Written, practical and oral tests. - Candidates with the middle secondary school were exempt.
1931	16 years old and no more than 35 years old	No qualification mentioned.	- Candidates with the middle secondary school were exempt.
1932	15 years old and less than 36 years old	4th grade of primary education.	- Written and oral tests, practices. Written tests or practices were eliminatory. - Candidates with the middle secondary school were exempt
1936-42	(Enrollments suspended)		
1942	16 years old and less than 28 years old	Middle secondary school (10th grade).	Written and oral tests. Mathematics was the subject of one of three tests.
1960	16 years old and no more than 28 years old	Middle secondary school (9th grade).	- Written (90 minutes each) and oral tests (15 minutes per subject). Arithmetic and geometry were the subjects of one of three written tests.

The amendments to the legislation made in 1931 and 1932, confirmed the conditions for admission to the schools set in 1930. In the reopening of schools in 1942 the qualifications were changed, returning to be required as a minimum qualification the middle secondary school, just as it was before 1930.

Throughout most of the period under review, the access to the profession was made through training schools. However, from the end of the 1930s onwards, access to the profession could also be done through an examination to which individuals with the 3rd grade of primary school could apply. This meant a throwback in the professionalization of primary school teachers.

In 1960, the conditions of admission to the Primary Teachers Training Schools were maintained but those who were in the profession for at least for 5 years without a course could be admitted with no entrance exam.

Mathematical topics of the entrance exams

Mathematical contents of the entrance exams in 1928 essentially included topics of arithmetic, geometry, and elementary algebra (table 2).

Table 2. Mathematical topics of the entrance exams in 1928. Source: Decree n.º 16.038, October 15, 1928

Arithmetic
- Whole numbers and respective operations. - Fractions, decimals, and respective operations. - Powers and root extraction. - Commercial calculation. - Legal system of weights and measures.
Geometry
- Fundamental notions of geometry. - Volume, surface, line and point concepts. - Axioms and fundamental postulates of geometry. - Plane Geometry: angle triangles, quadrangles and circumference. - Areas. - Straight lines measurement and its division into equal parts, construction line segments perpendicular, parallel. - Measuring angles and their division into equal parts. - Problem relating to the circumference. - Construction of regular polygons, scale, oval, ellipse, hyperbola, and parabola.
Elementary algebra
- Algebraic expressions. - Advantages of using symbols as a way to simplify and letters as a means of generalization. - Calculation of the numerical value of an algebraic expression, negative numbers, signs rules, algebraic operations on monomials and polynomials. - Study of algebraic fractions. - Study of 1st degree equations with one unknown. - Resolution of the 1st degree problems.

Contents for the entrance exams for the period under analysis were only stated in 1928. In the regulation for the Primary Teachers Training Schools, published in 1931, the topics for the exams were kept very similar to the ones of 1928. During the remainder of the period, these topics were not subject to regulation again.

Mathematics in the design of the courses

Although numerous legislative measures were adopted during the period from 1926 until 1936, some uncertainty about educational policy can be detected and the proposals for teacher education were not consolidated (Pintassilgo, 2012). As mentioned earlier, in 1930 Primary Normal Schools became Primary Teachers Training Schools. In 1936, enrollments were suspended (Decree n.º 27.279, 1936) and they would only be reopened in 1942. The schools were then reconfigured, put under control of the central government, and adapted to the values of the new regime. (Decree n.º 32.243, 1942). This situation remained until the end of the New State (Mogarro, 2001; Pintassilgo, 2012).

In 1960 we witness an extension of compulsory education and the approval of new syllabi for primary education, which required a professional development of teachers of higher education (Decree n.º 43.369, 1960). However, it did not produce fundamental changes in the structure and operation of teacher education (Pintassilgo, 2012). This regulation would last until 1974, the end of the period under study.

The course duration for primary teacher education has undergone changes over the period under study and table 3 synthesizes its duration, defined in legislation.

Table 3. Course duration in schools for primary teacher education – 1928-1960. Source: Decrees (1928, 1929, 1930, 1931, 1932, 1936, 1942, 1960).

Year	Course design
1928	Four years
1930	Two years (4 semesters)
1931	Two years (4 semesters)
1932	Three years (6 semesters)
1936-42	(Enrollments suspended.)
1942	One year and a half (3 semesters)
1960	Two years (4 semesters)

Initially, the course had duration of four years, inheriting the structure from the previous regime. Afterwards, the duration of the course ranged from a minimum of 1 year and a half (1942) and a maximum of three years (1932). Between 1936 and 1942 enrolments were suspended. It should also be noted that after the period of closure of these schools between 1936 and 1942, which led to the profession many untrained teachers, these schools became highly sought. In the period between 1943 and 1974, the primary school teachers trained in these schools increased from 810 in the school year 1943-1944, to 2792 in the school year 1969-1970.

Disciplines with mathematics contents

Table 4 summarizes the names of the disciplines relating to mathematics distinguishing those with a mathematical content from those focused on the ways to teach mathematics.

After the military coup that imposed the dictatorship, the structure of the course was changed in 1928. Initially, the course has a discipline with scientific mathematical contents and a methodology discipline, maintaining a structure that was common since the first courses in the mid-nineteenth century.

Table 4. Disciplines of schools for primary teacher education with mathematics contents per reform. Source: Decrees (1928, 1930, 1931, 1932, 1936, 1942, 1960).

Year	**Disciplines**	
	With mathematical content	**With didactical content**
1928	Elementary Mathematics	Methodology
1930		Teaching and practice at the application school
1931		Didactics; Practice in the application school
1932		Didactics
1936-42	(Enrollments suspended)	
1942	Drawing and educational handwork	Special didactics; Teaching practice
1960	Drawing and educational handcrafts	Special didactics group B (Arithmetic and Geometry, Geography, Natural Sciences and Educational Handcrafts)

Following a trend of lowering the mathematical requirements for educational courses and exams (Matos, 2014), the institution of the Primary Teachers Training Schools in 1930 removed the discipline with mathematical scientific content. At the same time, the practical character of the course was strengthened, with a higher workload of disciplines with educational content. In the reopening of the Primary Teachers Training Schools in 1942, mathematical scientific content was limited to geometry in the discipline of Drawing and Educational Handcrafts. Methods for teaching mathematics were taught in the Special Didactics discipline.

The reshaped course of 1960 strengthened the teaching methodologies either by separating Special Didactic A, specific for the teaching of the humanities, from Special Didactic B, with the teaching of natural sciences and mathematics, or by providing more class time for these disciplines. Special Didactic B was taught by a generalist teacher with specialization in mathematics teaching methodologies. This specialization does not mean that teachers had obtained any specialized training in mathematics teaching methodology. It just meant that those teachers ended up specializing in mathematics methodologies because they exercised exclusively, and did not have to teach humanities methodology, like happened before. Despite the changes made to the course in 1960, the syllabi were not changed.

Syllabi of the disciplines that relate to mathematics

The 1928 Normal Primary Schools Reform

Changes made to the Normal Primary Schools published in 1928 did not present new syllabi for the disciplines. Decree n.º 16.037, of October 15, 1928, only referred that the syllabi of 1919 (Decree n.º 6.203, 1919) should be used.
In those syllabi the mathematics contents were essentially addressed in two disciplines: Elementary Mathematics and Methodology. The Elementary Mathematics syllabus had three parts:

1st - Review of mathematical knowledge already acquired by the students, in order to enable them to teach the rudiments of arithmetic and geometry.
2nd - Development of the knowledge acquired in middle secondary school, so that students realize the educational value of elementary mathematics, their application in other branches of learning and its value in social action.
3rd - Development of methodological knowledge of the teaching of arithmetic and geometry in primary education.

The Methodology discipline essentially presented aspects related to the teaching methods and the use of didactic materials.

Table 5. Main topics of the disciplines with mathematical content – 1928. Source: Decree n.º 16.038, October 15, 1928.

Elementary Mathematics	**Methodology**
- Whole numbers and respective operations. - Fractional numbers and decimal numbers. - Potentiation, root extraction and logarithms. - Algebra notions. - Legal system of weights and measures. - Circular functions. - Notions of function and derivative. - Notions of descriptive geometry. - Cosmography. - Arithmetic and geometry methodology notions.	- Historical outline of mathematics teaching. - Teaching of mathematics and the parts that is divided: mental calculation; practical or economic arithmetic; mechanics; cosmography; geometry; metric system. - The importance of mathematics and its teaching in elementary school. - The essentially practical and experimental aspect that this subject must have in primary education. - Didactic materials.

The 1943 Primary Teachers Training Schools Reform

Despite the institution of Primary Teachers Training Schools in 1930, and all the restructuring that the course suffered as a result of this change, the syllabi of the disciplines were not published immediately. The closure of schools in 1936 led to a delay of the publication of those syllabi, which was done only in 1943 when schools reopened. The contents related to mathematics were essentially taught in Special Didactics. In this discipline arithmetic was approached from an educational point of view, starting with the analysis of the primary school syllabus. The program of the discipline of Design and Educational Handcraft also had some content that related to mathematics, specifically to geometry.

Table 6. Main topics of the disciplines with mathematical content – 1943. Source: Decree n.º 32.629, January 16, 1943.

Special Didactic	Design and Educational Handcraft
- Teaching methodology of whole numbers and its operations. - Teaching methodology of fractions and decimals. - Complex numbers: time measurements. - Proposition and problem solving. - Techniques for construction of diagnostic and prognostic tests.	- Line and line segment, perpendicular and parallel lines. - Measuring angles. Construction and angles division. Trace of the bisection of an angle without resorting to the vertex. - Polygons: construction of triangles and squares. - Draw a circle of given radius, passing through given points.

The syllabi for Primary Teachers Training Schools for 1943 valued essentially the didactic dimensions of mathematics and specific mathematical contents were mainly those from primary education. Thus, there was no mathematics discipline with scientific mathematical topics in the Teaching of Arithmetic, which became part of the Special Didactics discipline.
The redesign of the curriculum framework in 1960 split this discipline in two. The newly created discipline of Special Didactics B specifically included Teaching of Arithmetic and Teaching of Geometry. This reformulation included a greater workload of these disciplines, and the requirements of the teaching faculty for both Special Didactics disciplines. However, in this reformulation of study plans there were no changes of syllabi and still there was no mathematics scientific discipline.

Qualifications of the teachers of teacher education for primary schools

Table 7 details requirements for the faculty of the schools for the formation of primary teachers contained in the legislation. In 1928, teachers of Normal Primary

Schools were grouped under twelve groups of subjects, the 6th group being Mathematics and the 2nd Methodology. An analysis of the qualifications deemed necessary for the faculty of these schools shows two stages. At the first stage, most of the teachers who taught mathematics had degrees in mathematics from universities and in the Higher Normal Schools or had attended one year courses of Pedagogical Sciences in the Faculties of Humanities. In this first stage most of the teachers accumulated their work in secondary schools with the work on teacher education for primary schools.

Table 7. Required qualifications of teachers for schools of primary teacher education. Source: Decrees (1928, 1930, 1931).

Year	Qualifications
1928	Apart from specific qualification of the respective teaching group, in this case, specific training in mathematics, teachers should have qualification from Higher Normal Schools. Teachers in annexed schools should be primary school teachers.
1930	Teachers were recruited through public exams among graduates by the Higher Normal Schools with state exam in any teaching group of secondary teaching, and they had attended with success the general pedagogy subjects and experimental psychology from the Faculties of Humanities.
1931	Teachers of the Primary Teaching Schools should have approval on all chairs of pedagogical sciences of the Faculty of Humanities. Teachers of the 3rd group, which included Didactic, should be certified teachers, either from primary or higher education.

With the changes of the course design made in 1930, teachers who taught mathematics contents in Primary Teachers Training Schools were essentially primary schools teachers who had made Pedagogical Sciences in the Faculty of Humanities. By the end of this period, there were no longer mathematics teachers who were teaching mathematical contents.

Final considerations

With regard to admission at teacher education for primary schools, over the period it is essentially required the equivalent of the lower secondary school (9th or 10th grade). However, with the reform of 1930 and for a period of twelve years, it was only required the 4th year of elementary primary education, this was changed in 1942. It should also be noted that during the period between 1936 and 1942, when the primary teacher training schools were with enrollment suspended, it was possible to access directly to the profession only with the 3rd year of primary school.

In 1928, two disciplines stood out: Mathematics, aiming at the scientific and cultural preparation of teachers, and Methodology, mainly focused in the teaching of mathematics contents of primary education. The changes made to the training of primary school teachers in 1930 excluded Mathematics scientific discipline of teacher training in this level of education, going to focus only on the methodology. Later, with the changes made in 1942 and in 1960, the methodological character was reinforced with the creation of a new discipline: Special Didactic.

In 1928, the syllabus of mathematics discipline addressed content that went beyond what the future teacher would have to teach in primary education, in addition also address methodology content. There were also practical applications of mathematics content such as cosmography. Over the period in study, mathematics syllabi began to focus exclusively on methodological aspects. The curriculum framework and the syllabi established during the period under review meant a reduction to the elements considered essential to the profession, extinguishing many scientific training disciplines, like what happened with mathematics. The transformations that the Primary Teachers Training Schools had over the period under study emphasized its vocational character. It´s therefore legitimate to compare the status of graduates in these schools with the status of graduates in a technical school.

At the beginning of the period in study, the teachers were mostly graduates in mathematics, who accumulated the teaching in secondary schools with the teaching in the training of future primary school teachers. With the restructure in 1930, and especially with the changes in 1942, the teachers who teach mathematics content in teacher education for primary schools became essentially teachers with training in the primary school teaching, and no graduation in mathematics.

Acknowledgment. I thank José Manuel Matos, Cecília Monteiro and Francisco Pires for polishing the English of the present paper.

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Early geometry textbooks printed in Persian

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Abstract

The printing of mathematics textbooks in Persian did not begin until the second half of the nineteenth century. The initial stimulus to employ print technology appears to be the founding of Dār al-Funūn, an institution intended to provide instruction in the "new" scientific learning of Europe. Most of the earliest examples of vernacular printed textbooks were either translations of imported European textbooks or adaptations based on the model of these European textbooks. The examination of these early printed textbooks can serve as a case study of the kinematics of mathematical knowledge. Thanks to the digitization of a representative sample of nineteenth century printed geometry textbooks, we are able to trace some basic features of the evolution of these printed textbooks in Persian. Of special interest is the process of adapting European styles of the presentation of mathematical knowledge to forms that could be understood by Persian-speaking students educated in more traditional mathematical sciences. These efforts resulted initially in a "hybrid" presentation style – both the labels of diagram points as well as equations were printed in Roman script while the verbal elements of the text were presented in traditional Persian script.

Introduction

It was primarily in the nineteenth century that European mathematics began to have a significant impact on mathematics education in the Middle East. The first wave of infiltration involved the translation of popular European mathematics textbooks into Arabic, Ottoman Turkish, and Persian. These translations were made either by native speakers or, in some cases, by foreigners (often Christian missionaries) who had become skilled in the local vernaculars. Each group had a different and often competing agenda motivating its translation efforts. Native speakers were usually commissioned by political rulers to translate textbooks in order to assist in the process of educational and curricular reform for the purpose of strengthening the state against the increasing threats and incursions by European colonial powers. Christian missionaries also desired to reform traditional education, but for the purpose of proselytizing the populace and undermining what they perceived to be corrupt and repressive governments. Their translation efforts

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

were often supported by European governments who saw the introduction of modern education as an aid to colonial administration.[1]

Following European examples of printed textbook production, these translations were disseminated using recently introduced print technologies – usually lithography. This technology, in addition to its low production costs, preserved some of the visual features of hand-written manuscripts, helping to break down barriers against assimilation of the new mathematics. Several years ago, I prepared a preliminary survey of traditional geometry treatises printed in Arabic during the nineteenth century (De Young 2012a). In this study, I have pursued a complementary line of research, focusing now on the early printing of modern mathematics textbooks in Persian. My task has been facilitated by the large-scale digitization project of the Majlis Shūrā Library in Tehran, which has placed online digital versions of a representative collection of early printed geometry textbooks.

Dār al-Funūn

These Persian geometry textbooks from the nineteenth century were apparently initially intended for use at the Dār al-Funūn, a new educational institution founded in 1851 by Mīrzā Ṭāqī Khan (also known as Amīr-e Kabīr, the vizier of Nāṣir al-Dīn Shāh. The new academic institution was intended to be a kind of polytechnic institute in order to provide a cadre of military officers trained in the new sciences.[2] It was initially staffed entirely by Europeans recruited from Vienna, the capital of the Austrian Empire.[3]

Since this institution represented a new educational initiative, there was no tradition of textbooks on which the new instructors could draw. And since the students were, at least initially, ill prepared for study at advanced levels, the instructors found it necessary to teach fundamentals of science and mathematics before they could commence instruction in their specialized subjects. So almost from the beginning there was a demand for translations of European textbooks, as well as new

[1] My interpretation of the introduction of modern science and mathematics into non-Western cultures has been significantly influenced by the work of Marwa Elshakry (2007; 2008; 2010). The struggle to avoid a Western ethnocentrism in such discussions has seen an upsurge of study in recent years – see, for example, the essay review "Hybrid Science" by Winterbottom (2011) of *Brokered World: Go-Betweens and Global Intelligence, 1770-1820* (Science History Publications, 2009).

[2] Ekhtiar (2001) provides a succinct historical survey of the evolution of the institution, focusing on the influence of political events on the development of the school. Gurney & Nabavi (1993) cover much the same material, although focused more on the internal functioning of the institution and its administration.

[3] The choice of Austria was dictated by the deteriorating political and military relations between the Persian Empire and the two great powers of Europe, Britain and Russia. For a concise summary of the political events that ultimately inspired the foundation of the Dār al-Funūn and the introduction of modern science into Iranian education, see Slaby (2005).

textbooks modeled on those of the European tradition in the areas of technical instruction.[4] Some of the adaptations were prepared by European specialists appointed to the instructional staff and some were the work of the translators who assisted the Europeans (who, of course, were not often competent in Persian language). Among these were the textbooks on geometry and other mathematical topics that are our focus in this paper.

The early textbooks surveyed here reveal an unusual printing history. After the first geometry textbooks were printed shortly after the middle of the nineteenth century, there was a hiatus of more than four decades before new geometry textbook appear on the scene. This feature of the print history may reflect the institutional history of the Dār al-Funūn as a whole. Although initially supportive of the reformist educational efforts of the Dār al-Funūn, Nāṣir al-Dīn Shāh became increasingly suspicious of the school's leadership and by the early 1860s he had begun to withdraw his support for the new education. Although he did not close the school, his interests turned more toward the photography, music, and painting sectors. The military sector in particular was almost ignored and the number of students studying military subjects dramatically declined, even though the student population as a whole continued to grow. During this period, the number of foreign "experts" teaching at the school also declined considerably and its academic standards declined. By the end of the nineteenth century, the school had become less dependent on the personal patronage and whim of the Shah but now took on a new role as modernizing force in society (Ekhtiar 2001).

Printing in nineteenth century Iran

Since our focus is on printed textbooks, it may be useful to review some features of the history of print technologies in Iran during the nineteenth century. These print technologies had a direct impact on the presentation of mathematical information in the textbooks.

A typographic printing press, was introduced into Safavid Iran by European missionaries as early as 1629 (Floor, 1980), but since their purpose was essentially to proselytize, the technology had no impact on the spread of mathematical knowledge in Persian society. By the time a modern commercial printing press had been set up in Iran (in Tabriz, about 1816) using cheap and portable hand presses modeled on the printing technology invented by Charles Stanhope (Green, 2010,

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[4] Browne (1914, 157-158) lists treatises published by teachers at Dār al-Funūn, including textbooks on mathematical subjects such as algebra, geometry, arithmetic, as well as treatises on various natural sciences and military science. Unfortunately, he only lists authors and titles, but provides no bibliographic information. More recently, Nabavi (1990, 100-103) has also published a list of textbooks and translations used in the Dār al-Funūn.

305), numerous books had already been printed in Persian in India (Floor, 1990), including an edition of the first six books of Euclid's *Elements* in the recension of Naṣīr al-Dīn al-Ṭūsī in the Persian translation of Quṭb al-Dīn al-Shīrāzī (De Young, 2007; 2012b).

Both typographic and lithographic presses were imported into Iran during the 1820s. But the use of print technology did not become widespread until the proliferation of lithographic presses about the middle of the nineteenth century. Not only was lithography a cheaper process than typography (especially if the publication involved any illustrations), the technology also allowed printers to retain some of the aesthetic characteristics of manuscript copies, which is often cited as a reason why lithography rapidly surpassed typography for printing books in Arabic and Persian, especially in Iran and India (Scheglova, 2009, p. 12).[5] Robinson (1993) has discussed at some length the complex interplay of factors that encouraged the Muslim communities of India – and Iran – to adopt lithographic printing as the preferred way to disseminate ideas.

Some representative geometry textbooks in Persian

In this section we shall discuss a few representative geometry textbooks printed in Persian. Due to space considerations we will present some basic characteristics of a few typical textbooks to illustrate important features, both in terms of content and in terms of physical appearance or architecture of these early geometry textbooks printed in Persian. A more complete analysis is in preparation. Our goal is to present a case study of the transmission and assimilation of European mathematical knowledge into nineteenth century Iran using textbooks printed in Persia and the impact of this new mathematical knowledge on mathematics education.

This survey is limited to the sample of textbooks currently available for study online. Hopefully, additional textbooks will become available in future and will help to provide a more nuanced picture of the changing educational landscape. In this initial survey, I shall primarily point out features of each treatise that I consider important to the history of mathematics education in Iran. In future, studies of other centers of learning and publication in the Islamic world will help to reveal the unique features of the situation in Iran.

[5] Use of lithography was widespread across the Islamic world – witness the use of lithography to reprint the Pseudo-Ṭūsī recension of Euclid's *Elements* in Fez in 1293/1876 (De Young, 2012c, 283-284).

Hendese [Geometry] 1273[6]*/1856*

The earliest geometry textbook that I could locate was a Persian translation of an early edition of the French textbook, *Éléments de géométrie* of A.-M. Legendre (prior to its revision by A. Blanchet).[7] The translator is identified as `Abd al-Rasūl al-Isfahānī by the catalogers of the Majlis Shūrā Library although there is no title page in the digitized copy. The treatise was printed by lithography, a technology only recently introduced into the Persian Empire.

One of the most striking features of this Persian textbook is that all labels of the geometrical diagrams – including text references to these points – retain the Roman script of the original. Similarly, all equations are written in Roman script and so are read from left to right,[8] while the text itself is in Persian script, which is read from right to left. This use of bi-directional text must have introduced some difficulties, one may presume, for the first readers of the textbook.

The copyist's attempt to preserve such traditional aids as "catch words" at the end of each folio (which were intended to guide the reader onto the next folio) sometimes must have produced confusion rather than assistance. For example, when the next folio begins with an equation, the copyist has mechanically taken the last part of the equation (the part closest to the right hand margin) as the "catch word" rather than the first part of the equation (which, because it is presented in Roman script, lies on the left hand side of the line). Or sometimes the catch word is the first word of the first line of text, ignoring the equation(s) that preceded the text on the page.

One might interpret these features as a first step toward assimilation – much as mediaeval Latin translators from the Arabic had simply transliterated words for which they had no convenient Latin equivalent. But a more traditional convention may also be at work here. Medieval Arabic copyists often seem to regard the geometrical diagrams as discrete units. When space considerations necessitated rotating a geometrical diagram ninety degrees to the left or right, the Arabic letters labeling diagram points were rotated along with the diagram itself. Perhaps the preservation of Roman script in these diagrams and equations represents the tendency to see non-text elements as discrete entities that should be preserved intact.

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6 Although there is no title page, the date is indicated in a colophon (imitating the form of traditional manuscripts) following books one, three, and six.

7 The important differences between the text as penned by Legendre and its revision by Blanchet have been discussed by Schubring (2007). The initial sections of the treatise seem to be a fairly close translation of Legendre's textbook – or at least some parts of it. But unlike the typical French editions, the diagrams are inserted directly into the text for each proposition. But book V has material mainly from Legendre's book VIII. Moreover, in book VI the translator has diverged from the text of Legendre to focus on triangles and triangulation in a very practical sense, suggesting that in this way also the treatise could be considered a hybrid text.

8 All numerals, including fractions and exponents, however, are written using traditional Persian forms. Thus the formula for the area of a circle, for example, would include the Greek letter *pi*, the Roman letter *r*, and the Persian numeral two as an exponent.

Other features also suggest that the printers were struggling to assimilate the structures of printed European textbooks.

- Each page from books I – IV has a large, bold running header identifying the book. These running headers disappear after the first page of book V and reappear in book VI. The title used is *kitāb*, a literal translation of the French *livre*, representing a curious break with traditional Arabic mathematical discourse, which would have preferred the term *maqālah*.[9]
- Each page has a page number. Like all numerals, these are written in traditional Persian forms. Books I – II are paginated continuously, beginning from page 3. The page number is placed in the outer margin beside the first line of the text in the same size font as the text itself. Each of books III – VI are paginated independently, each new book commencing with page one. In book V the page numbers, centered in the upper margin above the text, replace the running header. In book VI, they return to the outer margin beside the first line of text.

At the same time, there are several traditional elements that have been preserved from the manuscript tradition:

- The treatise retains the traditional use of "catch words" at the end of each folio to guide the eye to the next folio.
- The treatise retains the use of traditional colophon forms at the end of most books. (The colophons have been omitted from books II and IV, apparently due to lack of space.)
- Key words in the text are sometimes highlighted by a line drawn over them. This technique is regularly used in manuscripts in the same way that italic typeface is used for emphasis in a modern book printed in Roman script. (In late mediaeval manuscripts, text could be highlighted using red ink, but this technique was difficult to adapt to print technology.)
- The text is written continuously with no break to indicate the transition to a new proposition. In fact, there are not even proposition numbers, which is unlike traditional Euclidean manuscripts in Arabic and Persian. This is also different from the style of Legendre's textbook in which each proposition is given a heading centered in an otherwise empty line.

Thus we see in this early printed geometry textbook a hybrid – a mix of new and old forms – to present the new geometrical science.

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[9] The same non-traditional terminology was used also by Cornelius Van Dyke when translating John Playfair's *Elements of Geometry* into Arabic (De Young 2014, 513-4). Since both translations appeared in print at almost exactly the same time, it seems unlikely that there can be any influence from one to the other. For that reason the choice of the same non-traditional terminology is especially striking.

`Ilm al-Misāḥat [Science of (Surface) Measurement] 1274/1857

The title is supplied by the catalogers of the Majlis Shūrā Library since there is no title page in the copy that has been digitized.[10] The title is very traditional and suggests a treatise devoted to more practically oriented applications of geometric principles, including calculation of areas and volumes, extending sometimes to include surveying.

The introduction informs us that the treatise is the work of Augustus Kržiž (1814-1886), an artillery officer and member of the first Austrian military mission to Persia (1851), recruited to help establish modern education in the Persian Empire (Storey 1958, 22).[11] We are told in the brief introduction that the text was first composed in French. It was then translated by Mīrzā Zakī Māzandarānī, an instructor at the Dār al-Funūn who translated several textbooks authored in French by Kržiž.

This textbook shares many features with the previously described example in addition to the fact that both were printed by lithography and neither seems to have been printed with a title page. The treatise appears not to be a translation of an already existing European textbook. Rather, it appears that Kržiž composed his own treatise on the model of existing European textbooks. In a few places, where the translator could not find an Arabic equivalent to the French technical term, he merely transliterated the term into Persian script and usually included also the original French as well.

This textbook is also a "hybrid" treatise in that the labels of geometric points in each diagram are in Roman script and equations are also in Roman script, so they must be read from left to right. The left-to-right orientation of equations is especially noticeable when the last line of a proposition is occupied by an equation. In such cases, the equation is placed flush with the left margin of the text, leaving the right side of the line devoid of text – which must have caused some discomfort for native speakers of Persian.

Unlike the previous treatise, though, this textbook has collected all the diagrams and placed them on large fold-out sheets at the end of the book, as was also done in some European textbooks of geometry (Barrow-Green, 2006, 20-21). Each diagram on the sheet is labeled with the proposition number written out in Persian script, which reads from left to right. But the diagrams are arranged on each sheet in ascending order from the left hand edge of the sheet, so that the

[10] Storey (1958, II, 22) gives the title as Kitāb-i ḥisāb ba-`ilm-i handasah [Treatise on arithmetic with geometry], but places the title in parentheses, suggesting that he did not find a formal title.

[11] Kržiž was one of several "experts" recruited from Vienna, the capital of the Austrian Empire. Little is known about his personal life, although his name suggests that his family was from the Czech or Slovak region. He is best known to historians for his detailed map of Tehran (1857). He is also credited with overseeing the installation of the first telegraph line in Iran, linking the Shāh's palace to the Dār al-Funūn (Gurney & Nabavi 1993).

diagrams are apparently intended to be read from left to right – an arrangement consistent with that in the original French edition.

Hendese Mutawassiṭe [Intermediate Geometry] 1316/1898

This short textbook on solid geometry is by Ghulām Hussein. I have been unable to locate any biographical information about the author apart from the statement on the title page that he was head of the division of arts and professor of mathematical sciences – presumably at the Dār al-Funūn, although this is not stated explicitly.

The treatise is clearly based on a European geometrical text, although its antecedent remains unidentified. Its European roots are seen in many places. For example, the treatise is divided into two books (called *kitāb*). At the end of each there is a selection of exercises (called *tamrīnāt*), exactly as one would expect to find in a modern mathematics textbook. Moreover, there are references to European mathematicians inserted into the text – Cavalieri (whose name the copyist has spelled "Cavalierie", suggesting that the origins of this treatise may have been French) and Riemann. Neither name is transliterated into Persian script but is copied in Roman script.

This short volume was not intended to be a stand-alone text. It was designed as a part of a set of geometry textbooks. This volume focuses on solid geometry. Unfortunately, the remaining parts of the set have not been digitized so that it is not possible at present to assess how completely the set covered basic geometrical topics.

At first glance, the textbooks seems a throw-back to the earlier printed Persian textbooks, since it retains the labels of geometrical points in Roman script, so that the text must be read in two directions. But the title page tells us that this is the fourth printing of the treatise, so the commingling of Roman script with Persian script may only represent inertia on the part of the publisher rather than reflecting an anachronistic tendency on the part of the author.

Uṣūl Awā'il Hendese [Principles of Elementary Geometry] 1317/1899

This short textbook on practical geometry was prepared by Mīrzā `Abd al-Ghaffār Khān b. `Alī Muḥammad Isfahānī (1255/1839-40 - 1326/1908), who proudly carried the title Najm al-Dawlah (Star of the Nation). He appears to have spent most of his career teaching at the Dār al-Funūn (Storey 1958, 22-23).

It has a title page, arranged like that of a typical European treatise, with title and author and printer information clearly displayed. There is a page number located at the top outer margin, but no running header. The scribe has included traditional "catch words" at the end of each page to guide the eye to the next page, a feature found in many manuscripts as well, but in manuscripts the "catch words" are typically only at the end of each folio. The last two pages contain a list of the topics covered in the treatise, but contains no page numbers, so it is not exactly an index or table of contents in the modern sense of the term.

The treatise is clearly based on European sources, although I have not yet been able to identify a specific progenitor. But even though it is based on a European model, there are two significant changes that set it apart from the earlier printed textbooks. First, the labels for geometrical points are now given in the Arabic alphabet and have been assigned according to the traditional *abjad* or alpha-numeric ordering.[12] Second, because the geometrical points are now labeled in Arabic script, equations (including radical signs) can be written from right to left and so correspond to the natural order of reading.

Uṣūl-i Hendese[13] *[Principles of Geometry] 1318/1900*

This textbook, a translation of Legendre's *Éléments de géométrie* in the revision of A. Blanchet, was also prepared by Mīrzā `Abd al-Ghaffār Khān b. `Alī Muḥammad Isfahānī and issued in a lithograph edition.[14] On the title page of the treatise, he is again described with the title Najm al-Dawlah. In addition, he is described as "teacher of all mathematical subjects in the Mubāraka School of the Dār al-Funūn."[15] His translation was re-issued posthumously in a third printing in 1333/1914. On its title page, `Abd al-Ghaffār is no longer mentioned as teaching in the Dār al-Funūn – probably because he has already died.

When the treatise was re-issued, it was entirely recopied – the original plates were not re-used. This is immediately clear from a cursory visual inspection. The earlier edition had a rather fussy border separating text from margin areas, but this border has disappeared from the later printing. The diagrams have also been redrawn. Although the essential geometrical features remain the same, the diagrams of the new edition often display a somewhat different metric from that found in the earlier edition.

It is instructive to compare this translation with the earlier translation in the treatise *Hendese*. Most notably, the use of bidirectional text has been abandoned.

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[12] Classical Greek did not have an independent set of numeral symbols but used instead letters of the alphabet. The Arabic alphanumeric system imitates the ordering found in Greek. See Wright (1971, I, 28B) for the standard Arabic *abjad* system and its variants.

[13] Storey (1958, 23) asserts that `Abd al-Ghaffār published a treatise with the same title as early as 1292/1875. I have not been able to see a copy of this treatise. It is presumably different from the textbook under consideration here because Storey gives it a separate entry in his bibliography.

[14] The translation included only the main text of Legendre, omitting notes, supplementary sections, etc. The translator apparently wished to focus only on the geometry. This emphasis helps to strengthen the specific educational aims of the translator and publisher.

[15] Storey reports (1958, 22) that Nāṣir al-Dīn Shāh also awarded to `Abd al-Ghaffār the title *Munajjim-bāshī*, or chief astronomer/astrologer. That title does not appear on the title pages of `Abd al-Ghaffār's geometry textbooks.

Certainly this more completely Persian rendition must have been easier to comprehend.

Uṣūl-i Hendese Dawre Ibtidā'iye [Principles of Geometry - Introductory Level] 1327/1909

This short introductory textbook consists of twenty two "lessons". Its author, Mīrzā Ridā Khān, was a professor in the Dār al-Funūn.[16] He was awarded the title of "Muhandis al-Mulk" (chief geometrician or chief engineer of the nation), as noted on the title page. According to the report of Browne (1914, 158) Riḍā Khān authored both elementary and secondary level geometry textbooks. Only this elementary textbook is currently available online.

The treatise is organized as a series of questions and answers.[17] The first "lesson" for example asks the following questions:

- What kind of science is geometry?
- What is a line?
- What is a plane area?
- What is a volume?
- What is a point?

Each question is followed by a brief answer. These answers are not focused on abstract mathematical knowledge but on practical applications of the mathematics. There are no demonstrations in a formal Euclidean sense. For example, we are introduced to the characteristics of a parallelogram and the author explains the basic procedures used to calculate the area of any parallelogram, but does not demonstrate that these procedures are mathematically valid, as Euclid would have done.

Calculations are always done using specific numerical examples – the author does not develop abstract formulae. It is interesting that the nature of *pi* is not explained nor is the symbol used in any equations or calculations. The author simply inserts an approximate numerical value for *pi* (3.14) when working out specific examples.

Like most lithographed geometry textbooks, this treatise contains features typical of a modern typeset book, such as page numbers and running headers at the top of the pages. But it also retains some manuscript characteristics, such as the colophon specifying the author, title, and date of copying. Interestingly, the copyist has abandoned use of typical "catch words" at the end of each folio.

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[16] Storey (1958, 23) attributes to him only a treatise on algebra.

[17] Abdeljaouad (2015) gives additional examples of this genre in Arabic.

Haftasad Mas'ile [Seven Hundred Questions] 1311[18] */ 1932-3*

This treatise represents the first example of a Persian geometry textbook produced by typography rather than lithography. The title page lists the author as Mīrzā Sayyid `Abd Allah Khān.

This treatise marks a distinct break with the kind of textbook that had been produced earlier. In fact, in some ways it is difficult to classify this treatise as a mathematical textbook. It consists of seven hundred calculation problems in the fields of arithmetic and geometry, numbered sequentially throughout the treatise. Only the last fifty problems deal directly with the traditional domain of geometry. The second section provides the answers. But the explicit steps of the calculations needed to solve these problems are not given.

Hendese Tarsīmī [Geometry of Drawing] 1313 / 1934-5

This treatise, also printed using typography, represents a more typical geometry textbook in the modern style. It is a textbook on descriptive or projective geometry, based on the work of Monge, who is mentioned explicitly in the introduction. Its author was Mīrzā Riḍā Khān,[19] who, in addition to his traditional title (Muhandis al-Mulk), is described as professor and author on higher mathematics.
Although the approach to geometrical knowledge is modern, we also note that some anachronistic or traditional visual elements are also evident in this typeset treatise. Most noticeable is the use of Roman script to label diagram points. This feature seems initially surprising since the practice had been abandoned in the later lithographed textbooks. Since the subject matter does not involve calculations, there are no equations which might require being read from left to right. In this treatise, since we are dealing only with labels of points, the Roman script labels become only symbols and perhaps make the problem of bi-directional text somewhat less difficult for readers. Additionally, each diagram is labeled with the Arabic letter *sīn* plus a numeral, a technique used in older lithographed textbooks (as in the *Hendese mutawassiṭe*, for example).

Concluding remarks

This brief look some early geometry textbooks printed in Persian has revealed several interesting features. Perhaps most importantly, it has given a glimpse into

[18] Iran had officially adopted the Jalali or solar hijra calendar in 1911. That this date should be interpreted as Jalali is clear because the author has mentioned the month (Farvardin) as well as the year.

[19] Browne (1914, 158) does not mention this work in conjunction with other works of Mīrzā Riḍā Khān. Perhaps it was first published after Browne constructed list listing of works printed by authors associated with the Dār al-Funūn.

the struggle to assimilate new mathematical knowledge across linguistic and cultural boundaries.[20] These textbooks set out to communicate elements of the new mathematics which had already become established in Europe. By looking at the historical development of these textbooks, we can recognize something of the struggle that occurred as educators and a traditional educational system tried to come to terms with the new forms of geometry and attempted to use the new geometry as part of a larger educational reform in the dying Persian Empire.

A part of this changing scene involved technology. Even though the earliest printing press in Iran featured typography, it was only after the introduction of the cheaper lithographic process in the first half of the nineteenth century that printed geometry textbooks began to appear in Persian.[21] From these early textbooks, we see that the marriage of print technology and geometry textbooks was not a steady and linear process. Rather it seems to have occurred in two waves – at least based on the sample of printed textbooks available online. And it was lithography that was used to produce the majority of geometry textbooks throughout the nineteenth century. In the sample of textbooks we have examined here, we find the use of typography employed to present geometrical knowledge only after WWI.

In terms of content, we notice a remarkably strong influence from Legendre's *Éléments de géométrie*, with at least two translations.[22] The indirect influence of Legendre is more difficult to assess, but was probably considerable. We know that Legendre's geometry was also translated into Arabic and Ottoman Turkish, although neither of these translations has been investigated systematically until now. This suggests that the French mathematical tradition and especially Legendre played an especially important role in establishing the new mathematics in the Eastern Mediterranean and the Middle East.

Finally, we should note that the earliest geometry textbooks printed in Persian preserved both the geometrical diagrams (along with their labels) and the equations in the text in their original Roman script. This use of bi-directional text must have introduced a problem of cognitive dissonance for the students who used these early textbooks. These intermediate "hybrid" texts were largely replaced with treatises produced completely in Persian in the second wave of textbook publication near the end of the nineteenth century.

Acknowledgment. Attendance at this conference was supported in part by a Conference Travel Grant from the American University in Cairo.

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[20] For additional examples, see Elshakry (2008) and Schubring (2000).

[21] The importance of lithography in Iran has repeatedly been noted. See, for example, Marzolph (2001, 13-18), who describes the technical difficulties of printing Persian text using lithography, although the primary focus is on book illustration, and Shcheglova (2009), who gives an overview of the history of lithography in Iran.

[22] Schubring (2007) has given us an example of how such a cross-cultural view of the diffusion of a single textbook can offer new insights into the history of mathematics education.

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The mathematical journals for teachers and the shaping of mathematics teachers' professional identity in post-unity Italy

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Abstract

The importance of the mathematical journals for teachers published in Italy in the second half of nineteenth century is multifaceted. Beyond their obvious role of spreading the mathematical culture, they created a net of contacts that fostered shaping the identity of Italian mathematics teachers and provided the terrain for initiatives such as the creation of the Italian association of mathematics teachers Mathesis. The aim of the present contribution is to outline the personalities of the founders to single out elements that contributed to shaping mathematics teachers' professional identity in the newborn nation.

Prologue

In mathematics education research teacher professionalism is viewed in terms of a teacher's professional identity, that is to say of a sense of self as a teacher of mathematics. According to Phillips (2007) identity is "the embodiment of an individual's knowledge, beliefs, values, commitments, intentions, and affect as they relate to one's participation within a particular community of practice; the ways one has learned to think, act, and interact." (p. 259)

The concept of identity is central in studying the development of the community of mathematics teachers after the birth of the new Italian nation. After the Congress of Vienna (1815) Italy was composed by some ten states. One of them, the Kingdom of Sardinia[1], was the core of the process of unifying the states of the Italian territory and establishing a unique state in 1861. The king of Sardinia became the king of Italy.

The policy of teacher education in the making of the new nation has been illustrated in (Furinghetti & Giacardi, 2012). It was mainly a legacy of the Kingdom of

[1] After the Congress of Vienna (1815) the Kingdom of Sardinia included Savoy, Piedmont, Liguria, the county of Nice, and Sardinia.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Sardinia adapted to the new situation generated by merging together the existing states of the Italian territory, which had different school organizations and different cultures. In their analysis the authors found the germs of some problems still existing in the Italian policy of teacher education, such as the lack of pedagogical content knowledge and of practical training in the classroom. Subject matter knowledge was considered sufficient for teaching. In some cases this knowledge was remarkable, as shown by the fact that in the list of the Italian mathematicians who died in the period January 1, 1861, to December 31, 1960 compiled by Tricomi (1962) about 100 of the 371 people cited had been secondary teachers for all or part of their career[2]. In spite of the inadequate policy of education, important figures of mathematics teachers, such as those portrayed in (Furinghetti, 2012; Furinghetti & Giacardi, 2012), emerged and contributed significantly to the professionalization of their colleagues. In the meanwhile the Italian mathematical research was developing and acquired a good international reputation.

In this context the first Italian journals addressed to mathematics teachers appeared. I consider them as a main support to the creation of mathematics teacher identity and in the same time the product of the perceived belonging to a community of practice. In some previous works I have studied the content of these journals with the aim of grasping their editorial policy, see (Furinghetti, 2006; Furinghetti & Somaglia, 1992). In the present paper I outline a concise prosopography of their founders with the aim of deepening information on the journals' character, on the climate that fostered the publication, and the motivations that urged the founders to begin the enterprise.

Introduction

Since the ancient times mathematicians felt the need of communicating their results. First the communication happened through private contacts and circulation of manuscripts. The spread of ideas received a fundamental boost by the invention of the printing press at the end of fifteenth century. Afterwards structured bodies such as academies, circles, and societies fostered contacts through meetings of the members and proceedings. The foundation of journals specifically dedicated to mathematical research is the last step in this path. After some ephemeral attempts carried out since the end of eighteenth century[3], two important journals, still existing nowadays, were issued: *Journal für die Reine und Angewandte Mathematik* founded by August Leopold Crelle (1826, Berlin), and *Journal de Mathématiques Pures et Appliquées* founded by Joseph Liouville (1836, Paris). Since then the number of journals dedicated to some

[2] See Appendix 4 in (Furinghetti & Giacardi, 2012).

[3] See (Preveraud, 2015) for some early American mathematics journals and (Verdier, 2009) for the early European mathematics journals.

aspects of mathematical activity grew so impressively that 182 mathematical periodicals were listed in *Catalogue of current mathematical journals* (1913). Of course, the locution 'mathematical research' has to be contextualized in the standard of mathematical research in those years. Taking the *Fortschritte* (*Jahrbuch*, 1871) as a reference we see that the topics treated in the research journals were: History and philosophy, Algebra, Number theory, Probability, Series, Differential and integral calculus, Function theory, Analytical geometry, Synthetic geometry, Mechanics, Mathematical physics, Geodesy and astronomy. We see that didactics[4] of mathematics does not appear officially in this list[5]. Of course, I am aware of the difficulties of identifying articles of didactics. They may deal with: revisiting elementary topics from an advanced standpoint, discussing methodological questions, studying national systems of instruction, designing teacher education programmes, reflecting on teaching/learning processes.

In spite of the difficult political situation, already before the unification the Italian mathematical milieu was rich with ferments (both scientific and social) that fostered the creation of the journals on mathematics or on history of mathematics listed in Table 1.

Table 1. The first Italian journals on mathematics or on history of mathematics

TITLE	FIRST ISSUE IN	PERIOD OF PUBLICATION
Annali di Scienze Matematiche e Fisiche founded by Barnaba Tortolini,	Rome	1850-1857,
continued as *Annali di Matematica Pura ed Applicata* edited by Enrico Betti, Francesco Brioschi, Angelo Genocchi, Barnaba Tortolini	Rome	1858-
Giornale di Matematiche ad Uso degli Studenti delle Università Italiane (aka *Giornale di Battaglini*) founded by Giuseppe Battaglini, Vincenzo Janni, Nicola Trudi	Naples	1863-1938/39, 1947/48-1965/67
Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche founded by Baldassarre Boncompagni	Rome	1868-1887
Rendiconti del Circolo Matematico di Palermo founded by Giovanni Battista Guccia	Palermo	1887-1917, 1919-1941, 1952-
Bollettino di Storia e Bibliografia Matematica (*Supplemento al Giornale di Matematiche di Battaglini*) edited by Gino Loria	Naples	1897
Bollettino di Bibliografia e Storia delle Scienze Matematiche founded by Gino Loria,	Turin	1898-1919,
continued as *Sezione Storico-Bibliografica* edited by Loria in *Il Bollettino di Matematica* founded by Alberto Conti	Bologna	1922-1943, 1947-1948

[4] In this paper I use the word "didactics" for referring to a topic explicitly linked to the teaching and learning of mathematics. With this meaning the word was already used in the fifth International Congress of Mathematicians in Cambridge (1912).

[5] In (Index, 1889), which was compiled during the *Congrès international de bibliographie des sciences mathématiques* (International congress of bibliography of mathematical sciences) chaired by Henri Poincaré, a more detailed list of topics outlines the content of mathematical publications of those years. Again "didactics of mathematics" or similar locutions do not appear as a separated topic.

To study the emergence of didactics as an autonomous topic I carried out an analysis of the contents of the three non-historical journals in Table 1, see (Furinghetti & Somaglia, 2005). We found that only *Giornale di Matematiche* published a few papers that may be clearly attributed to didactics[6]. In the other two journals some articles, reviews of books and solutions of exercises dealt with subjects linked to mathematics teaching, but they were not directly addressed to teaching. One early revealing episode of the inadequacy of these kind of journals for school milieu is the controversy about the first geometry textbook published in the unified country: it was Euclid's *Elements* edited by Enrico Betti and Francesco Brioschi (*Gli elementi di Euclide*, Successori Le Monnier, Firenze, 1967/1968). The book was not suitable to students both for the language used and the content. In *Giornale di Matematiche* the academic mathematicians debated this issue, while the criticism that, as reported by Natucci (1967), was present in the school milieu, had no places where it could be communicated, because there were no journals for mathematics teachers.

The lack of communication in the school world was felt and some ten years after the unification the first journal addressed to mathematics teachers was issued in Italy. Again, as it happened for the system of education, the start up was in Piedmont, which was the original kernel of the new nation. This event was not out of the blue: for example, Pizzarelli (to appear) reports on didactic journals published in Piedmont where mathematics was treated together with other subjects. Based on (Candido, 1904; Cavallaro, 1930) and my own investigation, I compiled a list of the early mathematical journals for teachers published after the unification of Italy, see Table 2.

Table 2. Early mathematical journals for teachers after the unification of Italy

TITLE	FIRST ISSUE IN	PERIOD OF PUBLICATION
Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario edited by Angelo Armenante, Eugenio Bertini, Davide Besso, Enrico De Montel, Luigi Pinto, Francesco Rodriguez, Leone De Sanctis	Rome	1873-1875
Rivista di Matematica Elementare, founded by Giovanni Massa	Alba (Piedmont)	1874-1885
Il Piccolo Pitagora founded by Alberto Cavezzali	Novara	1883, few issues
Periodico di Matematica, founded by Davide Besso title changed into *Periodico di Matematiche*	Rome	1886-1916 1921-1943, 1946-
Rivista di Matematica, founded by Giuseppe Peano	Turin	1891-1906
Bollettino dell'Associazione Mathesis, first editor Giovanni Frattini. It underwent changes in format, editorial line, and name	Rome	1896-

[6] An example of such papers is the article on the concept of function in teaching elementary geometry (1869, *7*, 131-136) by Davide Besso, a character presented in the following.

In this paper I consider these journals, except *Bollettino dell'Associazione Mathesis* which in the period considered was only a collection of announcements and news.

Starting the enterprise

The first journal of the list, *Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario* (Periodical of mathematical, and natural sciences for secondary teaching), is not mentioned by Cavallaro (1930). Candido (1904, footnote at p. 86 added to the previous article of 1903) reports that he has received from Gino Loria the information on a journal published in Rome in 1873 entitled *Rivista di Scienze Matematiche, Fisiche e Naturali* "che pare si sia arrestata al primo volume" (that seems has stopped at the first volume). As a matter of fact the journal existed, 12 monthly issues were published from June 1873. Its actual name was *Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario*[7]. The section "Annunzi di recenti pubblicazioni" (Notices of recent publications) of *Bullettino di Boncompagni* (years 1873, 1874, 1875, volumes *6*, *7*, *8*) and some notes of *Giornale di Matematiche*[8] mention this journal. The date of foundation has a great significance: this journal was born just after the declaration of Rome as the capital of Italy (1871), when making this town a center of high scientific culture was felt as an important mission by politicians such as Quintino Sella (Finance Minister in the years 1864-65 and 1869-73)[9]. Of course, revamping the University of Rome was an important step in this path. As for mathematics in 1872 Giuseppe Battaglini moved from University of Naples to University of Rome and then was its rector in 1873-1874. In 1873 Luigi Cremona started teaching at the University of Rome and at the Polytechnic School of Engineering. In the same year also Eugenio Beltrami moved to the University of Rome as a full professor of rational mechanics.

The Regio Istituto Tecnico (Royal Technical Institute), which is dedicated to Leonardo da Vinci, inaugurated in December 17 1871[10] in the presence of the

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[7] Despaux (2011, p. 169) in the Box 7.6 dedicated to mathematics journals indicates that this journal is still published. This is due to the possible misunderstanding with *Periodico di Matematica* founded by Besso in 1886.

[8] Y. (1873). Bibliografia, *Giornale di Matematiche*, *11*, 305-306. The birth of the journal is announced as an important event.

Hoüel, G. J. (1875). Remarques sur l'enseignement de la trigonométrie, *Giornale di Matematiche*, *13*, 72-79. The end of the journal is deprecated.

[9] The design of Sella about the reinforcement of Rome as a cultural center encompassed his action as a president (1874-1884) of *Accademia dei Lincei* (the most important Italian cultural institution). He gave a more modern structure to the organization, found a prestigious seat in a historical building, and extended the scope of the activities from Rome to the whole nation and abroad.

[10] In 1871 also the Liceo Classico "Ennio Quirino Visconti" was established.

Crown Prince Umberto, was the first school with a scientific-technological orientation in Rome. The Minister of Public Instruction Gabrio Casati had launched this kind of school on the model of the German Realschule with the aim of providing the new generations with education suitable to sustain the modernization of the country in the field of industry, trade and agriculture, see (La Redazione, 1873; Scoth, 2006).

In the cover of the debut issue (June 1873) of *Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario* we read that it was published "per cura dei signori [under the editorship of Messrs] A. Armenante, Prof. di Analisi Super. nell'Università di Roma - E. Bertini, Prof. di Matematica al Liceo e di Geom. descrittiva nell'Università di Roma - D. Besso, Prof. di Matematica nell'Istituto Tecnico - Dott. Enrico de Montel - L. Pinto, Prof. di Fisica al Liceo - Prof. F. Rodriguez, Preside dell'Istituto Tecnico - L. De Sanctis, Prof. di Zoologia e di Zootomia nell'Università di Roma". Four of the seven editors (Armenante, Bertini, Besso, and Montel) were mathematicians[11]. The most famous is Eugenio Bertini (1846-1933), who is considered to be one of the fathers of the Italian school of algebraic geometry. At the moment of the foundation of the journal he was a schoolteacher in Rome and lecturer in University of Rome, in 1875 he was appointed full professor at the University of Pisa. The other important character among the editors is Davide Besso founder of the journal *Periodico di Matematica* (see in the following).

Angelo Armenante (1844-1878) after being a navy officer studied mathematics in Naples. He started his career as a secondary teacher in Rome, Parma, and Chieti. He became professor of analytic geometry at the University of Rome.

Enrico de Montel (1846-1913) was a mathematician and economist who taught trading and financial mathematics in various schools and later on was full professor of financial mathematics in the University of Genoa. He edited the journal *Giornale di Matematica Finanziaria* founded in 1907 by the publisher Laterza in Bari and contributed to the *Giornale degli Economisti*[12].

Luigi Pinto (1846-1920) graduated in Pisa. From 1869 spent a period as a secondary teacher in the Liceo of Rome (Liceo "Visconti"). In 1880 he became full professor of mathematical physics at the University of Naples. There he was appointed as a rector in 1899-1901.

Leone de Sanctis (1840-1901) was a professor of Zoology and Zootomy in University of Rome. When the capital of the kingdom passed from Florence to Rome and the Archiginnasio Pontificio della Sapienza (the Pope's Institution) became the Regia Università (Royal University) he was appointed as a director of the zoological museum.

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[11] In the inside front cover of the debut issue there is the list of those who supported the project of the journal and promised contributions. First rank mathematicians such as Cesare Arzelà, Battaglini, Beltrami are present in this list. Among the contributors there is Cremona.

[12] Biographical notes taken from (Pareto, 1962, p. 104)

Francesco Rodriguez (1824-1903) was the *preside* (principal) of the Istituto Tecnico in Rome, see (Bruni, De Simone, Dionisi, & Sacconi, 1974, pp. 158-159). He had been a voluntary soldier in the first Italian war (1848-49) for independence against Austria. His university education was on mathematical physic. After graduating he was appointed as a principal in the Istituto Tecnico of Santa Marta in Milan (now "Cattaneo"). During this period he was member of the Board of Directors for the establishment of Milan Polytechnic. In 1871 he was appointed as a principal of the Roman Istituto Tecnico until 1884, when he had an important position (*Referendario*) in the State Council. As a scientist his main interest was on themes concerning geology. Rodriguez's political and organizational action was fundamental in the growing of the Institute[13] and influential in the development of the school system. He was at the forefront of the debate on the problems of his Institute and of the organization of technical institutes in general. About this issue he published articles in Italian newspapers such as *L'Opinione*. In his school he launched the publication of *Atti del* [Proceedings of] *Regio Istituto Tecnico di Roma*, and the series of volumes *Annuario*. (Annual Report). In these publications the teachers of the Institute contributed interesting papers and accurate reports.

In spite of being ephemeral this journal was important because it sowed the seed from which the idea of such a kind of journal developed. Its life was too short to allow an analysis of its nature, but the context that I have outlined through the biographical notes of the members of the Editorial Board hints at the mixtures of the aims and competences behind the enterprise. The Risorgimento spirit animating Rodriguez found continuity in the creation of this new means (the journal) for making the Italian school suitable to the changing times. This action shows that communication of ideas was becoming a fundamental means for the new society. The high scientific level of the other members of the Editorial Board as researchers in their disciplines was a warrant of the disciplinary knowledge transmitted through the journal. It is likely that this level has been the limit of the journal because the members of the Editorial Board became too involved in their academic career to continue the project. Only Besso, who was a schoolteacher until 1888, resumed the project some years later.

[13] An evidence of the good reputation of the Institute is given by many facts. Among the teachers there were prestigious figures, such as Battaglini, the founder of *Giornale di Matematiche* teacher from 1872 to 1875, according to his letter to Cremona (letter 24 in Palladino & Mercurio, 2011), and Luigi Marchetti who was appointed as a teacher of literary topics of the crown prince Vittorio Emanuele III. One of the sons of Quintino Sella was enrolled in this Institute (Bruni, et al., 1974).

The first non ephemeral journal

The journal *Rivista di Matematica Elementare* (Journal of elementary mathematics) was founded in Alba (little town in Piedmont) in 1874 by Giovanni Massa, who was the editor until 1878. In 1877 the journal was printed in Novara. In 1885 the journal ceased publication. Massa (1878, p. 245) wrote in his farewell note to the readers when leaving the editorship, that this journal was the only periodical publication on elementary mathematics in Italy. He claimed that the contributors were mainly schoolteachers (Massa, 1878, p. 246). Some of them are important characters in the history of mathematics teachers' professional development: Besso and Aurelio Lugli (in the following years editors of *Periodico di Matematica*), Francesco Giudice, one of the founder of the mathematics teachers association *Mathesis* in 1895 with Lugli and Rodolfo Bettazzi, Anselmo Bassani author of a remarkable textbook on geometry, Pietro Caminati, founder of *Il Tartaglia. Periodico di Scienze Fisico-Matematiche elementari per gli alunni delle Scuole secondarie pubblicato per cura del Prof. Ing. Pietro Caminati* in 1898, and Alberto Cavezzali (see below). Among the contributors there is a woman, Adele Capuzzo Dolcetta, see (Villani, 1915). The articles treated (mainly at elementary level) arithmetic, elementary number theory, combinatorial calculus, geometry, algebra; a small number of articles dealt with infinitesimal analysis (series) history of mathematics, and foundations, see (Furinghetti & Somaglia, 1992).

The journal contributed to the creation of a net of schoolteachers and sowed the seeds for communication of results in elementary mathematics. This mission is in line with the multifaceted personality and the activities of its founder Giovanni Massa (Alba, 9 May 1850 - Milan, 8 April 1918). He was a teacher of accountancy and bookkeeping in technical institutes for accountants (created with the Casati law in 1859). He was an important character in the development of these disciplines in Italy at the turn of nineteenth century, see (Coronella, 2007; D'alterio, 2008). In this field he founded some of the first journals: *Rivista di Contabilità* (Journal of accounting), *Il Ragioniere* (The accountant, edited with another scholar in the field, Vincenzo Gitti), *L'Allievo Ragioniere* (The pupil accountant) and *Il Monitore dei Ragionieri*. (The monitor of accountants). With Gitti published a pioneer complete treatise of accountancy[14]. He was one of the organizers of the first big exhibition of accountancy in the united Italy during the National Exhibition of Turin in 1884.

During his stay in Novara, Massa became involved in politics. He espoused democratic ideas close to radicalism. In 1900 he was elected deputy. He was the animator of the local associations of workers in Piedmont. He fought in favor of

[14] Gitti, Vincenzo, & Massa, Giovanni (1883-1892). *Trattato completo di ragioneria*, Milano-Novara: Tipografia della Rivista di Contabilità. Giovanni Massa editore.

universal suffrage, press freedom, freedom for associations, secular and compulsory public instruction, and progressive taxes.

Massa was very active in authoring textbooks of his disciplines and of mathematics. In Novara, where he was teaching accountancy and bookkeeping in the Technical Institute Ottaviano Fabrizio Mossotti and in the technical school Galileo Ferraris, he founded a publishing house. Here he published his journal *Rivista di Contabilità* and some hundred books. In 1882 united in society with a printing house (Tipografia Miglio) to have the possibility of publishing textbooks in the economic-scientific disciplines taught in technical and professional schools. To make more stable his publishing activity Massa moved to Milan. He enlarged his activity by publishing texts of various disciplines including pedagogy for normal schools (schools for prospective primary teachers) and textbooks for primary schools, see (Lacaita & Fugazza, 2013). Among the texts published there are textbooks on elementary mathematics for primary and secondary schools by Francesco Gastaldi, who in 1879 became the editor of *Rivista di Matematica Elementare* when the founder left the editorship. Massa published also textbooks for primary and secondary schools by Alberto Cavezzali and Cavezzali's journal (see below). This shows that, beside the commercial and economic goals, the publishing activity was also an efficient means for realizing the social ideals animating Massa such as solidarity, cooperation, communication, and inclusiveness.

Massa's ideals were animating other founders of journals. This is the case of Alberto Cavezzali (Reggio Emilia, 20 February 1848 - Bergamo, 29 October 1922). His journal *Il Piccolo Pitagora* (The little Pythagoras) had the subtitle *Periodico di matematica per gli alunni delle scuole secondarie e pei maestri elementari* (Periodical of mathematics for pupils of secondary schools and primary teachers). The first issue was published in March 15 1883, in 1884 the publication ceased. The publisher was Massa in Milan and the printing house was Tipografia della Rivista di Contabilità in Novara.

Cavezzali has been a principal in primary schools, and a teacher of various disciplines including mathematics in secondary schools. A primary school in Bergamo is dedicated to him. He is an interesting character in the history of the development of Italian school, in particular at primary level, see (Callegari, 1998). He was appreciated for his professionalism as a teacher or as a principal and for his innovative and humanitarian spirit. He wrote textbooks for primary and secondary schools (technical school, gymnasium, normal) and handbooks for teaching. He fought against the school dropout (very widespread in those years), launched important initiatives such as the first school meals for poor children, an outdoor school for slender children, and a library for primary teachers.

The journal still published: *Periodico di Matematica*

In 1886 the journal *Periodico di Matematica* (Periodical of mathematics) was founded in Rome by Davide Besso, who remained as an editor until 1890. As illustrated in (Furinghetti & Somaglia, 1992; Nurzia, 1993), the journal, still published, underwent many changes, including the name that in 1921 became *Periodico di Matematiche* ("matematiche", the plural form of "matematica" was used to stress the presence of different aspects of mathematics and its applications). This journal constitutes an important landmark in the history of mathematics teacher professionalization. It treated crucial issues of elementary mathematics, and published reviews of mathematical works. Among the authors there are the most important secondary teachers of the period. Only a few contributors were professional mathematicians. Among the most active authors there are Bettazzi, Giudice, and Lugli, the founders of the association *Mathesis*. During the years the bulletin of the association had links of various nature with *Periodico*, which at present is the official organ of the association.

The founder is Davide Besso (Trieste, 28 July 1845 - Frascati, 8 August 1906). Before he had been one of the editors of *Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario*, and contributor in the journal of *Rivista di Matematica Elementare*. This constant presence evidences his convinced support to communication among teachers and his genuine interest for problems linked to mathematics teaching. This interest emerges from his scientific production that, according to Marcolongo (1907) may be divided in two parts. A part concerns research on integral analysis, theory of linear differential equations, and equations of fifth and sixth degree[15]. Another part concerns elementary or historical questions. This part includes the articles published in the *Annuario* of his Istituto Tecnico, and in the first volumes of *Periodico di Matematica*. According to Marcolongo (1907) some of these publications are real jewels of mathematical elegance ("veri gioielli di eleganza matematica", p. 148) and are important.

After secondary education in Trieste, Besso studied in University of Pavia and in Pisa, where he graduated in 1866. From November of this year to 1870 he taught mathematics and sciences in technical schools (in Viadana and afterward in Perugia). In 1871 he succeeded in the competition for the chair in the Istituto Tecnico of Rome, that he left in 1888 when was appointed full professor of Infinitesimal Calculus at the University of Modena. In 1896 he retired for health reasons. He dedicated the last part of his life to spread popular instruction, by establishing many libraries in the region around his hometown Trieste. He was concerned with the idea of homeland. He bequeathed a conspicuous amount of

[15] Published in *Giornale di Matematiche*, *Atti* and *Memorie* of the *Accademia dei Lincei, Memorie della Reale Accademia di Scienze, Lettere ed Arti di Modena*, and *Jornal des Sciences Mathematicas e Astronomicas* of Francisco Gomes Teixeira in Coimbra.

money intended for works addressed to strengthen the sentiment of being Italian[16]. In the memories of one of his former university students he is described as a worthy and generous person, and a passionate teacher, see (Volpe, 1906). In spite of his humble attitude he was honoured with the title of Cavaliere della Corona d'Italia and some of his scientific papers were awarded cash prizes from Accademia dei Lincei and from the Ministry of Instruction.

A mathematician and his journal: Giuseppe Peano and *Rivista di Matematica*

The journal *Rivista di Matematica* deserves a particular place in my list of journals addressed to mathematics teachers for being the only one whose founder was not a mathematics schoolteacher. Its founder is Giuseppe Peano (Spinetta di Cuneo, August 27 1858 - Turin, 20 April 1932), a paramount mathematician in the field of infinitesimal analysis and of logic. In 1880 he graduated in Turin and began his academic career. In 1884 he received his qualification to be a university professor (libera docenza); in 1890 he was appointed as a full professor in infinitesimal calculus[17].

In 1889 Peano published the pamphlet *Arithmetices principia, nova methodo exposita* (Torino: Bocca), written in Latin, which contains the famous axiomatization of arithmetic and the introduction of a logical symbolism. This is the beginning of Peano's project based on the conviction of the need of a rigorous method, carried out through the use of symbols, which specify the logical structure of mathematical theories. This project is intertwined with his didactic project carried out in his journal and textbooks.

The journal *Rivista di Matematica* (Journal of mathematics) was founded in 1891. On the 76 authors (including the seven who contributed with reviews), 11 are foreign. This makes a difference with the other journals mentioned before, where foreign authors are practically absent. Letters of Georg Cantor and Gottlob Frege were published. Among the authors there are famous mathematicians and important characters in the history of mathematics teaching such as Bettazzi, and Giulio Lazzeri (in 1897-1934 editor of *Periodico di Matematica/Matematiche* and of the journal for students *Supplemento al Periodico di Matematica*).

Including this journal in the list of those addressed to mathematics teachers could be criticized. According to Grattan-Guinness (1986, p. 23) the aim of *Rivista di Matematica* is the publication of "of their [mathematicians and logicians around Peano] research papers". O'Connor and Robertson (1997) write:

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[16] At the turn of nineteenth century there was a movement asking the annexation to Italy of Trento and Trieste, towns belonging to Austrian Empire, (see Zuccheri & Zudini, 2007).

[17] See (Kennedy, 1980) for an account of Peano's life and work.

> In 1891 Peano founded *Rivista di matematica*, a journal devoted mainly to logic and the foundations of mathematics. The first paper in the first part is a ten page article by Peano summarising his work on mathematical logic up to that time.

Moreover, in the chapter dedicated to the professionalization of mathematics teachers of an important Italian encyclopedia of elementary mathematics Brusotti (1950) outline the ways that make a journal useful for teachers. On the basis of these considerations he provides a list of journals for mathematics teaching. *Rivista di Matematica* is not in this list, it is only mentioned (together with *Giornale di Matematiche*) as a journal that in the initial volumes contains some articles interesting for mathematics teaching.

Nevertheless I maintain that the reasons explained in the following may justify the inclusion of Peano's journal in my list of journal for teachers. Firstly Candido (1903; 1904), who was a schoolteacher during the period of publication of the journal, considers *Rivista di Matematica* useful for mathematics teaching. Moreover, and most important, in the short presentation of his journal (second page of the cover in the first issue) Peano himself claims that the aim of its journal is essentially didactic, since the journal especially deals with methods of teaching. He adds that particular attention is paid to reviewing treatises, and all publications concerning teaching. We must note that the word "teaching"' was intended in a broad sense including university teaching. In my studies on this journal, see (Furinghetti, 2006; Furinghetti & Somaglia, 1992), I found that many mathematical themes treated in the articles are pertinent to mathematics teaching. Many authors (among them Peano's pupils) were schoolteachers (even if they mainly contributed research papers). In some articles and reviews there are noteworthy considerations on mathematics teaching by the editor and his pupils. I also took into account the genuine interest of Peano for all teaching levels. He was author of textbooks, and booklets for teachers, see (Luciano, 2006; 2012). In his obituary written by the secondary teacher Natucci (1932) we read the appreciation of the mathematics teacher Alberto Conti, editor of the didactic journals *Il Bollettino di Matematiche e di Scienze Fisiche e Naturali* and *Il Bollettino di Matematica*, for the constant support to these publications. I add that some of his students became appreciated schoolteachers. He was involved in the activities of the Turin section of *Mathesis* and in organizing meetings for teachers.

All that said, it is true that after the first years the journal changed its nature (including the fact that Peano's universal language *Latino sine flexion* was used in the last volumes) and became a means for spreading Peano's logical programme. The idea emerging from *Rivista*, and from most Peano's didactic works, is that perfecting teaching methods means clarifying definitions and axioms, employing a mathematical language cleared from the ambiguities of the ordinary language. The journal can be considered as an attempt to bridge school and research.

As an epilogue

The data about the journals addressed to mathematics teaching considered in this paper allow some reflections. Firstly, the places of publication are revealing the terrain in which journals developed and the net of communication created through them. The Piedmont, where the first non-ephemeral journal was published, was a state with an established political organization and a tradition in education. Moreover at the University of Turin there was an active school of mathematicians, see (Giacardi, 2013). The first published journal and the most important still lasting journal were founded in Rome, the new capital, where important mathematicians were working in University and the initiatives for creating the nation were catalyzed.

Looking at the dates of foundation we observe that there is a remarkable continuity in the story of the mathematics journals addressed to teachers: when the first ephemeral journal published in Rome ceased *Rivista di Matematica Elementare* was founded in Piedmont. In 1885 this journal ceased and in the following year *Periodico di Matematica* was founded. The constant presence of Besso in all these enterprises is a further element of continuity. But we have seen that also other authors contributed to more than one journal. Then, not only the founders, but also other teachers really believed in reflection on their teaching and communication as means for refining their professionalism. The net created by the journals provided the terrain for the foundation of *Mathesis*.

Apart from the good disciplinary preparation, the biographies highlight some common aspects in the personalities of most founders. One is the sense of being Italian held by Rodriguez and Besso. Another is the social commitment that had various forms: Massa's political involvement, Cavezzali's concern for disadvantaged people, Rodriguez's effort at building the system of instruction. On the mixture of values such as solid knowledge, solidarity, communication, sense of belonging to a nation, the mathematics teachers of the second half of Nineteenth century shaped their professional identity.

Acknowledgment. I thank Leo Rogers for polishing the English of the present paper, Gianni Russo and Carla Callegari for providing documents.

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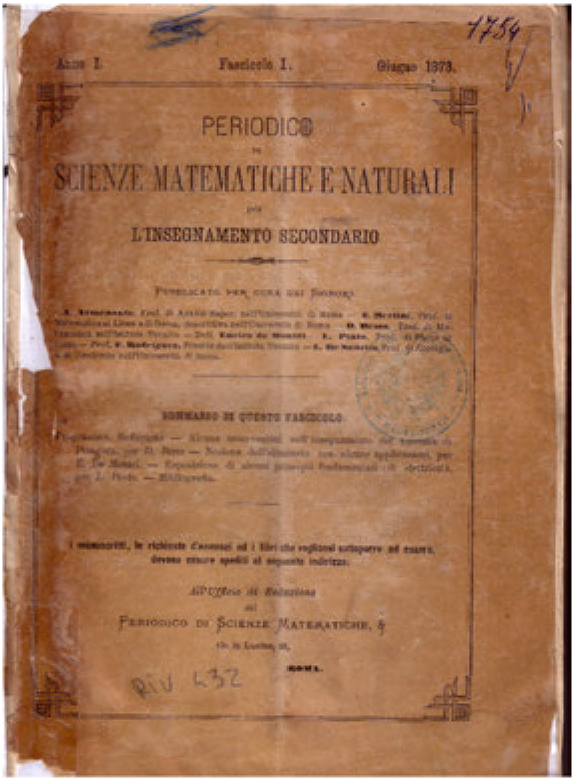

Fig. 1. The front cover of the debut issue (1873) of the journal *Periodico di Scienze Matematiche e Naturali per l'Insegnamento Secondario*

Teaching and dissemination of mathematics in Beppo Levi's work. From Italy to Argentina

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Mathematics is… more than anything else a way of thinking, it is a philosophy. (Levi 1941a, p. 7)[1]

Abstract

Beppo Levi (1875-1961) is a well-known Italian mathematician who gave significant contributions in various sectors of mathematics, but was also deeply interested in mathematics education. The scientific training in Torino, the teaching both in secondary schools and in different universities and the experience of his exile in Argentina, after the promulgation of the racial laws in 1938, marked his vision of mathematics and of mathematics education.

The aspects we intend to investigate are mainly the following: the epistemological roots of Levi's thinking in mathematics education; the methodological and didactic tenets that inspired his work; an example of Levi's thinking in mathematics education put into practice: the Abbaco da 1 a 20 *(1922), a little book addressed to the first year of primary school; and finally his educational and cultural project through the journal* Mathematicae Notae.

A biographical profile

It is rather unusual to find in the history of mathematics a whole group of high scientific profile mathematicians who were able to combine a strong interest in research with the effort to improve mathematics education. The Italian school of algebraic geometry offers a shining example of this attitude. (Giacardi, 2013)
Beppo Levi is among the mathematicians belonging to this School. He was born on 14 May 1875 in Torino, the fourth of ten brothers and sisters from a well-off Jewish family.[2] His scientific training benefited from the influence of both Corrado

[1] All the quotations of this paper have been translated by the authors.
[2] Concerning the scientific biography of Beppo Levi, see (Pla, 1962), (Terracini, 1963), (Coen, 1999).

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Segre, and Giuseppe Peano. In 1896 he graduated at the Turin University discussing a dissertation under Segre's direction on the resolution of singularities of algebraic surfaces, an important research topic of the Italian geometers at that time. In 1898 he completed his academic training by obtaining a diploma at the *Scuola di Magistero* (Teachers College) of the University of Turin (ASUT, Beppo Levi, *Carriera*). After some time as Luigi Berzolari's assistant (1896-1899) at the same university, he taught for six years (1900-1906) in various Italian secondary schools (*liceo*, technical institutes, schools for training female primary teachers): an experience which significantly influenced his thoughts on the teaching of mathematics and on its improvement. (Levi, 1946) In December 1906 he was appointed professor of Projective and Descriptive Geometry at the University of Cagliari, where he also performed the role of secretary and then chairman of the local section of Mathesis, the national association of mathematics teachers. The years following his graduation and the period of his stay in Cagliari were very important from a scientific point of view: Levi wrote approximately forty papers which were destined to leave a lasting trace and which were ranging from algebraic geometry to the theory of functions, to logic, from the theory of integration and of partial differential equations to the theory of numbers.

In 1910 he obtained the professorship of Algebraic Analysis at the University of Parma where he remained until 1928. In July 1914 the First World War broke out and those who didn't enlist stayed behind at the university, weighed down by the considerable workload; among these was Levi, who held courses of Analytic Geometry and Mathematical Physics in addition to Algebraic Analysis.[3] At the end of the conflict in November 1918, he became the Dean of the Faculty of science and in 1919 he founded the Mathematical Institute of that University. After the Reform of Giovanni Gentile (1923), when the Faculty of Science of Parma was reduced to the sole Faculty of Chemistry, Levi was appointed Dean and he was the only mathematician in the University. In Parma he devoted his energies not only to his Institute, but also to secondary teaching, creating a section of the Mathesis Association. Concerning his relationship with the teachers he wrote this to his brother Giulio Augusto:

> I have now founded the Mathesis [Association] here and it immediately took on an unexpected development: coming into contact with secondary school teachers my esteem for them grows enormously while the little I had for university teachers decreases. (B. Levi to G.A Levi, 17/18 June 1919, in Celli & Mattaliano, 2015, p. 282)

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[3] In 1917 he lost two of his brothers during the war, Decio and Eugenio Elia, a famous mathematician who gave important contributions in group theory, and in the theory of functions of several complex variables.

During this time, his many teaching positions stimulated him to extend the spectrum of his research to various fields of physics and to occupy himself with popularisation of science as well, through collaboration with the journals *Annuario Scientifico ed Industriale* and *L'Arduo, Rivista di Scienza, Filosofia e Storia*. In 1928 he moved to the University of Bologna where he taught various courses and where he carried out an impressive job within the Italian Mathematical Union (UMI) as the Treasurer (1931-1938) ad as a member of the Editorial Board of Bulletin of UMI (1929-1938) and of the Scientific Commission (1933-1938). He wrote 11 articles for the *Bollettino* but above all he reviewed 53 books, concerning various sectors of mathematics and physics, by authors from different countries. Luigi Berzolari, president of UMI, wrote to Enrico Bompiani:

> He is an ingenious person and has an extensive mathematical culture […] he has always carefully read every paper to be printed in the 'Bollettino', and if none of these contain errors, Levi should take all the praise: I can assure you that I will never come across another person who is so variously agile, patient and impartial as he is. (UMI Archive, L. Berzolari to E. Bompiani, Pavia, 7 February 1938)

This editorial work was some kind of apprenticeship for his future direction of scientific journals.

In 1938 because of the shameful racial laws, all the Jewish mathematicians were expelled from universities and schools and many of them preferred exile to remaining marginalized in Italy. Levi with the help of Tullio Levi Civita managed to obtain an invitation by Cortés Pla, Dean of the Universidad Nacional del Litoral in Rosario (Argentina), to direct the newly created Institute of Mathematics of the Faculty of Science.[4] He arrived in Rosario with his family at the beginning of November 1939 on a tourist visa and remained there until his death on 28 August 1961, even if in 1945 he had been reinstated at the University of Bologna (ASUB, 60, *Reintegrazione*, 21 July 1945). Levi was 64 years old, but he was enthusiastic and full of energy (Pla 1962, p. XIV). In Rosario Levi settled down very well as his wife Albina wrote: "Beppo comes home as happy as a youngster, often after having continued a discussion, which began at university, drinking coffee in a patisserie and, more often than not, using paper napkins to take notes" (L. Levi 2000, p. 66). Here he gave specialized courses on various questions of higher analysis and geometry, basic courses (after 1948) such as Analytic Geometry, Rational Mechanics, etc., and courses of Epistemology (1956, 1958-1960) in the *Escuela Normal* N. 1 *Nicolás Avellaneda* for teacher training. From 1939 to 1948 he directed the scientific journal of the Mathematical Institute *Publicaciones* and the series *Monografias*, but he devoted the majority of his energies to the *Mathematicae Notae*, a journal addressed to the students of the Faculty of Science.

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[4] See the letters of B. Levi to T. Levi-Civita, in (Nastasi & Tazzioli 2000, pp. 311-314).

The epistemological roots of Levi's thinking in mathematics education

Levi' interest in mathematics education was intense and continuative and it assumed different features in the two main phases of his life: the Italian period and the Argentinian one. The teaching of Segre and Peano, outstanding mathematicians also committed in education, most certainly played an important role both in awakening his sensitivity on teaching issues, and in the way of seeing mathematics education, as we shall see. The field experience, acquired through years of teaching in secondary schools, was just as valuable, as was his activity in the Mathesis Association in Parma and then in Cagliari.[5] No less important was the dialogue he had with his brother Eugenio Elia with whom he shared many reflections on the teaching of mathematics. (Celli & Mattaliano 2015, *Parte second*a, and *Parte terza*)

In order to understand Levi's educational thought it is important to know what mathematics was for him and the role that he attributed to it. In the years after his graduation, he became interested in the epistemological aspects related to his discipline. His writings show the prominence he gave to the questions "What is mathematics?", "Is it logic? Is it intuition?" and his answers, never trivial, are the result of meditations on the history of philosophy and science, with a focus on educational aspects.

He criticized both common people (and certain philosophers such as Benedetto Croce) who identified mathematics with numbers, formulas, or symbols, and mathematicians who reduced it to pure logical deduction. In this way, according to Levi, the discipline has been confused with its tools:

> Even though the mathematical theories appear to be the unequivocal development of the implications within a few initial propositions, the true mathematical spirit is shown exactly in the act of choosing these propositions and in choosing among their implications, the useful and interesting ones. (Levi 1940, p. 107)

In Levi's vision, mathematics is first and foremost a "way of thinking": *más que todo es un modo de pensar, es una filosofía.* (Levi 1941a, p. 7).[6] It is exactly for this reason why he believed mathematics plays an important educative role among the youth: it promotes the rise and development of certain intellectual abilities, first of all the *logical, observational and coordination* faculties. In particular the coordination faculties are absolutely necessary in creating an open and elastic mind; in fact, Levi affirmed: "coordination is recognising the similarities and differences among diverse ideas,

[5] See *Bollettino della Associazione Mathesis*: 1909, p. 32; 1910, p. 79, 1919, p. 152; 1920, p. 96, 137, 176; *Periodico di Matematiche*, (IV), 3, 1923, pp. 274-275; 4, 1924, pp. 371-373; 5, 1925, p. 390.

[6] He restated this idea many times. See for example, (Levi 1940, p. 102, 1941c, p. 37, 1945a, p. 5, 1954, p. 8).

and maintaining the connection between these alive, firm and spontaneous". (Levi 1908, *Opere* II, p. 621) Therefore, according to him, if logical faculties are indispensable for the teaching of mathematics to get an educational character, they are not, nevertheless, the only faculties to come into play. So it is no coincidence that he was interested in the interactions between logic, intuition and experience.

Methodological and didactic tenets

In spite of this peculiar vision of didactics, the way of understanding the mathematics teachings of Levi is based on a few precise methodological assumptions.

Deduction

As deduction is one of the main tools of mathematics and it guides students in mathematical reasoning, Levi stressed that they should learn deduction rules, recognise when they are applied in a proposition, and gain confidence in deduction, even if it is not an easy tool to be used. This confidence cannot be gained by authority or experimentally, but through a continuous analysis of the logical connections. However Levi also noticed that those teachers who appreciate mathematics only for its logical rigour and formalism, deprive the discipline of its intuitive aspect:

> woe betide if analysis is substituted with routine, bad habits. If a student recites a proof in a perfect manner, I feel it necessary to disturb him: he knows the main route well, but it is necessary to divert him into the fields so that he learns to find his own path or lane, it doesn't matter which. (Levi 1908, *Opere* II, p. 624)

In fact, according to him, deduction alone cannot constitute the whole process of learning towards the scientific truths: deduction is useless unless it is preceded by and completed with intuition. Echoing Klein's views, Levi maintained that deduction only produces new symbolic representations; to these a cultivated mind must learn to associate all the facts that they affirm. Only this complex association really indicates understanding. He speaks about intuition, and not experiment, as this latter allows us only to attest that, in the specific case under consideration, something happens – or at least appears to happen; on the contrary, from the particular experiment intuition only draws the occasion to conceive: "intuition does not verify, it abstracts, idealizes, distinguishes, unconsciously reasons". (Levi 1908 *Opere* II p. 625)[7]

[7] This is the reason why Levi criticised the Italian textbooks of so called "intuitive geometry", which often were "hybrid abridgement of deductive systematic treatment". (Levi 1907, *Opere* II, p. 542)

Relationship between experiment and intuition

So it is not surprising that the second pivotal point of Levi's thought is the relationship between experiment and intuition. According to Levi, an experimental teaching is exclusively informative teaching, and can present a grave danger with respect to the education of the mind. Experiment, he affirmed, can become a didactic tool only when it is used for bringing out hidden intellectual associations or relies on perfectly known associations.[8]

To clarify the difference, which seems rather subtle, between experimental and intuitive methods, we have chosen an example drawn from the very teaching experience of Levi. Students from a gymnasium (lower secondary school) were asked to intuitively prove that the diagonals of a rhombus (drawn on the blackboard) are not equal. The teacher, who relied on experiment, made the students measure the two diagonals with a rope or a rule and draw the conclusion from the measurements. Levi, on the other hand, proposed another kind of approach: "take a rhombus, consider it as an articulated quadrangle, imagine that its angles vary: while one diagonal shortens, the other gets longer". (Levi 1908, *Opere* II, p. 626) To those who might accuse him of simply making a different type of experimental observation, Levi replied that, on the contrary, this "is the pattern of a reasoning; and in any case it would never be an experiment of measurement, but a qualitative experiment whose character, compared to intuition, is very different". (*Ibid.*) The tool in this case plays a different role to that of the rule: the rule is a simple "artifact", while the articulated quadrangle is an object whereon a precise pattern of use is defined and which also allows 'limit cases' to be examined.

Socratic method and the relevance of exercises and problems

To stimulate intuition, according to Levi, the Socratic method of teaching can be very useful: with this kind of approach the teacher can accustom his students to recognise and to become acquainted with certain "mental schemes which are already present in our mind". (Levi 1908, *Opere* II, p. 625) That is why he attributed an important role to questions, exercises and problems to be addressed to students, as it is evident, for example, from the *Abbaco* and from the columns dedicated to these in the journal *Mathematicae Notae* (see § 5). Even if the levels of the two publications are very different, similar types of problems appear: not standard repetitive exercises, but problems which allow students to make connections between different concepts in different contexts, to take advantage of analogy, and which enable the teacher to make digressions in order to stimulate the curiosity in their students.

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[8] Some examples are particularly fitting and enlightening to illustrate Levi's vision: see Levi 1907, *Opere* II, p. 546 (see Furinghetti, 1992; Giacardi, 2009); Levi, 1938.

The teacher training

Levi's ideas regarding teacher training emerge rather clearly in an article of 1909, written at a time when a heated debate on this topic involved university professors, secondary teachers and experts in pedagogy. Mainly three points were under discussion: the distinction between a *scientific degree* to train researchers and a *didactic degree* to train teachers; the abolition of the degree thesis for future teachers; and the introduction of courses of history of mathematics, of didactic and methodology. Levi was definitely against the first two as he believed that for a secondary teacher a broad scientific culture was fundamental in order both to maintain his own teaching in contact with the ever-increasing needs of the mathematical culture, and to be able to guide the young minds. (Levi 1909, *Opere* II, p. 733). The teachers must have "autodidactic abilities" which will allow them to complete and adapt their cultural background throughout their career. (*Ibid.*, p. 735) Moreover he was firmly convinced that the degree thesis was very important for a future teacher as it makes the student "move away from guidelines of the course"; do an individual project; "leaf through some books and newspapers"; try out his critical sensitivity and gifts for discussions, which are all important features for a secondary school teacher too. (*Ibid.*)

Instead Levi agreed with the introduction of courses of history of science and of didactics, provided these courses be considered with the same dignity as the others, so that the students may choose freely. The didactic courses should be moulded on "Klein's seminars" and highlight links between elementary and higher mathematics. (*Ibid.* p. 734)

According to him the best mathematics teacher is "one who knows how to discover, within his own discipline, all the different nuances of life, from the simple subtlety of an observation or a result, to the triumphal success in facing an indomitable difficulty". (Levi 1947, pp. 67-77). He is one who is able to communicate the pleasure of discovery and a way of thinking to his students. Therefore it is no coincidence that in Levi's opinion, it was "impossible to teach how to teach because every didactic rule applied in a routine manner according to a precise pattern, suppresses the freedom of thought both in the teacher and the student and makes the first as ridiculous as the latter is intelligent". (Levi 1945b, p. 240)

In order to see how Levi's thinking in mathematics education was put into practice at different levels it is worth presenting two examples: the *Abbaco,* a little book of arithmetic addressed to children of first year of primary school (Coen 1998), and the journal *Mathematicae Notae.*

The *Abbaco da 1 a 20. Il primo libro d'aritmetica*

The *Abbaco* was drawn up during the years that Levi spent in Parma and, as his daughter Laura wrote (L. Levi, 2000, p. 30), originated from the didactic experience that Levi matured assisting his children, in particular his eldest, Giulio, who could be accepted at primary school directly into third year. It is a small book with only 60 pages, 44 of which are directed to the students, while the remaining are *Didactic Illustrations* for the teacher. Levi is author, editor and illustrator. Among the various aspects of the *Abbaco* which are worthy of mention, we will highlight only those we believe best illustrate the didactical thought of Levi, some of which have been overlooked until now.

In the preface, Levi announced that the aim of the book was to break with the traditional teaching of arithmetic; in fact he was against the exclusively oral lessons, that often were dedicated to reciting the numbers to 50 or 100 in chorus as if a boring singsong. According to him, the *Abbaco* should become "the ABC of arithmetic, in the same way that it would be impossible to imagine teaching to read without a spelling book". (Levi 1922a, p. 49)

Graphic and linguistic aspects

For this reason Levi was particularly concerned with the graphic aspects: flora and fauna patterns and objects that a child can recognise are portrayed; symmetry is used; and the lettering varies in size and colours (red and black).
In explaining the purpose of his book in the article published in 1922 in *Arduo*, Levi himself explained:

> I believe that an essential part of my book 'Abbaco' are the illustrations and the external shape of the pages: […] the eye and the mind of the pupil must pause on some pictures, on the symmetry of certain pages, in that intermediate stage between study and pleasure, distraction and observation with which children turn pages again and again in their first books until they know the position of every drawing and every line by heart. (Levi 1922b, *Opere* II, p. 851)

Also the presentation of the subject differs from that commonly adopted. In *Abbaco* Levi uses "narratives"(short stories related to daily life), he introduces many examples in order to highlight the sense of the calculations and to make sure students do not simply repeat the formulae by heart; he resorts to geometric shapes to visualise certain properties. He also pays particular attention to the language: he introduces few words which are easily understandable to the child, he distinguishes between number and figure, sum and addition, etc.; he uses verbal expressions with common repeated structure such as "after … comes", " the number after … is", etc., so that the child learns to recognise them immediately.

The influence of the research on the foundation of arithmetic

The research on the foundations of arithmetic and the teaching of Peano clearly influenced Beppo Levi.

He did not accept the commonly used definition of number as "the result of the counting operation": in fact he observed that this operation makes use exactly of the idea that natural numbers follow each other, so that the number is the means of the counting operation and not the result. In his opinion "counting is the implementation of a particular faculty of our intelligence, the faculty to *conceive by sequence*; and numbers are the elements of a sequence […] in which this faculty is put into effect". (Levi, 1922b, *Opere* II, p. 848) Contrary to the objective initiation into arithmetic, according to which the numbers are presented as grouping of objects (tokens, beans, etc.), Levi, gave the following example in order to display the aberrant feature of this method:

> an average child, when he has to buy 4 loaves of bread, will never compare each of them to the 4 pebbles that his provident mother put in his pocket: he will count with the 4 numbers "1, 2, 3, 4" that he has in his mind. (*Ibid.*)

As mentioned before, for Levi counting was an ability to "conceive by sequence", therefore he introduced the numbers according to the ordinal approach, which is inborn in the human mind, [9] and not cardinal, as was customary in the majority of the textbooks.[10]

In his *Abbaco* Levi is careful to illustrate the sequence of numbers, presenting other sequences that a child normally uses, such as time sequence. The first he introduces is the advancing of age which allows him to explain the meaning of the operator +1. In "3+1" – Levi notices – the two digits have a very different status: 3 is a number, +1 is an operator. (Levi 1922a, p. 56) This is the reason why he does not start with "1+1" but with "3+1", otherwise this difference would have been hidden!

Levi distances himself from tradition even when introducing the operation of addition, whose definition was usually based on the joining together of more groups of objects into one. In fact he follows the approach of Richard Dedekind

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[9] Twenty years later he wrote that he had found confirmation of the priority of the ordinal approach while reading a paper on the origin of the number systems of the American natives (Levi 1941b).

[10] We have consulted various widely-distributed textbooks – from the end of 19th century to the second decade of 20th century – for childhood and the primary school (G. Silvestri 1887; A. Stoppoloni & A. Tomei 1894, A. Vezzani about. 1905, C. Bevilacqua-Odetti 1907, A. Zaccaria 1907, A.&C. [Andorno & Cathiard] 1914, A. Zaccaria 1915, C. Ciamberlini about 1915), for lower secondary school (V. Amato 1913, E. Bortolotti 1914, A. Padoa 1923) and for normal school (by F. Del Chicca 1900, L. Predella-Longhi 1910, R. Bettazzi 1922). 8 of them adopted the cardinal approach 3 the ordinal one, 3 both.

and Peano. In *Arithmetices Principia* (1889) Peano, after assigning the well-known axioms of arithmetic, gives an inductive definition of addition (Definitio 18): be a and b two natural numbers, if $a + b$ is a number, but $a + (b +1)$ is not defined yet, then $a + (b +1)$ is the successor of $a + b$, that is $(a + b) +1$. Taking advantage of phrases such as "4 comes after 3", "4 is 1 more than 3", Levi arrives at the first sum "3 + 1 = 4". Similar observations are repeated to pass from addition "+1" to those "+2" and "+3". The formal scheme used by Levi is the following: b+1 comes after b, a+(b+1) comes after a+b. (*Ibid.*, p. 56)

Levi's intention was to create calculation schemes, which easily take root in the young student's mind. The scheme is initially illustrated step-by-step on some examples, subsequently the information gradually decreases, so the child is encouraged to proceed by analogy. Levi explained his point of view as follows:

By analogy, I say, and not by power of memory; it means that, if the child is no longer obliged to explicitly repeat the scheme in each case, he must constantly see it in his imagination. (*Ibid.*, p. 55)

The failure of the Abbaco

The *Abbaco* was not very successful, mainly because the Central Examining Board for textbooks chaired by Giuseppe Lombardo Radice included it among the "useful but not advisable for day-school textbooks" (see Ascenzi & Sani 2005, p. 414). The methodological approach was too different from the mainstream: it is significant the sarcastic comment by the writer and journalist Giovanni Papini:

> Recently a skilled Jewish mathematician, Beppo Levi, published a brand-new Abbaco, from which we can finally learn the true logical and scientific method to teach numbers to children. Here is a sample: 2 comes after 1, 3 comes after 2, 4 comes after 3…10 comes after 9, etc. And Einstein, of course, comes after Beppo Levi. (Giuliotti & Papini, 1923, pp. 51-52)

The Abbaco was later criticised with a far more scientific reason: Tullio Viola, for example, accused Levi of a lack of correlation between the concept of number and the more general one of set. (Viola, 1961, p. 516)

Other factors probably contributed to the failure of the *Abbaco*: Levi's attempts to obtain the authorisation to test the book in a primary school in Parma failed; [11] the book was published without the backing of a publishing house; furthermore, in 1925 Levi had signed the Benedetto Croce's Manifesto of the Anti-fascist Intellectuals.

[11] See the letter by B. Levi to G. Lombardo Radice (Parma, 13 May 1925, in which Levi also criticises the Examining Board's way of judging the textbooks saying that they should have limited themselves to analysing "scientific errors, linguistic mistakes, moral deficiencies". (Celli Mattaliano 2015, p. 129)

The *Abbaco* should have been part of a broader textbook publishing project, a "publishing dream", as Levi wrote to his brother Giulio Augusto, containing his own writings, his brother's, Eugenio Elia, treatises, school textbooks and also "a journal of movement of ideas pertaining to mathematics"[12], which, however, was never followed up.

Mathematicae Notae, the training of university students and dissemination of science

The publishing project of a scientific journal was not achieved in Italy, but in Argentina Levi was able to fulfill his dreaming through two journals, *Publicaciones* and *Mathematicæ Notæ*.[13] It is mainly in this second one that his educational project emerges. It has been issued in Rosario since 1941 as a periodic publication of the Instituto de Matemática de la Facultad de Ciencias Matemáticas and it is still active today. The aim of its creator, Fernando L. Gaspar, was to attract the students of the scientific faculty in order to give fresh impetus to research in Argentina. It is significant that the journal was given out free of charge to the students and that in order to stimulate their active participation, every year prizes – which consisted in scientific specialised or popularisation books – were awarded to those who correctly answered the proposed questions.

Levi, Director of the Instituto, ran the journal from its very beginning with the help of Luis A. Santalò until his death in 1961. In the *Prólogo* of the first issue he explained who the intended audience for the journal was and how it was to be organised:

> [it is addressed] mainly for the students of the Faculty that the Institute is connected to, but we hope to find some interest beyond the Faculty confines, maybe among the young people who are approaching this particular scientific area for the first time. I say particular, because mathematics […] interests the engineer, the chemist as a powerful tool for applications, and above all it is as a way of thinking, a philosophy. […] The "Mathematicæ Notæ" intends to awaken some interest towards mathematical thought; and intends to do so, as far as possible, in an eclectic and indirect manner. (Levi 1941c, p. 7).

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[12] B. Levi to G.A. Levi, Parma 17/18 June 1919 (Celli, Mattaliano 2015, p. 282). See also (Coen 1998, p. 87).

[13] In both editorial projects Levi was supported by a team of talented students and collaborators, such as P. Zadunaisky, S. Rubinstein, J. Olguín, F. Gaspar, H. Greppi, R. Laguardia, L. Santaló e M. Cotlar.

In accordance with this goal, the journal aimed to publish simple articles, although rigorous and not trivial, on various aspects of mathematics, without claiming to reach the highest levels of research, and to suggest problems that were suited to university students. Therefore the objective was twofold: the training of the young people and the dissemination and popularisation of the mathematical sciences.

During the twenty years (1941-1961) of his direction of *Mathematicae Notae*, Levi obtained the contribution of 47 authors of various nationalities,[14] he maintained good relationships with 168 mathematical institutions and established exchanges with 146 journals from 31 countries across the world. He himself published almost fifty articles[15] concerning mathematical analysis, number theory, logic and physics, and reviewed university treatises, books on the history of mathematics, texts related to mathematics education[16] for a total of 105. Moreover he carefully revised the papers received for publication:[17] he was really the life of the journal, which he called "a daughter that is always hungry". (L. Levi 2000, p. 82)

In order to understand how the educational aims expressed by Levi in the *Prólogo* were carried out, we only mention the columns *Ejercicios y problemas, Cuestiones, Flores y hojas*, that occupied a significant part of the journal. The first and most enduring of them was *Ejercicios y problemas*, which comprised exercises and problems related to various fields of mathematics (geometry, analysis, number theory, algebra, mathematical physics and from time to time, probability theory). The correct solutions delivered to the editorial office within the established deadline were published, often together with comments of the editor, including proofs of the properties used, additions of details or explanations or alternative solutions. In total, in the first ten volumes, 170 exercises and problems appeared. The column *Cuestione*s presented questions that required deeper knowledge and so were a good training for the students of the Faculty who wished to undertake scientific investigation. The third section *Flores y hoja*s ("Flowers and leaves"), stopped just after the third year. This section had contained more refined and difficult exercises such as finding gaps and errors within a reasoning. The lack of participation of the students was probably one of the main reasons why this column was reluctantly eliminated by Levi, even if he considered it the most noteworthy of attention. (Levi 1942, p. 2)

It was mainly Levi who prepared the problems and corrected the solutions with the help of Santaló, so his work was really impressive and testifies not only the

[14] Among these authors we mention: P. Capelli, M. Cotlar, R. Frucht, G. Fubini, L. Godeaux, B. Gross, T. Mihailescu, P. Pi Calleja, C. Pla, A. E. Sagastume Berra, L. A. Santaló, A. Terracini, G. Valiron, A. Wintner, etc.; see Coen 1999, p. XXXI.

[15] About 80% of these articles were concerning mathematical issues, while the remaining 20% were on historical, philosophical and biographical topics.

[16] Among the books on education we mention those by J. W. Young, V. I. Garcia Serrano, G. Pólya, R. Courant & H. Robbins.

[17] Cf. for example the letter by B. Levi to S. Sonnino, Rosario 1 July 1941, kindly given us by Salvatore Coen.

surprisingly wide-ranging scientific knowledge, but also his vision of mathematics education which should offer both a deep knowledge of the subject, and an unitary picture of science through an interdisciplinary approach, in order to stimulate young people's curiosity and encourage new research.

Conclusions

At this stage of our research we would like only to draw some provisional conclusions. Levi's didactic vision was focused on the idea of freedom and teacher responsibility (freedom of method, choice of textbooks, discretionary use of school programmes),[18] and was influenced by his *maestri* Segre and Peano from whom he did not only receive stimulus towards certain research areas (algebraic geometry, logic, etc.) but also didactic principles. From Peano he acquired his attention to language and rigour, and to the impact that research about the foundations of mathematics can have on education. From Segre he derived his vision of teaching as a mission, the importance of a strong scientific preparation for future teachers, the role of intuition in teaching practice and the significance of elementary mathematics from an advanced point in teacher training. Moreover the years of teaching in secondary schools of different levels and kinds allowed him to understand the need of diverse teaching methods according to the age of pupils, and to observe the defects of the Italian schools.

Levi's teaching in Italy certainly left a significant mark in the universities he taught in, however, it doesn't seem to have significantly influenced the official Italian attitude concerning the pre-university teaching, [19] even though he actively participated in the meetings and congresses of the Mathesis Association. The publishing context of the majority of his articles on teaching (*Nuovi Doveri, L'Educazione Nazionale* and *L'Arduo*), the scarce appreciation of the *Abbaco*'s educational program and the lack of success of the publishing plan are witness to this. Certainly one of the causes of this was the priority given to scientific research (Terracini, 1963), which absorbed most of his energy, although another reason could have been his independence of judgement towards the official spheres, especially during the Fascist period. (L. Levi 2000, pp. 24-25) In particular, the main complaints Levi had about the Gentile reform – that he called "Fascist reform and debatable

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[18] See for example Levi 1946 and the letters by B. Levi to G. Lombardo Radice, Parma 13 May1925, and by B. Levi to G.A. Levi, 17/18 June 1919, in (Celli & Mattaliano, 2015, pp. 128-129 and 279-285).

[19] Cf. B. Levi to G.A. Levi, Parma 17/18 June 1919 in Celli & Mattaliano 2015, pp. 279-285, where he speaks of his vision of politics and the academic world.

in many aspects" (Levi, 1946, p. 171) – was the lack of "free discussion and free development of all opinions".[20]

In Argentina, Levi above all rediscovered his dignity as a man and scientist, something Fascism denied him, he was able to develop some particular aspects of his personality, in particular his vocation as a "sower" and communicator of mathematical thinking, as well as his role as a guide for the younger generation who were entering research and teaching. In Rosario he had the opportunity to go more deeply into topics concerning mathematics education through new readings (George Polya, Richard Courant & Herbert Robbins, John W. Young, etc.) and through comparisons with Argentinian education system; he rediscovered the passion for training secondary school teachers; and he finally had the opportunity to achieve his dream of editing a journal on the "movement of ideas attaining to mathematics […] with a plan to spread a certain way of thinking; and then provide a little information on the bibliography".[21] He achieved this through his most extensive and lasting project, the journal *Mathematicæ Notæ*, a bridge he built in order to connect education and scientific research, a means to disseminate, through articles, reviews and obituaries, research works in international pure and applied mathematics, including those of the great Italian mathematicians.[22]

Acknowledgments. A special thanks to the Family of Beppo Levi, in particular Emilia Levi, Monica Resta and Paolo Momigliano Levi. Our most wholehearted thanks also goes to Salvatore Coen and Pompeo Vagliani for providing various documents, and to Kim Williams for help with the English text.

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[20] See B. Levi o G. Lombardo Radice, Parma 13 May 1925, in (Celli & Mattaliano 2015, p. 128).

[21] See B. Levi to G.A. Levi, Parma 17/18 June 1919 in (Celli & Mattaliano 2015, p. 282).

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Half a century of *Pythagoras*, a mathematical magazine for students and teachers

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Abstract

In 1960 Pythagoras, a mathematical magazine for students from high schools and colleges was founded in the Netherlands. In this article we retrace the origins of the founding of the magazine in the late 50s of the 20th century and its further development and influence in the ensuing half century.

Introduction: Post-war Dutch natural sciences and the Casimir Committee

During the Second World War in the United States of America and in Great Britain there had been a lot of scientific and technological development. In the Netherlands, from the beginning of the century one of the foremost countries in the natural sciences, by some historians of science called The Second Golden Age (Willink, 1998), the development of the natural sciences had come to a near standstill during the German occupation. In scientific circles there was fear of falling behind the developments abroad. In addition the Netherlands stopped to be a colonial power of importance after the loss of the Dutch East Indies – which became the independent state of Indonesia – causing fear of becoming a minor and unimportant country.

Thus there were enough reasons for an initial growth in government funding of the mathematical and natural sciences. At the same time it was necessary to change the curricula to suit the demands of a growing number of scientific disciplines. However, lack of economic growth forced the Dutch government in 1956 to a 'reduction of spending'.

In particular this development caused the astronomer Jan Oort (1900-1992) to act with the aim of stopping the impending reductions of the science budgets. A committee, the Committee Development for Research in the Natural Sciences, later called the Casimir Committee, was set up on December 9 1957 by the Secre-

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tary for Education Jo Cals (1914-1971) to advise the government. The main members of the committee were chairman and theoretical physicist Hendrik Casimir (1909-2000) and Oort. All the preliminary activities started before the launch by the Soviet Union of the satellite Sputnik 1 in October 1957. Sputnik 1 probably accelerated the political process, causing fear in the Netherlands of lagging behind the Soviet Union as well. The committee report appeared in October 1958, asking for a very substantial enlargement of the scientific budget, which without much discussion was approved of by the Dutch government. Fundamental research was one of most important points in the report. A few of the consequences were a tripling of the number of professors in the sciences, the building of new laboratories and a lot of money for research programs (Baneke, 2012).

Another important point in those years was also the great increase in the expected number of students, as a result of the 'birth wave' which started directly after the end of the war. The first numbers of extra students were to be expected to fill the universities in the beginning of the nineteen sixties.

New ideas on education of mathematics: The Dutch Committee for Education in Mathematics

In the nineteen fifties the teaching of mathematics in the Netherlands was still based on classical Euclidean principles. But there were compelling reasons for change. In the first place new fields of mathematics had been developed in many diverse scientific fields, e.g. technology, economy, econometrics, psychology etc. And further, over the years a deep gap had arisen between the 'proper' mathematicians working at universities and the teachers at secondary schools. The first group considered the second one not as 'real' mathematicians and moreover had no interest in didactics at all. These bottlenecks were ample reasons for action. So, in December 1953 the Mathematical Society of the Netherlands founded the Dutch Committee for Education in Mathematics (Dutch: *Nederlandse Onderwijs Commissie voor Wiskunde*, NOCW). Its first chairman was the philosopher Evert Beth (1908-1964), succeeded in 1955 by one of the then most prominent Dutch mathematicians Hans Freudenthal (1905-1990), who stayed on as chairman till 1975. We will use the name Freudenthal Committee. The committee had as task to contribute to the work of the International Commission on Mathematical Instruction (ICMI). The aims of the Freudenthal Committee were:

- Bridging the gap between scientist and teacher.
- Research into the role of the mathematician in the modern world.
- Furthering the interest for mathematics among young people, in particular students of the secondary schools and the schools preparing for university education.

Later extended with:

- Ameliorating the quality of mathematical education in the Netherlands and encouraging the practicing of mathematics and its applications.

A few remarks on Freudenthal's activities are relevant. In 1967 Freudenthal organized an international colloquium as an ICMI/NOCW activity. The title was "How to teach mathematics so as to be useful".

This started thinking about new developments in mathematical education, in which 'so as to be useful' led to mathematical instruction in contextful situations. The first International Congress on Mathematical Education in 1969 in Lyon was held under the presidency of Freudenthal. In connection with 'so as to be useful' a few remarks from Freudenthal's inaugural address are:

> Mathematics should be taught to be fully integrated by the learner, which means that he should enjoy it and know how to use it if need be. Mathematics should not be taught as an aim in itself but with a view to its educational consequences (Kleijne, 2004).

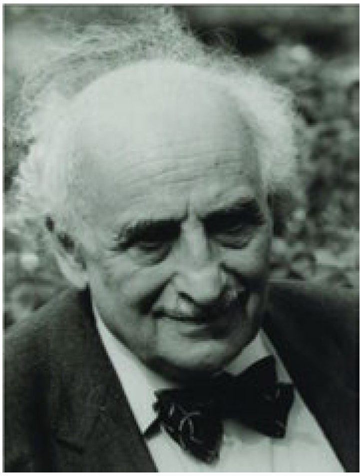

Fig. 1. Hans Freudenthal

The conception of *Pythagoras* magazine: Who would be the editors?

In the later years of the 1950's the Freudenthal Committee investigated the possibility of editing a mathematical magazine for young people, to stimulate the interest in mathematics among them. In particular Johan Wansink (1892-1985), who also was a member of the Freudenthal Committee and knew of developments in other countries, did a lot of work to achieve this goal. A few other European countries had preceded the Netherlands. In the minutes of the Freudenthal Committee (North-Holland Archives) of February 24 1959 we read that the member of the Freudenthal Committee F. Loonstra (1910-1989), a professor at the Technical University in Delft, wishes to discuss the possibility to edit a mathematical magazine for youngsters. Two months later, on April 21, there was an ample discussion concerning two ideas:

- Is there room for a mathematical magazine for youngsters?
- Would a Mathematical Olympiad in the Netherlands be a success?

Fig. 2. Johan Wansink

In Poland and Hungary Mathematical Olympiads already existed (in Hungary since 1900). The discussions had positive results and the Freudenthal Committee decided to find editors for the new Magazine. In the archives the first name in connection with the future magazine was Loonstra. Whether he was the first to bring forward the idea is not certain. It could as well have been Freudenthal himself or the secretary of the Freudenthal Committee Johan Wansink (1892-1985), who also was a great stimulus for the new magazine.

Anyhow, Freudenthal (North-Holland Archives) took the initiative to find editors. He contacted Hans de Rijk (1926-), a secondary school teacher at intermediate level, and Gerrit Krooshof (1909-1980), a teacher and assistant-principal at a secondary school in Groningen. Krooshof played an important role in the introduction in the Netherlands of a new method in 1968: Modern Mathematics, based on a Scottish method. After a lot of correspondence during the months of March and April 1960 a meeting between the two was organized and they started preparations for the first issue. Hans de Rijk, who probably was the first to be contacted by Freudenthal, hesitated, because he had no experience in pre-university classes. But with the experience of Krooshof at this level, they formed an excellent couple. Both of them had a keen interest in didactics.

Fig. 3. Gerrit Krooshof

Freudenthal himself was one of the editors by virtue of being chairman of the Committee. From the archives (North-Holland Archives, Freudenthal correspondence) it is clear that in the first years of the magazine Freudenthal wished to judge the articles to be published. But before presenting them to Freudenthal Krooshof judged most of the articles, so in effect he, in cooperation with De Rijk, decided on which articles were to be published. I could not find who proposed the name *Pythagoras* in the archives used.

As a special point of attention a few data about Hans de Rijk are interesting. De Rijk became a brother at the catholic Congregation St. Louis of Oudenbosch. Oudenbosch is a village in the south of the Netherlands. The congregation was started in 1840, but in 2015 only 21 brothers were still alive. No young members had taken the vows over the last forty years. De Rijk's name in the congregation was Brother Erich. Brother Erich became a teacher of mathematics and physics, but his interests were much broader. Later De Rijk left the congregation and married. When he was nine, he 'published' his first 'book' of four pages: The production of gases. In Oudenbosch he started the People's Observatory Simon Stevin. Among dozens of publications on diverse subjects he wrote about mathematics and art, calligraphy and the famous Dutch graphic artist Maurits Escher (1898-1972). They became good friends and as a result a lot of articles on the mathematical aspects of Escher's graphic work appeared in *Pythagoras* over the years. On Escher's work he wrote for instance 'Escher. Magician on paper' (Dutch: *Escher. Tovenaar op papier*). In cooperation with the Dutch mathematician Fred van der Blij (1923-) he established the society Ars et Mathesis, to further interest in the relation between art and mathematics. He published books using different pseudonyms, e.g. Bruno Ernst, Ben Engelhart, Ben Elshout, but all with the same initials as Brother Erich: B.E. (Ruttkay, 2000). As the reason for all the different pseudonyms De Rijk told in a movie made of his life in 2012: 'If I would publish a book on a completely different subject as my previous books, the reader would probably think 'I don't believe that he does know much about this new subject as well, so I won't buy it' (Klerk, 2012) His life motto is expressed by the Latin sentence: Nescius omnium, curiosus sum (I don't know anything, but I am curious to know everything).

Fig. 4. Hans de Rijk (Bruno Ernst)

First issues

The first two issues of *Pythagoras* appeared in 1961 without a cover. The success was immediately enormous. After the first issue there were already 9,000 subscribers, after the second one there were as many as 12,000.

After the publication of the first issues a contract was signed on November 14 1962 by Freudenthal and Wansink and a publishing company (Dutch: J.B. Wolters' Uitgeversmaatschappeij N.V., Groningen). A few details from the articles in the contract are:

- Four or five issues yearly.
- At least 64 and at most 120 pages yearly.
- A maximum of three editors.
- In the event of dropping below 5,000 subscribers new negotiations would follow.

In the introduction in the first issue, signed by G. Krooshof/Bruno Ernst, we read:

- The need of men and women who have learnt to deal with und can use mathematics in all branches of technology, science and trade is continuously increasing.'

- We hope that the *Pythagoras* magazine may reach two goals for its readers: arousing interest for mathematics … and beside that … inner joy … of thinking about mathematical problems.

The first of many hundreds of brain teasers that would follow read: 'Somebody has a square wooden tray that is 36 cm long. He must saw rectangles of 5 cm by 8 cm out of it. He asks:

- What is the maximum number of rectangles I can saw?
- In how many different ways I can do this?

Fig. 5. First issue in 1961: *Pythagoras* introduces itself. With: Flatlanders on a billiard ball

The success enabled the publisher to increase the yearly number of issues from four to six and the number of pages from 16 and 20 to 24 for the next four issues, a total of 132 pages, more than mentioned as maximum in the contract. Moreover the magazine was issued with a cover.

In the years to follow the number of subscribers increased fast, till an estimate of at least 30,000. The publisher saw profit in it, having a great influence in secondary schools in the whole country on the basis of having a large percentage of the market for schoolbooks.

Reasons for this immediate success were the absence of other magazines in the field and the fact that most schoolbooks were not extremely exciting.

Mathematical Olympiads

The second decision in connection with the popularization of mathematics for younger people taken by the Freudenthal Committee was the organization of a national Mathematical Olympiad. The first one was organized in 1962. Freudenthal considered it so important as to hold a press conference on January 19. The Olympiad was organized in two rounds. The first round was held in the schools, for 3,346 pupils from 284 schools (the later Nobel laureate Gerard 't Hooft took part). The best 60 were invited for the more difficult second round, in later years more than 100. In Eastern European countries national mathematical Olympiads had been held for many years. The first International Mathematical Olympiad was organized by Rumania in 1959. Great-Britain, France and Italy took part as first western countries in 1967, and in 1969 the Netherlands took part for the first time in the International Olympiad. The Dutch professor of mathematics Jan van de Craats, who also was an important editor for over a decade, became a member of the jury of the National Olympiad in 1973 and leader of the delegation to the International Olympiad in 1975 (Craats, 2000).

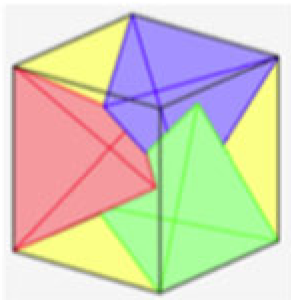

Fig. 6. Van de Craats' Tetrehedrons in a Cube

In the first years the results were poor and Van de Craats started practice sessions with the winners of the National Olympiad. In 1977 this led to a very successful participation in Yugoslavia with a fifth place out of 21 participating countries, and in 1983 in Paris with a seventh place out of 32 countries.

Van de Craats himself is a creative maker of intriguing problems. One of them is: 'How many regular tetrahedrons with edges 1 can you place into a cube with edge 1 without overlapping?'

Van de Craats started *Pythagoras*' own Olympiad in 1979, which continued till 1990. In March 1995 *Pythagoras* restarted the *Pythagoras* Olympiad with prizes and also the possibility to win a place in the second round of the National Olympiad. This Olympiad has been a fixed item in each issue of *Pythagoras* since then.

Different formats, periods of growth and decline, professionalization

Pythagoras stayed a success over the years. In 1970 there were 30,000 subscribers. A book was published by editor Bruno Ernst: *Pythagoras* Festival (Ernst, 1970), with the best articles from the first decade. Around 1980 the numbers of contributors diminished. In a document by De Rijk we read that he thought the 21st annual would be the last. The publisher Wolters-Noordhoff stopped in 1985. There was no profit anymore in *Pythagoras*, although the number of subscriptions was still around 10,000. Chief editor Bruno Ernst retired slowly during the 1980's, although contributing till 1991. A number of publishers in the ensuing years took over, although their influence in the secondary schools was little. The small number of editors became a problem too. The different publishers also led to a number of different formats and varying numbers of yearly issues. The number of subscribers diminished further till several thousand and the continued existence of *Pythagoras* was again in real danger a couple of times. During the 1980's there were periods when the magazine depended more or less on the activities of one editor only: in the beginning of the 1980's it was Hessel Pot and at the end it was Klaas Lakeman, although there were others that played an important role in keeping *Pythagoras* alive. As a typical *Pythagoras* article I would like to mention Pot's article from 1989 about the beautiful Theorem of Karel Petr (1868-1950), a Czech mathematician.

> Take an arbitrary polygon with n vertices. Describe on the edges to the outside n isosceles triangles with top angles $1*360°/n$. The top n vertices form again an n-polygon. Repeat the procedure with the top angles of $2*360°/n$, $3*360°/n$ up to $(n\text{-}2)*360°/n$ (with angles larger than 180° the triangles are directed inside). Then the last polygon is a regular one!

On the front page of the issue Petr's theorem for n=5 is shown. It was typical because the readers could use it to make beautiful drawings themselves, or try to prove the theorem for n=3 or n=4, or even for arbitrary n. For n=3 the theorem is known as Napoleon's theorem.

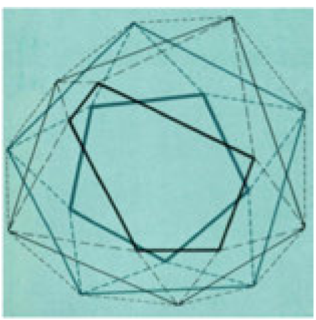

Fig. 7. Karel Petr's Theorem

The idea of the editors Lakeman and Van de Craats in the second half of the 1980's was that there should be more proper mathematics in *Pythagoras*. That has been the course ever since. Good mathematics and also some more difficult articles, understandable for the better students in the highest grades of the secondary schools. In 1990 Henk Mulder stepped in a second time, the first issue of the 30th annual in October 1990 mentioned only Hans de Rijk and Henk Mulder as editors. In the period 1991-1993 the couple Henk Mulder and Henk Huijsmans was the driving force of the magazine. There were a number of reasons for the decline of *Pythagoras* in a number of periods:

- During shorter or longer periods there was a shortage of (unpaid) editors.
- Other magazines on popular science had appeared on the market: Eos (1983-), Archimedes (1964-2003) and KIJK (English: LOOK; 1983-). Although Eos and KIJK were magazines with a broader scope than *Pythagoras* and described, more than explained, scientific developments, in cases of making a choice between more magazines the choice in those years could easily have been for a different magazine than *Pythagoras*.
- The profession of teacher became more demanding in those years, which prevented a lot of interested teachers to become active as an editor.
- After 1968 a major new system for all secondary schools in the Netherlands was implemented (the so-called Mammoetwet or Mammoth Law). The number of broad comprehensive schools grew enormously. In the education of mathematics a lot of lessons in these comprehensive schools were given by so-called 'teachers of the second grade'. They also gave the mathematics lessons in the lower classes to pupils who were taught up to that time by university trained teachers. The second grade teachers were mostly educated at non-university teachers colleges and reached a much lower level in mathematics. For those teachers the magazine *Pythagoras* was perhaps in some of the articles a bit too difficult. Their motivation to recommend *Pythagoras* to their pupils was consequently not overwhelming.
- In the last quarter of the 20th century there was a tendency in all teacher training programs to put more and more stress on didactical methods for use in the classroom. This diminished the time necessary for grasping a good knowledge of mathematics and could have been a reason of diminishing interest by new teachers in *Pythagoras*, which aimed solely on the mathematical content.
- There was a development in school books, from boring without interesting figures to full colour with a lot of side information and demanding extra questions.
- As a more general observation the economic situation in the Netherlands in the 1980's could be added. From the end of the 1970's and well into the 1980's the economic situation in the Netherlands was poor. There was no

> economic growth at all and unemployment rose to very high levels. Under the government, headed by prime minister Ruud Lubbers (1939-) a formidable program of spending cuts was executed, which hit many families severely. Lubbers could execute his program by an agreement between workers (no wage rise claims) and employers (more jobs and reduction of working hours). It took well into the 1980's before the economic growth returned to the old levels of the 1960's and the beginning of the 1970's. This process was later called the Dutch Miracle or the Polder Model. In the 1980's a lot of newspapers and magazines had consequently to deal with firmly lower subscription numbers.

Halfway the 1990's again there was a period of danger for the magazine. After a steady period under the guidance of Frank Roos, Marcel Snel, Jan Mahieu and Henk Huijsmans, the last two kept *Pythagoras* going till the end of the 35th annual in 1996. Then a complete new group of editors took over: Chris Zaal, Klaas Pieter Hart, Harald Haverkorn, Erjen Lefeber and Pier Sinia, shortly followed by René Swarttouw and Dion Gijswijt.

It was a sign for professionalization of *Pythagoras*.

Under the guidance of professional mathematician Chris Zaal Pythagoras started a new period of success. He managed to revise the financial situation. The chief editor and the magazine editor were payed for one day's work a week. The layout was professionalized. More colour was introduced. In 1998 Zaal managed to bring *Pythagoras* under the wings of the Mathematical Society. After a far higher asking price the Mathematical Society was able to buy the rights on *Pythagoras* from publisher NIAM in The Hague for 10,000 Dutch guilders (now € 4,500). The Mathematical Society did not aim at a profit with *Pythagoras*, so that improved the situation considerably. *Pythagoras* was in good hands. The number of subscribers started to grow slowly from a minimum of 2,000. In contrast to the large number of subscribers in the first years of the magazine, the number has been steady for a number of years on just over 3,000. Among the new aspects of *Pythagoras*, introduced by the new group of editors and their followers, were a yearly contest, special problems (Dionigma's), yearly themes, more guest authors, different production method (LaTeX), layout designers, use of logos, active promotion and series of articles on a special subject. More attention was given to the visual aspect of the magazine. To promote the magazine further a number of posters was published, which hang on the walls of many a mathematics classroom in the Netherlands. By many outside authors articles were offered for publication. A website was set up and a few books were published. The positive influence of the chief editors after Zaal, Marco Swaen, Arnout Jaspers and Derk Pik must be mentioned, and also of the magazine editor Alex van den Brandhof, who was mainly responsible for the bright and convincing layout of *Pythagoras* since 2001. In the following paragraph a number of the new developments are described in short.

New developments after 1995

Yearly contest

Each year a special set of related problems is given to the readers. They are given time to solve them individually or in groups in the classroom. They are given a couple of months' time to deliver their solutions, after which a number of prizes is awarded.

Editor Matthijs Coster devised a number of these contests. In particular he was very successful with his later called 'Coster numbers' (related to the so-called Friedman numbers):

> Write a number using all its digits exactly twice and +, -, *, / and parentheses (but no catenation of digits).

For instance: 256=(2*5+6)*(2*5+6). It turned out that there are infinitely many Coster numbers. The series of Coster numbers can be found in The On-Line Encyclopaedia of Integer Sequences (OEIS), with number A106007.

Another one of his contest problems was a puzzle consisting of nine pieces that could be put together forming the letter P. The problem is related to the Tangram puzzle, putting seven pieces together forming a square.

One of the most prolific users of *Pythagoras* in the classroom, the Belgian teacher Odette de Meulemeester, was able to put all of her pupils to creative work with the Pygram puzzles. The prizes were presented by chief editor Derk Pik and Coster, who travelled to Belgium to visit De Meulemeester's school. Her pupils wrote each letter in the word Pygram with the nine pieces.

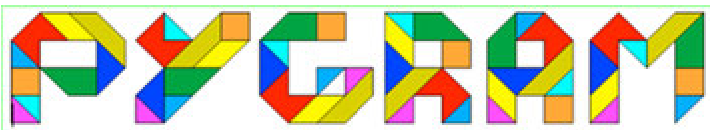

Fig. 8. Odette de Meulemeester's Pygram in Pygrams

Dionigma's

Between 1996 and 2009 Gijswijt published hundreds of creative puzzles and problems in *Pythagoras*, of which fifty were published in the book that appeared in 2011 at the occasion of the tenth lustrum of the magazine. These were coined 'Dionigma's' in the book, in honour of Gijswijt and with a hint at the enigmatic character of many of his problems. Two of them are:

- Is it possible to cover a square with sides of length 5 completely with three squares with sides of length 4? (It is, but can you prove it?)

- Given is a square of 12*12 unit squares. What is the largest number of strips consisting of eight unit squares that you can fit in (parallel to the edges and without overlapping)? (If you have an idea of the maximum, it is not easy to prove.)

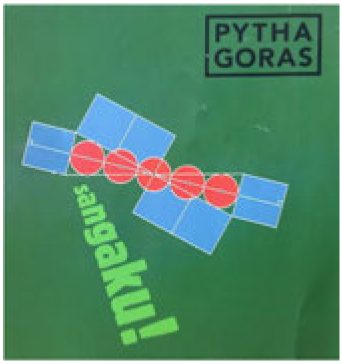

Fig. 9. Part of the Sangaku Poster: Which theorem is presented? Prove it!

Pythagoras promotion

In the course of the past decennium a number of mathematical posters was published, e.g.: Prime Numbers, Sangaku's, Platonic and Archimedean Solids. During the yearly National Mathematical Days for teachers many of them have been sold and now adorn the walls of many a classroom of mathematics in the Netherlands.

A further step in the public relations field, besides the website (www.pyth.eu), is *Pythagoras*' time line on *Facebook*, which presents a weekly problem (Nut to Crack), for which hardly any mathematical knowledge is required, especially for the younger ones, even at primary schools.

Half a century of Pythagoras

In 2011 at the occasion of its fiftieth anniversary ***De Pythagoras Code*** (Brandhof, 2011) was published, with a selection of the best articles and problems from half a century. The editors were Alex van den Brandhof, Jan Guichelaar and Arnout Jaspers.

In 2015 a translation of The *Pythagoras* Code, titled Half a Century of *Pythagoras* Magazine (Brandhof, 2015), was published by The Mathematical Association of America. The book includes, besides the translation, also a number of extra articles and problems.

Influence and future of *Pythagoras* magazine

One of the original aims of the Freudenthal Committee was:

> The advancement of doing mathematics, not from a specialist attitude, but from the perspective of society and the cultural value of mathematics (Kleijne, 2004).

The success of *Pythagoras* over the years has met this aim. Although the use of *Pythagoras* in classrooms has only been possible by the personal initiative of the teachers of mathematics, many thousands of pupils in the course of the years worked on problems and articles from the magazine. Thousands took a subscription after advice of an enthusiastic teacher.

The Olympiads had their influence too. A research by Van der Neut and Wansink into the first ten National Olympiads led to the result that many of the 100 prize winners chose a mathematical profession as a result of their success (Kleijne, 2004).

And an amazing number of current Dutch professors, university teachers and teachers in high schools and colleges in mathematics and natural sciences know the name of Bruno Ernst and mention with pleasure the joy they had in reading *Pythagoras*.

The future of *Pythagoras* will probably see a completely digital magazine. Ideas are worked out for a *Pythagoras* Junior, especially for primary school children. The concept of *Pythagoras*, proper mathematics and daring problems, is still, after half a century, bearing fruit.

Perhaps it is good to finish with words by one of the first two editors Hans de Rijk (Bruno Ernst):

> Mathematics is a luxuriant garden: high in the trees are the brilliant mathematicians, who have been toiling for years to reach the top. But down on the grass, within everybody's reach, there are beautiful flowers to pick as well.

Acknowledgements. I am very indebted to Jan van de Craats, Klaas Lakeman, Odette de Meulemeester, Hessel Pot, Marco Swaen and Chris Zaal for their information given during oral interviews. I thank Jeanine Daems for allowing me to use the notes of her contribution on the history of Pythagoras during a seminar in 2011, and Harm Jan Smid for information, preparing my contribution for the ICHME-4 conference in Turin. Special thanks for Martinus van Hoorn for scrutinizing the draught version of this article and suggesting many improvements. A lot of useful information I found in his article on 'Pythagoras and the competition element' (Hoorn, 2000). I thank my wife Ria Koene, professional teacher of English for 35 years, for her corrections.

Appendix A. Editors of Pythagoras over the years

Hans de Rijk (alias Bruno Ernst) (1961-1991), Gerrit Krooshof (1961-1966), A.F. van Tooren (1963-1975), A.B. Oosten (1966-1974), H.J. Engels (1968-1969), G.A. Vonk (1968-1980), A.J. Elsenaar (1969-1977), C. van der Linden (1969-1970), R.H. Plugge (1969-1972), W. Kleijne (1973-1982), Henk Mulder (1974-1985; 1990-1993), Jan van de Craats (1979-1990), Hessel Pot (1979-1987), Luc Kuijk (1983-1985), Klaas Lakeman (1983-1990), Leo Wiegerink (1983-1985), Henk Huysmans (1991-1996), Thijs Notenboom (1991-), Hans Oomis (1992-1993), Frank Roos (1993-1995), Jan Mahieu (1993-1996), Marcel Snel (1993-1995), Klaas Pieter Hart (1996-) Harald Haverkorn (1996-1997), Erjen Lefeber (1996-1998), Piet Sinia (1996-1997), Chris Zaal (1996-2008), René Swarttouw (1997-2007), Dion Gijswijt (1998-2009), André de Boer (1999-2001), Jos Brakenhoff (2001), Alex van den Brandhof (2001-), Matthijs Coster (2001-), Jan Guichelaar (2001-), Marco Swaen (2001-2007), Thijs Notenboom (2004-2005), Arnout Jaspers (2006-2012), Jeanine Daems (2007-), Willem van Ravenstein (2009), Paul Levrie (2011-), Marc Seijlhouwer (2012-), Derk Pik (2012-).

Appendix B. Publishers over the years

J.B. Wolters, Groningen (1961-1967), Wolters-Noordhoff, Groningen (1968-1985), Stichting IVIO, Lelystad (1985-1992), MEMO, Utrecht (1992-1993), NIAM, Den Haag (1994-1998), Koninklijk Wiskundig Genootschap, Utrecht (1998-).

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Kleijne, Wim (2004). NOCW 50 jaar. *Nieuw Archief voor Wiskunde 5/5*(4), 308-313.

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On the Russian national subcommission of the ICMI

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Abstract

The work of the International Commission on Mathematical Instruction at one time had a broad impact in Russia: one need only recall Russia's nationwide conferences of mathematics teachers, which were organized under the obvious influence of the ICMI, or the comparatively numerous publications devoted to the ICMI. However, the work of the Russian subcommission of the ICMI has clearly not received sufficient scholarly attention, nor has the subcommission's archival record (1909-1915), which the author has located, been examined and analyzed. In this paper, the author discusses the work of the subcommission on the basis of surviving documents and examines the biographies of its most active participants.

Introduction

This paper is devoted to the work of the Russian subcommission of the ICMI. The international movement met with a lively response in Russia and undoubtedly benefitted the development of Russian mathematics education, becoming important above all for reformers, of course, but also more generally facilitating the recognition of the problems confronting education and possible ways of solving them. Suffice it to say that the congresses of Russian teachers of mathematics (Doklady, 1915; Trudy, 1913), which became an important milestone in the history of Russian mathematics instruction, were organized under the influence of the work of the ICMI (Sintsov, 1913).

Although the literature on the first years of the ICMI is quite extensive, very little has been written about the Russian subcommission (Gushel', 2002, 2003), and even that has relied only on certain materials from publications that accompanied the work of the subcommission. Archival files, to this author's knowledge, have never been analyzed, although they are important for understanding what transpired. The present study relies mainly on a special file that has been preserved in the collection of the Scientific Committee of the Ministry of Education at the Russian State Historical Archive (Delo, 1909-1915), Another file, which to some extent, although not entirely, overlaps with this one, and which has been preserved in another Ministry collection, is also used (Ob uchastii, 1909-1916). We also offer

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

information on the biographies of certain active members of the subcommission, which provides a context for its work.

The Scientific Committee of the Ministry of Education and Its Chairman, Nikolay Sonin

The work of the Russian subcommission began with a letter from Felix Klein to the chairman of the Ministry's Scientific Committee, Nikolay Sonin, and therefore a word must be said both about the Committee and about Sonin.
The Scientific Committee was formed in 1856. The guidelines for its work formulated the following objectives:

> a) assessing courses at educational institutions; b) creating or reviewing programs for textbooks; c) reviewing and assessing textbooks; d) reviewing other books and manuscripts, and e) reviewing plans for presentations on academic, educational, and pedagogical topics and other similar projects (Georgievsky, 1902, p. 19)[1].

In fact, agencies with such functions had already existed earlier. Georgievsky (1902) begins his account practically with the Commission for the Establishment of Public Educational Institutions, which was established in 1782 and whose objectives included to some degree the ones listed above. Similar problems were also addressed by other commissions and committees established and dissolved by the Ministry (the Textbooks Committee of the Committee on the Organization of Educational Institutions, the Scientific Committee of the Main Academic Directorship, and others). However, it would be fair to say that it was specifically the organization that was created in 1856 that was the most effective, and that lasted longer than any other, too – until the revolution of 1917. Its long existence, however, was not without certain administrative reorganizations and reassignments, the most important of which took place in 1863 (which is why certain publications prefer to give this year as the founding date of the Committee).

By statute, appointments to the Committee consisted of representatives from different disciplines, including mathematical subjects. These representatives as well as the Committee's chair were chosen and approved by the Ministry of Public Education. From 1901 on, the chair was Nikolay Sonin.

Sonin grew up and received his education in Moscow, where he graduated from a gymnasium and from the University (in 1869), studying under Nikolay Bugaev, one of the major Russian mathematicians of that time. In 1873-74, he was

[1] This and subsequent translations from Russian are by the author.

sent to Paris, where he attended lectures by Liouville, Hermite, Bertrand, Serret, and Darboux. Even prior to this, however, in 1871, he defended a master's thesis, and shortly after that, in 1874, a doctoral thesis as well. In 1893, he was elected to the Russian Academy of Sciences. Before 1893, he taught at the University of Warsaw, where he became a professor and then a dean, but subsequently he moved to St. Petersburg, where he taught as a privat-docent at the Higher Women's Courses and the University. At the same time, he carried out important administrative duties, chairing examination commissions in various universities around the country. In 1899, he was appointed supervisor of the St. Petersburg school region, in other words, became the head of the system of all educational institutions run by the Ministry of Education in the northwest of the country. In 1901, retiring from this post, he became chair of the Scientific Committee and remained in this position until his death in 1915 (Posse, 1915).

During the last years of his life, Sonin was gravely ill – his official work file (O sluzhbe, 1915) consists mainly of letters to the effect that, due to the state of his health, he will be unable to fulfill his duties, or conversely, that he is resuming his duties. It ends (O sluzhbe, 1915, p. 18) with a document whose strict bureaucratic style might occasion grim humor: the note concerns the collection of money for a silver wreath from the Committee for its chairman's coffin, specifying precisely who gave 3 rubles and who gave 3 rubles 55 kopecks. But in 1909, Sonin was still in good health and in complete possession of his vast administrative, academic, and pedagogic experience.

The formation and first steps of the Russian subcommission

The decision to create an International Commission on Mathematical Instruction did not pass unnoticed in Russia. It was written about, for example, by Dmitry Sintsov (1908) of Kharkov, who participated in the fourth Congress of Mathematicians in Rome, noting that, following a proposal by D.E. Smith and Archenhold[2], the congress had decided to establish a commission "for the comparative study of programs and methods used for teaching mathematics in secondary schools in different nations" (p. 80). He also noted, however, that Russians were poorly represented at the congress: "Among Russians, in addition to Kharkov," he wrote, "only St. Petersburg was represented. [Here he lists a number of names.] But Moscow, Tomsk, Kazan, and Odessa were completely unrepresented."

[2] Sintsov had in mind the astronomer Friedrich Simon Archenhold, about whose role see Schubring (2008).

Therefore, the beginning of the work of the Russian subcommission may be dated from Klein's letter to Sonin of January 24, 1909, written in German, translated into Russian, and read at a meeting of the Scientific Committee on January 19 (Old Style). The letter stated:

> Highly Esteemed Colleague! You are, of course, already aware of the enterprise whose plans I am enclosing with this letter, in two copies. Having corresponded about the matter with my Russian colleagues, with whom I have the closest personal relations, I have the honor of requesting you, Esteemed Colleague, on behalf of the Central Committee of the International Commission on Mathematical Instruction, to see fit to take this matter into your hands, since it concerns Russia (Delo, 1909-1915, p. 21)

Klein then writes about what must be done, listing several key points. The most important of them are:

> 1) To secure the assistance of suitable individuals willing to work jointly with you as delegates from Russia. (I am informed by various correspondents that it would be advisable to choose all delegates from among St. Petersburg mathematicians.) 2) To organize a national conference with the aim of providing support to the delegates in their endeavors (p. 21). 3) To enter into communication with the government, in a timely manner if possible, so that it may subsequently not only confirm the delegates, but also put at their disposal the financial resources necessary for carrying out their work (p. 22).

The letter concludes as follows:

> My own request to you is that you not only accept my proposals in general, but also inform me of the delegates you have in mind. Then I, on behalf of the Central Committee, will send an official letter to your government. (p. 22)

The minutes of the same meeting on January 19 further state:

> During discussion of this letter, the Chairman remarked that he does not consider it possible to decline Prof. Klein's proposal only because, due to his position in the Ministry of Public Education, he does not believe that he has the right to do so. On the one hand, the dignity of Russia, as a great nation, demands that it involve itself actively in international cultural work; on the other hand, his position as chairman of the Scientific Committee offers special advantages in terms of the practical realization of the congress's propositions. More than anyone else, he can be informed about the state of affairs in different educational institutions, and more quickly than anyone else, he can obtain all necessary information (p. 22).

At the same meeting, on Sonin's proposal, the following member scholars of the Scientific Committee were selected to be part of the Russian delegation: B.M. Koyalovich, a professor at the Technological Institute, and K.V. Vogt, the director of a secondary education institution (a so-called *real'noe uchilsche* or "real school," to give the literal English translation, which was a Russian parallel to German Realschule). We will return to them below. Also discussed were possible expenses, the largest among which was a "support that will have to be provided to those delegates who will take part in the commission's meeting in 1911." It was noted, however, that "such expenses are unlikely to present any difficulty to the Ministry" (p. 23).

All of this, naturally, was made known to Minister Alexander Shvarts, who on February 4 (Old Style, here and below) approved and confirmed everything, but also made a remark that showed that his understanding of Russian reality was better than that of Klein and his Russian advisors:

> at the same time, I would ask you to discuss whether it might not be useful to have among the Russian delegates one of the members of the Moscow Mathematical Society, which has already been in existence for many years and which has demonstrated great interest in pedagogical questions pertaining to the teaching of mathematics in various educational institutions, and to apprise me of your decision (p. 24)

The file contains no direct response to this suggestion, and no new delegates were added, but steps were taken to involve individuals from different backgrounds in the work of the delegation (which had been, in fact, Klein's suggestion as well).

Even before this, Sonin wrote back to Klein and received a second letter from him, which informed Sonin that Klein had apprised Fehr of the makeup of the Russian delegation. This letter, along with all other new information, was discussed at a delegates' meeting on February 12. It was argued that information must come from individuals who were well informed. This could be "achieved in two ways: either by consulting the heads of departments or by directly approaching persons employed in these departments who are known to be acquainted with the matter and are fully trustworthy." It was pointed out that "the first way may lead to difficulties," since it might augment the role of departmental interests. (p. 56)

It was also decided to invite Scientific Committee member V.I. Sollertinsky and St. Vladimir University professor V.P. Yermakov to the next discussion, and also to follow up on the minister's suggestion by inviting various societies to become involved, including the St. Petersburg Physics and Chemistry Society.

The second meeting, on March 12, 1909, was attended by V.I. Sollertinsky and discussion focused on the makeup of the national subcommission. It was decided to invite two more professors (along with Koyalovich) form the St. Petersburg Technological Institute, Pavel Koturnitsky and Alexey Gatsuk; the inspector of St.

Petersburg's Fourth (Larinskaya) Gymnasium, Petr Kolesnikov; two representatives from military academic institutions, Generals Zakhar Maksheyev and Mikhail Popruzhenko; two teachers from the Women's Pedagogical Institute, Nikolay Mikhelson and Zakhar Vulikh; the director of the Mezhevoy Institute in Moscow, Vasily Struve (to provide information about land-surveying academies); and Nikolay Bibilin, to provide information about religious academies. Finally, the name of Distinguished Professor Konstantin Posse of St. Petersburg University was added to the list in pencil (p. 57).

Shortly after this, the delegation prepared a preliminary report about the formation of the International commission on mathematics education, accompanied by an announcement about the formation of the Russian subcommission. This text was published in offprints and also printed in the *Journal of the Ministry of Public Education* (Sonin, Koyalovich, & Vogt, 1909), and then reprinted in other journals. It ended with the following remark:

> In making this announcement, the Russian delegates express their firm conviction that they will receive energetic and active support in fulfilling their duties from professors and teachers of mathematics and related sciences (mechanics, physics) in educational institutions of various types and departments (p. 46)

And further, individuals wishing to participate in the preparation of reports were invited to get in touch with Sonin.

The offprints of this report were sent (on April 17, 1909) to a number of prominent figures in mathematics education (in accordance with the list previously discussed by the delegation), along with the following letter from Sonin:

> In view of the desirability of your active participation in the Russian national subcommission, the Russian delegation has instructed me to ask you to accept a membership in the subcommission. In addition, I consider it important to let you know that, for the delegation's preliminary considerations, it would be very important to receive from you as soon as possible some indication of those questions among the ones listed in the enclosed brochure concerning which we may hope to receive a comment from you for inclusion in the general report. (Delo, 1909-1915, p. 25).

On April 18, letters were sent to the deans of the mathematics departments at all universities (along with offprints of the preliminary report and the proposal to reprint it in university editions):

> In view of the desirability of the active participation in the national subcommission of representatives of the mathematical sciences residing in different parts of the Russian state and interested in the questions listed in the

> enclosed brochure, the Russian delegation has instructed me to invite the university's physics-mathematics department to appoint two members to the subcommission (p. 35).

During these same days (on April 16), Sonin also sent letters to all supervisors of school districts:

> In view of the desirability of enlisting as many collaborators as possible to fulfill the duty incumbent upon the delegation, I have the honor of humbly requesting you, Kind Sir, to assist us in distributing information concerning the aforementioned international undertaking by reprinting the enclosed brochure in the newsletters published by the Regional Directorate (p. 47)

In addition, in the same letter Sonin asked the supervisors to report how many hours are devoted to arithmetic, geometry, algebra, and trigonometry in teachers' institutes and seminaries, what programs form the basis of instruction, and which textbooks and problem books are being used (such surveys of educational institutions, however, had been regularly conducted previously as well).

Finally, on April 18 Sonin sent analogous letters to mathematical societies (in Moscow, Kazan, Kharkov, and Kiev), inviting each to appoint two members to the Russian subcommission.

Gradually, responses began to arrive. The Moscow Mathematical Society reacted almost immediately (letter of April 24 signed by the president of the Society, Nikolay Zhukovsky), having decided to postpone making a decision until its autumn meaning, and to reprint the brochure it had received in their journal *Matematicheskii Sbornik* (Mathematical Collection) (p. 52). The reply from the Kharkov district stated, not without irony, that it was not worth reprinting the report in newsletters since these were read only in school administrative offices, while all educational institutions received the *Journal of the Ministry of Public Education* (often in two copies), where the report was already printed. Therefore, the Kharkov correspondents proposed to print in newsletters only a notice about the report and information about its publication (p. 49). Sonin effectively agreed with this.

Those who had been invited to write reports for the most part agreed to do so, although the well-known Kiev professor V.P. Yermakov replied that he had no reports, but would like to read the reports prepared by Sonin, Vogt, and Koyalovich, who – he wrote – would be able to prepare such reports without difficulty, since they work at the Committee where all the relevant data is gathered (p. 34). In response to the invitation to nominate delegates, the University of Kharkov – which was the university of Dmitry Sintsov, one of the most active figures in mathematics education of that time – inquired whether their role would consist merely in the writing of comments on the questions posed in the brochure, or in a discussion of the report in its entirety (p. 39).

Sonin and colleagues reacted to such proposals rather decisively – they were discussed at a meeting on October 22, 1909. It was decided that Yermakov's response meant that he was declining to participate for the time being, while the Kharkov correspondents were informed that there was not a great deal to discuss:

> In the opinion of the delegates, there is no room either for the expression of subjective views or for debate in the execution this task [the publication of the report]; the delegation's general report must represent a mosaic, each of whose separate parts must be executed as thoroughly as possible. Under such circumstances, given that the general report will be submitted on behalf of the delegation to the international commission, it is difficult even to understand what a discussion of the report that the department finds so desirable might consist in. Resolved: to reply to Mr. Gruzintsev [the dean of the University of Kharkov] in this manner (Ob uchastii, 1909-1916, p. 14)

On the other hand, the contrary apprehension was also expressed: that the arriving delegates, although they would not meddle when they weren't asked, might not do what they were asked to do. In a letter to the Ministry, Sonin formulated this as follows:

> while letting us know that they had elected members to the subcommission, the departments failed to satisfy our request to inform us as to which questions the members whom they had elected intended to provide comments on. Such an omission leads the delegation to suspect that some of the elected individuals agree to accept membership in the subcommission without burdening themselves with obligations, and compels the delegation to seek clarifications concerning this matter from the elected members directly; it is quite likely that some members will reply to this query by renouncing their memberships (Ob uchastii, 1909-1916, p. 12).

On the whole, however, everything went quite smoothly, and on November 21, 1909, a wider meeting was held, at which many of the invited collaborators and delegates from universities were present. The meeting occasioned a discussion (not for the first time) about attitudes toward general questions that were being raised by the Preliminary Report. Sonin replied that certain questions of the general program were dubious – for example, the question of co-education or the complete elimination of exams "have no connection with the definition of mathematics education as such" (Delo, 1909-1915, p. 81). Some of those present did not agree with him: General Maksheyev reported that such questions would be discussed at meetings of the Pedagogical Museum. Sonin gave his approval and proposed to provide a discussion of questions of the second part in a supplement to the main report.

Also discussed was the schedule for completing all of the tasks and their distribution among different authors. All of this work, including this meeting itself, was financially supported by the Ministry.

The Ministry also funded the publication and the translation of the brochures that were being prepared – permission to publish the first of these, *Rapport sur L'Enseignement des Mathématiques dans les Ecoles de Finlande,* had already been granted on August 12, 1910. During the years 1910-1915, several more brochures came out, mainly in French, with the exception of Vogt (1911), which was written in German. Two hundred copies of each brochure were printed. They are contained in the file that we have been citing here (Delo. 1909-1915), but appear to be quite rarely found in libraries, for example, the author was unable to find all of them at the National Library of Russia in St. Petersburg, although at one time they must have been sent to what was then the main library in the country.

On the subsequent work of the Russian subcommission

Preparation and publication of reports, on the one hand, and participation in the work of the International Commission and its congresses, on the other, were the main tasks of the subcommission. Consequently, from a certain moment onward, the commission's file consists mainly of documents about the publication of brochures – the last of which was Mlodzievsky's (1915) brochure on women's education – and about work trips and their funding.

The congress in Milan[3] was supposed to be attended, from the Russian side, by Koyalovich and Vogt, but it was difficult for Vogt to travel at the beginning of the school year, according to Sonin, and Sonin had to ask the Ministry to send Sintsov to Milan instead, which the Ministry did (Delo, 1909-1915, p. 239).

Vogt was ill, and in 1913 he died, so on February 11, 1914, Sonin sent the Ministry a letter requesting that Actual State Councillor K.A. Posse, a distinguished professor at St. Petersburg University and a member of the Scientific Committee, who was already a member of the Russian subcommission, be given Vogt's place as a delegate. In the same letter, he requested that Posse be sent to the congress in Paris (p. 281).

After Sonin's death, Posse became the head of the Russian subcommission, and in this capacity, on April 15, 1915 he sent Minister of Education Ignatyev a request that Sintsov be approved as a member of the commission (delegate), since

[3] In 1911, a meeting of the International Commission took place in Milan. Other important meetings, which will be mentioned below, were held in Brussels in 1910 and in Paris in 1914 (see Furinghetti & Giacardi, 2010; Schubring, 2008).

he "has taken active part in the work of the Russian national subcommission" (Ob uchastii, 1909-1916, p. 63). The approval was received less than a week later (Delo, 1909-1915, p. 286).

The reports of Russian delegates about their work abroad, which were published in Russian journals at the time, and which survive in archival collections, reveal how what was happening in Western Europe and the United States was perceived in Russia. Without going into a discussion of the entire body of the surviving documents, let us consider a semi-private letter from Sintsov to Sonin (Sintsov, 1911a) about the congress in Milan, which overlaps with the published report about this congress (Sintsov, 1911b), but is still written in a noticeably different manner.

Sintsov not without pleasure takes note of the rivalry among delegations: that Germany was unable to do everything as it had planned, while France, which had shamed itself in Brussels, this time had come out on top, presenting five finished volumes (admittedly, not very large ones), so that "F. Klein consoled himself in private conversations only by the fact that the French delegates, especially C. Bourlet, themselves acknowledge the need for a sixth volume – of results – and that therefore even the French don't have everything finished yet." (Sintsov, 1911a, p. 1)[4]. He also notes personal rivalries, remarking, for example, that the "somewhat harsh presentation by Veronese, which criticized the entire work of the Commission en bloc, was explained by the fact that he had been passed up during the selection of delegates" (p. 2).

At the same time, Sintsov himself was optimistic (as he himself notes, by contrast with Koyalovich, who had a far more skeptical attitude to the proceedings). He agrees that "the whole work of the Commission amounted merely to a description of how things stand, and this is undoubtedly something very important and useful, but then it would be desirable to figure out also how things should be." But he at once notes that "this can hardly be done by a Commission, let alone an international one. It is rather a subject for journal studies and a topic for discussions at congresses such as our upcoming teachers' congress," and he further adds that it is the work of the future.

As for his own task in the subcommission, he saw it as follows:

> We have compiled a digest of what we have for the Commission, and we must ourselves use the Commission's work to acquaint Russian pedagogues with the character of teaching in the West: the enormous reports contain much that is of little use to them, and they are also almost inaccessible. At the congresses themselves, I see the role of our delegates thus far as being mainly informational in both directions. And I seek to use the congresses both to establish channels of communication with foreigners and to obtain

[4] In general, the ability to prepare reports on time was clearly considered very important. Another Russian participant in the Congress, Boris Koyalovich (n.d.), also devotes special attention to this.

> from them something useful for those organizations with which I have a connection (p. 2)

Accordingly, in the body of the letter he gladly names those individuals whom he invited to make presentations at the Russian congress of teachers of mathematics, those who promised to help him with books for the Pedagogical Library and for courses, those with whom he was able to arrange an exchange of books. Consequently, during the Milan congress he also called attention to:

> the revival of interest in pedagogical questions that is noticeable in Russia, and as if an awakening of the population of teachers, which expresses itself in the work of societies that are wholly or in part mathematical-pedagogical (math circles in Moscow, Warsaw, Riga, our mathematics society, and the one in Kiev), and finally in our own upcoming congress, which in essence serves the same purpose as the international commission (p. 1).

On certain members of the Russian subcommission

Sintsov (1867-1946) was perhaps the most active member of the Russian subcommission, and thus is possibly the best known today. A geometrician, a professor at Kharkov University, who on the whole successfully survived the changes brought about by the revolution, becoming a Ukrainian academician and dying in 1946, he has remained a recognized authority in pedagogy as well. Dissertations about his pedagogical work were still being defended as late as 1970 (Borovik, 1970).

Sintsov did in fact do a great deal to implement that program of information exchange which he wrote about. Apart from his speeches at Russian congresses and articles in journals (see in addition to already cited works Sintsov, 1910a, b, 1912a,b), he edited a Collection of Programs of Instructions on Teaching Mathematics in Western Europe (Sintsov, 1914). After the revolution, too, he attempted to maintain his international ties, even while expressing a rather skeptical attitude toward certain American innovations that were being introduced coercively at that time, such as the Dalton Plan (Karp, 2006).

Another member of the subcommission, K.A. Posse (1847-1928), was at one time famous at least among mathematicians, because he was the author of a popular textbook (for example, Posse, 1912). After graduating from St. Petersburg's Gymnasium No. 2, Posse enrolled at St. Petersburg University, where he studied under Pafnuty Chebyshev. His whole mathematical life was connected with this university, although he also taught at other institutions of higher learning in St. Petersburg. Ravdin and Sergeev (1992) note his active participation in community activities, including his membership in the committee of the Literature Foundation and his participation in the concerts of the St. Petersburg chamber music society. By 1917, Posse, an honorary academician since 1916, was already quite old, and adjusting to new revolutionary and post-revolutionary realities proved extremely difficult for him. His published letters of 1917-1918 (Posse, 1992) to the famous lawyer Tagantsev (whose son gave its name to the "Tagantsev conspiracy," an alleged plot fabricated by Chekists and used as a pretext to put the son himself and many others to death) and to the mathematician V.A. Steklov describe a reaction of unrelieved horror at the changes taking place. Posse died at the old age home of the Scientists' House (Ravdin and Sergeev, 1992).

Another member, Professor Boris Koyalovich (1867-1941), died in Leningrad (St. Petersburg) during the siege during WWII, probably of starvation. His life (first and foremost as a mathematician) is the subject of a study by N.N.'son (1973), which clearly relies on family archives. Several of his so-called "service record books" have also survived (Formuliarnye spiski, n.d.; Ob izbranii professorami, 1901-1914). The future mathematician was born in the family of history professor and actual state councillor Mikhail Koyalovich, and after graduating from the mathematics department of St. Petersburg University (Alexander Korkin is considered to have been his teacher) was kept on at the university to prepare for the position of professor; but two years before obtaining his master's degree, Koyalovich already began working at the St. Petersburg Technological Institute (in 1892). He continued working there, and after obtaining a doctoral degree, became a professor in 1903. In the same year, he became a member of the Scientific Committee. Note that his scientific research brought him together with Sonin, and he even developed some of Sonin's results.

Koyalovich wrote numerous reviews for the *Journal of the Ministry of Public Education*. They did not always elicit a positive reaction. The author of a famous problem book in algebra, Nikolay Shaposhnikov (1906), published a special pamphlet in reply to Koyalovich. Beginning with the remark that at one time "official pedagogical criticism, represented by the Scientific Committee of the Ministry of Public Education, could be regarded as a more or less normal phenomenon" (p. 1), he goes on to say:

> But the fundamentally harmful principle of the unimpeachable authority of what for this literature constitutes the supreme critical tribunal was not slow in leading to the development of absolutely abnormal excesses. Most recently, one of the representatives of elementary mathematical criticism on the Scientific Committee has been one Professor Koyalovich, who should have no place in a public organization of this kind. It cannot be allowed that an individual who is completely incompetent in a matter, and who himself has need – despite his title – of elementary philosophical-mathematical instruction, should influence the course of mathematics education in our homeland. (p. 1)

The passionate tone of N. Shaposhnikov, who had been offended by Koyalovich's criticism of his book on trigonometry, is sustained by Shaposhnikov over the entire forty pages of his pamphlet and is probably unfair, but demonstrates the intensity of conflicts in methodological circles. Mathematics educators, even quite successful ones, were far from unanimous in their views before the revolution, too.

We have no information about the way in which Koyalovich spent the first post-revolutionary years, but subsequently he continued working at the Technological Institute and other Leningrad (St. Petersburg) institutions of higher learning, including at one point the Pedagogical Institute as well. He was a member of the Main Board of Measures and Weights, and wrote and published textbooks. In 1928, he was awarded the title "distinguished scientist," and later, in 1932, he was given not an ordinary, but a so-called personal pension.

Karl Vogt (1845-1913), another member of the Russian delegation, was not a professor and possibly because of this his name remains completely unknown to the broader community of mathematics educators. At least the reference book by Borodin and Bugay (1979), which, albeit briefly, nonetheless does mention three other figures discussed above, says nothing about Vogt. However, information about him has survived in the institutions where he worked (for example, O naznachenii statskogo sovetnika K.V. Vogta direktorom uchilischa, 1899-1913). In addition, we should note the studies of German researchers (Amburger, n.d.) involved in gathering data about Germans who lived in Russia.

K.V. Vogt graduated from the physics-mathematics department of St. Petersburg University in 1871, and in 1872 he began working at the Gatchina teachers' seminary. Vogt was sent abroad to study the foreign experience of working in such seminaries, and in 1878 he was transferred to a six-grade pro-gymnasium, which subsequently became St. Petersburg's Gymnasium No. 8, of which he became the director in 1897. In 1899, Vogt became the director of the Second Real School. Ultimately, he reached the rank of actual state councillor and received numerous honors. It should be noted that he was clearly held in high regard at the Ministry, both as an organizer and as an expert on education – he was regularly invited to various commissions for improving education, formulating rules on various topics, discussing reforms, etc. His folder contains notes from Sonin with requests to organize reviews of various books received by the Scientific Committee (O naznachenii, 1899-1913, p. 39). During the last years of his life he was ill, but, although he had served for far longer than the 25 years required to obtain a pension, and consequently had to regularly obtain permission to continue working, he was not allowed to leave his job.

Discussion and conclusion

An examination of the work of the Russian subcommission gives rise to several sets of questions. First, in studying this work, we encounter many almost forgotten names and many completely forgotten documents. This author knows of no study that makes use of the materials prepared by the subcommission. Yet they are informative, clearly professionally prepared, and quite extensive. They were prepared in Russian (at least almost all of them – Vogt, perhaps, wrote directly in German), but by all appearances were not published in Russian (although it makes sense to continue searching for publications in local editions). Today, we should go back to them in order to form a better picture of pre-revolutionary mathematics education.

Above, we provided brief biographies of some members of the subcommission. The study of their lives and the lives of others who contributed to its work seems quite interesting and important. We have here a list, prepared by contemporaries, of the significant mathematics educators of the time. Of course, one should not automatically jump to the conclusion that these were in fact the most significant individuals in our field at that time, as clearly the composition of the list was biased – acquaintances and colleagues were often recruited. Nonetheless, all of the individuals named were notable figures, but our knowledge about them is scanty in general, and particularly so concerning their activities during the transitional revolutionary years. Meanwhile, some of them ultimately found themselves in demand, such as Sintsov or Posse's correspondent V.A. Steklov, in whose honor a Russian mathematics institute is now named; others wrote about the demise of Russia, like Posse; and still others emigrated, for example, Z.A. Maksheyev.

Another direction for research opened up by the study of the work of the subcommission is the reception of foreign experience in Russia. Historically, there have been times when such reception was extremely negative (Karp, 2006), but even in those cases when it was, on the contrary, positive, the way in which various foreign events are perceived in a country does not necessary coincide with the way in which these same events are understood by their foreign participants or observers. What part of the foreign experience proved to be the most interesting and relevant to Russians and how was it transformed by them? Any answer to this question must certainly draw on materials that pertain to the work of the subcommission and the groups and organizations initiated by it.

Going back to the facts cited in this article, we would argue that "Russians," that is, Russian mathematics educators, were far from unified. We see signs of struggles among different departments and different regions (which the Ministry as a whole and Sonin in particular tried not to aggravate), but the most important fact of all is that practically all of those who participated in the work of the subcommission belonged to only one of the existing groups. Granting that the ICMI's work was reformist in nature, we may say that the Russian subcommission brought together reformist generals – and generals not in some figurative sense, but literally, as very many of its participants were at least actual state councillors, a rank that corresponded to the military rank of major general.

The discussion between Koyalovich and Shaposhnikov in some measure reflected the opposition between official pedagogy, with its "unimpeachable authority," and as it were commercial pedagogy, which was powerful due to its massive success with readers. As we saw, the opposition was quite strong, but it was still contained within the bounds of the existing system. Even more notable departures from this official pedagogy may be observed at the Russian congresses of mathematics teachers to which the ICMI gave rise.

Only a few years later, the country underwent a revolution that was at least in its declarations aimed not only against ministry officials, but even against liberal local activists, whose traditions may be seen in the attitudes and feelings of Sintsov or Posse. The revolution ushered in a period of radical reforms in education, including mathematics education (Karp, 2012). Many of them derived from ideas promulgated by the ICMI, but the advocates of these reforms were now often completely different people from those who had convened for meetings organized by Privy Councillor Sonin or even at teachers' congresses.

Questions that had previously been considered at most pedagogical-technical – choosing the best teaching methods – or even simply bureaucratic (a great power should be involved in international cultural projects!), now became part of a social-political agenda, and someone who was an enemy of, say, the laboratory method in the teaching of mathematics could now turn out to be in a certain sense an enemy of the socialist state (for example, the well-known science educator Boris Raikov was sentenced to ten years in prison in 1930 for hostile views on education methodology (Raikov, 2011), although we know of no similar instances in mathematics education).

When analyzing the history of the reform movement in Russia and the USSR, its complexity and multifacetedness should be borne in mind. We are once again reminded of this fact by the documents of the work of the Russian subcommission of the ICMI.

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Changing direction: The "Second Round" of the School Mathematics Study Group

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Abstract

By the end of 1965, the School Mathematics Study Group (SMSG) – which was the largest and arguably the most influential of the new math projects in the United States of America – had completed the preparation, revision, and publication of 20 mathematics textbooks for Grades 1 to 12, along with a variety of supplementary materials for teachers and students. More than 5 million child-years had been devoted to studying from those textbooks. Concerned that the materials might soon become "frozen into a newly orthodox pattern," the SMSG Advisory Board began a process of planning and experimentation to design a new sequential curriculum for Grades 7 to 12 as well as to produce experimental instructional materials. Despite a considerable investment of time and money by SMSG, the second round had almost no effect on school practice. Very few teachers apparently knew about the materials, let alone taught from them. The present paper discusses how the SMSG second round began, what it attempted to do, and what it was able to accomplish. It ends with speculation on the reasons for its lack of impact, including observations on some ways in which it might have been ahead of its time.

There are no second acts in American lives.
– F. Scott Fitzgerald, notes for *The Last Tycoon*

Introduction

Active from 1958 to 1972, the School Mathematics Study Group (SMSG) was the largest and arguably the most influential of the so-called new math curriculum projects in the United States (Kilpatrick, 2012; Phillips, 2015). In February 1966, E.G. Begle, director of SMSG, published an open letter in the journal *Science* (Begle, 1966a) that was then cited in a *New York Times* article (Schwartz, 1966) under the headline "New Math Fears It Is Growing Old." A similar letter to "the mathematical community" appeared in the April *Mathematics Teacher* (Begle, 1966b). Both letters announced the launching of a new project to design, develop, and try out a new curriculum in school mathematics for Grades 7 to 12. For several years, the SMSG Advisory Board had been debating what further activities to undertake in curriculum development (Begle, 1968) and had decided that they

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

needed to take a "second look" (SMSG, 1966c, p. 9) at the secondary mathematics curriculum. Concerned that the materials SMSG had produced might become "frozen into a newly orthodox pattern," (Begle, 1966a, p. 632), the Board had appointed a steering committee in 1965 to get the new project underway.

Over the preceding 7 years, SMSG had produced 20 textbooks and many supplementary materials covering the curriculum from Grades 1 to 12. By 1966, the classroom use of SMSG textbooks had amounted to more than 5 million child-years (Begle, 1966a, 1966b). Now it was proposing to produce textbooks that would reflect "recent progress" in mathematics education and that would respond to what SMSG perceived as the increasing need for mathematics in society. To help the process get started, Begle's open letters posed a series of questions that asked readers to respond with suggestions regarding the school mathematics curriculum as well as additional questions that occurred to them. Those interested in school mathematics were asked to comment on such issues as trends in the way society was using mathematics, how more attention might be given to applications of mathematics, and how much mathematics should be recommended for all students.

A "catalogue" (Randall, 1966) of the several hundred suggestions Begle received ranged all over the map, from praise for the work of SMSG to strong objections to the work it had done. Almost every proposal one letter writer made was countered by another asking for the opposite. There were, however, numerous requests that the curriculum be made less theoretical and complicated, contain more applications of mathematics, and be better connected to the sciences and humanities. Many of Begle's correspondents volunteered to serve on committees and writing teams, sending him materials they had produced as examples of how the curriculum should look. They requested a better treatment of computer programming and the use of flow charts and algorithms, and greater integration of the separate strands of mathematics. They wanted teachers with a greater knowledge of the subject and a modified curriculum for students struggling with mathematics.

Conference on Secondary School Mathematics

Shortly after Begle's first letter was published, a conference was held in New Orleans, LA, to discuss the mathematics that might be included in the new curriculum (SMSG, 1966a). The 16 participants, who met for 5 days, included university mathematicians, high school mathematics teachers, and mathematicians from industry. The conferees' main recommendations were that the curriculum should contain: (1) frequent consideration of mathematical models of significant and interesting problem situations; (2) an introduction to those mathematical concepts that are important for general citizenship; and (3) in addition to traditional topics, some consideration of probability, logic, computing and flowcharts, and the concept of function. Much of the conference was spent in small groups studying individual topics that were then discussed and revised by the whole group. A draft conference report was circulated to the participants after the meeting, and their comments and corrections were used in preparing the final report.

The conference yielded two "main guiding maxims" (SMSG, 1966c):

> 1.The initial segment of the secondary school mathematics curriculum should be devoted to those mathematical concepts which all citizens should know in order to function satisfactorily in our rapidly expanding technological society. It was felt that capable students should be able to complete the study of this mathematics in three years or less, while the less able students might profitably spend four, five, or even six years completing the sequence.
> 2.The exposition of this mathematics for the average to slow-moving student will need to be satisfactorily developed if the project is to be a success. (p. 9)

SSM and SSAM

During the summers of 1966 and 1967, a group of mathematicians and high school teachers prepared detailed outlines of the program for the initial segment (SMSG, 1966b, 1967a), which became known as *Secondary School Mathematics* (SSM), and alternative proposals for a follow-up course (SMSG, 1967b), which became known as *Secondary School Advanced Mathematics* (SSAM). Experimental versions of 14 chapters for Grade 7 were written during the 1966–1967 school year and tried out the following year, when 14 chapters were written for Grade 8 (SMSG, 1969). The process continued with writing teams meeting during the summers of 1969, 1970, and 1971 and classroom tryouts during the academic years of 1969–1970 and 1970–1971. By the end of 1970, the first eight chapters for SSM were published in their final form. The remaining 20 chapters were published in final form in 1971, with each chapter accompanied by a teacher's commentary. The SSM program attempted to fuse arithmetic, algebra, and a concrete, descriptive geometry so that each supported the others. Mathematical modeling was introduced for the first time in American school text materials (Pollak, 1998) as a means of applying mathematics to real situations.

The SSAM materials consisted of eight chapters bound in five units to be followed by Part 2 of the textbook *Intermediate Mathematics* (SMSG, 1965). Students would study SSM textbooks for from 3 years or less up to 5 or even 6 years, and if they were still in secondary school would go on to take courses built from the SSAM materials followed by other courses using SMSG high school textbooks.

Henry Pollak (1985), who contributed to SSM and SSAM, described their content as follows:

> The second-round materials of the SMSG contained beginning work on computing, applications, and probability. Those materials were written somewhat closer to the English style of teaching in that different parts of mathematics were intermingled rather than presented as a solid year of algebra or a solid year of geometry. In the first round, the spiral approach, which had been preached for everything else, was used within each individual course, but not the intermixing of the different parts of mathematics. In the second round, we wrote totally different 7th and 8th grade books. Then we came to the conclusion that it wasn't possible to finish a total rewrite of high school mathematics, and if I remember correctly, there was a bridging volume written to take students from these new 7th and 8th grade materials back to a more traditional curriculum in the senior high school. (p. 236)

The bridging volume Pollak referred to comprised the SSAM materials (see Coxford, 2003, and SMSG, 1972, for details of the content).

Classroom tryouts

In 1967–1968, several hundred average or above average seventh graders from five Northern California high schools began studying the SSM materials (SMSG, 1972). Most remained in the program through the ninth grade, and some through the tenth grade. In 1970-1971, there were eight tenth-grade classes in seven high schools; they completed the SSM sequence in the fall and finished about half of the SSAM sequence that school year. Additional schools in Northern and Southern California used SSM in various settings, with feedback used to revise the text materials. Teachers involved in the tryouts attended biweekly seminars conducted by SMSG staff members, and the teachers' comments were also used in the revision process. When the project ended, a final report (SMSG, 1972) was produced on what had become known as the "second round" of secondary mathematics curriculum revision. A total of 54 people participated in the planning and writing of SSM and SSAM (SMSG, 1972). Details on the classroom tryouts, revision process, and implementation can be found in the final report.

Obstacles to implementation

A major obstacle to successful implementation of the SSM and SSAM materials lay in the attempt to develop a curriculum that would be suitable for all secondary school students. Although the argument was very appealing that all students should study the same mathematics, with the only variation being in the pace at which that was done, getting U.S. secondary schools to provide multiple mathematics courses at each grade from 7 to 12, preparing teachers to handle the new content in those courses, and arranging for students to take an appropriate course each year proved to be an almost impossible task.

Moreover, then, as now, U.S. secondary school teachers were not familiar with the applications of mathematics they were being asked to teach. They were, in general, poorly acquainted with ideas of mathematical modeling, computer mathematics, probability, and statistics. As pointed out by Henry Pollak (personal communication, June 28, 2015), the movement toward applications in the second round had been stimulated in part by SMSG's desire to respond to a 1962 criticism of the approach taken in the first round (see, e.g., Roberts, 2004). Among other complaints about the efforts of SMSG, the critics were arguing that students needed "a curriculum that had much more connection with the world than the first round of SMSG was reputed to have" (Pollak, 1998).

James Fey (1978) saw some irony in the timing of the second round, coming as it did just as criticism of the first round was building and "back to basics" was becoming a catchphrase:

> In retrospect, nearly all of the "New Math" curriculum and instructional ideas appear to have been rushed too quickly from planning to production, field test, and evaluation. Just as the second round of development efforts was beginning to produce some clever and balanced approaches to difficult problems, like unified structure in curriculum, the fundamental ideas were being rejected on the basis of hasty early efforts. (pp. 350-351)

The SMSG second round had begun too late to make much difference, as Art Coxford (2003) noted:

> It is unfortunate that the [second round] curriculum came so late in the life of SMSG …, since the backlash against earlier curricula rendered the new curriculum "dead on arrival." Already the commercial publishers were moving away from concepts *and* skills toward an emphasis on skills alone and emphasizing algebra and geometry organized in familiar packages, not integration. To my knowledge, few teachers knew of the existence of SSM or SSAM, and even fewer tried the materials. (p. 605)

Yet another obstacle faced by SMSG's second round efforts lay in the political realm. The 1970s saw not only the rise of perceptions that the new math projects had failed in their work but also, as Zalman Usiskin (1997) noted in a review of efforts to put applications into the school curriculum, the curtailment of funding for curriculum development by the National Science Foundation. The SMSG second round efforts had the misfortune of being launched at a time when "people were simply tired of 'new math'" (Pollak, personal communication, June 28, 2015).

Long-term impact

Today, the SMSG second round project appears to have vanished from the collective memory of U.S. educators. In a recent authoritative book, for example – *The New Math: A Political History* by Christopher Phillips (2015) – which concentrates on the work of the SMSG and makes extensive use of the SMSG Archives at the University of Texas, there is no mention at all of the second round. In a 1998 interview, Henry Pollak noted that "nobody remembers" the SMSG second round and that the materials produced almost "completely disappeared from human consciousness." Nonetheless:

> It was a very interesting and very modern development, way ahead of its time, this second round. And, as I say, it disappeared from consciousness. … Twenty years later people began, for the first time, to think seriously about teaching some modeling, of seriously seeing how and why mathematics connects with different aspects of the rest of the world.

Although progress has been very slow, mathematical modeling and applications began to find their way into the U.S. school mathematics curriculum between the 1960s and 1990s (Usiskin, 1997). For example, Pollak (2003) noted that "some of the high school curriculum projects in the middle 1990s recognized modeling as an essential ingredient of mathematical education" (p. 668). More recently, the *Common Core State Standards in Mathematics* (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010) included "model with mathematics" as one of eight "standards for mathematical practice." Pollak (2003) concludes his survey of the teaching of modeling with the optimistic observation:

> Since all students must learn mathematical modeling in order to use mathematics in their daily lives, as citizens, and on their jobs, a curriculum that emphasizes modeling can perhaps keep students together through most, if not all, of their elementary and secondary school mathematics education.

> Thus the universality of mathematical modeling can become a major unifying force in mathematics education and perhaps in society as a whole. (pp. 668-669)

Although invisible today, the SMSG second round efforts eventually contributed to the movement toward a more applied school mathematics curriculum in the United States.

The epigraph above from F. Scott Fitzgerald is often understood to be a bitter claim about the impossibility of second chances. Taken in context, however, it appears to have been meant as something once thought but no longer believed (Curnutt, 2013). Either way, it presents one common view of the American fascination with reinvention. For mathematics education, the lesson from the last half century seems to be that, given the complexity and ungovernability of the U.S. education system and the very slow pace at which new practices are adopted, each generation has at most one opportunity to make lasting changes in school mathematics.

Acknowledgment. Paper prepared for the Fourth International Conference on the History of Mathematics Education, Turin, Italy, 23–26 September 2015. I am grateful for their assistance to Henry Pollak at Teachers College, Columbia University, James Fey at the University of Maryland, and Carol Mead and Roy Hinojosa at the Dolph Briscoe Center for American History at the University of Texas.

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Mathematische Liefhebberye (1754-1769) and *Wiskundig Tijdschrift* (1904-1921): both journals for Dutch teachers of mathematics

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Abstract

In the northern Netherlands knowledge of arithmetic and practical mathematics became more and more important from the early 17th century. By the mid-18th century learning arithmetic had become more common and primary schoolteachers were expected to know mathematics. The rather strong interest in mathematics and the greater knowledge of teachers in the 18th century resulted in the publication of a monthly journal, Mathematische Liefhebberye (Mathematical Pastimes), between 1754 and 1769. Though the title suggests a periodical for mathematical recreation, the translation of the full title is: 'Mathematical Pastimes, with news of the French and Dutch Schools'. The content of this journal makes it clear that it was intended for teachers who taught mathematics, in primary schools or privately. One may conclude that already in the 18th century in the Netherlands there was a tendency to consider mathematics as an autonomous school discipline and also that there was the beginning of a spontaneous move towards professionalization of teachers.

After 1769, no comparable journals for mathematics teachers were successfully published for a long time. After 1815 mathematics very gradually became a more important subject in Latin schools, which officially belonged to higher education. Mathematics became an important school subject in the new secondary schools, which were part of legislation on secondary education in 1863. It was taught by mathematics teachers who had to have a degree in mathematics or engineering. In 1904 the first national periodical for mathematics teachers in secondary education was published, Wiskundig Tijdschrift (Mathematical Magazine).

Introduction

In the Dutch Republic of the 18th century arithmetic was not yet always part of the curriculum of the primary schools (Krüger, 2014). Some primary education was available in most villages and in all towns, with lessons in religion, reading, writing and also arithmetic. As the parents often had to pay some extra money in order for their children to be taught arithmetic, not all pupils received these lessons. Mathematics or mathematical sciences, such as accounting, Euclidean and

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practical geometry, algebra and navigation, could be learned through private tuition, by means of self-instruction or in 'French schools', privately owned small schools in which pupils learned reading, writing, arithmetic and also French language, and other subjects, such as mathematics, geography, needlework or fencing.

Many teachers of mathematics had been migrating into the northern Netherlands at the end of the 16th century, for economic reasons or as refugees from religious persecution or war activities (van Maanen & Vanpaemel, 2006). As the demand for instruction in practical mathematics remained a constant factor throughout the 17th and 18th century, teaching mathematics through private tuition or in small private schools had become a way of earning a living for a sizable group of people in the Dutch Republic.

So around 1750 there were indeed people earning some or all of their income through teaching mathematics; they may be considered as professional mathematics teachers. All the same, a distinct group of professionals in mathematics teaching did not yet exist (Schubring, 2015). However, there were symptoms of the evolution of such a group, without interference of the state.

Mathematics teacher as a professional

Definitions of professionals are usually based on the present-day situation concerning existing groups and their characteristics. The institutional component of professionalization, mentioned by Hoyle (2008) is very much derived from the professions of medicine and law, perhaps the oldest professional groups in Europe. Since the 19th century the concept of profession and of professionalization has broadened and has acquired a service component, also mentioned by Hoyle. The service component includes the more or less continuous improvement of knowledge, skills and commitment of the professional to serve the interest of clients, be they patients, students or other clients.

John (2008) mentions eight general characteristics of a profession. Translating these to the special case of the mathematics teacher as a professional they may be formulated as follows:

- the mathematical and teaching competency is based on empirical techniques and theoretical knowledge;
- mastery of these competencies require lengthy periods of mathematical education and training;
- specialised training is designed to both equip and socialise into the culture and symbols of the mathematics teachers;
- the mathematics teacher performs tasks that are inherently valuable to society and relevant to key social and human values;

- the teacher shows motivation to promote the educational success of the student(s);
- a long term commitment to the teaching of mathematics is noticeable and the teacher continuously up-grades her/his knowledge and skills;
- the teaching and related activities are characterised by a high degree of autonomy;
- the profession is guided by a well-developed code of ethics that guides practice and defines the profession's values.

Hoyle (2008) distinguishes between restricted and extended professionalism in teaching, though he admits that the terminology is unfortunate. It implies a hierarchy of teaching quality, instead of the intended distinction between two types of professionalism. The teaching of a restricted professional is intuitive, classroom focused and based on experience rather than theory. The restricted professional is sensitive to the development of individual pupils, an inventive teacher and a skilful class manager. But she/he is not so much interested in theory and does not habitually compare her/his work with that of others. The extended professional is aware of a broader educational context. This shows in collaboration with other teachers, an interest in theory and in current educational developments. The extended professional reads journals and educational books and considers teaching as a rational activity which is amenable to improvement (on the basis of research and development).

To research the development of the professionalization of mathematics teaching, one needs to take into account the historical context. The distinction made by Hoyle assumes the existence of educational research, publication of research results and the availability of journals and books on education. In past periods, such as the 17th and 18th century, educational research was not common, so in a sense the extended professional could not exist. Books and articles on education only started to become available in the late 18th century. Nevertheless, also in those periods, there were teachers of mathematics, or rather of mathematical sciences, who may be considered professionals, based on the criteria mentioned above. Examples in the Netherlands are Frans van Schooten Sr. in the 17th century and Laurens Praalder in the 18th century; in France in the 18th century Charles Bossut and Jean-Antoine Nollet (Confalonieri, 2015; Krüger, 2014). Those teachers fulfilled most of these criteria.

The role of communication

In order to establish a professional group there has to be communication between the individuals. During the 18th century, periodicals became available to more people, due to the diminished cost of production and the large number of publishers in the Dutch Republic. On mathematical sciences there were publications from the learned societies, available for members, periodicals on recreational mathematics and periodicals on general subjects, which also contained articles on mathematics (Beckers, 2003).

Furinghetti (2014) discusses the relevance of journals in mathematics education in relation to professional teachers associations:

- journals facilitated the founding of professional associations (USA, Italy);
- professional associations initiated the publication of journals (Germany, UK, France).

> [Journals became] an important means for transmitting ideas and information among teachers inside a country and proved of crucial importance in shaping the professionalization of mathematics teachers. [..] the readership was composed of teachers of the country of publication; most contributors were national and the action of teacher associations was mainly focused on dealing with national problems. (Furinghetti, 2014, p. 546)

The dissemination of ideas and information among teachers and thus facilitating the professionalization of mathematics teachers is an important role of professional journals.

There is some question whether there were indeed mathematics teachers before the early 19th century. Smid (2014) states that

> … from the time of the Renaissance, a number of people in Europe earned a living in teaching mathematics, mainly arithmetic for future tradesmen. Sometimes they were specialized in mathematics, like the so-called maestri d'abbaco or Rechenmeister, but most of them, whether they were appointed as schoolmasters or were independent entrepreneurs, taught other subjects besides mathematics. They considered themselves schoolmasters in general, not mathematics teachers specifically.

That changed after the French revolution; from the early 19th century on nations accepted responsibility for education, with an important role for mathematics in secondary education. So mathematics teacher became a profession for which one could become qualified by acquiring the right diplomas, after the study of a prescribed curriculum (Smid, 2014, pp. 579-580). This seems to imply that the profession of mathematics teacher is dependent on the existence of an educational system which is initiated, regulated and supervised by the state. The state also takes the initiative for teacher education, usually at first general teacher training for primary education (Schubring, 2015).

The Netherlands did not have formal teacher education until the 19th century. At the start of the 17th century the possibilities for the learning of arithmetic were such that Simon Stevin deemed it necessary to specify that the curriculum of the Dutch Engineering School in Leiden should start with basic arithmetic (Krüger, 2010). Gradually the number of private teachers in mathematics in the northern

Netherlands increased, partly through immigration. Some well-known private teachers from the 17th and 18th century were Samuel Marolois (1572-<1627), Sijbrandt Hansz. Cardinael (1578-1647), Jan Jansz. Stampioen (1610-1653), Abraham de Graaff (1635->1717) and Jacob Baert de la Faille (1716-1777). Simon Tyssot de Patot (1655-1738) taught mathematics at the Athenaeum in Deventer, Martinus Martens (1706-1762) taught mathematics at the Athenaeum in Amsterdam (Krüger, 2014). During the 18th century it became more common that candidates for a teaching position in primary schools were required by the local government to take part in a comparative examination. Especially in the western part of the country mathematics, including arithmetic, usually formed an important part of the examination. Competence in mathematics became an asset for prospective primary teachers; there also were teachers who specialized in mathematical sciences: mathematics teachers.

As a consequence in 1754, long before a national education system was established, the teaching of mathematics was considered of such importance that a periodical for teachers of mathematics (who also taught other subjects) and for mathematics teachers was published by a printer and bookseller, P. Jordaan, living in Purmerend, a small town in North Holland, in the western part of the Republic. The name of the periodical was *Mathematische Liefhebberye* (Mathematical Pastimes), a title which suggests a recreational magazine. However, its content and regular publication gave it the character of a professional journal.

Is in this period the beginning of professionalization of mathematics teachers as specified by John (2008) noticeable? What was the role of the journal published by Jordaan? Is the role of the state essential for development of the identity of professional mathematics teacher?

Mathematische Liefhebberye (Mathematical Pastimes)

Though the journal was known as *Mathematische Liefhebberye*, its full name was *Mathematische Liefhebberye, met het maandelijkse nieuws der Fransche en Duytsche schoolen in Nederland* (Mathematical pastimes, with the monthly news of the French and Dutch Schools in The Netherlands). Dutch schools were primary schools, usually council schools, in which children were instructed in religion, reading, writing and usually arithmetic. French schools were the private schools, in which more topics were offered, among them French language. The journal was published from 1754 until 1769. By the end of each year the monthly journals were collected into one volume. The university library of Amsterdam has a complete set of the volumes from 1754 to 1769.

Pieter Jordaan started publication in April 1754. As he stated in the Foreword, he wished to publish a monthly sheet with news from French and Dutch schools.

In order to encourage sales, he added some sheets on mathematics, as arithmetic and algebra were important topics for teachers in those schools.

> To all schoolmasters and lovers of arithmetic in the Netherlands.
> This concerns a monthly edition of the News of French and Dutch Schools, to which the publisher adds some pages concerning arithmetic and algebra, as these subjects are of importance to the teachers in those schools. (translation by author)

In November 1754 Jacob Oostwoud (1714-1784), a teacher and mathematician from Oostzaandam, became a major collaborator. He seems to have been the driving force behind the sheets on mathematics in *Mathematische Liefhebberye*. Each month there were some sheets with News, combined with quite a few sheets on mathematics. The journal was dispersed through booksellers in the different provinces.

News items were sent to the publisher by correspondents, usually teachers. As an example of the number of correspondents for the News there were 45 correspondents in June 1755. The news items were about vacancies, examination procedures and the questions of the comparative examination for a vacancy (which were different in each town), fulfilment of vacancies, a new insurance for widows of teachers (named 'love widow-insurance'), lists of hymns for each Sunday (as primary school teachers also were precentor[1] in the reformed church) and lists of school teachers in certain areas. The names of those who had taken part in the comparative examination for a post were also published.

The input for the mathematical part came from a number of collaborators. In 1755 Oostwoud started to publish letters written to and written by Dirk Rembrandtsz van Nierop (1610-1682). Van Nierop was a shoemaker, who became a well-known mathematician, navigation teacher and astronomer. He corresponded with famous scholars in the Netherlands, for instance Christian Huygens, René Descartes (who wrote most of his work while living in the Netherlands), Frans van Schooten Jr., Nicolaas Witsen and Johannes Hudde. He also corresponded with captains, lens-grinders, surveyors, publishers, ministers, governors, and 'connoisseurs of the mathematical arts' (Rijks, 2012). Van Nierop thus played a significant role in spreading knowledge and connecting practical knowledge and theory in the Dutch republic during the 17th century.

Mathematical content

The mathematical subjects in Mathematische Liefhebberye were arithmetic and algebra, both also applied to geometrical problems. The content consisted of some

[1] A precentor leads a congregation in its singing.

theory, with examples, problems to solve, solutions of problems from some months before, teacher examinations of different towns and once a year the solutions of exam questions. Solutions to problems were sent in by readers and also provided by the editors.

In 1754, from April until September, Jordaan published monthly articles on arithmetic and geometric progressions, a rather thorough treatment of these topics. In November 1754 Jacob Oostwoud became the main contributor. He first catered for the less knowledgeable reader by treating addition and subtraction, with worked examples on different currencies. This would be a topic suitable for a larger part of the population of primary school teachers. Soon afterwards Oostwoud published theory and problems on different levels, some simple, others requiring more advanced methods to solve, in order to cater for differences in skills of his readers.

The problems covered topics such as

- simple questions on arithmetic and algebra;
- modelling of a situation, resulting in (systems of) linear and quadratic equations;
- problems to do with trade;
- series;
- probabilities;
- proportionality, also in mechanics;
- mixtures of substances;
- distances travelled;
- plane and solid geometry;
- calculation of a maximum or minimum.

These problems often came from existing literature. In January 1758 Oostwoud mentioned the following authors of textbooks as sources for published problems: H. Meissner, co-founder of the *Mathematische Gesellschaft* in Hamburg (1690), M. Scharf, also from Hamburg, P. Halcke, C. van Leeuwen, Chr. Huygens, A. van Lintz, L. Praalder, L. van Ceulen, A. van Diepenbeek and S. Panser. He also mentioned the names of the contributors who had sent him these problems.

Regularly more than one way of solving a problem was presented, again with the sources of these solutions. That could be a professor in mathematics, such as Frans van Schooten (1615-1660) or Johan Lulofs (1711-1768), an author of mathematical textbooks such as Paul Halcke or Abraham de Graaf, or one or more subscribers.

Contributors to the journal

While Oostwoud was the main editor for mathematics, he managed to attract many contributions, who sent him mathematical problems and who sent in solutions. In principle a problem was only published if the contributor also sent a solution. Often these contributors were teachers at a Dutch or French primary school.

From April 1754 until December 1758 nearly 90 people sent in problems which were published. Some only sent one problem; some sent up to 60 problems. During 1758 about 50 readers sent in solutions to problems, and again, some sent only a few, others sent solutions to many problems.

In the solutions the contributors made use of arithmetic, including different types of numbers (trigonal, pentagonal, etc.), logarithms, algebraic equations with one or more variables and polynomials up to the fourth power, trigonometry and calculations in Euclidean geometry, plane and solid. They also occasionally used Newtonian fluxions to determine a maximum or a minimum. They did not use negative solutions of equations and rarely decimal notation; the contributors preferred using fractions.

Decimal notation, as described in 1585 in the *Thiende* (Tenth), was introduced by Simon Stevin in the curriculum of the Dutch Engineering School in Leiden as early as 1600. During the 17th century this notation was taught at Leiden to future surveyors, engineers and mathematics teachers (Krüger, 2010). In the 18th century one could find this notation in books for surveyors, i.e. Morgenster & Knoop (1744). However, it seems that decimal notation was not a part of the mainstream mathematics instruction in the 18th century.

The role of *Mathematische Liefhebberye* and its ending

The journal had several roles.

- It disseminated information on job vacancies and job requirements.
- It informed interested readers on the questions of the examinations and the solutions.
- It provided opportunity for teachers to improve their mathematical knowledge which was relevant to their career and to the quality of teaching. One aspect of this was the regular occurrence of comparison of different solution methods for a problem.

In December 1769 mr. Jordaan made public that this would be the last edition, the publication probably was no longer profitable for him.

One of the contributors, mr. Arnold Strabbe (1741-1805) attempted to start a similar periodical in 1770. It was named *Oeffenschool der Mathematische Wetenschappen* (Training School of Mathematical Sciences) but it only lasted one year, due to financial problems (Beckers, 2003). However, Strabbe, who was a mathematics teacher, author and translator of mathematical textbooks, did not give up easily. In 1778 he founded, together with a mathematics teacher, a reckonmaster and a surveyor, the national mathematical Society *Een onvermoeide arbeid komt alles te boven* (Untiring labour overcomes all). In 1782 the first volume of the mathematical journal of the Society appeared. Today the Society is known as the *Koninklijk Wiskundig Genootschap* (Royal Mathematical Society), its journal is *Nieuw archief der wiskunde* (New Archive for Mathematics). Though in the early years the members often were teachers, the Society did not attract the same involvement as *Mathematische Liefhebberye* used to do. During the 19th century academic mathematicians came to dominate the Society.

The law on secondary education of 1863

The law on secondary education of 1863 encouraged the establishment of schools for higher secondary education, the Hogere Burgerschool (Higher School for Citizens), usually called the HBS. The HBS was meant for those boys who would not enter university, but who were to study at the Polytechnic School in Delft or to take up higher technical or administrative positions in society (Krüger & Van Maanen, 2013). Mathematics, exact sciences and languages were important subjects at the HBS. The law also specified the qualifying examinations for teachers of mathematics, of physics, of chemistry, of commercial sciences, of political economy and of Dutch language, literature and history.

During the first 60 years of the 19th century there had been attempts by the national government to improve mathematics teaching in Latin schools[2], but they were only moderately successful (Smid, 1997; Smid, 2014). After 1863 the requirements for teaching mathematics at HBS and gymnasium were rather strict and the inspectorate saw to it that they were maintained (Krüger, 2014). So there was a fast growing number of highly educated professional mathematics teachers.

There were some regional journals for mathematics teachers, usually existing for only a few years (Beckers, 2003). Journals for prospective teachers who were studying for their exams did better, e.g. *De Vriend der Wiskunde* (The friend of mathematics), which was published from 1886 until 1916 and *Nieuw Tijdschrift der Wiskunde* (New journal of mathematics), published from 1913 until 1988. French journals were available as well, such as *L'Intermédiaire des Mathématiciens*.

[2] Latin schools provided entrance into the universities; they were classified as higher education, even if the pupils were quite young. After 1863 they became known as gymnasiums.

Wiskundig Tijdschrift from 1904-1921

The *Wiskundig Tijdschrift* (Mathematical Journal) was established in 1904 by three mathematics teachers of a HBS. They were F.J. Vaes (teacher at a HBS in Rotterdam), Chr. Krediet (also in Rotterdam) and N. Quint (teacher at a HBS in The Hague). At the start of the 20th century there were two national periodicals for mathematics teachers. One was the journal of the Mathematical Society, *Nieuw Archief voor wiskunde*; the other was *Vriend der wiskunde* (Friend of mathematics). Mr. Vaes stated in his introduction to the first volume that the Mathematical Society and its journal, with its strong emphasis on professional mathematics had lost its attraction for mathematics teachers. *Vriend der wiskunde* was a journal for those who studied for a teaching degree in mathematics. So the editors were of the opinion that there was room for a new journal, meant for all mathematicians, but with an emphasis on mathematics teachers. The purpose was to encourage more cohesion between mathematics teachers; inexperienced teachers should profit from knowledge of the experienced teachers. There was no intention to expand the content much outside the topics of the curriculum in HBS and gymnasium.

One could take a subscription or receive the journal through a bookseller, four volumes in a year. The first two years are taken into account for the following analysis. A list of subscription in 1905 and 1906 provides information on which people or institutes were subscribers. In 1906 there were 449 subscriptions, subdivided as in table 1.

Table 1 Categories of subscribers to *Wiskundig Tijdschrift,* 1904-1906

(former) teacher HBS, gymnasium or school for girls	137*
Teacher at university, technical institutes, military institutes	59
Primary school or teacher at primary teacher education	26
Booksellers	103
Unknown	80
Student, press, library	11
Others	33

*Of which 107 teachers at a HBS

The content consisted of mathematics and pedagogy; there were no items on vacancies or job requirements, but there was a section Questions and Answers. This section contained more general questions, which only needed a short answer. Such as 'Where do I find information on the method of Mascheroni?' During the first two years the mathematical content consisted of theory and problems to solve. There was some history of mathematics and a few items on pedagogy. The questions of several examinations were also included once a year. The exams concerned were for:

- teaching degrees in mathematics (upper primary school, lower secondary and higher secondary school);
- surveyors;
- school for civil engineers;
- HBS, the final exams;
- the veterinary school, admission exams;
- engineer of steam engines;
- water management.

The articles were written by Dutch authors and authors from abroad. In fact more articles were from foreign authors than from Dutch authors (Table 2). These articles were often but not always translated, so one may assume that the readers were able to read German, French and English.

Each journal also contained a number of problems, which readers were encouraged to solve. Solutions by readers, or by the editors, were published in later issues. On the front-page of the first volume 15 collaborators are named; according to the names with the articles, there were more contributors. The *Wiskundig Tijdschrift* existed from 1904 until 1921.

Table 2. Topics and number of articles by Dutch and foreign authors

1904-1906		
	Dutch	Others
Arithmetic	5	4
Algebra elementary	11	2
Algebra higher	5	-
Trigonometry	10	-
Plane geometry	11	40
Plane curves	7	12
(Descriptive) geometry	7	7
Curves 3d	5	3
Mechanics	7	4
Physics	3	-
Pedagogy	3	10
Others, reviews	6	19
Total	**80**	**101**

In 1921 the first association of teachers of mathematics and of science at gymnasia was established, followed in 1925 by the association of teachers of mathematics, mechanics and cosmography at the HBS. From September 1927 *Euclides* (Euclid), a journal for teachers of mathematics and science, was published, with J.H. Schogt and P. Wijdenes as editors and financed by the publisher Noordhoff. *Euclides* became the journal of the associations for teachers of mathematics in September

1940; at present, in 2016, it still is the journal of the Dutch Association of Mathematics teachers (NVvW).

Discussion and conclusions

Furinghetti (2014) points out the need for teachers within a country to share information. This became very noticeable in the second half of the 19th century, after nations were created and state-wide educational systems had become more common. Journals for mathematics teachers played an important role in transmitting ideas and information and spreading and improving mathematical knowledge. Moreover, at the end of the 19th and early 20th century there was a close link between those journals for mathematics teachers and the rise of professional associations of mathematics teachers.

However, long before national qualifications for mathematics teachers were formulated and also more than a century before there was a state education structure in the Netherlands, a journal for teachers of mathematics was published monthly during 15 years: *Mathematische Liefhebberye*. This journal provided a regular means of communication for and between teachers of mathematics, of whom an unknown number specialized as mathematics teachers. The journal gave information on vacancies and requirements for positions, it was a means of exchange of information and of spreading and improving mathematical knowledge among its readers. Thus already in the 18th century we find the characteristics mentioned by Furinghetti (2014).

Evidently in the Netherlands already in the 18th century there was a tendency to consider mathematics as an autonomous school discipline (Chervel, 1988; Matos, 2015). The teachers of mathematics, whether teaching in schools or specializing as private mathematics teacher, were eager to become more proficient in the teaching of the subject and also to receive information on the professional requirements in the various towns.

So do we see the beginning of professionalization of mathematics teachers in this period, mid-eighteenth century? If we consider the characteristics based on John (2008) most of those seem to be present among the subscribers of *Mathematische Liefhebberye*. The knowledge of mathematical theory useful for teaching, could be improved through the articles and problems treated in the journal. Theory on teaching had not yet been developed; in the 18th century teaching was very much a matter of experience. Mathematics teachers did have long periods of education and training, not at a university, but usually on an individual basis, during which time they also got to know the culture of their profession. Their work was seen as valuable to the society; the requirement to show proficiency in mathematics in order to get a teaching position is a testimony to this. Many teachers taught until they were too old to work or until they died, as we know from several sources, so definitely there was a long-term commitment to their work. They also were usually fairly autonomous, certainly if they only taught mathematical sciences. About the motivation to promote educational success of students we only know something of individual teachers, whose reputation survived the years. About the quality of teaching of the group of teachers of mathematics as a whole, not much is known. Considering the list by John, in the second half of the 18th century the beginning of a development towards professionalization is noticeable, made visible and stimulated by the journal *Mathematische Liefhebberye.* There was no involvement of local or national government in this. However, this development did not continue. The journal was too dependent on only a few people: the cooperation of the publisher and the efforts of a few mathematics teachers. A larger supportive structure was not yet present.

By the time the *Wiskundig Tijdschrift* appeared, in 1904, there was a strong system of secondary education, with a large group of highly qualified mathematics teachers, who had not yet formed an association. This journal had a more limited character; the emphasis of the journal was on mathematical content, information on professional matters other than content knowledge was scarce. The subscribers were people who had received higher education; the majority was or had been or intended to become a mathematics teacher. Mathematics teacher had become a profession in the sense mentioned by John (2008), nevertheless, the *Wiskundig Tijdschrift* was published only a few years longer than the *Mathematische Liefhebberye*; 17 respectively 15 years.

It is worth noting that professional associations of mathematics teachers and a professional journal for mathematics teachers (*Euclides*) existed at the same time, without a close connection between them. It took about 12 years before *Euclides* became the official journal of the professional association. It still has that position. So we may assume that the connection with the professional association has survival value for a professional journal, at least for mathematics teachers

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Mathematics and race in Turin: the Jewish community and the local context of education (1848-1945)

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Abstract

After having illustrated Jewish contributions to scientific and technical culture in Piedmont since the Risorgimento period, we will focus on the role played by some Israelite teachers of mathematics and we will outline the consequences of racial laws (1938) on the local milieu of the mathematical instruction in Turin.

Introduction

At the beginning of the twentieth century, Turin hosted the fourth largest Jewish community in Italy after Rome, Milan and Trieste. The leading town of Piedmont counted in fact 4,060 Israelis, 0.5% of the urban population. The reason for this concentration is clearly related to the history of Jew's diaspora. In Piedmont, the main Jewish settlements began in the 15th century and consisted of Jews who escaped from Provence, attracted by the perspective of moderately good living conditions. Through the 16th, 17th and 18th centuries, the most important Jewish communities (then called universities) were formed in Turin, Asti, Alessandria, Carmagnola, Casale Monferrato, Cuneo, Fossano, Moncalvo, Saluzzo and Savigliano. Jews lived together in specific areas (ghettos), and suffered numerous deprivations included the prohibition against entering schools and belonging to Arts and Trades corporations. At the end of the 19th century, in the frame of liberal movement, Count Cavour and the brothers Massimo and Roberto d'Azeglio pleaded for the extension of the constitutional rights of freedom and equality to oppressed minorities in the Sardinia Kingdom, including Jews and Waldensians. Eventually, King Carlo Alberto made a Parliamentary decision (*Statuto Albertino*) in July 1848 permitting the extension of all civil and political rights to the Jews. As a result of this emancipation, Jews began a new life: they could practice any profession or commercial activity and could participate actively in political life, which they did with great determination and success. It thus became inevitable that Jewish communities became depopulated by assimilation, and urbanisation. After a few years,

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

the families of outstanding prospective mathematicians like Corrado Segre, Beniamino Segre, Beppo Levi, Eugenio Elia Levi, Alessandro Terracini, etc. left their communities (Saluzzo, Ivrea, Asti) and moved to Turin[1].

Grown up in laic and emancipated families, in which "religion blended with the cult of the State" (in Italian, Morpurgo, 2016), the Turinese Jews constituted a transversal community from a political point of view. Among them were socialists and monarchists, anarchists as well as fascists such as the group of intellectuals that founded the journal *La nostra bandiera*[2]. For example, Emilio Artom (1888-1952), a teacher of mathematics at *liceo* Galileo Ferraris in Turin and husband of Amalia Segre (1891-1972), a private teacher of mathematics, remembered:

> Both my father and mother had such a strong sense of Italian patriotism, adhering totally to liberal-monarchic ideals. My father's devotion to the House of Savoy was limitless. My mother followed the same way of thinking and connected it to religion. I'll never forget how she taught us that whoever sacrificed their life for their country would go to Heaven, according to the Second Book of the Maccabees. [Emanuele Artom, *Diari 1940-44*, p. 156, transl. by the author][3]

A really distinctive element of Turinese Jewry was represented by their professional and occupational profiles: they were mostly working in the school milieu (as teachers of all disciplines, headmasters, school inspectors), in research (lecturers and full professors at university) and in the book industry (particularly for school, with famous publishing houses like Lattes and Rosenberg Sellier).

In particular, the weight of the Jewish contribution to scientific instruction in Piedmont was significant at every level of education. The presence of Israelite professors was outstanding in the realm of the University of Turin, so much so that the Turin mathematics 'school' drew the attention of the ministry, because of the "Jewish infiltrations"[4] that characterized it. The "tyranny" of the Jewish group, which included nine full professors out of a total of 75 scholars, was for example commented in local newspapers and denounced by an "anonymous fascist black shirt", in the early thirties:

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[1] About the history of Jews in Piedmont see for example (Maida 2001).

[2] About the Jewish community in Turin during the fascist dictatorship see (Levi, 1991; Levi, 1998; Ventura, 2002).

[3] "Tanto il babbo quanto la mamma erano dotati di un forte senso di italianità, e aderivano alle idealità monarchico-liberali. La devozione di mio padre verso Casa Savoia era illimitata. La mamma seguiva sentimentalmente le stesse correnti, e le collegava con la religione. Non dimenticherò mai che ella ci insegnava che, chi muore combattendo per la Patria, va in paradiso, secondo l'insegnamento del secondo libro dei Maccabei."

[4] Newspaper *La Stampa Torino*, 5-6 September 1938, p. 1; 13 October 1938, p. 2, 6.

> In Turin University's Mathematics Department, a few Jewish teachers (Freemason-socialists headed by the all-powerful Prof. Fubini) with a skill and Jesuitism of the worst kind resort to every measure possible to obstruct the work of the Regime. The Faculty is home to tyranny of all types: "favourites" receive special treatment and protection as those who will one day need to continue the villainous work of destroying the homeland, whilst those who can not enter the hallowed circle are oppressed, boycotted and attacked in countless ways. [Archivio Centrale dello Stato Roma, Ministero della Pubblica Istruzione. Fascicoli personali. Professori ordinari (1940-70) 3° versamento. Busta 214, *Personal dossier of Guido Fubini*, 7 October 1933, transl. by the author][5]

Due to the numerical relevance and the prestige of Jewish teachers of scientific subjects in Turin, the application of the racial laws in 1938 altered the local environment of education, because it irreversibly changed the composition of teachers staff, and the school population[6].

In light of these considerations, it appears interesting, first of all, to describe the socio-cultural premises that led to the development of a large and appreciated community of Israeli teachers in Piedmont; secondly to identify their common traits (obviously beyond their 'racial' identity) and finally to illustrate the disruptive effects that their expulsion had on middle and secondary schools in Turin.

'Educating is synonymous with emancipating, and emancipating is synonymous with educating'

In order to answer the previous research questions, we should remember that in 1848 the *Albertine Statute* granted Jews the possibility of enrolling in State schools of all types and levels. It was a concession of maximum importance, since exclusion from education had represented one of the most sinister forms of discrimination. Furthermore, in the eyes of the Savoy leadership, access to study was a key

[5] "Nella R. Università di Torino, Scuola di Matematica, pochi professori ebrei, social-massoni capeggiati dall'onnipossente prof. Fubini, con un'arte ed un gesuitismo della peggior specie si adoperano con ogni mezzo, per demolire quanto il Regime, con titaniche imprese, sta costruendo. In detta Facoltà si verificano inoltre soprusi di ogni risma: sono favoriti i protetti, i discepoli che dovranno un giorno continuare la opera infame, disfattrice della Patria, e sono oppressi, boicottati, danneggiati in ogni modo quelli che essi sanno di non potere attirare nella loro cerchia. Una severa minuziosa inchiesta metterà alla luce del sole quanto ho avuto l'onore di esporre all'Eccellenza Vostra. Vecchia Camicia Nera, anonima suo malgrado per evidente necessità."

[6] About the consequences of anti-semitic decrees in Italy see (Israel & Nastasi 1998; Israel 2010). As far as concerns the application of racial laws in Turin see (Rinaldelli 1998; Cavaglion & Romagnani 2002).

tool in the process of integration[7]. As the pedagogue Domenico Berti was used to affirm, for explaining this political strategy:

> Educating is synonymous with emancipating, and emancipating is synonymous with educating [Domenico Berti, in *Atti della Società di Istruzione ed Educazione* … 1849-50, p. 724, transl. by the author][8].

Until the promulgation of *Albertine Statute*, Piedmontese Jews determined to pursue their studies had only counted on two alternatives: emigrate to States (like the Tuscany Grand Ducky) where less restrictive norms were in force, or attend the Jewish schools.

Among the first Piedmont Israelis that undertook studies in mathematics abroad, in order to became a teacher, Simeone Levi (1843-1913) stands out[9]. Born in a small country village near Turin, Simeone received his first education at home and in 1849 he enrolled in public school, where his unusual talent distinguished him. In 1861, having obtained a diploma at *liceo* Gioberti in Turin, Levi enrolled in the faculty of mathematics at the university of Turin, thanks to a scholarship assigned to him by baron Franchetti von Rotshild. As the degree course in pure mathematics did not start, in 1864 Levi was forced to move to Pisa, where he came into contact with Enrico Betti. After completing his degree with full marks, Levi returned to Turin and started to teach at the technical and accountancy school in Tortona, publishing the handbook *Complementi di aritmetica ed algebra* (Turin, Paravia, 1871). This text appears very modern in comparison to the more widespread books of the time, including contents like a generalization of Newton's formula of the binomium according to Betti, linear systems, continuous fractions, numerical and approximate calculus, the theory of numbers and the first concepts of probability. The originality of Levi's handbook emerges even more if we think that many of the topics dealt with had made their first appearance only few years before in Angelo Genocchi's calculus lectures at the Turin university (1867).

Apart from such few exceptions, most Jews born in the period 1845-1855 had to rely on the school network inside the ghettos[10]. Most of them pursued technical and scientific studies in order to obtain positions as surveyors, accountants or primary school teachers. In 1848, the Jewish educational web in Piedmont consisted of four main institutes (in the towns of Turin, Asti, Casale and Acqui) and a large group of small schools, established in various towns of Piedmont (Cuneo, Fos-

[7] For an overview of the contributions of Jews to scientific instruction and education in Piedmont from 1848 to 1920 see (Luciano 2013).

[8] "Educare è sinonimo di emancipare, ed emancipare è sinonimo di educare."

[9] Levi's biography is sketched in (Arian Levi & Viterbo 1999; Levi 2001).

[10] On the history of Jewish schools in Turin in the second half of nineteenth century see (Maida 2001, p. 70-90).

sano, …). In Turin, Colonna Finzi College consisted of a kindergarten and a primary school. Children of both sexes were admitted, free if they came from poor families. The math taught in these schools was reduced to the first four rules of arithmetic and to basic notions of plane and solid geometry. However, bearing in mind the occupational perspectives of the majority of the students, in the fifth grade some notions of accounting and simple and double entry bookkeeping were usually introduced.

There was also a wide and ramified network of Jewish professional and technical institutes for both girls and boys. Among the main features of the teaching offered in these schools for arts, crafts and trades was the tendency to assign a major role to applied mathematics, chemistry and technology. In effect, these competences were considered essential for giving truly modern culture to new generations, so as to allow young Jews to excel in Piedmontese society. The professional and practical character of the scientific instruction offered in these institutes had as a consequence that few of their former students pursued the studies at university in the pure mathematics, physics and natural sciences courses.

Jewish schools enjoyed a good reputation during the *Risorgimento*, that is the period (1815-1861) characterized by the movement for Italian unification that culminated in the establishment of the Kingdom of Italy (1861). As time passed, however, this system declined rapidly and many of these institutions were reduced to being schools *of* and *for* the poor, attended by pupils coming from very observant or needy families. In particular, this school network was damaged by the change in mentality that affected much of emancipated Piedmont Jewry, above all by the belief that equality of rights began from equality of cultural opportunities[11]. Israelite journals and magazines tried (without success) to steam the assimilation process, for example by reporting on the successes achieved in the 1890s by the young scholars Corrado Segre (1863-1924), Beppo Levi (1875-1961), Gino Loria (1862-1954), Azeglio Bemporad (1875-1945), Gino Fano (1871-1952), Ida Terracini (1870-1964) and Costantina Levi (1870-?), in their studies and careers[12].

For their part, after 1848 young Jews upheld themselves in State schools and universities. In Turin, at the *licei* d'Azeglio, Alfieri, Gioberti and at the technical institute G. Sommeiller, the first enrolments date from the 1850s[13]. The first licenses to teach math and science in technical institutes and normal schools were awarded in 1868. What oriented the first generations in the choice of the course of studies was a complex plot of family and economic factors, as well as cultural models. In brief, we can say that a large number of young Jews was dedicated to scientific studies, but the majority of them only completed the first two years of the university

[11] See (Artom 1913; Colombo 1925).

[12] For example see the sections of news entitled *Torino* in the journal *Il Vessillo Israelitico*, XL, 1892, p. 265; XLI, 1893, p. 252; XLII, 1894, p. 36-37; XLIII, 1895, p. 242; XLIV, 1896, p. 208, 244, 279; XLVII, 1899, p. 109-110, 155; XLVIII, 1900, p. 160.

[13] See (D'Orsi 2003, p. 175-197; Liceo Gioberti et alii 2012, p. 113-116).

curricula and, having obtained the license (*licenza*), went to the school of application (*Scuola di applicazione*), aspiring in careers as architects or engineers. The very testimonies of many mathematicians born in the Sixties confirm the impression that the so-called 'humanitas scientifica' was a category of thought that was not common to Jewish families and that scientific culture was considered above all in relation to occupational prospects that it opened. Segre, Levi, Fano, Loria, Padoa, Artom described the many generational clashes that opposed them to their fathers: the latter pushed them to a career in engineering or finance, while they were inclined towards pure research only undertaken out of love for knowledge.

The situation changed after 1880. From this date onwards the number of Jews who took up advanced scientific studies was constantly rising: more than 150 Israelite students graduated in pure mathematics, physics and natural sciences between 1848 and 1938 at the Turin University. Most of them became teachers or headmasters in State schools of all levels. As a result of this massive phenomenon, the city's educational system underwent substantial changes.

The 'Small School' of Corrado Segre

The Piedmontese Jews that entered the university of Turin and the world of education – during the *Belle Epoque* and until the racial laws – had some social and cultural features in common. Regarding the first aspect, they were men and women perfectly integrated in local society, whose religious and racial identity was generally ignored by colleagues and pupils. Furthermore, this group shared a peculiar university training.

They could be defined 'the small school of Segre', in the sense that they constituted a network of disciples of Corrado Segre that did not carry out research in advanced geometry under his direction (and so they were never listed among the members of the 'Italian school of algebraic geometry'), but attended Segre's lectures of higher geometry and his lessons at the *Scuola di Magistero* (teacher training school). They had Segre as a point of reference, a common leader (*Maestro*) and, in many cases, Segre was the supervisor of their degree dissertations and of their qualification thesis as teachers. They kept in touch with him – before and after studying at university – also because of the common affiliation to the Jewish community (they were Segre's nephews, relatives, family friends, etc.)[14]. The legacy of

[14] For example, the Emilio Artom's library includes some offprints by C. Segre with the handwritten note: "To my beloved disciples Emilio Artom and Amalia Segre, in loving memory of their professor". The cataloguing of Artom's library, recently discovered by Erika Luciano in the cellars of the department of mathematics of the university of Turin is currently underway. The Artom's library comprises more than 240 offprints and booklets.

Segre's mentorship is evident in the textbooks and articles published by Emilio Artom, Alice Osimo (1886-?), Elsa Bachi (1894-1972), Alberto Levi (born in 1874, Beppo Levi's brother-in-law), etc.[15] Through his courses at the teacher training college, Segre instilled his main didactic beliefs in these disciples: the principal objective of teaching should be the development of the powers of induction, visualization and reasoning; the first approach to mathematics should be experimental and intuitive; the concepts of function and transformation (in line with the ideas of Felix Klein) should be introduced at an early level.

At the same time, although assimilating the methodological assumptions defended by the 'Italian school of algebraic geometry', this group of teachers did not limit themselves to inherit the legacy of Segre's methodological thought. Indeed, almost all these Jewish perspective teachers attended one or more courses by Peano in infinitesimal calculus, advanced analysis or complementary mathematics (*matematiche complementari*). So they had an opportunity to learn the meaning and use of his logical symbols and to meditate on the introduction of the research on the foundations of arithmetic and geometry into teaching and textbooks. In addition, many of the members of the 'small school of Segre' maintained fruitful and long-lasting exchanges with the rival 'school of Peano', by attending the *Conferenze Matematiche Torinesi*, a lecture series for teachers that Peano and his coworkers Tommaso Boggio and Matteo Bottasso started in 1915[16].

As a result, the Jewish teachers belonging to the 'small school of Segre' were able, in their teaching practice, to balance and blend in an harmonic synthesis the two types of approach: the synthetic, intuitive and constructive one, dear to Segre, and the hypothetical-deductive, abstract and formal one, promoted by the Peanians[17]. In this sense, Emilio Artom, Alessandro Padoa, Elsa Bachi, Vittorina Segre, and many others appear like 'bridge' figures between the two famous Turinese mathematical 'schools', commonly depicted by historiography as antagonistic and impermeable regarding collaborations and mutual influences.

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[15] See for example the issues on math education asserted in the prefaces of the following textbooks: Prosio, Pietro and Artom, Emilio (1927, 1934), *Elementi di fisica, ad uso degli Istituti tecnici e magistrali*, Torino: Chiantore; Artom, Emilio and Osimo, Alice (1935), *Aritmetica, geometria, algebra ad uso delle scuole d'avviamento professionale: Vol. I-III*, Rocca S. Casciano: Cappelli; Artom, Emilio and Osimo, Alice (1938), *Geometria, ad uso delle scuole medie inferiori*, Rocca S. Casciano: Cappelli; Elsa Bachi (1926), *Testo di geometria piana e solida ad uso delle scuole industriali, commerciali e medie inferiori*, Torino: Paravia.

[16] About the *Conferenze Matematiche Torinesi* see (Luciano & Roero 2008, p. 145-149; Luciano & Roero 2010, p. XI-XVIII, 1-212).

[17] See, for example, the methodological instances supported by E. Artom in *Elementi di aritmetica ad uso delle scuole secondarie inferiori*, Bologna: Cappelli, 1922, p. 5-11, 100-111.

A gender balance: Jewish female students and teachers

Among the distinctive features of this community of Jewish teachers of scientific subjects is the fact that it counted many women[18]. Having grown up in a context which offered a true equality of educational opportunities, Israelite girls were in a privileged position compared to their classmates. In Turin, since the pre-unity period they had been admitted to attend Jewish schools and they had counted on the support of some welfare structures like the *Israelite Women's Charity Society*, which helped young women to undertake a job and to qualify as schoolteachers, supported them with scholarships, and set up prizes for deserving pupils. Besides, in Turin there were no fewer than 5 boarding schools for girls. At these latter institutions the teaching were actually aimed at providing girls with competences to manage a household efficiently (for example basic notions of home economics, hygiene, pharmacopoeia, etc.), but there were also some cases of young female Jews who, after studying in these schools, became teachers, headmistresses, inspectors, supervisors of summer camps or orphanages etc.

The result of female Jewish schooling was important. In the faculty of sciences, female Israelite students constituted 6% of the graduates in the period of 1892-1938. The majority of them, after having obtained the degree and the qualification, entered the world of school.

In this female universe, Ida Terracini stands out. Born in Asti, Ida attended kindergarten and primary Israelite school in that town, then enrolled at *liceo* Alfieri in Turin and obtained a degree in pure mathematics at the university in July 1892. She was the first woman to take a degree in mathematics in Turin. Soon afterwards, she was qualified to teach and founded a boarding school for Israelite girls in Turin, also open to students of other religions. In addition to directing this private college, she was also a math teacher, and then a headmistress, in many local public schools. No less emblematic is the career of Costantina Levi, who was the second woman to achieve a degree in mathematics at the university of Turin. A disciple of Segre and Enrico D'Ovidio, Levi showed good talent for geometrical research, to the point that she was invited to publish an abstract of her dissertation. She was a 'pillar' of the Turin's school *milieu*, teaching for more than 40 years at *liceo* Alfieri.

The racial laws

If the Jewish presence represented an aspect numerically and qualitatively so important in the local context of scientific education, it is not surprising that the

[18] For a first survey of the community of Jewish female teachers of scientific subjects in Turin see (Luciano 2014).

process of aryanization imposed by the fascist regime on the Turinese schools had dramatic consequences: 23 teachers were dismissed from their positions; more than 380 students were expelled[19]; the use of textbooks by Jewish authors was prohibited in all State institutes (this is the so-called procedure of *Bonifica libraria*)[20]. Racial laws threw the Turinese Jewish teachers into incredulous disarray. For the majority, who had often declared that they were of Jewish origin but of no declared religious belief, this move constituted political rather than racial discrimination. Between 1938 and 1943, the teachers of mathematics mentioned before opted for exile, escaped in the mountains, or were deported and killed in extermination camps. At *liceo scientifico* Galileo Ferraris, Emilio Artom was expelled amidst general indifference of his colleagues[21]. The same fate concerned many former disciples of Segre and Peano: Ugo Levi (born in 1903, a teacher in Saluzzo); Amalia Segre Norzi (1898-1943, a teacher in San Remo), etc. A tragic destiny of deportation and death affected two other disciples of Segre and Peano: Vittorina Segre (1891-1944) and Annetta Segre (1897-1944). Graduated in 1914 and in 1918 respectively, both had been supervised by Segre for their qualification thesis in the teacher training school. Indeed, they had also maintained scientific exchanges with the team guided by Peano, by attending the *Conferenze Matematiche Torinesi*. Well integrated into the national scientific community, they were appreciated members of the society of the teachers of mathematics Mathesis and of SIPS (*Società Italiana per il Progresso delle Scienze*). They were both arrested in Liguria and deported to Auschwitz, where they died in winter of 1944[22].

The aryanization did not omit minutiae such as the removal of geographical maps and scientific instruments drawn by Israelite authors. Moreover, the ministry imposed a didactic program which was "frankly racist". Despite a bit of upheaval, Turin's schools conformed to these rules. For example, *licei* d'Azeglio, Alfieri and the Technical Institute Sommeiller agreed to actively participate in the *Racial Exhibition* of 1940[23].

The racial laws stated that in the presence of a sufficient number of pupils, Jewish communities could set up their own para-state schools under the leadership of an Aryan commissioner. In Turin, Israelite schools were not to be set up *ex novo* but

[19] *La Stampa Torino*, 30 August 1938, p. 7; 2 September 1938, p. 5; 3 September 1938, p. 6; Liceo Gioberti et alii 2012, p. 43-46; 117-124, 167-170.

[20] On the procedure of book cleansing see (Fabre 1998).

[21] Archive of *liceo scientifico* Galileo Ferraris, *Stato del Personale*, Emilio Artom *ad vocem*.

[22] On the deportation of Vittorina Segre and her stepdaughter Betty Foà see (Veziano 2007, p. 194). Both Vittorina and Annetta Segre had been animated by strong patriotic beliefs; in particular Annetta had worked in the *Laboratorio delle studentesse per i combattenti* during the first world war (see http://www.grandeguerra.unito.it/items/show/188).

[23] Cf. Turin State archive: *Minutes of teachers riunions* of *liceo ginnasio V. Alfieri*, 27 February 1940; letter of the headmaster of Technical Institute G. Sommeiller to the ministry of *Demorazza*, 26 February 1940, digitalized in http://www.uciimtorino.it/fonti/documento_4_parte2.htm.

the ancient Colonna-Finzi College was to be reformed[24]. The mathematician Alessandro Terracini (1889-1968), former full professor of geometry at the university of Turin, together with his brother Benvenuto (1886-1968), an eminent linguist[25], investigated the modifications to be introduced in order to make the Colonna-Finzi College more laic and similar to the standards of national institutions:

> I therefore believe that the problem of middle schools, which I mentioned already yesterday, is very important and notwithstanding new difficulties should be our priority to resolve in the best way possible, in order to guarantee our children the very best education possible. My thoughts regarding this are currently: a) the need for our schools to be organized with programmes which fully observe the government regulations b) the consequent need to limit the confessional party as much as possible c) I don't know whether or to what point existing buildings and equipment can be used but it's necessary to have a physics cabinet …. [A. Terracini to B. Terracini, 8 September 1938, in Lore Terracini (ed.) 1990, p. 448, transl. by the author][26]

Regarding this, in Turin the idea of offering the students an eminently practical and professional scientific instruction, as offered by the Risorgimento Jewish schools, was rejected. Colonna-Finzi College was thus restructured, flanking accounting courses delivered by Ester Levi with a classical *curriculum* of studies (*liceo classico*). In Colonna-Finzi high school the mathematical courses were entrusted to Bonaparte Colombo (1902-1989) and Ugo Levi (born in 1903), and the science courses to Olga Viterbi Beer (born in 1890) and Marisetta Fubini Treves (1908-1973); in the middle school, the courses in mathematics and natural sciences were held by Adelaide Diena (1900-1981). For these teachers – who all arrived from State institutes – the racial laws were not only to determine dramatic awareness of

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[24] See (Corinaldi 1988; Bonino 1999, p. 65-92).

[25] Particular attention should be given to the experience of A. Terracini who, in the years 1938-48, for the first time devoted himself in a priority and systematic way to teacher training and to publishing textbooks of mathematics. In the last months he spent in Turin, before his exile in Argentina, Terracini accepted the invitation to publish an algebra textbook for *licei classici*, under a false name. The handbook, which bears the signature of Francesco Tricomi, is in some respects unusual. Terracini here fully developed the theory of real numbers according to Dedekind's construction, respecting a demand for logical-deductive rigor that would seem extraneous to a member of the 'school of Segre' and that would seem to better reflect the rigorist approach typical of the 'school of Peano'.

[26] "Ritengo dunque che il problema di una scuola media, a cui ho già accennato ieri, sia molto importante e che, se non si frappongono nuove difficoltà, sia un dovere impellente di risolverlo, e nel modo migliore, per non far mancare ai nostri figli nel limite del possibile quell'istruzione di cui non devono mancare. Le mie osservazioni in proposito al momento sono: a) necessità che la scuola sia organizzata con programma in nessuna parte inferiore a quello governativo; b) conseguente necessità di limitare al minimo la parte confessionale; c) non so se e fino a che punto si possano utilizzare locali e impianti esistenti ma occorreranno strumenti di fisica…"

their 'identity' but also a sort of return to their roots, in the framework of a working experience in contact with an entirely Jewish staff and class. As far as contents, these teachers were not allowed to deviate from the official syllabi. Instead, they rebelled via their cultural and ethical choices, continuing to adopt non-fascistized textbooks and encouraging pupils to continue developing critical free-thinking skills.

Conclusion

The Turinese teachers did not (and could not) remain detached from the upheavals of the education system following the establishment of fascist dictatorship. In particular, the racial politics imposed by the regime collided with the existence of peculiar local dynamics and socio-cultural traditions. This is what emerges when analyzing the part played by the Jewish network of mathematics teachers in Piedmonts' educational system. In this case, the regime was faced with a very consistent community, fully integrated and characterized by some cultural landmarks, in particular by its being a bridge between two styles of research and teaching: that developed by the 'school of Segre' and that promoted by the Peanians. By emitting racial laws, the regime decisively altered the local context of scientific instruction in Turin. The abrogation of those discriminatory decrees in 1945[27] did not stem their impact and did not succeed in completely restoring the Jewish presence in the Turin school world.

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[27] Some of the decrees concerning the abrogation of racial laws are dated 1944 but they were enacted by the Senate of the University of Turin on 24 May 1945. The Purge Commission (*Commissione per l'Epurazione*) of the University of Turin was nominated on 9 July 1945. Its President was the eminent anatomist and histologist Giuseppe Levi. In Turinese secondary schools the Purge process proceeded a bit more slowly. For example, at Liceo Galileo Ferraris Emilio Artom was reintegrated on 30 October 1945, but because of health reasons he did not resume his position.

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Precision and approximation mathematics for teacher education: The lecture course of Guido Castelnuovo and the influence of Felix Klein

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Abstract

In 1913/14 Guido Castelnuovo gave a series of lectures at the University of Roma entitled "Matematica di precisione e matematica delle approssimazioni".

The lectures by Castelnuovo were explicitly inspired by a course delivered by Felix Klein in 1902, with the title "Anwendung der Differential- und Integralrechnung auf die Geometrie: eine Revision der Prinzipien", which would be edited again in 1928 with the title "Präzisions- und Approximationsmathematik" as volume III of the series on Elementary mathematics from a higher standpoint.

In the same period Castelnuovo was the author of the mathematics curricula for the new Modern Lycée. The course on "Precision and application mathematics" shows his point of view on mathematics, which he shares with Klein, and presents a "higher point of view" on the programs of the Modern Lycée.

Introduction

At the beginning of the 1900s Guido Castelnuovo turned his attention toward methodological, didactical, historical and applicative issues. He was an active member of the ICMI (International Commission on Mathematical Instruction) and of the Mathesis, the Italian association of mathematics teachers (see Furinghetti & Giacardi, website). He was also the author of the mathematics curricula for the new Modern Lycée and he taught the first University courses devoted to pre-service training of mathematics teachers.

In 1913/14 Castelnuovo gave a series of lectures at the University of Roma entitled *Matematica di precisione e matematica delle approssimazioni*. The notebook of his lessons has been recently published on the website of the Accademia Nazionale dei Lincei (Gario, website).

The course taught by Castelnuovo was explicitly inspired by a course delivered by Felix Klein in 1902, with the title "Anwendung der Differential- und Integralrechnung auf die Geometrie: eine Revision der Prinzipien", which would be edited a second time in 1928 with the title "Präzisions- und Approximationsmathematik" as

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

volume III of the series on *Elementary mathematics from a higher standpoint*. The book has been recently translated into English and edited together with a new edition of the two first volumes (Klein, 2016).

Castelnuovo's course was delivered as the first part (in the first semester) of the general course on "higher geometry". Notwithstanding this title, the courses of higher geometry were also devoted to topics, such as non-Euclidean geometry and probability, directed to those mathematics students who were to become mathematics teachers in high school. Indeed, some years later these same topics became part of Castelnuovo's course of *Matematica Complementare*, explicitly devoted to the training of teachers.

Why such an interest for approximation mathematics (that is, mathematics for applications) in the training of pre-service teachers?

Surely, the 1900s were a period in which most countries started reforming the mathematical programs in their secondary schools. The teaching of mathematics had to include new topics, like differential and integral calculus, with a wider application to the life sciences. Many reforms took place in this direction, such as those carried out in France and in Germany. Italy participated in the international debates, even if the Italian representatives (for instance within the ICMI) frequently had different positions. For example, Italian representatives took part in the two important debates held within the ICMI concerning "*La rigueur dans l'enseignement mathématique dans les écoles moyennes*" (Fehr, 1911) and "*L'enseignement scientifique et l'enseignement technique moyen*" (see in particular Bourlet, 1910).

Felix Klein played an important role in this context; already in the last decade of the 19th century his ideas became widespread and were particularly appreciated in Italy (see Gario, 2006; Bernardi & Menghini, 2003). We also recall, with reference to the international contacts, that Klein became president of the ICMI from the year it was founded in 1908, and Castelnuovo was among the Italian representatives.

But the reason for the interest of Guido Castelnuovo and Felix Klein in applications, and thus for the importance attributed to a course in "Precision and approximation mathematics", lies not only in the international movement, rather it arises from profound (and older) personal convictions.

The role of the applications of mathematics: the point of view of Felix Klein

As is well known, Klein had shown interest in the applications (and in the teaching) of mathematics already in his *Erlanger Antrittsrede* of 1872:

> mathematics exists not simply for its own sake, it also exists in order to serve the other sciences, as well as for the formal educational value that its study provides (see Rowe, 1985, p. 132, translated by the author.)

In this context, Klein also stresses the importance of intuition, creativity and logic, and thus the importance of the connection of mathematics with "intuition-oriented disciplines" (see Rowe, 1985, p. 138).

These ideas are taken up again in the 6th of Klein's American Conferences (Klein, 1894), indeed entitled *On the mathematical character of space-intuition and the relation of pure mathematics to applied sciences*:

> From the point of view of pure mathematical science I should lay particular stress on the heuristic value of the applied sciences as an aid to discovering new truths in mathematics. Thus I have shown (in my little book on Riemann's theories) that the Abelian integrals can be best understood and illustrated by considering electric currents on closed surfaces. In an analogous way, theorems concerning differential equations can be derived from the consideration of sound-vibrations; and so on (p. 46).

So, mathematics is strictly connected to other disciplines, it is not a set of arbitrary constructions but it arises from the need of understanding the problems and of discovering solutions.

As for mathematics teaching Klein states:

> It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is of course nothing but a universal pedagogical principle to be observed in all mathematical instruction. (Klein, 1894, p. 50)

Of course, in the long period between the *Erlanger Antrittsrede* and the new edition of the third volume of *Elementary Mathematics from a higher standpoint*, Klein's opinion about the role of applied mathematics changed a little (Schubring, 1989), and also applied mathematics itself changed, but most of the motivations for studying this branch of mathematics are still valid.

The role of the applications of mathematics: the point of view of Guido Castelnuovo

In his paper on *the educational value of mathematics and physics* (Castelnuovo, 1907), which is a kind of *manifesto* of his thought, Castelnuovo stresses the importance of observation and of experimental activities, the usefulness of the continuous comparison between reality and abstraction and the importance of applications to "highlight the value of science". Moreover, he maintains that heuristic procedures should be favoured for two reasons:

> the first, and the most important reason, is that this type of reasoning is the best way to attain to truth, not just in experimental sciences, but also in mathematics itself […] The other reason is in the fact that this is the only kind of logical procedure that is applicable in everyday life and in all the knowledge involved with it. (Castelnuovo, 1907, p. 336, translated by the author)

Of course this vision is linked to the way in which mathematics should be presented, at all levels. For instance, Castelnuovo is convinced of the importance of teaching mathematics and physics together. Moreover, his interest in linking to applications indicates an interest in the way in which concepts arise, and thus in the *history of mathematics*. For instance, with respect to the close relationships between mathematics and physics:

> It is not necessary that the mathematics teacher constantly anticipates the theoretical knowledge needed by his physics colleague, but it is sometimes better, following the historical development, that the physics teacher gives a hint of a notion, which can later on be precised and developed by the mathematics teacher (Programmi Ginnasio - Liceo Moderno, 1913, Istruzioni, translated by the author).

For both Klein and Castelnuovo applied mathematics leads not only to a major consideration of the historical processes in which a concept was born, but also of to an interest in intuition: the analysis of the mathematical properties that applied mathematicians take for granted when studying certain phenomena leads to a comparison between properties that can be grasped by intuition and properties that can be understood only in the theoretical field of abstract mathematics.

The training of teachers

The position of Klein with respect to *teaching* and the *training of teachers* is expressed quite clearly in the American conferences. Instead, we don't find many references to these aspects in the prefaces of his course on *Precision and Approximation Mathematics*: indeed, the re-editing of this lecture course as a teacher training course was completed after Klein's death. We read in the preface of 1902 that the text had been written with the purpose of filling the gap that had been created between pure mathematicians and applied mathematicians, and this purpose is somehow repeated in the preface to volume 1 in 1924, presenting the new project that involved the re-editing of the three volumes for the teacher training courses:

> The work of 1902 was designed to bridge the gap between the needs of applied mathematics and the more recent investigations of pure mathematicians. ….the combining of volume a) with b) and c) will meet the approval of all who appreciate the growing significances of applied mathematics for modern school instruction (Klein, 1924, preface, translated by the author).

A reference to teaching in schools can be found anyway in the preface to Volume III of 1902, although we find nothing explicit about his general vision on applications:

> At the same time, with this lecture course, I complete those discussions that I often presented on the methods of mathematics teaching, particularly in higher education (see for instance Jahresbericht, vol. 8, 1898-1899). My opinion is still that the teaching for beginners and for those students who want to use mathematics only as an auxiliary means for other studies has to make a naïve use of the intuitive approaches; the conviction that this is necessary for pedagogic reasons, considering the disposition of the majority of students, has become noticeably stronger in the last few years, here and abroad.
>
> But I am no less convinced (something I never omitted to state) that – in accordance with today's development of science – such a form of teaching will not suffice for the training of the professional mathematician in high school; rather, that here must prevail, in addition to the facts of intuition, the central importance of the modern concept of number and of the far-reaching related developments (Klein, 2016, Vol. III, preface to the 1902 edition).

Unlike Klein, Castelnuovo states clearly at the beginning of his lectures course that the teaching and learning of mathematics would be more successful if it included, besides the logical procedures that lead to the theorems, also the way in which concepts are formed starting from observations, and how these can be verified in practice.

As we can see from these prefaces, Klein's volume on *Precision and Approximation Mathematics* was not born as a book for the training of teachers, but its usefulness in this regard slowly became clear.

This was clear in Castelnuovo's mind when he used its contents in his lectures of 1913/14 and this was clear for Klein when he decided to add the volume to the two other volumes of Elementary Mathematics from a higher standpoint.

But for Castelnuovo there was a more specific reason for giving these lectures in 1913/14, as is shown in the following section.

The modern lycée

In 1911 a Law of July 21st established the *Liceo Moderno*, which effectively started in 1913. In this lycée, which was a hybrid of the modern lycée and the scientific lycée provided by the Commissione Reale in 1908, the preparation for university studies (not necessarily of a scientific nature) was achieved through the study of Latin, modern languages, and the sciences (Marchi & Menghini, 2013)
Its mathematics programs were ascribed to Castelnuovo (see Castelnuovo, 1912) and to the Inspector of the Ministry Mineo Chini. The *Liceo Moderno* lasted three years, following the classical *Ginnasio*, and its first year coincided with the first year of the classical lycée. It therefore presented new contents only in the last two years, after the common teaching of rational arithmetic and rational geometry. In these two years, mathematics was presented as an appropriate language for describing natural phenomena.

> The renovation of the mathematics of the 17th century is linked to the blooming of the natural sciences. Within this context, the teacher will have to explain how the fundamental concepts of modern mathematics, particularly the concept of function, are implied by the observational sciences, and – being then rendered precise by mathematics – have in turn had a positive influence on the development of the latter (Castelnuovo, 1912, p. 124, translated by the author).

In the second year, in addition to traditional topics, there was also the study of approximate measures and operations on them. Cartesian coordinates were introduced, and their use was recommended for the graphical representation of functions. These were to be introduced through the phenomena that were described in the various courses of physics, chemistry, biology, and economic geography. In addition, a step towards orthogonal coordinates in space was suggested, but these were only used when studying crystallography in natural science courses. Polynomial functions of the first and second degree and hyperbolic functions were studied with emphasis on their physical interpretation. Logarithms and the logarithmic function were studied as well. In the final year, trigonometry included the study of circular functions, followed by an introduction to the concepts of limit, derivative, and definite integral, as well as their geometric and mechanic interpretations.

As we have seen, a part of the programs is concerned with approximation mathematics and its heuristic nature. Castelnuovo's course on precision and approximation mathematics is therefore very much aimed at training the teachers of *his* modern lyceé, which he hopes will become more and more widespread. We will see this better looking at the content of the course.

The contents of the course and their relation to the programs of the modern lycée

A first part of the university course of Castelnuovo concerns a question that is not developed as such in the corresponding course of Felix Klein (even if there are different hints to the problem, also in Klein's 6th American conference). Castelnuovo poses the question: "Within which limits can the results of mathematics be verified in practice?" In fact, in many cases the conclusions drawn with a purely logical reasoning can neither be verified by intuition nor by practical experiences.

To answer this question we can divide geometrical theorems into three classes: theorems that can be verified in an exact manner, theorems that can be verified in an approximate manner, and theorems that cannot be verified in practice.

Theorems of the first class are for instance *Euler's formula for the polyhedra*, or the theorem concerning the existence of non-orientable surfaces, as the Möbius strip. These statements are either absolutely true or "grossly" false.

Theorems of the second class concern geometrical properties (as measures of angles or segments, parallelism, …) that can be verified only in an approximated manner. Exactness has no meaning in these cases, even when using particular instruments. With the best procedures one can succeed in measuring a segment with a precision up to 10^{-7} m (this example is also made by Klein). So we can only verify approximately that the sum of the angles of a triangle is 180 degrees. Only the axioms can guarantee the correctness of the theorem.

In relation to this point we read in the programs:

> Recalling that lengths and angles are measured in practice with meter and protractor, the teacher will tell the students that any measure is affected by an error, which can be reduced using better measuring instruments but cannot be suppressed.
>
> … in the more developed applied sciences a limit to the error is fixed, and, if this limit is respected, the measure is considered in practice as exact.
>
> Approximated measures will naturally lead the teacher to speak of the operations on decimals representing approximated measures… through the comparison between approximated and exact measures the question arises of the existence of a common measure, leading to the concept of incommensurable magnitudes. These are linked to irrational numbers… (Programmi Ginnasio - Liceo Moderno, 1913, Istruzioni, translated by the author).

So, approximations are the first step for the formal introduction of rational numbers.

Coming back to the *university course*, we find that the best example of theorems of the third class when Castelnuovo speaks about the existence of *incommensurable* segments.

All this serves as an introduction to the theme of the course: on the one hand logical reasoning cannot always assure the result of a practical experience, on the other hand intuitive or empirical procedures are not sufficient to justify mathematical results.

Therefore, following Klein's ideas, Castelnuovo distinguishes between *precision mathematics and approximation mathematics*, where the first includes all the propositions that can be logically deduced from the axioms of geometry or of analysis – obtained by abstraction from experience; while the second includes the results that can be obtained from experience with a certain degree of approximation.

These two kinds of mathematics are strictly connected, but not coincident. There are results that can be taken from one field into the other, but there are theories in precision mathematics (such as that of irrational numbers), which do not make sense at all in approximation mathematics. This should not lead to a discredit of approximation mathematics. Precision mathematics has an "economic" value, in that its instruments help in the solution procedures of a problem.

With two different examples (taken resp. from astronomy and chemistry) Klein and Castelnuovo explain the use that practitioners make of the term *commensurable* (as they never meet incommensurable quantities): two magnitudes are commensurable if their ratio can be expressed by "small" numbers.

Castelnuovo then gives a brief historical account on how the concept of function has developed (this part exists but is much briefer in Klein, 1902); he starts from the line seen by Euclid as "length without breadth" or as boundary of an area, then passing on to its definition as a trajectory of a moving point or, more precisely, as locus of points satisfying a given condition. He mentions the different definitions of Descartes, Leibniz, Euler, Dirichlet, up until today's definition of locus of points whose coordinates – for instance Cartesian coordinates – satisfy a given equation $y = f(x)$ (if it can be expressed in terms of y).

What is important is that "given x, one can – through a finite number of arithmetic operations – obtain y with an arbitrarily fixed degree of approximation".

Here too we can see the link to the programs of the *Liceo Moderno*, which pose an enormous value on the concept of function, necessary if

> we really want that the high school students are inspired by mathematics and understand something of the greatness of its structure (Castelnuovo, 1919, p. 5, translated by the author).

The successive part of Castelnuovo's lecture course is completely taken from the course of Felix Klein, and concerns the relation between *empirical curve* and *idealized curve*. This is the part that really corresponds to the *Introduction* of the programs of the *Liceo Moderno*, which states that the origin of the concept of function is situated within the observational sciences and is then rendered precise by mathematics.

The idea was already present in Klein's 6th American conference:

> In imagining a line, we do not picture to ourselves "length without breadth", but a strip of a certain width. Now such a strip has of course always a tangent (Fig. 1); i.e. we can always imagine a straight strip having a small portion (element) in common with the curved strip; similarly with respect to the osculating circle. The definitions in this case are regarded as holding only approximately, or as far as may be necessary.

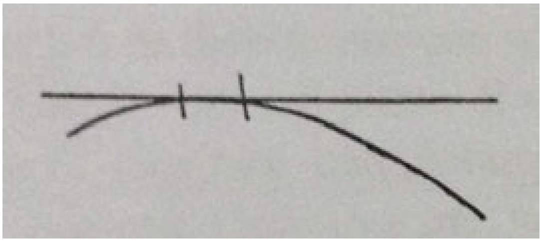

Fig. 1

> The "exact" mathematicians will of course say that such definitions are not definitions at all. But I maintain that in ordinary life we actually operate with such inexact definitions. Thus we speak without hesitancy of the direction and curvature of a river or a road, although the "line" in this case has certainly considerable width. (Klein, 1894, p. 98)

These considerations are taken up again in the lecture course of 1902 (Klein, 1902) and the question then posed by Klein is: how can we determine y as a function of x by means of an empirical curve?

Castelnuovo expresses in his lecture course the reason for considering the relation between empirical and idealized curve: the development of the concept of function that we described above has as a consequence that – if a curve is the set of points satisfying a relation of the kind $y = f(x)$ – such a curve may lack intuitive aspects such as continuity, existence of a finite number of maxima and minima, existence of the first and second derivatives. So we need to examine which restrictions we have to give our idealized curve $y = f(x)$ so as to obtain that it corresponds to our intuitive concept of empirical curve.

> An empirical curve (and observational sciences only have to do with empirical curves) does not yield y as a function of x in the sense of precision mathematics, rather it represents relations of the kind $y = f(x) \pm \varepsilon$, where f is a function of x and ε is a positive number varying as x varies and subject to the only condition not to exceed a certain value δ. Klein calls the analytic object kind $y = f(x) \pm \varepsilon$, a function stripe and says consequently that an empirical curve does not define a function but a function stripe, (see Gario, webpage, translated by the author)

In relation to this, we read in the instructions to the programs of *Liceo Moderno*:

> It is better to introduce the notion of function considering again the phenomena described in the lessons of physics, chemistry, biology … The distinction will be explained between functions that are defined for discrete sets of values of the variable (whose graphs have a partially arbitrary shape and could also be represented by polygonal chains) and functions defined for all values in certain interval (whose diagrams are often traced by recording instruments).
>
> Analysing the curve the intervals will be determined in which the function is increasing, has a maximum, … (Programmi Ginnasio - Liceo Moderno, 1913, Istruzioni, translated by the author).

The concept of an abstract curve, geometrically defined, is associated to certain properties that belong to the idealized curve $y = f(x)$ only if f satisfies certain conditions:

1. The concept of continuity of an empirical curve expresses the empirical fact that between two points of the curve there is a sequence of points which are so close that in any, however small, interval of the curve there are infinite points of the sequence, that is: the curve is connected. Thus the function $f(x)$ representing the idealized curve must be continuous in the neighbourhood of any of its points. Castelnuovo also recalls that the continuity of $f(x)$ in the interval (a, b) means that f can take on any value in the interval between $f(a)$ and $f(b)$. The continuity of $f(x)$ in its point x_0 can be expressed by $lim_{x=x_0} f(x) = f(lim_{x=x_0} x) = f(x_0)$.
2. The existence of maxima and minima requires a further restriction to the condition expressed in 1. We can consider, for example, the function $y = \sin\frac{1}{x}$ (Fig. 2). This function is in fact continuous for every value of x, except 0. But the maximum $y = 1$ is reached for infinitely many values of $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \frac{1}{4\pi}$, ... in the interval (0, 1).

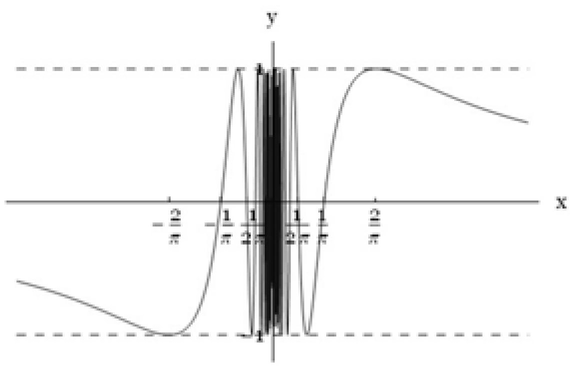

Fig. 2

The second example is given by the function $y = x \sin\frac{1}{x}$ (Fig. 3). This function is continuous also in $x = 0$, where it takes on the value 0. It has an infinite number of maxima and minima in every finite interval containing the origin.

We require, on the contrary, that our idealized curve has a finite number of maxima and minima in a finite interval.

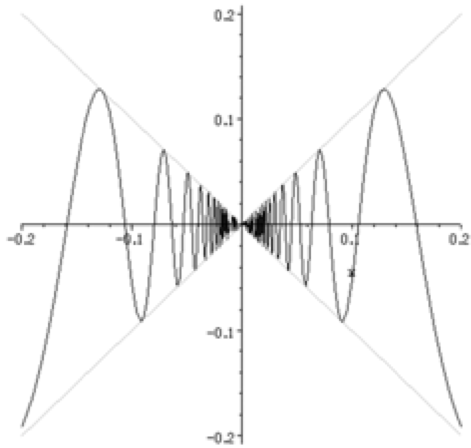

Fig. 3

3. An empirical curve is considered to have a slope in every point, which is the direction of a material point moving along the curve when all the forces and constraints, to which it is subject, cease and the point can satisfy the laws of inertia.

This way we can imagine the tangent to an empirical curve at every point. But it is not easy to define such a tangent. An example is given by experimental sciences (this example is not in Klein's text):

> Let us consider the graph traced by the very subtle pen of a recorder, which – due to the inertia of the instrument, traces horizontal strokes. We will obtain a scale-curve, which has only tangents parallel to the x-axis. It is also evident that these tangents don't give us any indication on the velocity of variation of the phenomenon, that is instead indicated by the slope of the lines joining point on the curve which are not too close, for instance two vertices of convex angles, or of concave angles (see Gario, webpage, translated by the author).

For an empirical curve the slope of the tangent is not given by the derivative $\frac{dy}{dx}$, but by the difference quotient $\frac{\Delta y}{\Delta x}$, where Δx must be large enough with respect to the width ε of the curve, but small enough with respect the considered part of the curve.

Still using the geometrical language we can define as the tangent to an empirical curve at a point an "adherent" straight line that passes through the point and leaves the neighbourhood of the point on a same side. This definition is valid for all points, with the exception of a finite number of them.

> So the idealized curve $y = f(x)$, obtained by abstraction from the empirical curve, needs to have a tangent at any point; that is, the function $y = f(x)$ needs to have a derivative at any point. This condition is not a consequence of continuity – even if for a long time mathematicians were convinced of this. An example of the fact that this is not true is given by the function $y = x \sin\frac{1}{x}$, which is continuous in $x = 0$ but does not have a derivative at this point, because $\frac{f(h)-f(0)}{h} = \sin\frac{1}{x}$. And when h tends to 0, $\sin\frac{1}{h}$ can take an infinite number of times any value between +1 and -1 (see Gario, webpage, translated by the author).

Once the characteristics of the curves that do *not* have an *intuitive* character have been established, the mathematician can of course engage in their study. The functions mentioned above, as $y = \sin\frac{1}{x}$, are for instance *idealized* curves. In particular, both Klein and Castelnuovo devote much space to the study of the *Weierstrass* function (in fig. 4 the approximating curves drawn in Castelnuovo's notebook), that surely represented a significant example of a function which is continuous but never differentiable:

$$y = \sum_{n=0}^{n=\infty} b^n \cos(a^n \pi x)$$

Other topics, also contained in Klein's lectures, are only listed without further developments in Castelnuovo's notebook: the Peano curve (another example of non-intuitive idealized curve); polynomial interpolation (in particular, Klein stresses the finite series expansions and their capacity of approximating an analytic function), the harmonic analyzer, etc. This last example is interesting. Klein describes in detail this device that calculates the Fourier coefficients. This instrument is surely no longer up to date (it was constructed by the Swiss Gottlieb Coradi toward the end of the 1800s), but the description of its functioning has a strong didactic value and helps the understanding of the concepts involved.

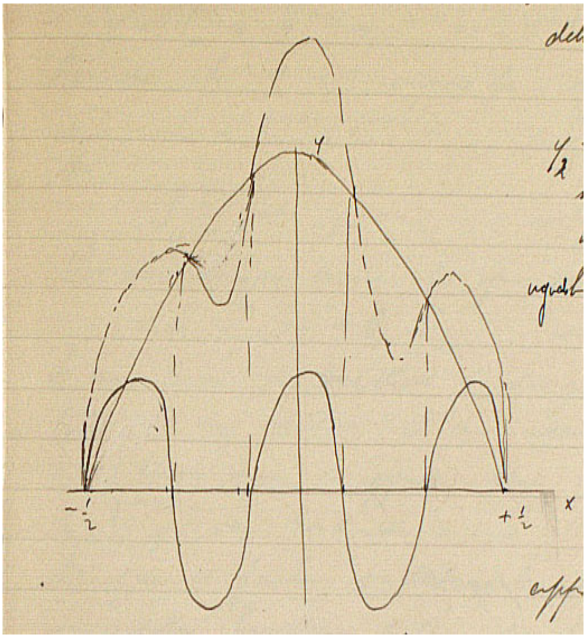

Fig. 4

Conclusions

In building his ideological framework on mathematics education, Castelnuovo is also influenced by Felix Klein, who claims that pure mathematics is connected to the other sciences: it is not a set of artificial and arbitrary constructions, but it arises from the need to understand the profound meaning of the problems. This should not be seen as an attempt to limit the studies of pure mathematics. Fearing a separation between pure and applied mathematicians, which would lead to applications not being developed from a sound basis and to the isolation of the pure mathematicians, Klein proposes an approach to mathematics that is less abstract and constantly refers to applications.

In line with Klein's ideas, Castelnuovo considers the Italian school system to be too abstract and theoretic. Instead, teachers should encourage their pupils to use imagination, observation, logic. Castelnuovo speaks in favour of applications, of observations and of experimental activities, of a continuous comparison of reality and abstraction.

From Castelnuovo's note book on *Precision mathematics and application mathematics* clearly emerges his attention to those chapters of abstract mathematics that more often appear in applied sciences. He includes in precision mathematics all the propositions that can be logically deduced from the axioms of geometry or of analysis – obtained through abstraction from experience; while the second includes results that can be obtained from experience with a certain degree of approximation. These two kinds of mathematics are strictly connected, but not coincident. He then examines the two different concepts of a curve given by pure mathematics and by approximation/intuition: the idealized curve and the empirical curve.

> This is the main fault of the doctrinaire spirit pervading our school. We teach there to distrust approximations, which are reality, to take as idol perfection, which is illusory. … There is no better way to reach the scope than combining at every step theory with experience and science with applications…
>
> If, for the sake of culture, we stifle in our pupils the practical sense and the spirit of initiative, we lack the greatest of our duties (Castelnuovo, 1912-13, translated by the author).

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The standardisation of the place of problems in French geometry textbooks during the 19th century

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Abstract

The place of problems in 19th Century French geometry secondary education textbooks will be studied in this article. It is part of a PhD work studying the notions of problems and methods in 19th century French geometry textbooks (Moussard, 2015). The questions tackled here are: where are the problems located in the textbooks? What is their relative importance in comparison with the theorems? How are the problems organized? What are the institutional prescriptions concerning the practice of problem solving?

We will see that the organization of problems changes profoundly during the 19th century, and that these changes reveal modifications to the very conception of the notion and the role of problems in textbooks. Finally, we will see how the conception and organization of problems in geometry textbooks became widely standardised by the end of the period.

Institutional and mathematical contexts

To understand our subject properly it is necessary to introduce some elements concerning the early 19th century institutional and mathematical contexts, and to say a word about the corpus.

An important consequence of post-Revolution reforms in French education was to give mathematics a significant place, with its own teachers, curricula and schedules. Secondary education was the one place of higher studies and a required passage to accede to liberal professions and to the *Grandes Écoles*, the most famous of which was the *École Polytechnique* (Belhoste 2001). Nevertheless, the level of French educational establishments was heterogeneous, the best schools being concentrated in the main cities, and particularly in Paris. These Parisian schools were often the principal concern of the administration which wrote rules and curricula with them in mind.

From a mathematical point of view, the beginning of the 19th century saw a renewed interest in geometry problem solving and methods, as shown in the publication of some important books such as Monge's *Géométrie descriptive*, Carnot's *Géométrie*

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

de position, Servois's and Mascheroni's *Geometry of the ruler and of the compass* respectively, Poncelet's *Traité des Propriétés projectives des figures*, Michel Chasles' *Aperçu historique* and his *Traité de géométrie supérieure* (Chasles, 1852). Magazines like the *Annales de Gergonne* or the *Correspondance sur l'École polytechnique* dedicated a significant place to problems. In all these publications, new methods were developed and applied to problems. In particular, the so-called "Géométrie rationnelle" claimed to be able to compete with analytic methods in problem solving.

To carry out this research, a large corpus of secondary education geometry textbooks has been analyzed, which up until now has been very little studied. This point leads to the question of how to characterize a textbook. Though contemporary research shows that the frontier between research books and textbooks in the history of mathematics is not easy to set (Barbin & Moyon 2013), elements were exhibited to characterize this particular editorial object by Gabriel Sauter and Miguel Somoza (Sauter & Somoza 2001, p. 15-24). Many of the books reviewed here are in fact explicitly intended for teaching. They usually had a systematic presentation of the contents in successive sections. Their structure was adapted to the intellectual maturity of a scholar audience, and finally they complied with official regulations.

A split from the Euclidean heritage

We shall begin with the study of elementary geometry and will come later to analytic geometry.

Euclid's *Elements* remained a strong reference at the end of the 18th century in France for anyone who wanted to write a geometry textbook, though this does not mean they were used as textbooks. In fact, they were not used in some of the main mathematics teaching places, such as the military schools (Dhombres 1989, p. 55-78).

Once the axioms and demands are settled, Euclid's *Elements* contain two types of propositions: problems and theorems. The problems are constructions with a ruler and compass, while theorems enunciate the properties of figures. These two types of propositions alternate in Euclid's books with respect to the deductive order. The theorems are established on figures, which construction has been produced upstream, and the proof of the problem's solutions use the theorems which have been proved upstream. We understand to what extent the problems, here constructions, are essential for the deductive order (Moussard, 2014). For that matter, the very first proposition is a construction, that of the equilateral triangle.

Now what was the place of problems in geometry textbooks in use around 1800 in France? Three main textbooks were in use in the 1800's (Moussard 2015, p. 112): those of Bézout (Bézout, 1762), Legendre (Legendre, 1794) and Lacroix (Lacroix, 1798). None of the three adopted Euclid's organisation of problems.

Legendre undertook the writing of his *Elements* willing to come back to more rigour, meaning the rigour of deduction and of definitions:

> Geometry elements are blamed for being less rigorous […] There is none where one was able to prove all propositions in an absolutely satisfactory way. Sometimes the authors assume things that are not contained in the definitions; sometimes these definitions themselves are defective (Legendre 1794, p. v)[1].

Concerning the problems, Legendre separated them from the theorems, placing the problems at the end of the successive eight books of his *Elements,* thus making the construction of the figures unnecessary to the deductive order of propositions. So, despite the fact Legendre referred in his preface to the rigour of the Ancients, he abandoned the linkage between problems and theorems absolutely essential in the Euclidean work.

Bézout wrote his *Elements* for the Marine officers, and he was soon asked to adapt them for all army corps. As a matter of fact he wrote the artillery version of the book (Alfonsi, 2011). He had in mind the military application of mathematics.

The propositions of the geometry textbook were organized into three levels: principles, that are propositions of first rank importance, then consequences, that are second rank propositions deduced from the previous ones, and finally applications (Moussard 2015, p. 114-117). Problems of construction belong to the last category, that of applications.

Lacroix wrote his *Elements* "guided by experience" as a teacher in the newly created *Écoles centrales*. His pedagogic concern led him to alternate theorems and problems in the textbook, as the latter enlighten the former:

> The constant use of providing students with problems, made me feel the disadvantage in presenting an entire section of theorems, and in displacing further the problems that succeed. This arrangement, although strange, to say no more, which exposes the problem when the theorem on which it rests, and would have enlightened or confirmed, is already erased from memory, deprives the reader of the means to build his figures with some care (Lacroix, 1805, *Essai sur l'enseignement*, p. 334)[2].

[1] This and all following translations are mine. Original text is : « On reproche aux éléments de géométrie d'être peu rigoureux […] Il n'en est aucun où l'on ait réussi à démontrer toutes les propositions d'une manière absolument satisfaisante. Tantôt les auteurs supposent des choses qui ne sont pas contenues dans les définitions ; tantôt ces définitions elles-mêmes sont défectueuses ».

[2] « L'usage constant de proposer des problèmes aux élèves, m'a fait sentir l'inconvénient qu'il y aurait de présenter une section entière de théorèmes, et de renvoyer après, les problèmes qui en sont la suite. Cet arrangement, au moins très singulier, pour ne rien dire de plus, qui fait paraître le problème quand le théorème sur lequel il repose, et qu'il aurait éclairci ou confirmé, est déjà effacé de la mémoire, prive le lecteur des moyens de construire ses figures avec quelque soin ».

Lacroix clearly opposed Legendre's organisation of problems. Problems of construction were placed immediately after the theorems necessary for their resolution, and contributed to the better understanding of those theorems.

As we can see, the conception and place of construction problems in these three textbooks were very different. They were placed along the textbook as applications of the theorems in a military purpose in Bézout; they were separated from the deductive order of the propositions of geometry in Legendre; they followed the theorems in order to enlighten them in Lacroix. None of these three textbooks adopted the Euclidean deductive structure alternating theorems and construction problems, and the construction of the figures was not needed for the proofs of the succeeding theorems any more.

In the 1820's, a textbook author of the next generation, Alexandre Vincent (Vincent, 1826), encountered the biggest difficulties of setting the place of problems in his textbook.

Vincent's cours de géométrie élémentaire

Alexandre Joseph Hydulphe Vincent studied at the *École Normale Supérieure* and passed his *Agrégation* in 1820. He wrote the *Cours de géométrie élémentaire* when he was a teacher at the *Lycée* of Reims. It was therefore very likely intended for students of the *classe de Mathématiques élémentaires* preparing for the entrance examination to the *Grandes Écoles*, among which the most famous was the *École polytechnique*. The textbook was rather successful, and counted for a while as one of the few serious competitors of Legendre's[3]. Vincent, who had Lacroix as a teacher, showed a strong pedagogical concern. In particular, he stated in his textbook a major interest for problems.

The notion of problems according to Vincent

Firstly, from a strictly quantitative point of view, the problems were more numerous than the forty or so appearing in Bézout, Legendre or Lacroix's textbooks, reaching here 150 problems in the 1832 second edition. Secondly, various types of problems were identified: "graphic problems" designated construction problems, and "numeric problems" designated particular numerical applications. Such numeric problems were not to be found in Lacroix and Legendre textbooks, but rather in the so-called *Géométrie pratique* books. Indeed, Vincent borrowed some of his numeric problems from the *Cours de mathématiques* that replaced that of Bézout in the special military school of Saint-Cyr (Puissant et al., 1813).

[3] Le correspondant, Paris: Douniol, vol. 77, 1869, p. 533.

A third type of problem was mentioned by Vincent that he called "theoretical problems", "which solution provides new general knowledge". The author did not provide any example of such problems whose resolution led to a theorem. The following problem taken from the later textbook of Antoine Amiot may illustrate what Vincent had in mind:

> The sum of the perpendiculars drawn from a point inside an equilateral triangle to its sides is constant. How should this theorem be modified for a point exterior to the triangle? (Amiot, 1850, p. 70)[4].

Vincent decided to put the "theoretical problems" together with the theorems, and noticed that "there is no proposition that shall not be possibly transformed, by inverting the order of the ideas, into a question to be solved, and vice-versa". The Euclidean distinction between problems and theorems had definitely vanished, as the problem was from then on thought of as a question to be solved, and no longer restricted to a geometrical construction.

According to Vincent, problems are particular applications of theorems, whether graphic or numeric. Such a status does not diminish their fundamental pedagogical role: "it is important that pupils be skilful in applying a knowledge that, if its practice came to be neglected, would become more or less useless".

Where should the problems be placed?

Vincent was very much concerned with where he should place the problems in his textbook. He undertook to reconcile two apparently contradictory purposes: one purpose is to keep the problems grouped by "analogy", as are the theorems, which is a thematic analogy. The picture on the next page shows the criteria of Vincent's classification: the themes concern either the nature of the figures (lines, circles, polygons), the nature of the relation between figures (perpendicular, parallel, tangent), or the nature of the magnitude under consideration (length, area).

The second purpose was to submit the problems to the pupils throughout the textbook and immediately after the necessary theorems used for their resolution had been exposed. To avoid the dissemination of the problems, in accordance to the first purpose, Vincent used cross-references to indicate which problems might now be solved.

Vincent was also concerned in bringing problem solving into the textbook as soon as possible. This led to the main modification in the second edition, according to Vincent. He dealt with the properties of the circle at the same time as those of the lines. Vincent split with the order of simplicity of figures, introduced by

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[4] « La somme des perpendiculaires tracées d'un point pris à l'intérieur d'un triangle équilatéral sur les trois côtés est constante. Comment faut-il modifier l'énoncé du théorème pour un point extérieur au triangle ? ».

Arnauld (Barbin, 2009) and adopted by Lacroix in his *Elements*, to confront the pupils with problems earlier on.

CHAPITRE PREMIER.

Problèmes Graphiques.

Numeros.			Pages.
309...337.	§ I.	Problèmes sur la Construction des Perpendiculaires, des Parallèles, des Angles, des Polygones, du Cercle, etc.	257
338...366.	§ II.	Problèmes sur les Lignes Proportionnelles et sur les Aires	280
367...388.	§ III.	Problèmes sur les Contacts	308

CHAPITRE II.

Problèmes numériques.

389...398.	§ I.	Problèmes sur les Lignes Proportionnelles	329
399...410.	§ II.	Problèmes sur les Aires	333
411...423.	§ III.	Autres Problèmes sur les Aires	338
424...431.	§ IV.	Problèmes sur l'Aire du Cercle	345

Fig. 1. Vincent's organisation of problems.

Increase of the number of problems

The number of problems in textbooks increased widely over the following decades (Moussard, 2015, p. 246-256). We have seen that Vincent included three or four times more problems in his textbook in comparison to Lacroix for example. Though, this still remains few in comparison with the lists of hundreds of problems that can be found later. They appeared in two types of books: collections of problems and textbooks.

Collections of problems: a new editorial object

The first important corpuses of geometry problems appeared in France in specifically dedicated books of collections of problems. There were not so many of such books, and they were far less numerous than the publications of geometry textbooks for example. We have established the exhaustive list of these collection books for the first half of the century. Authors are: Jean-Guillaume Garnier (Garnier, 1810), Antoine Reyaud (Reynaud, 1833), Jean-Marie Duhamel (Reynaud and Duhamel, 1823) and Eugène Catalan (Catalan, 1852), all professors related to the

École polytechnique, Georges Ritt (Ritt, 1836) who was graduated from the *École Normale supérieure*, Jules Planche (Planche, 1845) and Nicolas Percin (Percin, 1848) who graduated with the *Agrégation*.

These books, all written by higher education teachers, intended to complete and enlighten the elementary geometry textbooks and to prepare for the entrance examination to the French *Grandes Écoles* as well as to the *Concours général*, a competition highly valued by the Parisian establishments.

The criteria of the organisation of problems were different from one book to another. Plane or space, elementary or analytic geometry were most often separated. Within elementary geometry, the eight books of Legendre's *Elements* provided a widespread organisation. Problems were sometimes organised according to the type of proposition (theorem to demonstrate, construction problem to solve or geometric loci to determinate), or to the nature of the figure. Finally, a distinction was sometimes made between determinate and indeterminate problems, that is, problems with a single solution or various ones.

Lists of unsolved problems became systematic

Alongside these collections of problems, from the 1840's on, textbooks proposed lists of unsolved problems located either at the end of the book or at the end of the successive sections. The first ones to do so were Pierre Finck (Finck, 1838), who graduated from the *École polytechnique*, and was a professor and examiner at the special military school, and Auguste Idoux (Idoux, 1842), also a professor. After 1845, such lists appeared almost systematically. Later on, some authors chose to publish the solutions of the problems proposed in the lesson book in a separate book as did Antoine Amiot and Desvignes (Amiot and Desvignes, 1858) or Guilmin (Guilmin, 1862), all professors, Amiot having graduated with the *Agrégation* and Guilmin from the *École normale supérieure*.

Modification of the notion of problems

The increase of the number of problems came along with a modification of the notion of problems itself. Vincent distinguished problems and theorems: in his opinion problems were practical, whether graphic or numeric, and theorems were theoretical. This distinction between practice and theory did not continue, whereas the conception of problems as "questions to be solved" gained upper hand, especially as mathematic problem solving played a major role in examinations for the entrance to the *Grandes Écoles* (Belhoste 2002, p. 141-175) and after 1864 for the *Baccalauréat*, increasing the importance of problems in the textbooks and giving them a more institutional character.

What happened in this new context with the conception of problems as geometrical constructions? As a matter of fact, the term "problem" covered at this stage two notions. One was a reference construction with instruments, like the construction of the tangent to a circle through a given point, or the construction of the fourth proportional to three given lines. Such constructions were considered as belonging to the corpus of propositions – together with the theorems – forming a geometry textbook. They were often treated in specific chapters of the textbooks dedicated to constructions, in accordance with the curricula. The second conception of a problem was a question to be solved, either a construction problem, or a theorem to demonstrate, or a locus to determinate, or a numeric problem. Such problems were progressively designated as "exercises". They were grouped together and followed the theory, an organisation that prevailed in most textbooks at the end of the century.

Institutional prescriptions

We come now to study the institutional prescriptions concerning problems in geometry teaching. The university was re-created by Napoléon in 1808 as the one corporation that gathered together all secondary education (Prost, 1968). The first curricula simply recommended specific textbooks: those of Lacroix and Legendre for elementary geometry for example. In the 1830's, for the necessity of a uniform preparation to the *Concours général* in the Parisian *lycées*, detailed curricula were written. In the following years, such detailed curricula became widespread, still referring sometimes to specific textbooks.

In 1852, the so-called "bifurcation" reform separated secondary education in two sections, one of literature and one of sciences (Hulin, 1989). Two years later, the public instruction minister Hippolyte Fortoul sent detailed instructions on how they must run the classes to the professors. Concerning sciences, the text recommended simplicity and clarity, it quoted at length the textbooks by Clairaut and Bézout, and recommended the usage of one written by Lacroix. It took precise positions on how certain key points of elementary geometry should be taught. Attention was paid to the relation between theory and its applications and the geometric constructions were given some importance in particular in the sciences section.

In 1865, the "bifurcation" reform was abandoned, and the old order with a united classic secondary education restored. Two features of the new reform concerned problems: the conviction that problem solving develops reasoning faculties and the attention paid to the preparation of the *Baccalauréat* and of the *Grandes Écoles* competitive examinations, in order to face the competition of denominational education.

> Specifically in the *classe de Mathématiques élémentaires*, the main purpose of education is the preparation for the *Baccalauréat* […] Exercises shall be numerous and always corrected carefully. The professor shall insist on the problems discussion, as this exercise is the most beneficial to familiarize pupils with the true scientific methods and to give their mind suppleness and inventive creativity (Minister Victor Duruy, 1865).

In parallel with the reunification of classic secondary education, a special secondary education was created for professional careers. The institutional prescriptions were to remain close to practical considerations. Exercises occupied a place of prime importance, and consisted, in geometry, in calculations of areas and volumes, in geometric drawings with instruments, and in descriptive geometry.

Textbooks may be intended for one or several of those streams. The Rouché and Comberousse textbook (Rouché and Comberousse, 1866) was clearly intended for the *classe de Mathématiques élémentaires*. The André textbook (André, 1871) was intended for both classical literary education and special education. Both textbooks mentioned in the title the great number of problems they contained.

In the 1880's, Girod's textbook (Girod, 1881) was intended for all levels and displayed many practical applications. Combette's textbook (Combette, 1887) was clearly specific for the *classe de Mathématiques élémentaires*. Here again they both contained hundreds of exercises.

Problems and methods

The notions of problems and methods are interlinked. The attention paid to problems was accompanied in some textbooks by an attention paid to the notion of method (Moussard, 2012) and this had an impact on the conception and organisation of problems.

Gabriel Lamé's notion of particular method

In 1818 Gabriel Lamé, who graduated from the *École polytechnique*, wrote a book (Lamé, 1818) in which he envisaged to classify the problems according to the method employed for their resolution. This led to the original conception of "particular method" that can be used to solve a large though indeterminate set of problems (Moussard, 2015, p. 326-337). The mention of particular methods appeared in the textbooks of professors Georges Ritt (Ritt, 1836), Jules Percin (Percin, 1848), Antoine Amiot (Amiot, 1850) and Charles de Comberousse (Comberousse, 1861), who graduated respectively from the *École normale supérieure*, the *Agrégation* and the *École centrale*.

As an example of such "particular method", the geometric loci method provides a solution to the problem of inscribing an equilateral triangle in three given concentric circles (Amiot and Desvignes, 1858). Having set one vertex *A* on the first circle, the second vertex belongs to the locus of point *B* such that the third vertex *C* belongs to another circle and *ABC* is equilateral. The intersections of this locus with the last circle are the possible positions of point *B*.

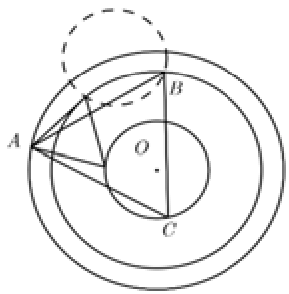

Fig. 2. To inscribe an equilateral triangle in three given circles.

The introduction of rational geometry methods

After 1852 and the authoritarian institutional reform we already mentioned, textbooks followed scrupulously the curricula. Though they encouraged the practice of problem solving in the classroom, they did not mention anything about problem solving methods.

Later on, between 1865 and 1870, three textbooks were published that claimed to introduce Michel Chasles' *Traité de géométrie supérieure* (Chasles, 1852). They applied the theories of the new born *Géométrie rationnelle*, to problem solving up to then mostly absent from textbooks, in order to provide the reader with efficient tools for examinations. Authors are Jacques Lenthéric (Lentheric, 1865), a professor in Montpellier, Charles Housel (Housel, 1865) and Luc Millet (Millet, 1870), who both graduated from the *École Normale Supérieure*.

Then, around 1880, three textbooks were published in France which were fully dedicated to problem solving methods in elementary geometry. They contain, of course, a large number of problems together with their sometimes multiple solutions. The authors are Adolphe Desboves (Desboves, 1872), who graduated from the *École normale supérieure*, Julius Petersen (Petersen, 1880), a Danish mathematician, and Friar Gabriel Marie (Frère Gabriel Marie, F.I.C., 1882), a member of the congregation of Christian schools brothers.

Analytic geometry

The situation with analytic geometry is rather different than that of elementary geometry. Before the expression "Géométrie analytique" was used, the "Application de l'algèbre à la géométrie" was thought of as a problem solving method. Hence the method was demonstrated through problems. In their textbooks Bézout (Bézout, 1766) and Lacroix (Lacroix, 1798) showed how the method works on a succession of problems chosen and organised in order to illustrate all its different aspects, subtleties and difficulties.

After 1800, analytic geometry textbooks contained few problems. The major interest was given to the study of second degree curves and progressively surfaces. After 1848, analytic geometry textbooks included some problems – meaning here questions to be solved – without solutions, a new phenomenon quite contemporaneous in elementary geometry. Their number reached 162 exercises in Briot and Bouquet's 1863 4th edition of their *Géométrie analytique* (Briot and Bouquet, 1863).

Two textbooks were marked out by the great number of construction problems with the solutions they displayed: Georges Ritt's *Problèmes d'application de l'algèbre à la géométrie* (Ritt, 1836) and Charles Jacob's *Application de l'algèbre à la géométrie* (Jacob, 1842). These textbooks treated the whole analytic geometry corpus, but focused on the geometrical resolution of construction problems, meaning here by intersecting second degree curves.

For example, given two points *E*&*F* and a circle *A*, find the point *M* of the circle such that angles *AME* and *AMF* are equal. The solution is obtained with the construction of a hyperbola (Ritt, 1836, p. 124).

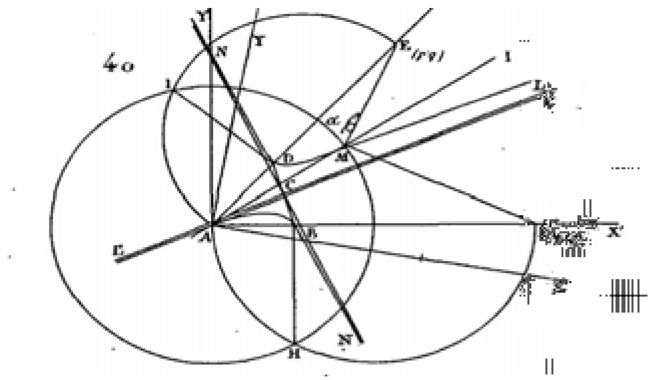

Fig. 3. Determination of a point by intersecting conics

The authors also proposed simpler solutions to problems that could be solved with a ruler and compass. A striking example is the classic problem of drawing a circle tangent to three given circles. It is obvious that the centre of the circle is at the intersection of two hyperbolas, the foci of which are the centres of the given circles (Ritt, 1836, p. 150).

After 1865 the interest in problem solving was renewed in analytic geometry. From that time on, the exams of the *Concours général* and of the *Grandes Écoles* were mostly oriented toward analytic geometry problem solving. As a consequence, textbooks new editions included more and more problems, as for example those of Briot and Bouquet, and of Sonnet and Frontera (Sonnet and Frontera, 1854).

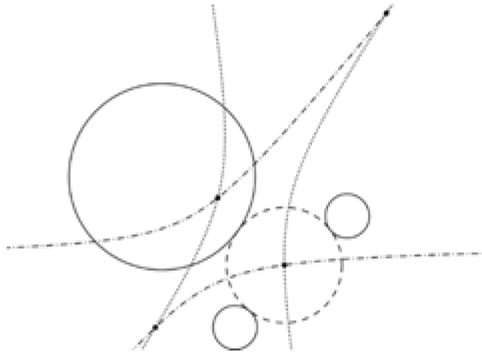

Fig. 4. Centres of the circles tangent to three given circles.

Furthermore, Georges Salmon's textbook *A treatise on conic sections* published in London in 1855 (Salmon, 1855) exposed new methods in analytic geometry together with a wide range of problems. Textbooks dedicated to problem solving and modern methods were written by professors Louis Painvin (Painvin, 1866) and Eugène Jubé (Jubé, 1866), graduated from the *Agrégation* and professors Emile Boquel (Boquel, 1872), Henri Koehler (Koehler, 1886) and Claude Rémond (Rémond, 1887), graduated from the *École polytechnique*.

Conclusion

After 1865, the place of the problems in elementary and analytic geometry textbooks is standardised. They appear in lists at the end of the successive sections of the textbooks, their solution is not exposed, and they are very numerous.

Our study of textbooks shows how during the 19th century problem solving becomes an essential part of geometry teaching in France. The usage of a new word, "exercise", renders the conceptions at work: the study of geometry shall include moments of exercise solving. Consequently, after 1865, almost all textbooks have exercise lists at the end of their successive sections.

The declared intentions of the authors that include problems in their textbooks were of different types. They were academic for the preparation of the *Concours général*, the *Baccalauréat* or the *Grandes Écoles* examinations. They were pedagogic when it comes to enlighten a theory with problem solving. They were epistemological when they consider problem solving as an essential part of mathematic activity. They were professional concerning the training of future qualified workers.

Finally, problems have a strong interaction with considerations about methods.

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The problem section of *El Progreso Matemático*

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Abstract

Many journals devoted to Mathematics appeared all over the world during XIX century. In countries where research was not an important activity, these journals contributed to introduce and disseminate new mathematical ideas and to improve the mathematical education of the society. This was the case of El Progreso Matemático, which was published in Spain during the periods 1891-1895 and 1899-1900. It was the first Spanish journal solely devoted to Mathematics and, among its sections; it included starting from the 7th issue a problem section where problems were proposed to the readers who then submitted their solutions. In this work, we will give an overview of this problem section focusing mainly on the proposers and the topics covered by the problems and, to a lesser extent, on the solvers. As a case study, we will present three problems involving the Fibonacci sequence and we will try to understand why the proposer sent them.

Introduction

During XIX century "the number of journals dedicated to Mathematics grew impressively" (Furinghetti, 2014, p. 545). They responded to the need to communicate and disseminate research results and new mathematical ideas. Nevertheless, in some peripheral countries, where mathematical research was not a very important activity, periodicals were important in order to introduce new ideas and to improve the mathematical education of the society (Alpaslan, Schubring & Günergun, 2015).

On January 20, 1891 the first Spanish journal solely devoted to Mathematics appeared. It was called *El Progreso Matemático*. In this paper, we focus on the problem section of this journal. In particular, we want to present a brief overview of this section focusing mainly on the proposers and the topics covered by the problems and, to a lesser extent, on the solvers.

The structure of the paper is as follows. In the next section, we provide some historical and mathematical context regarding the XIX century in Spain. After that, we will present the founder of *El Progreso Matemático*, the mathematician Zoel Gar-

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

cía de Galdeano and will give some general remarks about the journal. Then, section 4 will be devoted to present the main facts about the problem section and in section 5; we will approach a case study involving some problems about the Fibonacci sequence. Finally, some conclusions and final remarks will be presented.

Spain during XIX century: Historical and mathematical context

El Progreso Matemático was published during the last decade of the XIX century. It might be interesting to provide some context, both historical and mathematical, in order to understand the situation of the country at that time.

The history of Spain during XIX century was rather agitated. By no means can we aspire to summarize this important period in a few lines, but at least we can present some facts to illustrate this convulsion:

- The century began with a quite long war on independence against France (1808-1813). This war was very destructive both materially and ideologically.
- There were also wars related to inheritance rights, the Carlist wars (1833-1840 and 1846-1849) and territorial wars against Morocco (1859-1860) and USA (1898).
- The Spanish Empire, if such still existed at this point, disappeared during this century. The independence of the American colonies began while the Independence war was happening in Spain and ended with the independence of Cuba and Puerto Rico in 1898. Philippines were ceded to the USA in 1899.
- Up to six constitutions were enacted during this period: 1812, 1834, 1837, 1845, 1869 and 1878.
- Very different types of governments took place. From absolute monarchy to republic passing through a provisional government and closing the century with the Bourbon restoration.

This political and social instability had an impact on education and, in particular, on mathematics education. There were 25 mathematical syllabi in less than 75 years (Ausejo & Matos, 2014, p. 286) and many different educational regulations appeared. The most important was the so-called Moyano Law from 1857. This law organized education at all levels, from primary education to university, and it was the basis of the Spanish education system for over 100 years.

At the beginning of the century, mathematical activity was mainly concentrated in military academies (Peralta, 2009). Nevertheless, it progressively spread during the century to more academic forums. Several engineering schools appeared on the second third of the century and the Royal Academy of Exact, Physical and Natural Sciences was founded on 1847. The Faculty of Science, with a section devoted to exact sciences, was created at Madrid University in 1857.

In any case, very little original research was carried out during this period. According to Sánchez Ron (1992, p. 58) "the area where Spanish physicists and mathematicians of that century [XIX century] moved was, with very few exceptions, that of teaching[1]". Therefore, we find a huge number of mathematics textbooks published all over this century, both original and translated (mainly from French authors like Lacroix, Francoeur, Cirodde, Rouché & Comberousse, Briot, Serret, etc.).

We must also remark the figure of the sowers. With this term, attributed to Gino Loria (Ríos, 1988), we refer to four mathematicians: José de Echegaray, Zoel García de Galdeano, Eduardo Torroja and Ventura Reyes Prósper. They were committed in the importation and dissemination of "modern" Mathematics into Spain and they were also involved in curricular and pedagogical reflections (Ausejo, 2008).

Zoel García de Galdeano and *El Progreso Matemático*

This section is devoted to briefly present the figure of Zoel García de Galdeano, already mentioned before and the journal that he founded on 1891. There are several works devoted to present various aspects of the life and works of García de Galdeano. We cite, for instance the papers by Ausejo (2010), Hormigón (1983; 1991; 2004) or Rodríguez (1964).

Fig. 1. Portrait of García de Galdeano from the Spanish National Library

[1] "El ámbito en el que se movieron los físicos y matemáticos españoles de aquella centuria fue, con muy pocas excepciones, el de la enseñanza" (Translation by the autor).

García de Galdeano was born in Pamplona in 1846. Orphan of an army officer, he had a very complete education. He graduated as agricultural engineer, he was teacher of upper elementary education and he earned a master's degree in humanities and later (1871) in mathematics. He taught mathematics at the high schools of Ciudad Real, Almería and Toledo. He finally held a professorship at the Faculty of Sciences of Zaragoza, first of analytic geometry (1889-1896) and later of mathematical analysis until his retirement in 1918.

This background helps to understand the wide view of mathematics that García de Galdeano had. As an engineer, he was aware of its practical importance. As a teacher at many levels, from elementary school to university, he was acquainted with the difficulties of learning and teaching mathematics. As a professor, he was concerned about the necessity to introduce new ideas in Spain and to develop original research.

García de Galdeano was very active, publishing several papers on both mathematics and its teaching. He also played a main role in the rising Spanish mathematical community. He was, for instance, president of the Spanish Royal Mathematical Society and of the Academy of Sciences of Zaragoza.

It is also worth mentioning that García de Galdeano was also very active at an international level, in particular according to the Spanish standards of that time. He attended the first ICM in Zürich (1897) were he presented the communication 'L'unification des concepts mathématiques'. He also attended the International Congress on the Bibliography of Mathematical Sciences (Paris, 1899), where he became a member of the Comission Permanente du Repértoire. He also was a member of the Comité de Patronage of *L'Enseignement Mathématique* when it was founded in 1899. In the first issue of this journal, García de Galdeano published the 16 pages paper 'Les mathématiques en Espagne' (Furinghetti, 2003).

Fig. 2. Top of the front page of the first issue of *El Progreso Matemático*.

Here we are interested on the role of García de Galdeano as founder, editor and director of the journal *El Progreso Matemático* (Hormigón, 1981). On January 20, 1891 the first issue of this journal was published. It was published in two periods, first from 1891 to 1895 and later from 1899 to 1900, for a total amount of 73 issues. This was the first journal solely devoted to mathematics appearing in Spain since, in words of

García de Galdeano, "it is a striking fact that in Spain, where many journals intended for various purposes are published, there is none whose sole purpose is the dissemination and the development of the mathematical sciences[2]".

Hence, the main goal of García de Gadeano was to introduce and disseminate "new" mathematical ideas into Spain. He was also interested in exchanging this publication with foreign journals in order to create an international mathematical library (that still exists today). To a lesser extent, the journal was also intended to publish some original works.

The wide view of mathematics and the aim at internationalization are clearly demonstrated if we have a look at the list of official collaborators of the journal. We find representatives from the army, from the university or from engineering schools and high schools. Moreover, these collaborators came from seven different countries: Spain, France, Portugal, Italy, Belgium, Germany and Russia.

The general structure of the journal was organized around four sections. The main section was devoted to present articles and memoirs of mathematical content. There was also a bibliographical section were reviews of books and journals were published. A third section was devoted to publish articles about philosophy, pedagogy and history of mathematics and, finally, there was a varieties section with miscellaneous content.

The problem section: An overview

As we have already mentioned, *El Progreso Matemático* included a final section with a diverse content. In the 7th issue of the journal (July 20, 1891), García de Galdeano transcribed two problems taken from *Journal des Mathématiques Elémentaires* while he was reviewing its last issue. Nevertheless, there was no apparent intention of continuity on his action.

However, in the next issue (August 20, 1891), the editor apparently changed his mind and the problem section of the journal was officially born. García de Galdeano, in the presentation of the section, points out the interest of "not neglecting other points of view. One of these is the originated by unsolved problems […], as an incentive to the action of the readers[3]".

Hence, this section seemed to be important in order to engage the readers and to promote their active participation in the journal. This importance is even clearer

[2] "Es un hecho sorprendente que en España, donde tantos periódicos se publican, destinados a los fines más diversos, no exista uno cuyo objeto exclusivo sea la propaganda y desenvolvimiento de las ciencias matemáticas". *El Progreso Matemático*, 1, p. 1 (Translation by the autor).

[3] "No descuidar otros puntos de vista. Uno de estos es el que originan las cuestiones por resolver […] como aliciente que ponga en juego la actividad de los lectores". *El Progreso Matemático*, 8, p. 205 (Translation by the autor).

if we observe that, from that moment on; only three issues of the journal did not contain problems or solutions.

During the first issues, the problems that appeared in the section were mostly borrowed from the journal *Nouvelle Correspondence Mathématique*. There were also problems taken from other publications, like *Journal des Mathématiques Elémentaires* or *Nouvelles Annales de Mathématiques*. Progressively, we find problems proposed specifically for *El Progreso Matemático*. Finally, in the 13th issue (January 15, 1892), we find the first problem proposed by a Spanish collaborator, J.J. Durán Loriga, and the first solution was published (written by Ángel Bozal, a student from Zaragoza).

Table 1. List of proposers of the problems appearing in the section

NAME	COUNTRY	NUMBER OF PROBLEMS
Van Aubel	Belgium	74
Brocard	France	42
Neuberg	Luxembourg	40
Lemoine	France	35
Gillet	France	23
Retali	Italy	20
Durán Loriga	Spain	20
Laisant	France	14
Catalan	Belgium	9
Sollertinsky	Russia	9
Pirondini	Italy	7
Cesàro	Italy	4
Fontené	France	4
Schiappa Monteiro	Portugal	4
Guimaraes	Portugal	3
Dmtrieff	-	2
Droz-Farny	Switzerland	2
Wolstenholme	England	2
Meyer	Germany	2
Aubry	France	1
Gob	-	1
Chevallier	France	1
Edmunds	-	1
Quetelet	Belgium	1
D'Avillez	-	1
Ripert	France	1
D'Ocagne	France	1
Rankine	Scotland	1
Nicolaides	-	1
Breton	France	1
Riemann	Germany	1
Vecchio	Italy	1
Zahradnik	Czech Republic	1
Luzón de las Cuevas	Spain	1
Jiménez Rueda	Spain	1
Torroja	Spain	1
De Alba	Spain	1

During the existence of the journal, 351 problems were proposed: 260 in the period 1891-1895 and 91 in the period 1899-1900. From these problems, 97 remained unsolved when the journal disappeared (53 from the first series and 44 from the second). Thus, we see that the second period was less active in the number of solutions received. Only a 20% of problems from the first series remained unsolved while this percentage increased to a 48% in the second series. This fact possibly illustrates some of the reasons that ultimately led to the disappearance of the journal.

In some cases, more than one solution to a problem was published. From the 254 solved problems, 89 were solved more than once. Four problems received up to 6 different solutions and all of them were published.

Foreign collaborators proposed most of the problems that appeared in *El Progreso Matemático*. In fact, only five Spanish mathematicians submitted problems: J.J. Durán Loriga, J. Luzón de las Cuevas, C. Jiménez Rueda, E. Torroja and L. de Alba. Moreover, only Durán Loriga can be described as an active collaborator, with the proposal of 20 problems. The other four mentioned mathematicians proposed only one problem each.

Among these foreign proposers, the most active were H. Van Aubel (74 problems), H. Brocard (42 problems), J. Neuberg (40 problems), E. Lemoine (35 problems), J. Gillet (23 problems) and V. Retali (20 problems). We find more than 30 different proposers, with names like C.A. Laisant, E.C. Catalan, E. Cesàro, G. Pirondini, etc. In Table 1, we present the full list. If possible, we have provided the country (according to current geographical divisions) where each proposer was born. In some cases, we have not been able to find information about the authors of the problems.

We must point out that some of the names that appear in Table 1 do not correspond to "actual" proposers. Riemann, for instance, died in 1866 and Quetelet in 1874. Those cases correspond to problems transcribed from other journals and attributed to the original proposer.

We can also find problems that appeared without being attributed to any individual proposer. One of them was attributed to Leonardo de Pisa (Fibonacci) in the solution, while in some cases García de Galdeano claimed that they were problems proposed at the Concours Académique at Grenoble (1878) or Lille (1874-1877).

Table 2. Topics covered in the section

TOPIC	NUMBER OF PROBLEMS
Geometry	308
Number theory and combinatorics	22
Algebra	10
Mathematical analysis	6
Differential equations	3
"Real life" problems	2

Regarding the content of the problems, the vast majority involved geometry. Nevertheless, they covered a rather wide range of topics, from number theory to differential equations (Oller, 2012). Table 2 shows the list of topics and the number of problems for each of them.

This list of topics clearly illustrates the likes of that time. The problems of geometry included conics, constructions using straightedge and compass, geometric loci, or the very popular at the time geometry of the triangle. Geometry problems mostly dealt with plane geometry. In fact, we find only 25 problems involving spatial geometry.

We have pointed out that there were 37 different collaborators that proposed problems for the section. If we turn to problem solvers, the number increases to 42. It is interesting to point out that only 15 problem proposers also submitted solutions. Moreover, the most active proposer (Van Aubel) only submitted one solution and that it was to solution to a problem proposed by himself. This implies that we can identify three clearly different profiles among the collaborators of this section: collaborators that mainly proposed problems, collaborators that mainly solved problems and collaborators that both proposed and solved problems.

Hence, Van Aubel is a clear example of the first profile, proposing 74 problems and solving only 1. On the other side, we find Ripert, who proposed only 1 problem but submitted solutions for 9 different problems. Finally, an example of collaborators that both proposed and solved problems could be Brocard, who submitted 42 problems and 55 solutions.

Table 3. List of collaborators in the section

NAME	PROPOSED PROBLEMS	SOLUTIONS SUBMITTED
Van Aubel	74	1
Brocard	42	55
Lemoine	35	8
Gillet	23	2
Retali	20	52
Durán Loriga	20	2
Sollertinsky	9	81
Pirondini	7	1
Fontené	4	1
Schiappa Monteiro	4	33
Guimaraes	3	2
Droz-Farny	2	21
Ripert	1	9
Jiménez Rueda	1	16
De Alba	1	1

In table 3, we present the number of proposed problems and solutions submitted for those collaborators working in both directions. The three different profiles are clearly identified.

A case study: The Fibonacci sequence

The small number of problems devoted to topics different from Geometry, allows us to have a closer look at them. Here, we will focus on some problems about number theory and, in particular, on the only three problems that involved the well-known Fibonacci sequence. The same person (Charles Ange Laisant) in two consecutive issues of the journal (October 1899 and January 1900) proposed the three problems. See Table 4.

Laisant was born in Basse-Indre in 1841 and had an agitated political life that took him from a radical republicanism to anarchism before his death in 1920 (Lamandé, 2011). He was president of the French Mathematical Society (1888) and in 1894; he founded the Journal *L'Intermédiaire des Mathématiciens* together with Lemoine. He was also the founder in 1899, with Henri Fehr, of *L'Enseignement Mathématique*. This activism and statements like "measure a science by its utility is nearly an intellectual crime[4]" (Laisant, 1907, p. 121) clearly show his concern about the teaching and the popularization science.

Table 4. Statements of the problems involving the Fibonacci sequence

286	In the Fibonacci sequence. $u_0 = 0, u_1 = 1, u_2 = 1, u_3 = 2 \quad \ldots\ldots$ All the terms u_{16n} are multiples of 987.
296	Given the Fibonacci sequence. $u_0 = 0, u_1 = 1, u_2 = 1, u_3 = 2 \quad \ldots\ldots$ A new series is considered: $u_0, u_1x, u_2x^2, \ldots, u_nx^n, \quad \ldots\ldots$ and it is sought the condition for it to be convergent. If x is equal to 1/k, with k an integer, find the sum of the series.
297	Given two integers u_0, u_1 with the same last digit a, we form the sequence $u_0, u_1, u_2, \ldots, u_n, \quad \ldots\ldots$ Defined by the relation $u_n = u_{n+1} + u_{n+2}$. We want to find the last digit of a term u_n for a given n.

As a mathematician, Laisant was not specially brilliant or original, but he was quite prolific (Lamandé, 2011, pp. 286-287). His first publications date from 1867, ten years before he obtained his doctorate. He published several papers in journals like *Nouvelles Annales de Mathématiques* (21 papers), *Bulletin de la Société mathématique de France* (29 papers) or *Nouvelle correspondance mathématique* (16 papers) and 3 books.

Among the works by Laisant it is possible to find out papers that show some interest on recurrence sequences in general and on the Fibonacci sequence in particular. These works are:

- 'Sur les séries récurrentes dans leurs rapports avec les équations' (Laisant, 1881).

[4] "Mesurer une science à son utilité est presque un crime intellectuel" (Translation by the author).

- 'Les deux suites Fibonacciennes fondamentales $(u_n)(v_n)$' (Laisant, 1920).
- 'Observations sur les triangles rectangles en nombres entiers et les suites de Fibonacci' (Laisant, 1919).

Moreover, the classic book *Théorie des nombres* by Édouard Lucas was published on 1891. In that book some questions related to these problems are treated and Laisant may have been acquainted with it.

Hence, the motivation to propose these problems seems quite clear. We are going to have a closer look at problem 286 as an example. The 1920 paper 'Les deux suites Fibonacciennes fondamentales $(u_n)(v_n)$' presents a table with the values of u_n and $v_n=u_{2n}/u_n$ from $n=1$ to 120, together with some conjectures arising from it. For instance, we read (p. 53)[5]: "We can easily observe that u_{5n} is always a multiple of 5, that v_n is never a multiple of 5, that u_{12n} is a multiple of 9 and that the same holds for v_{12n+6}"

The statement of Problem 286 is very similar and is supported by the values given in Laisant's table $u_{16} = 987$, $u_{32} = 2178309 = 2207 \times 987$, etc. Of course, this paper was published 21 years after the appearance of problem 286 but since 1920 was the year of the death of Laisant, we may assume that he already had these ideas in mind. In any case, we have been able to trace the problem 286 to Laisant's work.

Concluding remarks and further work

In this work, we have tried to give an overview of the problem section of *El Progreso Matemático*. We think that this section of the journal points out how García de Galdeano wanted to stimulate the mathematical activity among its readers and, consequently, among the members of the Spanish mathematical community of his time. It also provides, in our opinion, a good example of his concern on internationalization. This is especially clear if we look at the nationality of the most active collaborators in the section.

The problems that were submitted to the section and those that were (o were not) solved also give information about the topics that were most popular. In particular, Geometry was mainstream (nearly 88% of the problems) and the other topics covered in the problem section were nearly anecdotal.

We have been able to identify three profiles among the collaborators of the problem section, depending on whether they mainly submitted problems, solutions or both.

[5] "On peut facilement constater que u_{5n} est toujours un multiple de 5, que v_n n'est jamais un multiple de 5, que u_{12n} est un multiple de 9 et qu'il en est de même pour v_{12n+6}" (Translation by the author).

Finally, we have showed with some examples that the problems can provide information about the interests and motivations of the proposers. Sometimes the statements can be traced back to the proposer's own mathematical work.

Some ideas for further work may include the following:

- It would be interesting to compare this journal with other journals published in Europe during the same period. Some research questions related to this point are being investigated in the project CIRMATH (cirmath.hypotheses.org).
- We could try to study the possible influence of this journal over other mathematical journals that appeared in Spain after the disappearance of *El Progreso Matemático*. In particular, it seems interesting to find out what happened to the unsolved questions. Were they published somewhere else?
- Most of the collaborators that mainly proposed problems were rather known names, but this is not the case for those that only submitted their answers. It might be interesting to focus on these collaborators and try to identify them.

The author belongs to the research group "S119-Investigación en Educación Matemática", partially funded by Aragón Government and the European Social Fund.

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The teaching of mathematics in the Italian artillery schools in the eighteenth century

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Abstract

The artillery schools were established in concomitance with the European wars which followed each other in the first six decades of the XVIII century. In most European states, this period of conflicts corresponded with greater interest in the training of troops and, in particular, with new attention placed on the training of officers of the so-called "armi dotte".

Developments and improvements in military technology derived from the need to form new cadres who were able both to exploit the morphological features of the landscape through a knowledge of topography and plan works of fortification and make best use of artillery. In order to acquire this knowledge a thorough grounding in mathematics was required. Three important Italian military schools were established in this period: in the Kingdom of Sardinia, the Regie Scuole teoriche e pratiche di Artiglieria e Fortificazione of Turin (1739), in the Kingdom of Naples, the Real Accademia or Scuola Matematica (1745) and, in the Republic of Venice, the Militar Collegio of Verona (1759). The management of the military schools was entrusted to engineers and mathematicians: Ignazio Bertola and Alessandro Papacino D'Antoni in Turin, Nicola De Martino and Vito Caravelli in Naples, Anton Maria Lorgna and Leonardo Salimbeni in Verona.

Introduction

Those who chose a military career had always been taught mathematics. For some time, however, in-depth studies of geometry and mechanics useful for military applications were obtained by means of personal study of the great treatises of mathematics like the *Cursus mathematicus* (Paris, 1634-1637) by Pierre Hérigone (1580-1643), and the *Cursus seu mundus mathematicus* (Lyons, 1674) by Claude François Milliet Dechales (1621-1678). Since the formation of the ruling class was in the hands of the ecclesiastical colleges in Catholic Europe, particularly the Jesuit ones, it is here that we must look for the first institutionalized forms of mathematical teaching destined to the future officers. Only from the beginning of the

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

XVII century did the institutions devoted to the formation of military cadres begin to appear.[1]

Between the end of the XVII and the beginning of the XVIII centuries, artillery and engineers corps were inserted into the most important European armies, whose militarization, after several decades, led to the emergence of a markedly scientific type of military school. Moreover, the publication, in 1737, of *De l'attaque et de la défense des places*, written by Marquis Sèbastien Le Prestre de Vauban (1633-1707), on the new techniques in the construction of fortifications, made clear the need to create schools of formation for these technicians which belonged to the army itself in the form of a new corps with special expertise, instead of using experts coming from a civil engineering background.

The introduction of the eighteenth century technical-military schools is also to be found in the riding and fencing schools which had mushroomed in the last decades of the sixteenth century in Europe, particularly in France, the Rhenish region and in some German-speaking countries which adopted the military reforms activated in the Low Countries by Maurizio di Nassau, Prince of Orange (1567-1625), President of the State Council of the future Republic of the United Provinces (1584) and Captain General of the Dutch Army (1588-1625) (Bianchi, 2011, pp. 150-151).

In the first six decades of the XVIII century, Europe was devastated by three great continental wars and the Seven Year War. As Winston Churchill wrote, the latter could be considered the "first world war" in history, as it extended as far as the colonies. In order to deal with the demands of the situation, the most important institutes for the formation of officers, particularly the engineers and artillery corps, were set up in less than thirty years. The military schools in various European countries were organized in accordance with the first experiments of French artillery schools (such as Douai, Metz, Strasbourg):

- 1731, France: École de Artillerie;
- 1739, Kingdom of Sardinia: Regie Scuole Militari teoriche e pratiche di Artiglieria e Fortificazione;
- 1741, Woolwich (England): Royal Military Academy;
- 1745, Kingdom of Naples: Real Accademia or Scuola Matematica;
- 1748, France: École Royal du Génie de Mézières;
- 1751, Spain: Escuela de Mathematica con el titulo de Artilleria;
- 1752, Neustadt (Austria): Militärakademie von Maria Theresia;
- 1759, Venetian Republic: Militar Collegio of Verona.

In Italy, management of the military institutes was always entrusted to engineers and mathematicians. In this field, the military institutes present in the Kingdom

[1] The results demonstrated in this article derive from wider studies on the teaching of mathematics in the military schools in eighteenth century Italy, made possible thanks to a grant awarded by the Fondazione Filippo Burzio of Turin in 2014.

of Sardinia, the Kingdom of Naples and the Republic of Venice were known for their excellence.

This article proposes to examine how these three military institutes were organized and how mathematics contributed to their development; such schools are still today the main schools of the Italian army, that is: *Scuola di Applicazione dell'Esercito* in Turin; *Scuola Militare "Nunziatella"* in Naples, and *Accademia Militare* in Modena. The role of mathematics in the curriculum will be examined as it was often more extensive than the teaching of mathematics in university courses of those days. Indeed, differential and integral calculus was introduced into these schools practically from the beginning as it was considered a refined tool with which to solve mechanical problems linked to war.

School of Artillery and Fortification of Turin

In the Duchy of Savoy there were plans for a school of mathematics for the formation of engineers and artillerymen as early as the time of Vittorio Amedeo I (1587-1637). Studies included: arithmetic, algebra, Euclidean geometry, trigonometry, longimetry, planimetry, stereometry, fortification, mechanics and practical applications.[2]

The formation of officer cadres of the Duchy (and then of the Kingdom of Sardinia) began to appear in the period of peace which followed the Treaty of Utrecht of 1713. With Vittorio Amedeo II (1666-1732), the artillery was militarized and the Cannon Battalion was set up. On 20th December, 1726, the *Regolamento di ciò che si deve insegnare nelle scuole per la teorica pratica dei cannonieri, bombisti e minatori* was issued. The introduction of this regulation meant that access to positions of command was not achieved only by royal concession, but candidates were required to demonstrate intellectual and scientific skills. The following year, military engineers were also admitted to the battalion. However, the term of artillery cadets does not officially appear until 1739 in the regulation of the *Scuole Militari teoriche e pratiche di Artiglieria e Fortificazione*.

Although that year is generally considered the one in which these schools started to exist, it may be conjectured from the following title *Corso d'Aritmetica Dettato nell'Accademia d'Artiglieria Dal Luogotenente V.A.C. Sinser Direttore della medema* [sic] *l'anno 1736* that for at least three years before they had been called *Accademia d'Artiglieria*.[3] Sinser (or Sincer) was the nickname given to Vittorio Amedeo Conti, who commanded the House of Savoy artillery in the siege of Cuneo in 1744. Sinser was the author of the *Trattato di Geometria Pratica Geodesia, Transfigurazione de piani*

[2] Montù (1934) reports in the *Memoriale* transcribed by Leschi (1994), vol. 1, p. 112.
[3] Biblioteca Reale of Turin, ms. Saluzzo 569.

uso degl'Instromenti Matematici, Trigonometria, e Mecanica Esposti nell'Academia d'Artig.ª Dal Ten.te V.A.C. Sinser Direttore della medema L'anno 1737.[4] The position of Director that the author attributes to himself in the manuscript has not been confirmed through any other documentation. The academy may have been a school within the battalion, or the manuscripts may have been prepared with a view to the Artillery School that Ignazio Bertola (1676-1755) was planning to set up (Comba & Sereno, 2002, vol. 2, pp. 36-39).

The Polish War of Succession (1733-1735) had, in fact, brought to light the need for a school for the formation of officers for artillery and engineering. The king's first engineer, Giuseppe Francesco Ignazio Bertola, proposed a military school to form a corps of engineers competent in artillery matters when he presented a project as early as 1736.[5] In that year the construction of a new arsenal was begun to guarantee the independence of the State of Savoy through the supply of arms and technically advanced cannons. On 16th April, 1739, Charles Emanuel III established that the entire Artillery Battalion (composed of eight companies of artillerymen, one of bombardiers, one of miners, one of diggers and another of workers) should be concentrated in Turin, and he further set up within the battalion the *Regie Scuole Militari teoriche e pratiche di Artiglieria e Fortificazione*, whose activity was to last about eighty years. These schools were under the supervision of the *Gran Maestro di Artiglieria* and were divided into two schools: *Scuola Teorica* and *Scuola Pratica* (Carasso & Gaidano, 1939, p. 15).

The *Scuola Teorica*, managed by a general director, was divided into a general school and six specialization schools. They were entrusted to a teacher of mathematics and two supply teachers, and the subjects studied were mathematics, artillery and fortification; in the first five specializations the subjects studied were: technical skills for artillerymen, bombardiers, miners, diggers and workmen; in the sixth: design of figures, architecture and topography. In the *Scuola Pratica*, under the supervision of the Commanding Colonel of the Artillery Battalion, cadets were taught how to fire cannons, throw bombs, build batteries and bridges. Lessons in the artillery schools took place in Italian, and, initially, the courses probably lasted seven years. The first general director was Count Bertola himself. The teaching staff included Professor Antonio Banzes, teacher of mathematics and theoretical artillery; the two engineers, Francesco Domenico Michelotti and Carlo Andrea Rana,[6] supply teachers of mathematics and theoretical artillery; Captains Persol

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[4] Biblioteca Reale of Turin, ms. Saluzzo 571.

[5] Giuseppe Francesco Ignazio Bertola, Antonio Bertola's son to his second wife, was a teacher of mathematics at the *Reale Accademia*. He worked alongside his father in the siege of Turin (1706) in the planning of military architecture. As a teacher he wrote a manuscript for the students of the Academy entitled: *Li primi sei libri della Geometria d'Euclide esposti da Giuseppe Ignazio Bertola Professore delle Matematiche Nella Reale Accademia di Torino Consecrati all'Altezza Reale di Carlo Emanuele Principe di Piemonte* (Folio, ff. 156, Torino, 1717) preserved in the Biblioteca Nazionale of Turin.

[6] Carlo Andrea Rana (1715-1804) was both civil and military architect (nominated Royal Architect by the House of Savoy) as well as a topographer. He taught in the artillery schools for forty-one

and Sinser, officers for the teaching of theory; Captains Dansigny and Dulach (from 1784 Captain D'Antoni) officers for the teaching of practical training; Captain Bozzolino and Lieutenant Tignola, supply officers; Captain Blavet, teacher of field activity; teacher of design Filippo Brambilla and supply teacher, Gioacchino Brambilla; secretary to the general director, responsible for supplying books and instruments, Giuseppe Gaetano Oselletti (Leschi, 1994, vol. 1, pp. 116-117).

Following the death of General Bertola in 1755, a new sovereign act was issued transforming the schools of theory and practical training into formative institutes for cadets, and sanctioning their division. For each one a director was nominated under the direct supervision of the *Gran Maestro di Artiglieria*. The school for theory was entrusted to Alessandro Vittorio Papacino D'Antoni (1714-1786),[7] while Count Ludovico Birago di Borgaro was put in charge of the school for practical training. With the reform of 1775, the study course was divided into three periods lasting seven years: in the first two periods, the teaching subjects, taught over five years, were common to both students of artillery and engineering; in the third period the subjects were specific for the professional specialization. In the first period, pure mathematics was studied: arithmetic, literal equations, Euclidean geometry, geodesy, plane trigonometry, solids, conical sections, and stereometry; in the second period mixed mathematics was studied: speculative mechanics with the study of ballistics, statics, hydrostatics, aerometry and general principles of hydraulics. The lessons on theory took place in the morning, whereas practical training was carried out in the afternoon.

On 26th September 1755, Carlo Andrea Rana, a teacher of mathematics with Bertola, worked alongside a brilliant young scholar, Giuseppe Luigi Lagrange (1736-1813), who was the son of Giuseppe Francesco Lodovico, Treasurer of the Office of Public Works and Fortifications in Turin. The young assistant probably held this position until 1759, and wrote, for these students, two manuscripts which marked a turning point in teaching towards differential and integral calculus, which had not previously been introduced into university teaching courses of Turin. The first manuscript was *Meccanica*, a subject taught by Lagrange between 1758 and 1759.[8] Unfortunately there is no trace of this manuscript. The second one, instead, a copy of which is preserved in Cesare Saluzzo collection in the Biblioteca Reale of Turin, is entitled *Principj di Analisi sublime*

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years and he wrote several handbooks with Captain Bozzolino and D'Antoni. These books were translated into French, German and English. Some manuscripts are preserved in the Biblioteca Reale of Turin: *Dell''architettura militare per le Regie Scuole d'Artiglieria, e Fortificazione* (1756-1759), (Brayda, 1939).

[7] D'Antoni, born in Villefranche-sur-Mer on 2th May, 1714, from a family with a military background, in 1731, as a voluntary soldier, entered the artillery of the King of Sardinia, taking part in various military campaigns during the Polish War of Succession (Balbo, 1805); reference of *Dizionario Biografico degli italiani* by Paola Bianchi.

[8] In Borgato & Pepe (1987), p. 28, clearly stated by the testimony of Prospero Balbo in *Vita del Commendatore d'Antonj* (1805) Lagrange taught Mechanics in the Artillery Schools in 1758 and 1759.

dettati da La Grange alle Reggie Scuole di Artiglieria, and was devoted to analytical geometry and differential calculus.[9]

In 1764, the name of the two schools was changed to *Scuole Teoriche Militari d'Artiglieria, e Fortificazione* and *Scuole per la pratica*. The most important novelty was the decision to make analytical geometry a compulsory subject. In 1765, the general management of the artillery schools was entrusted to D'Antoni; with him were Giangiuseppe Francesco Blavetti (director of practical training) and Ignazio Andrea Bozzolino (director of theory).[10] D'Antoni's nomination was reconfirmed in 1769, a position he held until his death in 1786. The following reform (1777) set out that the students of the school had to have a rudimental knowledge of mathematics (arithmetic, plane and solid geometry, the rules of literal calculus) in order to be admitted to the institute.

These schools lasted right up to the arrival of the French troops. The first attacks, begun in 1792, by the French fleets on the Sardinian coast had the effect of speeding up the courses of both the theory and practical schools of artillery and fortification which had been activated in 1787 in order to bring them rapidly to a conclusion. The course initiated in 1793, instead, was carried out in alternate phases to be compatible with the military campaigns; the students were able to attend the school only during the winter as they were called to join the war during that summer (Vichi & Zambrano, 1993, p. 13). In the summer of 1798, the French Directory had decided the occupation of Piedmont was completed, an indispensable bastion for the army fighting in Italy which was under threat from a new Austrian-Russian offensive. Charles Emanuel IV was therefore obliged to abdicate on 8th December and flee to Cagliari the next day (he arrived on 3rd March, 1799).

Artillery Schools in the Kingdom of Naples

The first artillery school in the Kingdom of Naples was the *Real Accademia* or *Scuola Matematica* established by Charles of Bourbon on 10th September, 1745, who, in his decree, declared that it was necessary to train the military forces in mathematics, a science on which military successes mainly depended.[11]

Organisation of the new school was entrusted to Nicola De (Di) Martino (1701-1769), who, at that time, was working in Spain as secretary to the Embassy.

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[9] Published in Borgato & Pepe (1987), pp. 45-198.

[10] Bozzolino wrote *Dell'architettura militare per le regie scuole teoriche d'artiglieria e fortificazione. Libro secondo in cui si tratta dell'attacco e della difesa delle piazze regolari* (Torino, Stamperia Reale, 1779). The volume is the second of the work in six volumes *Dell'architettura militare per le Regie scuole teoriche d'artiglieria e fortificazione* by Alessandro Vittorio Papacino d'Antoni.

[11] «dalla quale scienza principalmente dipendono i più felici successi delle operazioni della Guerra» (Castronuovo, 1970, p. 6).

The new school, located inside the Panatica Palace in Saint Lucia, had to prepare not only the students of artillery but also those of engineering. The professional formation of the cadres had to include attendance at both theory, which was mainly introductory lessons, and practical training. The two-year study course, was compulsory for officers and artillery cadets, but was also open to other members of the army and the nobility who had passed an exam which was proof of their rudimentary grounding enabling them to follow the lessons. Theory in mathematics, physics, design and fencing took place within the main building of the institute, whereas practical training was held in Molesiglio, the shipyard and the small fort of Vigliena. The former course was held from November to June, July being kept for holidays, and then practical training was carried out in August, September and October. First year subjects included: arithmetic, practical geometry (including plane trigonometry), mechanics, plane and solid geometry (including spheres, cylinders, and conical sections) and analytical geometry; in the second year: regular and irregular fortification, attack and defense of strongholds, theory and practical training in artillery, statics and hydrostatics. Nicola De Martino published three texts for these students: *Nuovi elementi di geometria piana* (Napoli, Palombo, 1746);[12] *Nuovi elementi della geometria pratica* (Napoli, Di Simone, 1752); *Trattato dell'equilibrio e del moto dei corpi* (Napoli, Di Simone, 1753).[13]

In 1754, a *Scuola speciale* or *Accademia del Corpo degli Ingegneri militari* was set up for the formation of the corps of Military Engineering officers. From 1759, alongside the two academies there was the *Real Brigada dei Cadetti di Artiglieria* which most likely represented the organization of the students of the two schools. Although little information on this company is available, it is likely that its study courses were very similar to those of the two academies already under way (Leschi, 1994, vol. 1, p. 483). In 1769, given the similarity between the studies of mathematics and theoretical and applied sciences, the two academies were joined to form the *Reale Accademia Militare*, with official headquarters in the Panatica Palace. The running of the Academy was entrusted to Brigadier Luca Ricci, while Vito Caravelli (1724-1800), professor of mathematics at the *Accademia di Marina*,[14] was nominated director of sciences. The institute was organised into four-year courses, the school year being divided into two periods, beginning on 5th November and finishing 4th July. Teaching was organised as follows: in the first term of the first year there was arithmetic and plane geometry; in the successive term elements of

[12] In accordance with what is reported by Caravelli in his *Elementi di Aritmetica* (1771), De Martino must have published, in 1748, a volume of *Aritmetica* which cannot be traced nowadays (Gatto, 2010, p. 89).

[13] The publication of a second volume of this work on mechanics was interrupted due to a mistake. The second part was then published in 1781 with the *Trattato dell'equilibrio e del moto dei corpi composto da Niccolò Di Martino regio precettore e Maestro di Matematica di Ferdinando IV […] ristampato per uso dell'Accademia militare del Battaglione Reale Ferdinando* (Napoli, Milo).

[14] The first military school in the Kingdom of Naples was the *Accademia di Marina* founded in 1735. Pietro De Martino (1707-1746), Nicola's brother, was called to teach mathematics.

algebra; the second year started with solid geometry, logarithms, rectilinear trigonometry, and was followed by conical sections in the second phase; the third year included practical geometry, statics, hydrostatics and hydraulics; the fourth year proposed artillery, regular and irregular fortifications, strategies of attack and defense of strongholds. Lessons officially began on 1st February of 1770. The director of sciences, besides carrying out duties as coordinator and teacher, was in charge of the publication of treatises. From 1770 to 1772, Caravelli republished and completed, for these students, the *Elementi di Matematica*[15] already compiled for the students of the *Accademia di Marina*. In 1773, integrating his studies on theory with experimental data collected by the captain of the cadet brigade, Francesco Zito, Caravelli published, in two volumes, the *Elementi di artiglieria, composti per uso della Reale Accademia Militare* (Napoli, Raimondi). In 1776, he published further seven volumes entitled *Elementi dell'architettura militare composti da Vito Caravelli* (Napoli, Raimondi).

Even if the *Reale Accademia Militare* and the *Brigada dei Cadetti di Artiglieria* provided for the professional formation of cadets and officers, almost all the cadets of the armies continued to be formed by the regiments. In order to create a teaching entity providing a suitable formation for the young men wishing to have a military career, in 1771, a selected corps of cadets called the *Real Brigata* was set up, which from the year 1772 was renamed the *Battaglione Real Ferdinando*. In 1774, the *Brigata* and the *Reale Accademia Militare* were amalgamated with the *Battaglione Real Ferdinando* thus creating the new *Real Accademia militare del Battaglione Real Ferdinando* open to cadets from all armies. In accordance with the study plan of the military college of 1785, the teaching of mathematics was distributed as follows: II and III years (practical arithmetic), IV year (arithmetic and plane geometry), V year (algebra, solid geometry and logic), VI year (conical sections, trigonometry and practical geometry), VII year (mechanics). There was, moreover, an "extraordinary" class for the study of geometry and differential calculus aimed at the formation of teachers for the academy itself (Leschi, 1994, vol. 2, pp. 949-950).[16]

Professor Nicolò Fergola (1753-1824) was temporarily nominated for the chair of geometry and differential calculus (Leschi, 1994, vol. 1, p. 513). The course was to be based on the treatises by Vito Caravelli, Benjamin Robins and Papacino D'Antoni. In 1786 there came the publication of the *Trattati del calcolo differenziale e*

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[15] *Elementi di matematica composto per uso della Reale Accademia militare dal professore di Fisica sperimentale e Chimica e Direttore delle Scienze della medesima Vito Caravelli*, Napoli, Raimondi, 1770-1772: *Elementi di aritmetica* (vol. 1, 1770), *Elementi di geometria piana* (vol. 2, 1770), *Elementi di Algebra* (vol. 3, 1770), *Elementi di geometria solida* (vol. 4, 1770), *Della dottrina de' logaritmi* (vol. 5, 1771), *Elementi delle sezioni coniche* (vol. 6, 1771), *Elementi di geometria pratica* (vol. 7, 1771), *Elementi di meccanica* (vol. VIII and vol. IX, 1772); these last works were composed in four books (dynamics, statics, hydrostatics and hydraulics).

[16] Gatto (2010, p. 90) presents this organisation of studies divided into ten classes for the Royal Military Academy of 1769.

del calcolo integrale per uso del regale collegio militare (Napoli, Raimondi) written by Caravelli and his former student Vincenzo Porto (1747-1801). The work, composed of 304 pages, is divided into eight chapters devoted to algebraic integrals of differential quantities of first degree of one variable, integration of logarithmic and exponential differentials, integration of differentials containing more than one variable and the use of integrals of first degree of one variable in order to calculate the volume and surface area of solids of revolution. This treatise is not about derivatives or differentials of trigonometric functions; geometrical applications are limited to conics, cycloids and logarithmic curve (Amodeo, 1924, p. 169).

In 1787, the *Reale Accademia del Battaglione Real Ferdinando* underwent changes and assumed a new name – the *Reale Accademia Militare*. It was housed in the Pizzofalcone, in the building of the former Jesuit Novitiate adjacent to the Nunziatella church, from which the institute took its name of "Nunziatella" military school. Following the proclamation of the Neapolitan Republic (1799), the school changed its name to the *Nazionale Accademia Militare*. With the restoration of the monarchy, following the collaboration of some teachers and officers of the Republic established with the French invasion, the academy was temporarily closed; however, the king allowed the school to continue with a reduction in the number of students. Afterwards, first under the name *Real Convitto Militare* (1801), then as *Real Accademia Militare* (1802), the "Nunziatella" once more took up its function as an institute.

The Military College of Verona

With the outbreak of the Seven Year War (1756), the Senate of the Venetian Republic, entrusted General William Graham of Montrose to come up with a plan for the reconstruction and modernisation of the land army. The Scottish general's vast plan included the creation of a military college for young cadets. Verona, which was located in the very centre of the Veneto State, was the city chosen to host this new institute which aimed at forming a substantial number of military engineers who constituted an established part of the army.

The *Militar Collegio* of Verona was set up on 1st September, 1759, and inaugurated on 3rd September. The Veronese castle of Castelvecchio was chosen to house the college. The establishment of the college took place almost at the same time as the foundation of the first stable nucleus of artillery (1757), and some time before the creation of the corps of engineers (1770). A retired officer, Captain Tommaso Pedrinelli[17] was called to run the school. The study course lasted seven

[17] The first study on the Verona College was carried out by Barbarich (1908); more recent studies are to be found in Baroni (1985); Farinella (1991); Curi (1992-1993); Leschi (1994), vol. 1, pp. 189-226; Premi (2009a); Premi (2009b).

years, divided into two three-year courses which included the following subjects: numerical and literal arithmetic, theory of geometry («I primi sei libri degli Elementi di Euclide coll'undicesimo e duodecimo, non che i Teoremi più scelti di Archimede, e le Sezioni Coniche di Apollonio»),[18] practical geometry (longimetry, altimetry, planimetry, stereometry), trigonometry, algebra, conical sections, mechanics, statics, hydraulics, systems of fortification and artillery. At the end of the first three years, the twenty-four students were to be sent to two different classes; engineers or non engineers. The other disciplines included the Italian language, Latin and French, design and fencing.[19]

Pedrinelli was also a teacher of mathematics in the first class alongside the ensign and engineer, Francesco Benoni, who also covered the role of inspector. Twenty-four young men were admitted to the college, aged from fourteen to twenty years (sons or nephews of officers or nobles). Most of the students only knew how to read and write. The first years of the Institute were very uncertain; the organisation and aims had to be better outlined. Furthermore, the tasks and professional duties of a corps of technicians had to be more precisely set out. The first years of the college carried on without any articulation of a rigid study plan and lessons; this created great internal tension. The reforms initiated by Doge Alvise IV Mocenigo in the University of Padua in that period, aimed at secularizing teaching by releasing it from Jesuit domination, drew the attention of the Senate, which initiated a series of reforms in order to relaunch the Military School of Verona (Barbarich, 1908, p. 230). In 1763, the *Libro de' Doveri per il Collegio Militare di Verona fatti estendere dell'Illustrissimo ed Eccellentissimo Girolamo Zuliani Savio di T. F. alla Scrittura ed approvati dal Sovrano decreto dell'Eccellentissimo Senato* (Verona, Merlo) and the *Piano generale degli studi da farsi in un sessennio nel pubblico collegio militare di Verona, fatto estendere da Alvise Tiepolo, Savio di Terra Ferma alla Scrittura, approvato dallo Eccellentissimo Senato* (Venezia, Pinelli) were published (Barbarich, 1908, p. 234). The same year Alvise Tiepolo nominated as mathematics teacher for the second class, the young captain of engineers, Anton Maria Lorgna (1735-1796).[20] Pedrinelli's

[18] Details taken from Pedrinelli's introductory speech on inauguration day, published in *Per la Pubblica Scuola Militare di Verona Solennemente Aperta nel giorno 3 di Settembre 1759* …, In Verona, 1760, Per Dionisio Ramanzini Librajo e Stampatore a San Tomio, Con Licenza De' Superiori (Leschi, 1994, vol. 2, pp. 245-258; anastatic copies).

[19] The first plan on the organisation of study courses may be found in the *Terminazione degli Eccellentissimi Signori Ferrigo Renier Savio alla Scrittura e Alvise Tiepolo Savio Uscito addì 25 Febraro 1758 m. v.*, Merlo, Venezia, 1758, analysed by Curi (1992-1993), pp. 126-129.

[20] Among the studies on Lorgna we may remember Jacoli (1877); I cento anni dell'Istituto Tecnico Commerciale e per Geometri "Anton Maria Lorgna" di Verona (1969); Piva (1985); Accademia di agricoltura scienze e lettere di Verona (1986); Farinella (1993); Piva (1993); Anton Maria Lorgna scienziato ed accademico del XVIII secolo tra conservazione e novità (1998). Lorgna is also the reference of the *Dizionario Biografico degli italiani* by Ettore Curi. In particular, regarding Lorgna and the *Militar Collegio* see Sbardellati (1937), Farinella (1991) and Curi (1992-1993).

incompetence as director of the institute as well as a teacher of mathematics contributed to the decline of the college's internal life. In 1764, other decrees followed, which nominated the governor, Francesco Ferro, entrusting him with the discipline and military command of the college, and established that the institute should be open exclusively to the sons and nephews of officers or nobles, aged between twelve and fourteen years with good school education. Moreover, it was decreed that after successfully completing the study course they should be sent, according to ascending order of merit, to the corps of engineers, artillerymen and troop, respectively. The real reform of the college came in 1765 following a visit to the college on the part of judge Marc'Antonio Priuli who, availing himself of the opinions of trusty men like Lorgna and Giuseppe Torelli, a man of letters and mathematics from Verona, dismissed Pedrinelli from the institute and had a new *Terminazione Statutaria* as well as a new *Libro de' Doveri* drawn up.[21]

Lorgna and Torelli had been pupils of Giovanni Poleni, who was the teacher of mathematics in the University of Padua from 1719-20 to 1760-61. The topics of his lessons were essentially about Euclidean geometry (first year), mechanics, which included statics and simple machinery, and the applications of the movements of water or the movements of animals (second year), geometry applied to optics, perspective, and mathematical geography, i.e. the sphere, and military architecture (third year). Depending on the study year, also elements of plane trigonometry and theory of geometry of conical sections were also taught. There were no provisions for the study of differential and integral calculus.[22]

The arrival of Lorgna, who had contributed to further highlighting the limitations of the college, gave a dominant role to mathematics by transforming it into the real fundamental and essential nucleus of the school system. In 1770, the *Corpo regolato degli Ingegneri* and the *Reggimento Artiglieria* were set up, which were open to the students of the military college. The same year, Lorgna was promoted to the rank of Lieutenant Colonel. In 1777, the college was reformed as far as length of courses and teaching staff were concerned. In accordance with the new regulations, the head teacher of mathematics, who continued to be Lorgna, also had the post of director of the school. The permanence of cadets in the college was extended to eight years. The study plan included a six-year course in common, at the

[21] The *Terminazione Statutaria del Collegio Militare di Verona*, the *Libro de' Doveri per il Collegio Militare di Verona* and the *Terminazione Institutiva del Governatore del Collegio Militare di Verona*, *fatta estendere Dall'Illustrissimo, ed Eccellentissimo Signor Marc'Antonio Priuli I savio di T.F. alla scrittura uscito, Ed approvata da Sovrani Decreti dell'Eccellentissimo Senato de dì 28 Novembre 1764 e 7 Marzo 1765* (Venezia, Pinelli, 1765), may be found in Leschi (1994), vol. 2, pp. 259-293 (anastatic copies).

[22] Giovanni Poleni (1683-1761) had studied at the *Collegio dei Somaschi della Salute* in Venice. Afterwards, he took up juridical studies. Following Jacob Hermann's arrival in Padua (1707), he began to study mathematics. In 1709, he published his first work which gained him the chair of Astronomy at the University, the *Miscellanea* (Venezia, Pavini). It contains three dissertations: one on barometers and thermometers, one on *machina aritmetica* and one on conical sections and solar clocks (Pepe, 2013; Pepe, 2016).

end of which the students were divided into two classes: the first twelve classified made up the first class, while the remaining were assigned to the second class. The study of mathematics was concentrated in the first four years: in the first year subjects included arithmetic, algebra and design; the second proposed algebra, elements of plane and solid geometry, solution to problems by means of algebra and architectural design; in the third, applied algebra, logarithms and logarithmic calculus, trigonometry, elements of perspective geometry, topography, topographic design; in the fourth year conical sections, mechanics, topographic survey on the land, topographic design, and perspective design. From the fifth year, study included technical disciplines and revision of mathematics already taught.

Lorgna, following the death of Sergeant General Giovanni Carlo Pagnelli Cicavo, governor of the college (1779-1780), on 1st March, 1780, temporarily took up the position of director of the institute. In 1785, the nomination was permanent and in the same year Lorgna brought about changes in the school's regulations as presented in the *Leggi del Collegio Militare di Verona esposte dal Cavaliere Anton Mario Lorgna Colonnello degl'Ingegneri, Governatore e Direttore di quell'Istituto*. The length of the course, in fact, was returned to six years and the distribution of the study of mathematics was re-adapted to the first four years with a clearer description of the teaching contents: the first year provided for the study of the first four books of plane geometry; in the second year plane geometry was continued using the fifth and sixth books of Euclidean geometry, and studies included solid geometry, plane trigonometry, logarithms and the first elements of algebra; the third year was devoted to algebra up to 3rd and 4th degree equations and conical sections; in the fourth year trigonometry of spheres was studied (Leschi, 1994, vol. 1, pp. 219-220).

Lorgna, besides being Director of the College, was entrusted with many other tasks by the Republic of Venice: from the regulation of the river Adige (1786), to works on the river Piave (1783); from the restoration of the walls of the city of Crema (1772) to that of the Legnago Fortress (1792), to mention but a few. He was summoned also by other States where his technical knowledge was required. As a scholar he was eclectic, his interests ranging from chemistry to the physics of fluids, from topography to mathematics, from astronomy to meteorology, he continued to produce works on scientific matters that were being studied by the leading scholars of the day. He was therefore in contact with some of the most important scientific figures in Italy and Europe; he exchanged correspondences with Euler, the Bernoulli brothers, Giuseppe Luigi Lagrange, Joseph-Jérôme Le Français de Lalande, Marie-Jean-Antoine-Nicolas de Caritat marquis de Condorcet, and Antoine-Laurent de Lavoisier, among others. To each of them he sent the results of his research and studies on relative matters, which were always well-received. His renown increased as a result of his studies and intense research works as he received recognition and awards from the most important Italian and foreign academies; for example, he became a member of the Royal Society of Sciences in

London, the Royal Academy of Sciences in Paris, the Royal Academy of Sciences and Letters in Berlin and the Imperial Academy of Sciences in St. Petersburg.

His scientific activity reached its peak in 1782 with the institution of the *Società Italiana delle Scienze* or *Accademia dei XL*, whose aim it was to keep science in Italy up to date with Europe. The Academy, which is still operating, brought together the forty best Italian scientists. They had no official base and the Forty never met in public. Their task was to produce an unedited scientific memoir to be published every two years in the Acts of the Society. The fact that these volumes reached French, German, Swedish and Russian scientists is a testimony to the vitality of Italian science at that time (Piva, 1985; Piva, 1993; Curi, 2006).

Notwithstanding Lorgna's scientific openings to contemporary analytical methods,[23] the study course of the *Militar Collegio* of Verona remained anchored in traditional arguments: arithmetic, plane and solid geometry, theorems of Archimedes, conical sections of Apollonio, trigonometry and algebra. In 1785, under the direction of Lorgna, the following subjects were added: the use of logarithms, algebraic solution of 3rd and 4th degree equations, analytical theory of conical sections and the study of perspective, but not the study of differential and integral calculus. The example of Poleni and the presence of Torelli had, in fact, markedly conditioned the cultural milieu of Verona.

In accordance with the regulations of 1791, the director of studies and teacher of mathematics was Leonardo Salimbeni (1752-1823), son of Zuane Salimbeni, who was General commander of the troops of the Republic and Director of the College of Verona between 1778 and 1779. In 1794, Salimbeni took over from Lorgna, who was ill, as director of the college, a role which he maintained up to the closure of the institute, which came about in the summer of 1796, with the entrance of the French troops into the city (1st June). On 16th July the last students of the school left the Veronese castle for the very last time. Napoleon transferred the seat of the Verona school to Modena, and Leonardo Salimbeni was nominated director of the new *Scuola Militare del Genio e dell'Artiglieria*.

[23] Lorgna wrote widely on differential calculus and the analytical methods; among the printed works we find: Opuscula tria ad res mathematicas pertinentia (Veronae, Ex Typographia Ramanziniana, 1767); Opuscula mathematica et physica (Veronae, Typis Marci Moroni, 1770); Specimen de' seriebus convergentibus (Veronae, Typis Marci Moroni, 1775); De casu irreductibili tertii gradus et seriebus infinitis exercitatio analitica (Veronae, Typis Marci Moroni, 1776); De functionibus arbitrariis calculi integralis dissertation (Petropoli, Typis Academiae Imperialis Scientiarum, 1781); among the contributions to the Memorie di matematica e fisica della Società Italiana: Nuova investigazione della Somma generale delle Serie and Ricerche intorno al Calcolo integrale dell'equazioni differenziali finite (in the first volume, 1782); Indagine nel Calcolo integrale (in the second volume, 1784); Sopra l'integrazione della formula $Qdx + Py^2x \pm dy = 0$ (in the third volume, 1786); Delle variazioni analitiche finite (in the fourth volume, 1788); Calcolo delle variazioni finite nella Trigonometria piana e sferica (in the seventh volume, 1794); the list could go on if we were to add the texts published in the Acta Academiae Scientiarum Imperialis Petropolitanae and the Mémoires de l'Académie Royale des Sciences de Turin, as well as the unedited contents of the correspondence preserved in the Biblioteca Civica of Verona (Jacoli, 1877, pp. 43-74).

Conclusion

With the introduction of the military schools in the XVIII century, mathematics became the cornerstone of technical-scientific teaching, and remained important also in the following century in the technical schools and institutes, established in the most industrialized Italian cities, among which Turin.

The school of artillery and fortification in the Kingdom of Sardinia, opened in 1739, paved the way for the organization of future institutions in the Kingdom of Naples (1745), and the Republic of Venice (1759). The creation of these schools, qualified to give military cadets a professional formation under the direction of important mathematicians, implied the adoption of evolved teaching programmes that, in some cases, like Turin and Naples, included the study of differential and integral calculus which, for most of the eighteenth century, were not part of the university courses of these cities. The curriculum of the Verona Military School, however, excluded the study of Leibniz and Newton's methods, as did that of Padua University which was still attached to the courses of Giovanni Poleni. In other cities, like Pavia and Bologna, instead, a different situation was to be found; the university reforms of the XVIII century aimed at elevating mathematical studies thanks to the presence of mathematicians who promoted the new method in Italy.

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On the relationships between the geometric and the algebraic ideas in Duhre's textbooks of mathematics, as reflected via Book II of Euclid's *Elements*

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Abstract

The present article explores the relationships between the geometric and algebraic ideas presented in Anders Gabriel Duhre's mathematics textbooks. Of particular interest is Book II of Euclid's Elements *as presented by Duhre in his textbook on geometry from 1721. We consider in detail Duhre's two versions of Proposition II.5, dealing with straight lines cut into equal and unequal parts, as well as the two proofs of the propositions that he presents. Duhre's formulations are slightly different from traditional geometric formulations, as he moved away from a purely geometrical context towards an algebraic one. Duhre established Proposition II.5 using algebra in Descartes' notation as well as in the notation of Wallis and Oughtred. Duhre´s reason for introducing algebra in Book II of Euclid's* Elements *was to obtain convenience in calculations, as well as the possibility to generalize results to different kinds of quantities.*

Introduction

Anders Gabriel Duhre (1680-1739, or possibly 1681-1739) was a Swedish mathematician and mathematics teacher. He was the son of the Circuit judge Gabriel Duhre in Waksala outside of Uppsala. In 1695 he became a student at Uppsala University, where he studied for the astronomy professor Pehr Elvius (1660-1718), but he left the university during the year of the plague 1710. In 1712 he was a student of the Swedish scientist, inventor and industrialist Christopher Polhem (1661-1751) at his school Laboratorium Mechanicum in Stjärnsund. For a few years he then taught mathematics to engineering students at Bergskollegium – a central agency with the task to lead and control the mining and metal processing – and to prospective officers at the Royal Fortification Office in Stockholm. In 1723, after receiving permission from the parliament, Duhre opened his own school, Laboratorium Mathematico-Oeconomicum, in Ultuna outside Uppsala. This school is the precursor to the Swedish University of Agricultural Sciences,

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

which is located in the same area. Duhre's school was a technical school, economically based on farming operations, where young boys were taught theoretical and practical subjects. Of particular interest is that mathematics was taught in this school; for example, it is known that infinitesimal calculus was for the first time taught in Sweden in Duhre's school. The mathematics teaching at the school was not located at Ultuna, but at Uppsala, close to the university (Rodhe, 2002).

Even though Duhre never received a position at Uppsala University, he was an important and influential person within the Swedish mathematical society. He had knowledge of modern mathematics that was not taught at the university, and among his students were several of the mathematicians to be established during the 1720s and 1730s – among others Eric Burman (1692-1729), Anders Celsius (1701-1744) and Mårten Strömer (1707-1770) – all of whom would later become professors of mathematics. As students at Uppsala University they turned to Duhre to learn more on modern mathematics, which they apparently did not have the opportunity to do at the university. Samuel Klingenstierna (1698-1765), the most important and internationally most well-known Swedish mathematician during the 18th century, was probably not a student of Duhre, but he was recommended by Duhre to study, among others, Charles Reyneau's (1656-1728) book *Analyse demontrée* (1708) on differential and integral calculus (Rodhe, 2002).

Duhre taught in Swedish and planned early on to write mathematical textbooks in Swedish, in order to introduce the Swedish youth to the new and modern mathematics. He contributed with two textbooks in mathematics – one in algebra and one in geometry. The first book, *En Grundelig Inledning til Mathesin Universalem och Algebram* (A Thorough Introduction to Universal Mathematics and Algebra), was edited by his student Georg Brandt (1694-1768) and published in 1718. In this textbook, which is based on Duhre's notes from his lectures at Bergskollegium, modern algebra according to René Descartes' (1596-1650) notation is presented, as well as examples from Isaac Newton's (1642-1727), John Wallis' (1616-1703), and Bernard Nieuwentijt's (1654-1718) theories from the late 17th century.

The second textbook, *Första Delen af en Grundad Geometria* (The first part of a founded geometry), was published in 1721 and was based on Duhre's lectures held in Swedish at the Royal Fortification Office. He probably planned a second book on geometry, but this was never realized. Duhre's book on geometry is the most advanced textbook in mathematics in Swedish during the 18th century. It is a voluminous book of about 600 pages, which distinguishes itself from previous books on geometry by not being based on Euclid's *Elements*. Instead, most of the book treats infinitely large and infinitely small quantities. Duhre also takes advantage of the theories he earlier presented in his book on algebra. Of particular interest in his book on geometry is his use of algebra in the geometrical context as presented via parts of Book II of Euclid's *Elements*.

After war, plague and bad harvest, Sweden in the 1710s was a devastated country in great need of supply, science and new ideas. Duhre was convinced that knowledge of the new mathematics, together with the physics derived therefrom, would provide an increased prosperity to the country. In the introduction to his book on geometry, Duhre wrote that his motive to teach and write in Swedish was to make it possible for talented students, who due to poverty had no experience in foreign languages, to study mathematics (Duhre, 1721). This was an important step to implement Duhre's vision. Probably his two books were used at Bergskollegium and at the Fortification office at least until 1723 when he opened his own school. It is not known if the books were used at Laboratorium Mathematico-Oeconomicum, even though it is likely that some of the modern mathematical ideas presented in his books were also taught at this school.

Duhre's school project ended in personal disaster when he in 1731 had to leave the school. By economical prompting the governor Johan Brauner (1668-1743) managed to get the Parliament to transfer the lease of the farm, where the school was located, to him. Duhre was left in poverty, but he still had ideas on educational initiatives, and wanted to start similar schools in the whole country. This was however not realized and he died in 1739 (Hebbe, 1933).

Duhre was the precursor of many modern ideas in mathematics as well as in the technical education and in the rationalization of farming. His work contributed to the constitution of the professorship of economics at Uppsala University in 1740 (Hebbe, 1933). Duhre is known as a great inspirer, and due to his teaching and his two books on algebra and geometry, Swedish mathematics in the 1730s had become just as advanced as in most countries in Europe (Rodhe, 2002).

Book II of Euclid's *Elements* and the relationships between geometry and algebra

Book II of the *Elements* attributed to Euclid contains 14 propositions on plane geometry and it raises interesting questions regarding the relationships between geometry and algebra. For example, during the 1970s there was a rather heated debate about whether the Greeks presented a kind of algebra in some of their geometry, and especially Book II of Euclid's *Elements* was discussed as an example of Greek algebra hidden behind a "geometrical veil". In 1975 Sabetai Unguru argued that the claim that Euclid was a "geometric algebraist", handling geometrical notions but actually practicing common algebra, was incorrect and based on an anachronistic reading of ancient Greek texts in the sense that they were translated into a modern algebraic notation; according to Unguru algebra was imposed on the Greek texts rather than discovered in them. In 1978, the leading mathematician André Weil dismissed Unguru's critique by accusing Unguru of not knowing enough mathematics, claiming, without much justification, that Euclid just used a somewhat cumbersome notation in his algebra. Nowadays Weil's claim is instead sometimes regarded as a scandal in the field of the history of mathematics (Öberg, 2011, p. xxv; Corry, 2013, p. 638).

In a paper presented at the HPM 2008 satellite meeting of ICME 11 in Mexico City, Gert Schubring discussed the use of historical material in the teaching of mathematics. He exploited the debate on the existence of "geometric algebra" in Greek mathematics provoked by Unguru, to initiate a methodological debate on the use of sources that have been modernized and distorted for didactical reasons. For several essential reasons, such as for example conceptualization, notation, language and epistemology, this modification of sources constitutes a common practice in projects making use of history of mathematics in the teaching of mathematics. The question is which degree of distortion can be claimed to be legitimate for the aim of teaching (Schubring, 2008).

In the first chapter of his book on geometry Duhre stated and proved eight of the propositions of Book II of Euclid's *Elements*; he did however not include the first two and the last four of the propositions attributed to Euclid. Later in the same chapter he stated the propositions again, now also including the first two, in an alternative way. This is probably the first time parts of the *Elements* were published in the Swedish language. However, Duhre has earlier not been acknowledged for the publishing of the first Swedish edition of parts of Euclid's *Elements*. Previously the first Swedish edition has been attributed to Duhre's student Mårten Strömer (see, for example, Heath 1956, p. 113). In 1744 Strömer published a Swedish translation of the first six books of Euclid's *Elements*, in a traditional geometrical context.

Nevertheless, neither Duhre nor Strömer were the first to publish the *Elements* in Sweden. Already in 1637 the Swedish mathematician Martinus Erici Gestrinius (1594-1648) had contributed with a commented edition of the *Elements* in Latin. Gestrinius did include algebra into his geometry, at least in Propositions 4, 5 and 6 of Book II. He did this by associating the propositions with three different kinds of quadratic equations, before showing how the equations can be solved in three different ways: rhetorically, with tables, and geometrically. Thus, Gestrinius used the quadratic equations to illustrate the propositions; that is, he utilized algebra in order to illustrate geometry (Pejlare & Rodhe, 2016). Duhre probably had studied Gestrinius' edition of the *Elements*, since it was used at Uppsala University. Also Christopher Clavius' edition of the *Elements* from 1574 was used at Uppsala University, and was most likely known by Duhre. The wording of the propositions and proofs of Book II of Clavius and Gestrinius are very similar, but Clavius did not include any algebra. However, as we will see in the following section, Duhre's presentation of the propositions and proofs of Book II is very different from Gestrinius' version, as well as from Clavius' version, and the traditional geometrical formulations and proofs of Euclid.

Formulation and proof of Proposition II.5

In order to illustrate how Duhres' formulations of the propositions of Book II, as well as his proofs, differ from Euclid's, we will investigate one of the propositions in detail. The proposition we will consider is Proposition II.5, which in Duhre's edition is the third proposition, dealing with straight lines cut into equal and unequal parts. A traditional formulation, attributed to Euclid, of this proposition is as follows:

> **Proposition II.5:** If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equal the square on the half (Heath 1956, p. 382).

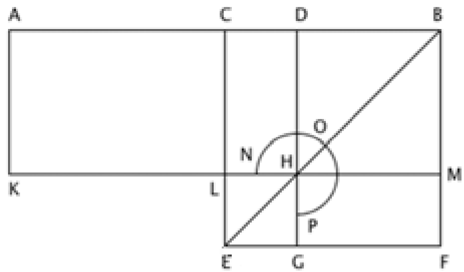


Fig. 1. A visualization of Euclid's Proposition II.5

According to the proposition and referring to Fig. 1, the straight line AB is cut into equal segments at C and into unequal segments at D. The rectangle $ADHK$ together with the square $LHGE$ equal the square $CBFE$. We can easily approve this proposition, as the segment AC equals the segment BF and the segment AK equals the segment BD and thus the rectangle $ACLK$ equals the rectangle $BFGD$.

Duhre's first formulation of Proposition II.5, translated into English, is as follows:

> **Duhre's first version of Proposition II.5:** If something whole is divided into two equal parts and then into two unequal parts, then the *product* of the unequal parts together with the *square* of the difference between one of the

equal and one of unequal parts is equal to the *square* of the half of the whole (Duhre 1721, p. 20).[1]

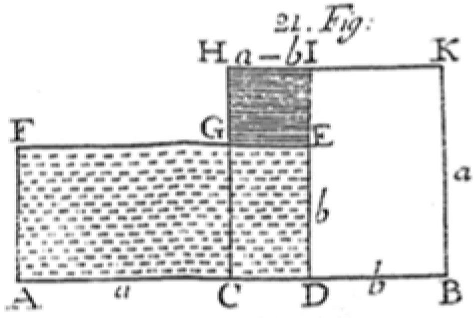
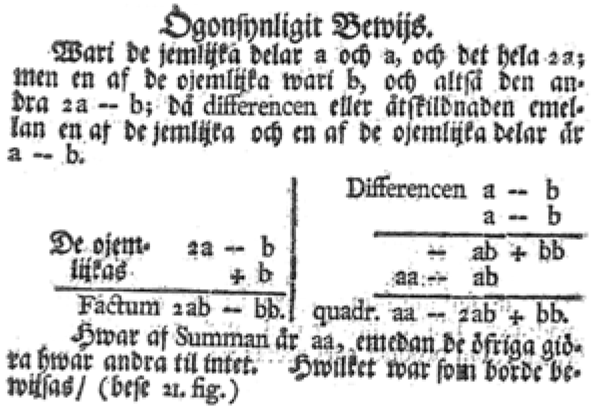


Fig. 2. Duhre's visualization of Proposition II.5 (Duhre 1721, p. II).

As we can see, Duhre's formulation of the proposition is slightly different from the traditional formulation. When Euclid used the concepts *straight line*, *segment*, *rectangle,* and *square on the straight line*, Duhre used the concepts *something whole*, *part*, *product,* and *square of the difference.* This indicates that Duhre moved away from a purely geometrical context and considered the proposition also in an algebraic context. This becomes even clearer as we consider Duhre's proof of the proposition. The traditional proof of Euclid is purely geometrical, but Duhre's version is, even if it refers to Fig. 2 above, purely algebraical (see Fig. 3):



Ögonsynligit Bewijs.

Wari de jemlijka delar a och a, och det hela 2a; men en af de ojemlijka wari b, och altså den andra 2a -- b; då differencen eller åtskilnaden emellan en af de jemlijka och en af de ojemlijka delar är a -- b.

	Differencen a -- b
	a -- b
De ojemlijkas 2a -- b	-- ab + bb
+ b	aa -- ab
Factum 2ab -- bb.	quadr. aa -- 2ab + bb.

Hwar af Summan är aa, emedan de öfriga gjöra hwar andra til intet. Hwilket war som borde bewisas/ (bese 21. fig.)

Fig. 3. Duhre's proof of Proposition II.5 (Duhre, 1721, p. 20).

[1] "Om något helt warder fördelat uti twänne jemlijka delar och der näst uti twänne andra ojemlijka måtte de ojemlijkas *product*, jemte *quadraten* af den åtskilnad som är emillan en af de jemlijka och en af de ojemlijka delar wara jemlijk emot *quadraten* af bemelte helas halfpart."

Duhre lets the equal parts be a, the whole is $2a$ and one of the unequal parts is b. Using algebra he now shows that $(2a - b)b + (a - b)^2 = a^2$.

This proof is very different from the geometrical one we know from Euclid. Instead, Duhre established the proposition, as well as the remaining seven propositions he included in his book, using algebraic ideas in Descartes' notation. Thus, Duhre's proof could be seen as a proof of an algebraic identity where he performs operations on algebraic expressions. Duhre motivated his choice of using algebra in the following manner:

> Here would have been an opportunity to prove the preceding propositions according to Euclid, which is both certain and beautiful; but as the *Method* to prove through symbols is more *universal* such that it for the sense reveals the unchangeable truth of these propositions, with the assurance that they do not only refer to *lines*, but also to *surfaces*, solids, and everything that belongs to the word *quantities*; thus the great advantage that therein consists can be observed (Duhre, 1721, p. 24).[2]

Reading this quote it becomes clear that Duhre knew of the Euclidean geometrical proofs of the propositions of Book II, but he thought that this method – to use algebra – is much more general since the quantity he refers to as *something whole* does not have to be a straight line but could also be another kind of quantity, such as a surface, a solid, or something else. Thus, even though he refers to a figure (Fig. 2) where *something whole* actually is considered to be a straight line, this does not have to be the case.

Proposition II.5 in Wallis' and Oughtred's notation

William Oughtred (1574-1660) was one of the first mathematicians to exemplify theorems of classic geometry using algebra (Stedall, 2002). He demonstrated all of the 14 propositions of Book II of Euclid's *Elements* in his *Clavis mathematicæ* from 1631 with his analytical method, which means that he used François Viète's (1540-1603) algebraic notation, as presented in Viète's symbolic algebra, or the *Analytical Art*. During the end of the 16th century Viète was inspired by Diophanto's work and used capital letters instead of abbreviations as symbols for the unknown and known entities.

[2] "Här hade man fuller haft tilfälle at efter Euclidis sätt bewijsa föregående föreställningar hwilket är både säkert och wackert; men såsom *Methoden* at bewijsa igienom kiennetekn är mera *universal*, så at den för förnuftet uppenbarar dessa föreställningars oföränderliga wisshet med försäkran at de icke allenast sträkia sig til *linier*, utan och jämwäl til *superficier*, fasta kroppar och alt det som hörer under ordet *quantum*; altså kan man förmärkia den stora fördehl som der uti består."

Duhre formulated and proved eight of the propositions of Book II of the *Elements* using algebra in Descarte's notation before he mentions John Wallis and William Oughtred. Duhre refers to Chapter 26 of Wallis' *A Treatise of Algebra* from 1685, where Wallis used Oughtred's notation to present the first ten of the 14 propositions of Book II of Euclid's *Elements*. Duhre considers this notation to be both "clear and convenient for the sense" (Duhre, 1721, p. 26), and thus he proceeds in presenting these 10 propositions of Book II in a similar way as Wallis had done. Duhre's second formulation of Proposition II.5 – in Wallis' and Oughtred's notation and translated into English – is as follows:

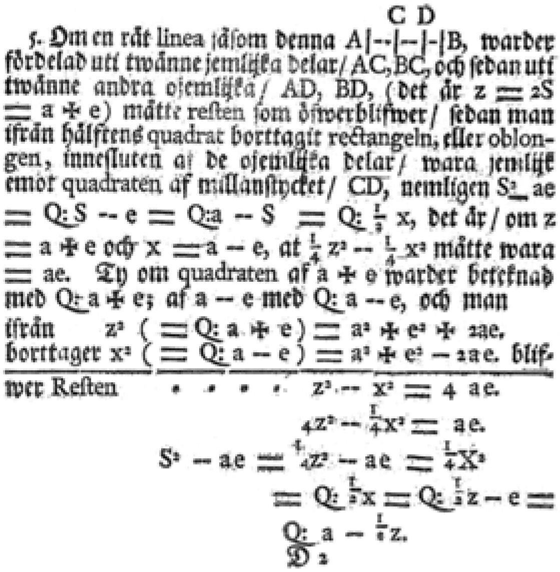
> **Duhre's second version of Proposition II.5:** If a straight *line* such as AB is distributed into two equal parts AC, BC and into two unequal parts AD, BD, (that is $z = 2S = a + e$) then the rest that remains when from the square of the half the *rectangle*, or the *oblong*, contained by the unequal parts has been removed be equal to the *square* of the middle part CD, that is $S^2 - ae = Q{:}S - e = Q{:}a - S = Q{:}\frac{1}{2}x$ (Duhre, 1721, p. 27).[3]

Thus, Duhre lets the straight line $AB = z$ be distributed into two equal parts S and into the two unequal parts a and e, that is, $z = 2S = a + e$. The symbol Q stands for the squaring of the expression, that is $Q{:}S - e = (S - e)^2$. The proposition claims that the rest that remains when the rectangle contained by the unequal parts has been removed from the square of the half, that is $S^2 - ae$, equals the square of the middle part, that is $(S - e)^2 = (a - S)^2 = (\frac{1}{2}x)^2$, where $x = a - e$ is twice the middle part.

We notice that Duhre's language has changed in this second presentation of the proposition. When he in his first presentation uses the concepts *something whole* and *product*, he in this latter presentation uses the concepts *straight line* and *rectangle*. This indicates that Duhre now moves back towards a more geometrical understanding of the proposition. Nevertheless, he still uses the concept *part* instead of the concept *segment*, illustrating a difference from the geometrical understanding of the proposition attributed to Euclid. Moreover, in Duhre's second proof of the proposition an algebraic language is used.

We will now consider Duhre's second proof of Proposition II.5 (see Fig. 4), in which he utilized the notations of Wallis and Oughtred:

[3] " Om en rät *linea* såsom denna AB, warder fördelad uti twänne jemlijka delar AC, BC, och sedan uti twänne andra ojemlijka AD, BD, (det är $z = 2S = a + e$) måtte resten som öfwerblifwer sedan man från hälftens quadrat borttagit *rectangeln*, eller *oblongen*, innesluten af de ojemlijka delar wara jemlijk emot *quadraten* af millanstycket CD, nemligen $s^2 - ae = Q{:}S - e = Q{:}a - S = Q{:}\frac{1}{2}x$."



C D

5. Om en rät linea såsom denna A|--|--|-|B, warder fördelad uti twänne jemlika delar/ AC, BC, och sedan uti twänne andra ojemlika/ AD, BD, (det är z = 2S = a + e) måtte resten som öfwerblifwer/ sedan man ifrån hälftens quadrat borttagit rectangeln, eller oblongen, innesluten af de ojemlika delar/ wara jemlikt emot quadraten af mällanstycket/ CD, nemligen S² -- ae = Q:S -- e = Q:a -- S = Q: ½ x, det är/ om z = a + e och x = a -- e, at ¼ z² -- ¼ x² måtte wara = ae. Ty om quadraten af a + e warder beteknad med Q: a + e; af a -- e med Q: a -- e, och man ifrån z² (= Q: a + e) = a² + e² + 2ae, borttager x² (= Q: a -- e) = a² + e² -- 2ae. blifwer Resten . . . . z² -- x² = 4 ae.

¼z² -- ¼x² = ae.

S² -- ae = ¼z² -- ae = ¼x² = Q: ½x = Q: ½z -- e = Q: a -- ½z.

D 2

Fig. 4. Duhre's formulation and proof of Proposition II.5, utilizing Wallis' and Oughtred's notation (Duhre, 1721, p. 27)

First Duhre shows that $\frac{1}{4}z^2 - \frac{1}{4}x^2 = ae$. He does this by using $z = a + e$ and $x = a - e$ and calculating

$$Q{:}\, a + e = a^2 + e^2 + 2ae \quad \text{and} \quad Q{:}\, a - e = a^2 + e^2 - 2ae.$$

This implies that

$$z^2 - x^2 = a^2 + e^2 + 2ae - (a^2 + e^2 - 2ae) = 4ae.$$

Thus $\frac{1}{4}z^2 - \frac{1}{4}x^2 = ae$, and since $S^2 = \frac{1}{4}z^2$, Duhre now concludes that

$$S^2 - ae = \frac{1}{4}z^2 - ae = \frac{1}{4}x^2 = Q{:}\frac{1}{2}x = Q{:}\frac{1}{2}z - e = Q{:}\, a - \frac{1}{2}z,$$

which establishes the proposition.

Even though Duhre in the formulation of the proposition partly relied on a geometrical interpretation, the proof he performed is entirely algebraic. Duhre claims that he follows Wallis and in fact he uses some of the symbols that Wallis used in *A Treatise of algebra* in 1685. For example Wallis, as well as Duhre, used the symbol Q to indicate the squaring of an expression. Duhre also used the same letters as Wallis, even though Wallis used capital letters and Duhre usually used lowercase letters.

Concluding remarks

Duhre was primarily an educator and his importance in the Swedish history of mathematics lies in his ability to transferring modern mathematics to the following generation of Swedish mathematicians. His textbooks on algebra and geometry were written in Swedish, which was important to reach a wider audience in Sweden.

One interesting question is why Duhre used algebra in his presentation of Book II of Euclid's *Elements*. With algebra Duhre could obtain convenience in calculations, since complicated expressions can be transformed into simpler ones. With algebra geometrical results can also be generalized to different kinds of quantities, since unknowns do not necessarily have to be, for example, lines. Throughout his book on geometry Duhre gave many examples of how algebra can be used to solve geometrical problems. The book is concluded with the following statement:

> Now it is unnecessary to give more examples to demonstrate the usefulness of algebra in geometry, and how those in previous chapters given linear demonstrations can easily be shown by algebra; Therefore there is every reason to do so. (Duhre, 1721, p. 561)[4]

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[4] "Nu synes onödigt at med flera exempel wijsa Algebrans nytta wid Rätliniska Geometrien, samt huru de i föregående Capitlen stående lineariska demonstrationer lätteligen igenom Algebra kunna uplösas; Emedan man af dessa der til kan hafwa en fullkomlig Anledning."

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Mathematics textbooks for teachers training in Spain in the second half of 19th century: The metric system implementation

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Abstract

The establishment of the Metric System (MS) in 1849 imposed a series of changes in the curriculum of mathematics in Spain. These changes happened in the educational system during the nineteenth century, both for primary and secondary education. The new curricula demanded variations in the teachers' preparation and training, who needed to know such modifications, mainly for its teaching in primary education. These reforms have been recorded in the textbooks of mathematics for those levels.

This paper presents the results of a study on the treatment given to the MS in mathematics textbooks in Spain during the nineteenth century. The training of teachers in the Spanish Normal Schools, that took place during the period 1849-1892, is emphasized. The training of teachers in regards to metrology was guided by scientific principles and immediate application in the most frequent activities of society. Exposure of the MS was associated with historical, legal and social impact. These aspects provided teachers with a comprehensive training on the system and the context in which it was implemented.

Introduction: The metric system in Spain

The old Spanish system of weights and measures, prior to the implementation of the Metric System (MS), was established by a variety of local units, without a common pattern to rule its relationships. With the advent of the nineteenth century, a process in Spain for the unification of the more than 400 year-old Castilian measures came to its end. This unification, which preserved the traditional anthropometry, implied a unification of the values and proportions of the several units for linear measure, area, capacity and weight hitherto diverse and arbitrary throughout the Kingdom (Aznar, 1997; Puente, 1982).

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Internally, the Spanish Government was challenged to secure and enforce its power over the people and the regional authorities. The economy was propelled to recover and reorganize itself through improvements in agriculture and industry. Socially, a change emerged against the domination of economically stronger classes. Common people demanded equality of contract, equity to restore confidence in commercial practices and eradicate outrages to which for years they had been subdued.

After many attempts to have a unique metrological system, in 1801 the King Carlos IV established a unified system of weights and measures (Pragmatic of January 26, 1801). It was characterized by use of duodecimal and binary divisions, and the use of common names among Spanish villages of the time. The Pragmatic collected the historical tradition of previous attempts of unification in Spain and established a Spanish system of measures that put aside the new one originated during the French revolutionary period. King's intention was to avoid social conflicts caused by ideological and political differences with France, which already had a unified system of measures – the MS – adopted in 1799. This Pragmatic closed a chapter of trends of unification based on traditional measures units with a Castilian nomenclature.

The unification of measures through the adoption of the MS meant an improvement of relations with other European countries; it ensured equal trade treatment between citizens; it allowed better control by the government in taxation, trade and business, and it also became the solution for eliminating commercial injustices to the population and to the small business, in general.

The Weights and Measures Act of July 19, 1849, marked the legal starting point in the process of the introducing the MS in Spain. Its introduction led to adopting a syntax that would relate all the different units as a legal language. The Act also fought against the limited understanding of the new system, which was one of its greatest and difficult battles.

We have bounded the study of the implementation process of the MS into three historical stages, considering both political and educational criteria. The first stage, named Legal Enactment, State Insertion, and Educational Outreach (from 1849 to 1867), begins with the enactment of the law of 19 July, 1849. This stage establishes the insertion of new weights and measures in state agencies and also includes the moment when should start teaching of the new units of weights and measures in all educational institutions of the State (January 1st, 1852). Here we chose the first textbooks published for teaching the MS to children and youth in all Primary and Secondary education institutions, and the teachers training in normal schools. The second stage, called Initiatives to Promote the Dissemination of the MS (1868-1879), corresponds to the period of widespread initiatives of the MS in Spain. It is extended from the establishment of the mandatory use of the MS for all individuals until the publication of the relevant regulations for the development of the law of weights and measures. The latest includes the mandatory use of the MS in the Peninsula and the possessions of Spain in America, Asia and Africa. The last step, named stage of Legality and Enforceability (1880-1892), involves the compulsory of the use of the units of weights and measures of the MS. It began at 1880, when – by royal decree – was declared the non legal use of any unit of weights and measures different from those established in the new system of measures. The process of change ended at 1892 when the adjustment to one system, single and definitive, of weights and measures in Spain was established.

The Spanish Government decided to increase the necessary infrastructure for the implementation of new metrology (Picado & Rico, 2012). Noteworthy that were the provisions for complying with Article 11 of the Law about teaching of the MS in Primary educational institutions.

> En todas las escuelas públicas ó particulares, en que se enseñe ó deba enseñarse la aritmética ó cualquiera otra parte de las matemáticas, será obligatoria la del sistema legal de medidas y pesas y su nomenclatura científica… (Ministerio de Comercio, Instrucción y Obras Públicas, 1868, p. 2)[1]

Implicit to the elaboration of Primary Education textbooks, teaching children about decimal-metric units in educational establishments was one of the most relevant and effective activities for implementing the MS. However, despite its inclusion in the curriculum and its treatments by mathematics textbooks, the MS had to deal with methodological obstacles for its complete adoption. The teaching of traditional weights and measures was not entirely abandoned; the older Spanish measures were still part of the social reality of the people. Teachers did not have school materials enough in addition to the recent texts and the scarce of new collections of weights and measures in primary school greatly complicated the practical application of the latest units.

Curriculum changes and the metric system

Political and social context in Spain, when MS was introduced into the classrooms, lead to an organization of teaching and regulation of curricula. As Aznar (1997) says:

> Cuando el sistema métrico irrumpe en las aulas de España, sobre todo en la enseñanza primaria, la organización del aparato escolar puede contarse ya

[1] [The authors of this paper have translated all quotations] In all public and private schools where arithmetic, or any other area of mathematics, is taught or should be taught, it will be compulsory teaching of the legal system of weights and measures and its scientific nomenclature… (Ministry of Commencer, Instruction and Public Works, 1868, p. 2)

> como uno de los más grandes éxitos de la burguesía liberal española que desde 1833 ha venido emprendiendo importantes reformas educativas… (p. 298)[2]

Specifically, for the new MS teaching were needed several changes to catch the mathematical meanings of the new units concepts of weights and measures. The analyzed documents (curricula and textbooks) show particularities that describe and define new trends on the ways to present the MS as a mathematical structure by means of explicit definitions, concepts and procedures. Also, considering the different representations that display and express these notions and with the several modes of use in context for the decimal-metric units. For example, the order of presenting mathematical concepts into the textbook; the relation between concepts and the MS; the use of graphic and symbolic representations for decimal-metric units; and the presentation of scientific, commercial and social situations to show the senses, practical uses and benefits of the new system.

Added to this, some cognitive contents and instructional orientations were identified; for example, how apprehend decimal-metric units conceived (rote or practical learning), what were the learning limitations of the students when they worked the MS, and what were the methodologies for the MS and the opportunities for its learning, suggested by the authors of the textbooks (mostly teachers).

Goal of the study

In general, our goal consists of studying the conceptual foundations that gave way to a new curricular regulation in the Spanish Mathematics Education, designed to incorporate the Metric System over the period 1849-1892, and the following curricular changes subsequently caused, both in learning the schoolchildren as in teachers training (see Picado, 2012).

Particularly, this paper highlights the treatment given to the MS in mathematics textbooks elaborated for training of teachers in Normal Schools in Spain during the period 1849-1892.

In the following, we describe some conceptual particularities around the MS, their link with the mathematical concepts and procedures, and certain didactical specificities identified into the textbooks for teaching the MS.

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[2] When the Metric System breaks into the classrooms in Spain, especially in primary education, the organization of the School system can be counted as one of the greatest successes of the Spanish liberal bourgeoisie, which since 1833 has been undertaking important educational reforms… (p. 298).

Information sources

The study included the selection and analysis of documental sources. The selection of textbooks included a search, localization, revision, and classification process of the mathematics textbooks linked to the teaching of the MS in Spanish Normal Schools in the second half of the XIX century. This process included the definition of specific criteria for the selection, testing of the authenticity, and the legitimacy of the text, and the assessment of the exactness of the content.

To locate the textbooks we considered the finding of sources in previous studies and the catalogues in some documentation centers. The previous studies provided a preliminary list of documents; the studies were choosen from Aznar (1997), Carrillo (2005), del Olmo, Rico and Sierra (1996), Picado (2009) and Vea (1995). These texts selected and analyzed were edited for teaching arithmetic or dissemination of MS in Spain during the nineteenth century. Documentation centers consulted and visited were the National Library of Spain, the Library of the University of Granada and several electronic catalogs. The textbooks analyzed in this study are available in the electronic catalog of the National Library of Spain (www.bne.es).

The considerable number of textbooks located led to the need of more precise criteria to control and reduce in convenient and reasonably manner the sample on which the analysis stage was raised. The first phase of selection considered five criteria: date and place of publication, title with MS name, purpose, availability and originality of the text. Representativeness of the text into the historical stages, the author, textbook content and style of the document are four additional criteria we have considered for the second phase of selection (see Picado & Rico, 2011).

The selection process allowed us to find 21 textbooks for teacher training in the Normal Schools in the period 1849-1892. Fifteen of them were published in the years covered by the first historical stage, one textbook in the second stage and five textbooks in the third stage. Along with checking the originality of the document was carried out a first review of its content. This test, based on defined categories of analysis, allowed us to identify the representativeness of the content and to do the final selection of the sources. Thus, taking into account the criteria described, five edited textbooks for teacher training were selected. Table 1 describes selected textbooks.

Table 1. Textbooks for teachers training in Spanish Normal Schools

Author	Year of printing	Title
Joaquín Avendaño	1852	*Elementos de aritmética* [Elements of arithmetic]
Juan Cortázar	1856	*Tratado de aritmética* [Treaty of arithmetic]
Ramón de Bajo e Ibáñez	1877	*Nociones de aritmética y algebra* [Notions of arithmetic and algebra]
Luis Sevilla González	1890	*Explicación de artimética. Arreglada al programa de la escuela normal superior de maestras de la provincial de Murcia* [Explanation of arithmetic. Arranged to the program of the Normal School of teachers from Murcia]
Antonio Marín y Rus	1892	*Programa de aritmética para uso de las alumnas de la normal de maestros de Málaga* [Program of arithmetic for use of the students of the normal of teachers from Malaga]

The analysis of the textbooks was carried out based on several interrelated techniques from which the conceptual analysis, the content analysis, and the didactical analysis were singularized (see Rico & Fernández-Cano, 2012; Picado & Rico, 2011; Picado, 2012). These techniques allowed us to establish analytical categories to characterize the mathematical content and some relevant aspects of the didactical content from these textbooks.

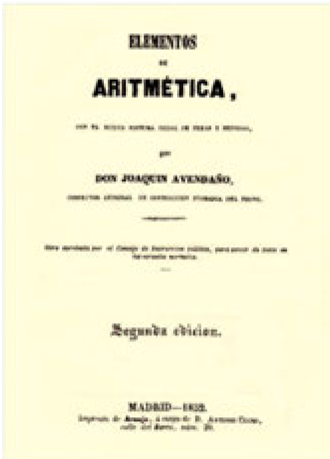
The established categories responded to nine domains of analysis. These are: *Preliminaries*, includes historical introduction, previous knowledges, legal data, and social impact; *Finalities*: types of purposes and type of learning (rote or practical learning); *Objectives*: explicit or implicit goals; *Concepts*: definition of number, fractional number, decimal fraction, rational number, decimal number, Decimal Numbering System, magnitude, types of magnitude, unit, quantity, measure, old system of weights and measures, Metric System, meter, basic units, multiples and submultiples, currency unit. *Procedures*: arithmetical operations, rules for conversions between units; *Types of representations*: verbal, symbolic, graphic, tables, others; *Contexts*: natural, scientific, commercial, mathematical, social, and technical contexts; *Learning limitations*: difficulties and errors; and, *Tasks*: examples, exercises, task sequences, and didactical materials (see Picado, 2012). As example, we present the analysis of one selected textbook. Then, we show results about the five textbooks analyzed.

Analysis of the textbook "Elementos de aritmética" by Joaquín Avendaño

The text corresponds to material approved by the Public Instruction Council. It was destined to the teaching in Normal Schools, published in Madrid in 1852 (Figure 2).

Preliminaries, objectives, and finalities

The textbook recognizes legal aspects linked to the Spanish weights and measures: The real order of January 26th, 1801, established by Carlos IV, which attempted to unify the traditional metrology in Spain; and the law of July 19th, 1849 which implemented the units of the metric system. The older system is widely presented highlighting the units and the equivalences.


ELEMENTOS

ARITMÉTICA,

DON JOAQUIN AVENDAÑO,

Segunda edicion.

MADRID

Fig. 2. First page of the textbook Elements of arithmetic

The rating of these Elements shows the relevance of the foundations and principles of arithmetic in this work linked to Pestalozzi's doctrine, as we mentioned "when title of a document dedicated to initial teaching includes 'Elements of…', it is detected a Pestalozziana influence" (Picado, 2009, p. 65).

It is important to point out that a stated object, as part of the elaboration of the textbook, is not explicitly declared. From the point of view of the learning objectives, the formative, political, cultural, and social intentions are highlighted because the textbook is destined for the training of teachers to use specific contents and developed capacities in an instructional center, particularly the Normal Schools.

The textbook is organized in three parts to cover general themes: calculation, measurements and problem solving. Each one of these units is structured in sections that highlight specific topics. The ideas are written using a narrative style.

Concepts

Just as in the majority of textbooks analyzed, the quantity and the units are conceived from the Newton (1761, 1802) and Euler (1810) notions. The author follows a classification for the number (integer, abstract, and concrete) without giving a general definition, useful to link with the previous mentioned notions.

The exposition of the Decimal Numbering System (DNS) is linked to the presentation of the numbering that highlights the writing, reading, and the forming of numbers (orders). This is perceived at the end of the instructions to build numbers: "The fundamental principal of this enumeration is that ten units of any order form one unit of the immediate superior order. The base of this system is ten and its name is the decimal system" (Avendaño, 1852, pp. 8-9). The presentation of the common fractions comes after the exposition of the arithmetic calculation operations with integers. The fraction is defined as "any number lower than the unit" (p. 31) whose origin is located in the inexact division of numbers (the division with rest). This highlights that the teaching tradition is located far away from Newton's ideas, who presents the fractions as result of the measurement.

In the same manner, the highlighted relationship that is established between the improper fractions and the concept of fractional number (that corresponds to a mixed numbers): "if the numerator is greater than the denominator, the fraction is called fractional number (…) is therefore a fractional number, or an improper fraction, as it is generally called" (Avendaño, 1852, pp. 31-32).

Regarding the decimal fractions, decimal numbers, or simply put decimals – signals the author – they are defined as the "compound fractions of parts that go ten by ten times less than the unit" (Avendaño, 1852, p. 51) keeping the term of decimal numbers to the decimal fractions preceded by one or several whole units. This statement establishes and highlights the link between the decimal fractions and the DNS.

> La sucesión de las fracciones decimales siguen el mismo principio de nuestra numeración, en la cual toda cifra colocada á la derecha de otra vale diez veces menos que esta. Así las fracciones decimales se forman considerando la unidad dividida en diez partes iguales… (Avendaño, 1852, p. 51)[3].

The exposition of the MS is preceded by an approach of the traditional metric system in Spain. This information includes notions such as to measure, measurement unit, magnitude, and measurement classes. Linked to the notions of quantity and unit, the action of measuring is conceived as "to look how many times a quantity contains another one of the same type, that it is taken by a unit of measurement", understood the latter as "a known quantity, taken as term of comparison between quantities of the same type, whose relationship want to be expressed in the form of a number" (Avendaño, 1852, p. 67).

The statements work as a complement of the initial ideas. They relate the quantity and unit with the act of measuring that at the same time provide the notion of magnitude as a measurable quality in certain quantities (genres) and whose result is a number.

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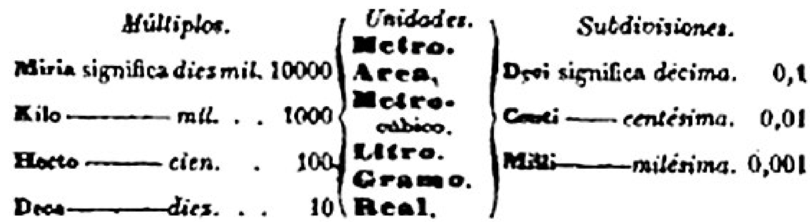
[3] The succession of the decimal fractions maintains the same principal of our numeration, which considered the number to the right of another number, ten times smaller than this. As such, the decimal fractions are formed taking into account the unit divided in ten equal parts.

The MS is "the new metric system in Spain [defined in this manner] because the meter is its base [also known as] the legal system due to its use; is prescribed in all public acts, and it is mandatory to every person since January 1st, 1860" (Avendaño, 1852, p. 73).

The meter as a unit of measurement is described from a scientific and etymological conception. It is the base of the new metric system from which all the measurements are derived, and its length is verifiable in time and place (country).

Previously to the exposure of the MS's main units, the text recognizes measurement classes (understood as the magnitudes that can be measured by some quantities). These are the linear dimensions, surface, volume or solidness, and capacity, the weights and the currency to measure monetary issues. For these, the principal units of meter, square meter and area, cubic meter and liter, gram and the "real" (Spanish currency) are defined. In terms of the variations, it has been specified that, by law and the smallness of the gram, the usual weight unit is the kilogram. "The multiples and subdivisions of each of the units are referred to the decimal metric system in a way that each unit, ten times greater than the one of an inferior order is ten times less than the one of a superior order" (Avendaño, 1852, p. 73).

With the introduction of the superior and inferior units to each of the main units, the author clears up the relationship between the DNS and the MS. Even better, the influence exercised by the decimal numeration in the definition of the new concept of weights and measurements. Its nomenclature comes from the Greek and Latin words, just as Figure 3 shows.



Múltiplos.		Unidades.	Subdivisiones.	
Miria significa diez mil.	10000	Metro.	Deci significa décima.	0,1
Kilo — mil.	1000	Area.	Centi — centésima.	0,01
Hecto — cien.	100	Metro cúbico.	Milli — milésima.	0,001
Deca — diez.	10	Litro.		
		Gramo.		
		Real.		

Fig. 3. Multiples and submultiples (p. 74)

The presentation of multiples and submultiples becomes wider and particularized for each one of the considered measures. In this manner, they are defined in terms of the main unit and are presented with their correspondent equivalence with their previous counterpart in Castilla's system and are excluded some because of the singularities of their size.

The currency system has as the main unit the "real" for which a multiple, the Isabel's doubloon (100 "reales") and a sub-multiple, the decimal "real" are defined. In addition, other currency permitted is annexed with their non-decimal equivalences.

Procedures

The shown procedures are centered in arithmetic. The operations are described with numbers "complex"[4] and "non-complex", where you would expect at least a partial use of the decimal-metric units, are reserved to the older units. This does not allow the presentation of examples that can illustrate the usefulness of the MS in different contexts. That happens in the third part of the text (dedicated to problem solving), where the author employs the recent metric units for posing problems and solving them, related to the proportionality and other rules, with a diversity of daily life situations (Figure 4). Also, in this text, it could be found procedures to the writing and reading of metric numbers.

Representations and contexts

The representation of concepts is mainly gathered by its verbal expressions. The symbolic and tabular modes are also used. These two last modes are identified in numeric expressions, abbreviations, calculation signs and the organization of information in tables and using symbols.

There is a wide use of specific situations for the new metric concepts, in natural, commercial, and social contexts, both in problems as in examples. Frequently, to introduce new weight units and measurements, are commonly used the mathematical operations, the mention of new concepts and geometric figures; usual phenomena are the exposition of physical and natural conditions, trade situations dealing with cost, buying, and selling of products.

2. **Problema XXIV.** *Un comerciante desea cambiar paño por colonía: 2 metros de paño valen tanto como 3 metros de casimiro y 5 metros de casimiro tanto como 7 metros de colonía. ¿Cuántos metros de colonía recibirá el comerciante por 60 metros de paño?*

Segun la enunciacion del problema, diremos:

Un metro de paño vale $\frac{3}{2}$ metros de casimiro, y un metro de casimiro vale $\frac{7}{5}$ metros de colonia: luego un metro de paño valdrá los $\frac{3}{2}$ metros de $\frac{7}{5}$ de colonía ó $\frac{21}{20}$ de colonía.

Los 60 metros de paño valdrán por consiguiente 60 veces $\frac{21}{20}$ de colonia ó sea 126 metros de colonia.

Fig. 4. The meter into a trade problem (p. 107)[5]

[4] In this textbook, the "complex number" (or "denominated number") is the number formed by units of the same "species". For example, 8 Kg, 3 Dg, 2 g (read this as 8 Kg with 3 Dag and with 2 g; this is, as a just one number).

[5] The resolution includes an author's calculation error.

Tasks

The tasks include the exercises solved. Additionally, examples that reinforce the theoretical ideas presented are used following the most common sequence identified up to now: theoretical ideas and examples.

Textbooks for Normal Schools: The training of teachers

The analysis of the textbooks used in the Normal Schools has permitted the recognition of particularities in the conceptual, procedural, and didactical characterization of these documents. Table 2 presents information on the conceptual complexity into the five textbooks, along the three historical stages. Table 3 shows the procedural complexity.

Table 2. Conceptual Content in the Textbooks for Teacher Training

Stage 1: Textbooks edited from 1849 to 1867	**Stage 2: Textbook edited from 1868 to 1879**	**Stage 3: Textbooks edited from 1880 to 1892**
Facts		
Terms: zero, one, two…, nine; ten, one hundred, one thousand, ten thousand, tenth, hundredth, thousandth, millionth. Deca, hecto, kilo, miria, deci, centi, milli. Half, third, fourth, fifth… Addition, multiplication, subtraction, division.	*Terms:* zero, one, two, three … ten; higher / lower than; deca, hecto, milli, miria, deci, centi, milli; tonne, quintal; ten, hundred, thousand unit, million unit, trillion…; tenth, hundredth, thousandth, millionth. Addition, multiplication, powers, subtraction, division, roots.	*Terms:* zero, one, two, three… ten; one hundred, one thousand, million; deca, hecto, milli, miria, deci, centi, milli. Addition, multiplication, subtraction, division.
Notations: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; ¾, ½, 1/10… (fractional); 8 2/3 (mixed); 0031; 0,666… (decimal expansion); x, +, -, :, =;10, 100, 1000, 10000.	*Notations:* 1, 2, 3, 4, 5, 6, 7, 8, 9, 0; ½, 7½… (fractional, mixed); +, -, :, x, (·); =, >, <; (ʻ), (,); ( ); 6^2, $(6+1)^2$; m, dm^2, Km^3, ml, dg, p, Tp, Qm; 10, 100, 1000; $\sqrt{129}$; 0,1; 0,001,…	*Notations:* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; +, -, · , :, =, <, >, ( ), ¾, ; 10, 100, 1000, 10000; 2’45, 0’222,… 2
Conventions: Metro as the ten-millionth part of the quadrant of the terrestrial meridian. Names for main units (meter, square meter, area, liter, gram, cubic meter). The types of units and the frequency of the orders of the numbering system [(u, d, c), (um, dm, cm)], [(uM, dM, cM)]… Reading integers and	*Conventions:* Meter: word derived from the Greek, ten-millionth of the meridian quadrant passing through Paris. Definition of the main and usual units from meter. Name of the main units: meter, area, cubic meter, liter, gram, peseta. Grades of units and the frequency of the orders of the numbering system [(u, d, c),	*Conventions:* Meter: ten-millionth of the meridian quadrant passing through Paris. Definition of the main and usual units from meter, name for the main units (meter, area, liter, gram, cubic meter). The types of units and the frequency of the orders of the numbering system [(u, d, c), (um, dm, cm)], [(uM, dM,

Stage 1: Textbooks edited from 1849 to 1867	Stage 2: Textbook edited from 1868 to 1879	Stage 3: Textbooks edited from 1880 to 1892
decimals numbers. Table for adding, subtracting, multiplying. Placing the terms of a operation. Formation of multiples and submultiples with Greek and Latin voices and words.	(um, dm, cm)], [(uM, dM, cM)]… Place value. Reading and writing numbers integers and decimal numbers. Placing the terms of an operation. Formation of decimal multiples and submultiples from Greek and Latin words, and decimal equivalents.	cM)]… Place value. Reading and writing integer and decimal numbers. Multiplication tables (Pythagorean). Placing the terms of an operation. Formation of decimal multiples and submultiples from Greek and Latin words and their decimal equivalents.
Results: ten units form a new higher order; each division by ten forms a lower order; equivalence tables	*Results:* ten units form a new higher order; equivalence tables; square units increase and decrease every 100 units; cubic units increase and decrease every 1000 units	*Results:* ten units form a new higher or lower order; square units increase and decrease every 100 units; cubic units increase and decrease every 1000 units; equivalence tables
Concepts		
Quantity, unit, number, magnitude, measure; relationship between quantities; comparing quantities; expression of the quantity, number (whole, "broken" or fraction, decimal broken, others); DNS; length, area, volume, capacity, weight, currency, time; MS; meter, square meter, area, liter, cubic meter, gram, real.	Magnitude, quantity, size, unit, measure, number (commensurable and immeasurable, integer, fractional, decimal fraction, concrete number, homogeneous, others); DNS; MS; meter, area or square decameter, cubic meter, liter, gram, peseta; length, area, capacity, volume, weight, currency	Meanings of quantity (continuous and discrete), unit, number (unit, expression, amount), number classification (whole, broken, others; simple or compound, abstract, concrete); comparing quantities; DNS; meter, liter, gram (from Greek), peseta, cubic meter, area (from Latin).
Structures		
Natural numbers; broken numbers; decimal broken; MS; every kind of measure	Natural numbers; fractional numbers; MS; species of measures (main unit, multiples and submultiples)	Natural numbers, broken numbers; DNS; MS; species of measure (main unit, multiples and submultiples)

In relation to the teachers training, in the three stages the presentation of the concepts of quantity, unit and number were identified with some particularities into the presentation of the magnitude and the measurement.

In the textbooks there was not a formal or a specific presentation of the DNS. This was exposed as part of the conventional numeration. Regarding the MS, particularities related to the presentation of this structure in each one of the historical stages were identified. The system was a new structure derived from the DNS and the establishment of a logic unit of length: the meter. This condition prevailed while it was promoted and generalized. When the 19th century ended, it was emphasized in the regulatory descriptors that reflect the last governmental dispositions for their definite implementation and mandatory usage. Also, it is of importance to highlight the uniformity in the presentation of the gram as the main unit to the weighing, and, in a less unified manner, the omission of the cubic meter and other volume measurements for primary education. These particularities based the teachers training on scientific knowledge, specifically for primary instruction, and shown the interest of highlighting the real usefulness of the MS in science and society.

Just as in a mathematic structure, the MS was composed of a fundamental unit, main and secondary units (multiples and submultiples) for each one of the measurement species considered. These units "interacted" from their own metric derivations and the established equivalencies through the decimal fractions, which are used in the DNS. The latter allowed highlighting the link that is established between these systems (DNS and MS). In like manner, other structures were recognized in the presentation of the natural and fractional numbers with which the metric-decimal units of the new system were linked to.

Based on the facts, is perceived an etymological approach to the presentation of multiples and submultiples had, that was complemented with the decimal equivalences drawn from their meaning. The arithmetic operations were the basis of the calculation using metric numbers. In the textbooks, there was found a dichotomy regarding the literal notations. It was in the second stage where abbreviations for the presentations of main units, multiples and submultiples, were found. The numeric notations follow the same pattern throughout the period.

An interesting aspect for this educational level was the continued use of the old units for weights and measurements in calculation tasks and their presence in the content of the textbooks until the end of the period. Its usage was constant until the mandatory stage of supporting the establishment of the new system, together with the difficult process of incorporation and acceptance of the MS in the Spanish society of the 19th century. The incorporation of data about the origin and benefits of the system was a common component in the textbooks. Just as in the dispositions regarding its adoption and legalization, these data provided teachers the information that the political and educational system had implemented for their students.

Table 3. Procedural Content in Textbooks for Teacher Training

Stage 1: 1849-1867	**Stage 2: 1868-1879**	**Stage 3: 1880-1892**
Abilities		
Writing and reading of metric numbers; operations with "complex numbers"; reduction of units	Reduction of units. Operations with metrical numbers	Reading and writing of metrical numbers. Reduction of units, establishment of equivalences
Reasoning		
Figurative: for expression of square meter and cubic meter		Figurative: for expression of square meter, area, and cubic meter
Strategies		
Problem solving	Problem solving	Problem solving

The textbooks focused their attention on procedures, alternatively in the stages, for the calculation of the metric units, their reading and writing and the reduction of units, without a consistence presentation in the period for this teaching level. However, the development of skills and the use of strategies for problem solving stood out.

A common characteristic from the textbooks was constituted by the systems (modes) of representation. Through the use of verbal, symbolic, and tabular modes, the authors presented the concepts and procedures related to the MS. The graphic representations were not used in the textbooks (see Table 4).

Table 4. Representations, Contexts, and Tasks in Textbooks

Stage 1: 1849-1867	**Stage 2: 1868-1879**	**Stage 3: 1880-1892**
Representations		
Verbal, symbolic, tabular	Verbal, symbolic, tabular	Verbal, symbolic, tabular
Contexts		
Natural: physical and natural elements (alternative). Mathematical: arithmetical calculation, use of geometrical concepts. Commercial: costs, buying and selling. Social.	Natural: physical and natural elements. Mathematical: arithmetical calculation, areas and volumes of geometrical bodies. Commercial: buying and selling. Technical: distances, construction and textile industry. Social: pay taxes, historical-political stories.	Natural: physical and natural elements (gram). Commercial: buying and selling. Mathematical: arithmetical calculation. Technical: construction, textiles and topographical situations.
Tasks		
Examples, exercises. Sequence concept-example.	Exercises for using metric units. Sequence: concepts, exercises. Using metric units in algebraic tasks.	Examples and exercises; refresh tasks

The presentation of concepts and procedures that were part of the new system was introduced and recognized through situations that were framed in the natural, commercial, mathematical, and technical domains, with some approximations to the social context, especially in the first two initial stages. The explanation and application of mathematical concepts and skills, form part of the training of teachers and was boosted through examples, exercises, and evaluative questions that prevail in the first two historical stages.

Lastly, the political, formative, cultural, and social aims were recognized in the textbooks, in each one of the stages. The teaching and the learning of certain mathematical contents for primary instruction was projected. The textbooks designed to the teachers training level have some specificities in the use of decimal metric units primarily through rote learning. In spite of being documents thought for a professional training in the field of education, the difficulties and the errors of learning identified in these textbooks could require attention for future teachers.

Last considerations

Analyzing the implantation of the MS in mathematics textbooks for teachers training in the Spanish Normal Schools has allowed us to recognize particularities that characterize the growing development of this mathematical structure in the second half of the nineteenth century (1849-1892).

The presentation of facts, concepts and structures maintained its uniformity in the three defined stages. This has highlighted a wide exposure of arithmetical and metrological concepts during the period.

In the two stages between 1849 and 1879 the MS was presented as a novel structure. For the last two decades of the nineteenth century, the emphasis was on highlighting the rules of its incorporation. The MS was structured from the DNS and the meter. It consisted of main units, multiples and submultiples for species of length, area, volume, capacity and weight. The exposure of the gram as the main unit was consistent throughout the period. The kilogram appeared as one of its multiples. The cubic meter was approached for its instruction in higher levels of education (professional degrees).

The preparation of teachers in metrological matters was guided by its scientific fundamentals and their immediate applications in the most frequent activities of society. However, during the period, the weights and measures of Castile maintained their presence in the texts. That is, teachers were trained both in the MS and the older Metrology.

The explanation and justification of the MS were associated with historical and legal aspects and with its social impact that gave comprehensive training to teachers on the system and the contexts in which it was implemented.

The reduction of units and problem solving were the skills and strategies that consistently appeared in the texts. Others followed a discontinuous pattern in the different stages.

The modes of representation used were verbal, symbolic and tabular. The use of the MS was illustrated with mathematical, commercial and natural situations in the three stages. The use of technical context has been identified in the generalization and compulsory stages, and the social context in the enactment and generalization. Thus, the period 1869-1879 presented the greatest diversity of situations. Tasks were focused on the basis of presenting examples and laying out the exercises.

The elaboration of the textbooks was guided by four purposes: political, educational, cultural and social. That it means that the textbooks were edited responding to political requirements; they were aimed for teaching and learning of mathematics, for disseminating the mathematical culture of the time, and for promoting the use of the MS in social activities. The specific goal of the authors was to develop a text to train teachers to teach the metric system by means of the arithmetic.

They did not have explicit common references on how to understand learning. However, it is easy identified a shared methodological trend, the rote learning.

Recognition of limitations was raised in the first stage, such as difficulties in establishing correspondence between systems, which pointed to the conflict in implementing a new metrological system while still using a different one.

In summary, textbooks for the training of teachers facilitated clarity and security in presenting the MS. With a wide conceptual foundation that was accompanied in the three stages by procedures, examples and exercises, illustrating their usefulness. They provided external ideas that helped its understanding, emphasizing, from the beginning, the legal and social benefits it entailed. Treatment of the MS for teachers level was consistent and successfully.

Acknowledgment. Miguel Picado thanks to the Scholarship Board of the National University of Costa Rica and the Incentive Fund of the National Council for Scientific and Technological Research of the Ministry of Science and Technology of the Republic of Costa Rica.

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Teaching of mathematics in educational journals of Turin (1849-1894)

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Abstract

Since the first half of the 19th century, the most renowned intellectuals in the Savoyard Kingdom put effort into facing up the backwardness of the educational system and in particular into enhancing technical-scientific studies. Educational journals were one of the ways to achieve these aims (Chiosso, 2013). *From 1849 to 1894 in Piedmont the* Società d'Istruzione e di Educazione*, a Turinese association of politicians, professors and teachers, which represented a unique case in all of Italy, published the first successful educational journals of the country. Since the first periodicals dedicated solely to the teaching of mathematics flourished just in the 1870s, our purpose is to investigate the role of the teaching of mathematics in the journals of this association, by showing the knowledge and working habits between primary, technical and secondary schools.*

We examine how Savoyard Kingdom stood out for its educational journals, which were free from the censorship contrary to other Italian territories and which payed attention towards the elementary and secondary scientific teaching. We also analyse the situation after the Italian Unification, when a conspicuous number of 'practical educational journals' for elementary schools, which provided weekly articles about the teaching of mathematics, were published in Turin. This situation changed when in the 1880s the mathematical journals for teaching blossomed.

Introduction

Studies concerning the first half of the 19th century in Savoyard Kingdom[1] have already underlined the role of intellectuals and professors of the University of Turin in promoting technical-scientific education (Pepe, 2012; Roero, 2013). Since the promulgation of Boncompagni (1848) and Casati (1859) laws, the educational system underwent deep changes, adopted in all of Italy after the Unification (1861). Specific programs were offered for every school level and grade, which

[1] After the Vienna Congress (1815) the Savoyard Kingdom included Savoy, Piedmont, Liguria, county of Nice, and Sardinia. It was the core of the process of the creation of the new Italian state.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

included a heavy focus on scientific disciplines concerning both the contents and the methodological approach.[2]

Nevertheless all these improvements weren't at first effective, because of the underdevelopment of the country (Chiosso, 2013). Before 1848 the 69% of the citizen in Piedmont was illiterate (Morandini, 2003, pp. 52-63) and most of the literates attended schools steeped in humanistic disciplines and rote learning. Teachers were mostly unprepared. The first teacher training schools were established in 1844 in Turin and, even if they were at the forefront of education methodologies in Italy and admired by many intellectuals, they were not able to overcome the need of all the territory.

Mathematics teaching in primary schools was restricted to elementary arithmetic operations and in secondary schools to basic notions of arithmetic, algebra, and plane and solid geometry. Four years after the Boncompagni law, the educationist Domenico Berti (1820-1897) stated that "in many of our secondary schools teaching is the same as in 1772" (1852, p. 318). Schoolbooks that circulated at that time in the territory were in many cases not updated and often full of mistakes (Giacardi & Scoth, 2014, p. 210, 212). To contrast these issues, educational journals proved worthy (Chiosso, 2013).

In this paper we analyze three journals published in the second half of the 19th century in Turin, dedicated to primary, secondary and university teaching. Firstly, we show the comparison with the other Italian Kingdoms as regards educational periodicals. Secondly, we investigate the role of mathematics teaching in these journals.

The Italian context: the first educational journals in the 1840s

Since the 1830s journals addressed to elementary teachers and "family men" started to circulate in different Kingdoms of Italy. One of the first Italian educational journal, the *Guida dell'Educatore* (1836-1845), was published in Florence (Grand Duchy of Tuscany). It spread the educational ideas of the renowned educationist Raffaele Lambruschini (1788-1873), and continued in the successful journal *Letture di famiglia* (1849-1883). At the same time in Venice a similar journal, *L'Istitutore* (1836-1837, 1851-1866), was published for the elementary and technical teaching.

These journals promoted the improvement of elementary teaching, by illustrating new methods in theoretical and practical articles, and by describing favour-

[2] On the mathematics curricula for primary, secondary and technical schools between the Boncompagni and the Casati laws see (Pizzarelli, 2013); on the following period see (Giacardi, 2006); (Pepe, 2013); (Giacardi & Scoth, 2014), (Pepe, 2016).

able examples of foreign educational systems. The teaching of arithmetic and plane geometry had the same space as the one of reading, writing and religion.

These were the main characteristics of the educational journals, which spread since the 1840s firstly in Tuscany and then also in the Papal State and in the Savoyard Kingdom. The political context in other Kingdoms of the Italian territory was less favorable: in the Kingdom of Lombardy-Venetia, despite an intense debate on education among intellectuals, and the birth of a considerable number of periodicals, until the 1850s there weren't educational journals, because of the censorship of the Austrian Empire. Likewise in the Kingdom of the Two Sicilies, even though advanced ideas circulated for long time, in 1843 education was under control of the church and the press was censored by the government, especially after the revolutions of 1848 (Chiosso, 1989).

The first educational journal in Rome was *L'Artigianello* (1845-1848), devoted to elementary teachers and young artisans. It achieved success in many schools, where in some cases it replaced textbooks. A long series of lessons about plane geometry for practical use was published for three years (1845-1847), providing topics written in a more simple language than that of the common textbooks, and full of examples and illustrations. But the journal lasted for short time, because of the return of the censorship after the 1848.

In the Savoyard Kingdom a favorable environment was set since the 1830s, thanks to new reform policies and the social commitment of many intellectuals and scientists. The first successful educational journals from Turin were the *Letture Popolari* (1836-1842), *Letture di Famiglia* (1842-1847) and *L'Educatore Primario* (*EP*, 1845-1848) for primary teaching. They contributed to spread the new educational ideas by Heinrich Pestalozzi (1746-1827), which became the basis of the advanced teacher training schools held in 1844 in Turin.

In particular, as regards mathematics education, the editors of these journals focused on the mission of changing the custom of teaching from a dogmatic, superficial and parrot-like approach, into one more linked to everyday life, to intuition and to reasoning. Besides theoretical articles by educationists, teachers provided examples from their practices in classroom. Some of them dealt with elementary arithmetic (e.g. mental calculations, fractions, the use of the abacus) to be used in everyday teaching in primary schools.

The decimal metric system for weights and measures was a topic particularly important. It was introduced in the Kingdom in 1845 and became compulsory in 1850. The journals of this period contributed to popularize the new system and to spread the conversion tables, through educational articles and favorable reviews of recent updated textbooks.

The journals tried to differ from the schoolbooks, by applying new methodological ideas and by proposing topics of arithmetic and geometry in the form of 'maieutic dialogues' between teacher and pupils. They also focused on promoting observations, practical problems and the use of everyday objects, in order to bring step by step to the discovery of mathematical properties and to the abstract reasoning (Berti, 1849). Some of the ideas were particularly innovative. In 1845 a teacher promoted the use of colored strips of different size to introduce the operations with fractions (*EP*, 1, pp. 527-536, 546-550).

Mathematics teaching topics for secondary schools and universities were rarely available in the educational journals. An example is an article on the physics applications of the inverse-square law, published in 1845 by *L'Educatore Primario* (pp. 417-424).

The journals of the first teachers' association in Piedmont

After the Albertine Statute in 1848, the government of the Savoyard Kingdom extended and granted the freedom of the press and of association, in contrast to the majority of the other Italian Kingdoms. In 1849 this brought to the birth of the first teachers' association in Italy, which represented the entire scholastic system in Piedmont, called *Società d'Istruzione e di Educazione* (*SIE*, 1849-1893) (Morandini, 2003, pp. 210-230; Pizzarelli, 2013).

Its success was testified by the increasing number of members (from 1250 in 1850 to 4019 in 1923), which were the two seventh of all the teachers in the Kingdom in 1855 (Pizzarelli, 2013, p. 32). The peculiarity of the society was the presence of representatives from every educational level and grade (teachers and professors of Turin University from different Faculties, both scientific and humanistic; principals and deans of public and private schools; university students, etc.) and of politicians (past ministers of Public Education, functionaries of ministerial scholastic councils, etc.). There were members of institutions, like the Turin Military Academy and the Academy of Sciences, churchmen, lawyers, engineers, architects, etc. For our purpose it's interesting to notice that the honorary president was the mathematician Carlo Ignazio Giulio (1803-1859), professor of rational mechanics and dean of the University (1844 to 1848), who put effort into modernizing the scientific education in Piedmont and in 1845 founded the first technical schools for workers in Turin (Roero, 2013).

The association organized yearly general conventions in different cities of Piedmont, which involved most of the local population. There were sections for every level (primary, technical, secondary, university), dealing with common problems: programs, textbooks, teaching methods, materials, salaries, etc. The meeting's reports were published in the *SIE*'s journals: the bi-weekly *Giornale della Società d'Istruzione e di Educazione* (*GSIE*, 1849-1852), and the weekly *Rivista delle Università e dei Collegii* (*RUC*, 1853-1854) and *L'Istitutore* (1852-1894).

The *GSIE*, which involved the collaborators of the previous journals of the 1840s, is considered the most distinguished educational periodical at that time in the Kingdom (Chiosso, 2013, p. 293) and it was addressed to every educational level. The board members changed every year and were equally chosen among primary and secondary teachers, and professors coming from the five Faculties of the University (Medicine, Law, Philosophy, Mathematical and Physical Sciences and Theology). The *RUC* and *L'Istitutore* were founded after the *GSIE* ended and followed specific topics: the former was devoted to secondary and academic teaching and the latter to primary and technical teaching.

The *GSIE* was born after the Boncompagni law. The editors tried to keep teachers, professors and also parents updated on the new rules and programs, by editing laws and official newsletters by the Ministry of Public Education. At the same time they published many critical articles about the local educational system. The main topics were about the reorganization of the curricula for each discipline, the comparison between primary and secondary programs, the need for high quality textbooks and innovative teaching methods. They were often supported by statistical and historical news and by reports on European educational systems, overall France and Prussia. Thanks to the presence of politicians among the members, the *SIE* succeeded into influencing some of the scholastic laws that constituted the basis of the Casati law (Pizzarelli, 2013).

As regards mathematics in the secondary level, editors privileged an action of monitoring and denouncing on the educational system shortcomings (e.g. rote learning, imbalance between humanistic and scientific disciplines, inadequate textbooks), instead of opting for a more practical educational support. In particular the journal stressed the need for scientific textbooks written in Italian and not in Latin and for a link between primary and secondary level.

The *GSIE* was particularly interested to promote technical-scientific studies, which were introduced in the educational system in 1848. The new schools provided a deeper scientific teaching than that provided by schools of humanities. They addressed to artisans, traders, managers and engineers, which were professions very important for the recent industrial developing of the country. The editors underlined the importance of these schools, overall during congress' debate, and promoted the use of the *Elémens de Géometrie* by Alexis Claude Clairaut (1713-1765), whose translation by Giulio in 1850 was officially approved for technical schools by the Ministry and had different reprints.

Moreover the journal applied pressures on the government to establish a polytechnic school in Turin, following the French and German examples. The goal was accomplished in 1852, when the *Regio Istituto Tecnico di Torino*, later *Regio Politecnico* (1906), was created.

Most of the educational articles for elementary school aimed at applying the new educational programs in class and supporting teachers with the day-by-day routine. There were some proposals of dialogues on mathematics, regarding overall the operations with integer numbers and fractions, and the metric system. We remark a particular commitment of the editors in the teaching of the nomenclature of the first elements of plane and solid geometry, which was considered marginal by comparison with arithmetic in the previous years. A new textbook about elementary geometry, written in form of maieutic dialogues by a member of the *SIE* (Rayneri, 1851), was published in several issues by the *GSIE* in 1852 and it had many reprints in the following years. Thanks to this innovative form of presenting mathematics topics, the journal provided something different from the textbooks and immediately usable for unprepared teachers.

What emerged from the journal was also a need to consider new teaching methods, based on intuition and observation. The accurate descriptions of solid and plane figures, compared with common objects, were preferred to abstract and mnemonic definitions. Different educational materials were promoted, like the abacus and the wooden and cardboard scale models of geometric solids, also little stones and dices to learn counting.

They put attention to elementary mathematics textbooks too, by underling the importance of a gradual approach and clarity, and of guidebooks for teachers, which were rare at that time. It emerged also the need for a books' list selected by the government in order to make uniform the Kingdom's education and to contrast the overflow of low quality books. It's interesting to notice that in 1853 the government, inspired by the *SIE*'s initiative, created award contests for the best textbooks.

The *SIE* was known and admired in the North-Central Italian territories. Intellectuals and educationists from these countries were in the *SIE*'s list of members. In particular the Kingdom of Lombardy-Venetia had a strict relationship. In 1861, after the liberation from the Austrian Empire, the *SIE* established a twinning with a similar association of primary teachers from Milan, the *Pio Istituto dei Maestri di Lombardia*, which published the periodical *L'Educatore Lombardo* (1857-1860), then *L'Educatore italiano* (1861-1885). In the previous years in Milan similar educational journals were born, like *L'Educatore* (1850-1853), for elementary and technical schools, and the *Rivista Ginnasiale* (1854-1859), which defended and promoted the humanistic secondary studies.

Journals for secondary and university teaching in the 1850s

The experience of the *RUC*, even though short-lived,[3] is particularly important, because it was the very first Italian educational journal devoted to secondary and university teaching of humanistic and scientific disciplines, where wide space was given to scientific topics and advanced mathematics.

No specific journals addressed to mathematics teaching existed at that time in Italy. In 1850 in Rome Barnaba Tortolini (1808-1874) published the mathematical

[3] The journal ceased in 1854, because of the inconstancy of many authors.

journal *Annali di Scienze Matematiche e Fisiche* (1850-1858), continued as *Annali di Matematica Pura ed Applicata* (1858-).[4] This journal was focused mainly on research in pure and applied mathematics, rather than on teaching.

Regarding technical-scientific studies, the Milanese journals *Il Politecnico: Repertorio Mensile di Studj Applicati alla Prosperità e Cultura Sociale* (1839-1844, 1859-1869) by Carlo Cattaneo (1801-1869) and since 1865 by Brioschi, and *Il Giornale dell'Ingegnere Architetto Civile e Agronomo* (1853-1868) by Raffaele Pareto (1812-1882) should be mentioned. Their goal was to spread the recent results of the scientific research applied to different technical fields, and only since 1869, when they merged into *Il Politecnico* (1869-1927), secondary and superior technical education became the main topic (Lacaita, 2012).

Therefore the *RUC* was an innovation among the educational periodicals of that time in Italy. Its main goal was to monitor the level of the university students and of secondary teachers in Piedmont and to offer updates upon the most relevant local, national and international news in the scientific fields.

The editor board was directed by Carlo Cadorna (1809-1891), past minister of public education, and the members were secondary teachers and professors, who took on scientific and humanistic sections according to their competences. For example, *Mathematics and mechanics* section had as compilers a professor of technical schools in Turin, and Genocchi, professor of algebra e geometry at the University of Turin since 1856.

The periodical reported weekly news about the current educational situation and scientific academies in Italy and in many foreign countries in the world (France, England, Austria, Russia, etc), and bibliographic reviews. But the true innovation, that set it apart from the previous educational journals, was the remarkable number of sections dedicated to sciences. A monthly review, entitled *Scientific publications*, provided news on scientific discoveries, related to mathematics, astronomy and scientific instruments. There were reports on ancient measurements of lands, on the astronomical calendar of the Egyptians, and on the discovery of an Arabic translation of the Commentary upon the X Book of Euclid's *Elements* made in 1850 by Franz Woepcke (1826-1864). Readers could also find summaries of memoirs published in local, national and international scientific academies, institutes and societies, pertaining mathematics, astronomy, chemistry, geography, meteorology, industry, etc.

One of the most interesting sections of the journal concerns the weekly meetings of the *Società delle Conferenze sull'Istruzione Tecnica* (*SCIT*), an association born in 1853 in Turin to promote advancement in technical-scientific education at all levels. Many members of the association were actually professors, among whom the mathematician Genocchi. He was among the promoters in the *SCIT* of mathematics programs

[4] Angelo Genocchi (1817-1889), Enrico Betti (1823-1892) and Francesco Brioschi (1824-1897) joined Tortolini in the editorial board (Bottazzini, 2000, pp. 71-84).

for primary, secondary and technical schools, in which topics and methods were deeply specified (*RUC*, 1853, pp. 113-115). They were particularly advanced and innovative. For example for technical schools they proposed to introduce combinatorial analysis, algebra's applications to geometry for the graphical representation of the laws of size variations, and the theory of orthogonal projections applied to building and machinery used in factories (pp. 168-169).

Genocchi's collaboration with the *RUC*, during the most prolific period of his activity on number theory, gave many important contributions to the periodical concerning recent mathematical innovations. In his opinion some of his works, published between 1853 and 1854, were apt for secondary and academic mathematics teaching, thus their summaries found their place on the journal. An example is the memoir about the transformations of multiple integrals (1853b), which he decided to publish for the clarity of the language used. Also Genocchi's notes on the convergence of Binet's series (1853a) and on Euler's theorem (1854a), published by the *Bulletin of the Brussels Academy*, and an educational article on numerical approximations (1854b) appeared among the reviews of the *RUC*. According to the author, the simplicity of the contained demonstrations was accessible to secondary students and also suitable for textbooks. Among the other mathematics topics at an academic level, the *RUC* published the recaps of some memoirs on Calculus and Geometry, like the proprieties of the gamma function and the minimal surface bounded by a skew quadrangle.

The *RUC* was a precursory initiative that, concerning the aims, the audience and the link with both the national and international scientific community, resembled the successful mathematical periodical *Giornale di Matematiche: ad Uso degli Studenti delle Università Italiane* (Naples, 1863-1967), edited by Giuseppe Battaglini (1826-1894) until 1894. In fact its goal was to offer news and tools, which could be useful both to academic courses and advanced research, on national and international ground. Nevertheless the mathematics contents and the structure of the two journals were absolutely different: the *RUC* was mostly linked to the educational journals, the *Giornale di Matematiche* to the mathematical ones.

'Practical educational periodicals' for elementary schools after the Italian unification

Despite the loss of political importance due to the government being moved in 1864 to Florence, Turin played a leading role in the Italian educational publishing industry after the Unification. This was due to the successful experiences of the previous decades and to a substantial presence of dynamic protagonists in the educational field. Among them, we find renowned writers of textbooks for elementary classes, like Giuseppe Borgogno (1820-1879) and Vincenzo Scarpa (1836-1912), who penned books on arithmetic, geometry and the decimal metric system, which had a great circulation in Italy (Chiosso & Sani, 2014). After the 1870s other Italian cities, overall Milan and Rome (the new capital city), Palermo and Naples (Chiosso, 1997, pp. 7-8), took on a notable role in the educational periodicals' field.

L'Istitutore, which became the *SIE* house organ, has been considered "the first true educational periodical with a national reaching" (Chiosso, 1997, p. 366) and "one of the most renowned journals for schools since 1850, not only in Piedmont, but in all of Italy" (Romano, 1925, p. 761). It became a model in Italy for 'practical educational periodicals', which aimed at improving teachers' knowledge and providing materials to be used directly in class, such as exercises, exams' themes and also homework. This type of journals spread in Italy starting from the 1870s. In Piedmont there were: *La Guida del Maestro Elementare Italiano* (1864-1897, 3000 copies) by Giovanni Parato (1816-1874), and *L'Osservatore Scolastico* (*OS*, 1865-1899, 1500 copies) by Borgogno, which had the same style and aims as *L'Istitutore*, and in many cases also the same collaborators. They referenced each other and they had a strong national circulation. In 1873 these three journals from Turin were among the first four Italian educational weeklies per number of copies (Ottino, 1875, pp. 23-40).

L'Istitutore picked up the legacy of the *GSIE*, by keeping up the tradition of being – as said by one of the editors – "a training ground where to discuss everything useful for elementary education".[5] In fact *L'Istitutore* diminished the political critique sections, in respect to those of the *GSIE*, and devoted itself more on helping elementary teachers with their day-by-day practice, offering advices on single disciplines, equally divided between humanistic and scientific.

The inspiration was offered by French educational periodicals, such as *L'Instituteur: Journal des Écoles Primaires* (1833-1840) and the *Manuel Général de l'Instruction Primaire* (1832-1940), which underwent a great surge after the promulgation of the Guizot law in 1833 (Chiosso, 2013, p. 301).

Mathematics teaching in the educational journals in Turin

Despite a diminished participation of scientists and university professors among the collaborators, by comparison with the experience of the *GSIE*, many articles of the journals for elementary schools were about the teaching of mathematics. They tended mainly to focus on methodological issues, for example by promoting the use of educational tools (abaci, dices, geometric solids, …), and on how to overcome the common difficulties, such as to explain the operations with fractions. Long articles about arithmetic were written, looking like supplements to textbooks. Some educational articles had a more ambitious scope, dealing with

[5] Museo Naz. Risorg. Italiano di Torino, *Fondo Berti*, F. Paoli to D. Berti, 30th August 1853.

geometric progressions, shadows theory, unit of measurement and the monetary system of the Japanese empire.

They promoted the history of sciences too. From 1853 to 1855 *L'Istitutore* devoted a weekly section, entitled *Historical conversations*, to articles on the lives and works of mathematicians and physicists from the past, like Nicolò Tartaglia, Evangelista Torricelli, Giambattista Beccaria, Giuseppe Luigi Lagrange, etc.

The reviews of textbooks tended to almost exclusively promote the books written by the editors and produced by the typography of the very same periodicals. An emblematic example is the *OS*, which was free for all teachers who asserted to choose for their classes the textbooks written by the director Borgogno.

Notably, *L'Istitutore* didn't took part in the fierce debate on the so-called "return to Euclid" (Giacardi, 2006, pp. 1-8; Pepe, 2013, pp. 149-151) that raged in the mathematics community around the 1870s. Among the mathematics textbooks reviews just the *Elementi di Geometria proiettiva: ad uso degli istituti tecnici del Regno d'Italia* (1873) by Luigi Cremona (1830-1903) was mentioned. This detachment of the educational journals from the debates on the teaching of mathematics was due to their major interest towards elementary education and also to the absence of a reference editor linked to the Faculty of Physical and Mathematical Sciences in these years.

L'Istitutore offered news and opinions on textbooks approved by foreign ministries, mainly the French ones. For example they suggested texts by Jean-François-Adolphe Dumouchel (1804-1870) for elementary schools, like *Problèmes et exercices de calcul sur les quatre opérations fondamentales, les nombres décimaux, les fractions, etc., appliqués à l'arithmétique élémentaire* (1844), especially recommended for the variety of problems.

The Italian translations of French works were quoted too, such as the ones of the *Traité elementaire de trigonometrie rectiligne et spherique et d'application de l'algebre à la geometrie* (1798) by Sylvestre François Lacroix (1765-1843), whose different translations circulated in Italy since 1813. Moreover in 1883 *L'Istitutore* published in many issues the Italian translation of the first book, and part of the second one, of *La Langue du Calcul* (1798) by Etienne Bonnot de Condillac (1714-1780).

Mathematics exercises: the questions and answers' section

The most important news as regards educational journals in this period in Piedmont was the renovated educational section, devoted to practical exercises, which was the favorite by the audience, as emerged from the correspondence published at the end of each issue.

L'Istitutore, for example, offered educational materials both inside the journal and in its supplement *Didattica per le Scuole Elementari* (1855-1894). Readers had a weekly column to take inspiration from, in which there were exercises, problems and homework, especially arithmetical ones, bespoke for every elementary school year level, as well as the text of the exams being taken in various schools. The algebraic exercises for secondary and technical schools were in compliance with the official programs. Solutions were generally enriched by detailed explanations on how to face the problems, sometimes including also different methods and considerations on how to treat the most difficult issues in class.

Influenced by the positivist ideals spreading in Italy at that time (Roero, 2013, p. 354-359), editors tried to stimulate the reasoning, the use of educational tools and the application of mathematics in everyday life. For example, in 1873 the *OS* proposed the "inventive problems", i.e. data-less arithmetical exercises on programming and foretelling revenues for a commercial enterprise.

The editors of the educational sections pretty frequently promoted their textbooks too and inserted problems related to them. This happened for example for *L'Istitutore*, where the editor of the exercises section from 1866 and the director of the journal itself from 1871 to 1873 was Eugenio Comba (1841-1874), teacher of mathematics in technical schools in Turin and author of successful arithmetic books for elementary and secondary schools (Chiosso & Sani, 2014, n. 646). He decided to add something more difficult to the exercises proposed by the periodical, by including problems involving fractions and the rule of three (for secondary schools), and theorems on triangles and simple algebraic equations (for technical ones).

Some of the exercises were taken from French journals too, like the *Bulletin de l'Instruction Primaire* (Paris, 1854-1858), and from textbooks like the *Leçons d'algèbre* (1833) by Louis Lefébure de Fourcy (1787-1869), professor of differential and integral calculus in Paris, thus underlining the liaisons with France.

Starting from 1854, *L'Istitutore* and then the *OS*, began to host also the "questions and answers" section: easy questions about arithmetic, algebra and geometry for elementary and secondary schools, proposed to the audience as competitions or short enigmas. In the following issues the journals used to publish the best answers by the readers. They were mostly teachers and students of teacher training schools and, even though the majority came from Piedmont, some of them came from other Northern and Central Italian cities, thus proving the national circulation of these periodicals.

The educational journals didn't proposed innovative and advanced mathematics exercises and methods, meaning they never quite reached the advancements in the teaching of mathematics that grew ever-important in the 1880s (Giacardi, 2006).

New educational needs: the birth of the mathematical journals for teaching in the 1870s

Between the 1870s and the 1880s the educational context changed and improved. The 'practical educational periodicals' of the previous decades no longer met the needs of teachers, who went from asking for ready-to-use materials to more intellectually stimulating contents (*La Scuola Italiana*, 1880, pp. 86-87). Periodicals for secondary education got powered up: teachers' associations' bulletins and journals for the teaching of specific disciplines started to flourish. Mathematics, in particular, was one of the most involved disciplines during this period.

It is well documented that since the beginning of the Unification of Italy the mathematicians' community payed attention to the issues concerning education and teaching, by actively getting involved in politics, in associations and in the educational publishing house.

The debate on Euclid's *Elements* for the secondary schools, which involved well-known mathematicians, set the stage for a general rethink on the mathematics teaching, pertaining programs and textbooks, which had to comply with the innovations in the mathematics research, and more rigorous methods. The effects of these changes reflected in the journals of the 1880s.

The first mathematical journal for teaching published in Italy was the *Rivista di Matematica Elementare* (Alba, Novara, 1874-1885). Many similar journals were issued in the following years, mainly devoted to secondary teaching (Candido, 1903; Furinghetti, 2017 to appear; Furinghetti & Somaglia, 1992 and 2005; Salmeri, 2013). One of the most successful was the *Periodico di Matematica* (Rome, 1886-1916, 1921-1943, 1946-), which promoted the creation of the first association of mathematics teachers in Italy, the *Mathesis* in 1895 in Turin (Giacardi, 2005).

The collaborators of these journals were mainly secondary teachers and professors. In this period professors taught in secondary schools at the beginning of their academic career, like Cremona, but it also happened that secondary teachers run an academic course, like Rodolfo Bettazzi (1861-1941).

The new mathematical journals for teaching aimed at making uniform the mathematics teaching in the Italian territories and at overcoming the inadequacy of the majority of the textbooks, circulating in the schools. They also tried to introduce the recent advancement of the research in the teaching, pertaining overall Calculus and Geometry, and also the Foundations of mathematics. They fostered the comparison among methodological ideas and experiences, and provided solved exercises, riddles and historical curiosities too.

The educational journals applied themselves just into the elementary teaching, by restricting the mathematics issues to weekly elementary arithmetic and geometry exercises. Few mathematical journals were devoted to elementary schools, like *Il Bollettino di Matematiche e di Scienze Fisiche e Naturali* (Bologna, 1900-1917) by Alberto Conti (1873-1940).

Conclusion

From the 1840s to the 1870s in Italy the teaching of mathematics found its place only in educational journals devoted to different disciplines, mostly for elementary schools. They aimed at overcoming the backwardness of teaching methods and textbooks, by promoting foreign educational systems, and by providing practical hints and new methodological ideas.

In this context the Savoyard Kingdom stood out from the rest of Italy, thanks to the freedom of the press and the local educational publishing houses' involvement. In the 1850s journals from Turin didn't hinge on the ideas of a single educationist, as in the Grand Duchy of Tuscany, but on a heterogeneous and politically influential teachers' association, the *SIE*. Its educational journals, the *GSIE* and then *L'Istitutore*, had a social and political role: they contributed to level out the education in the Kingdom, by promoting the innovations of the Boncompagni law, and to involve a great part of the population in the education's problems. They also helped spreading the debate on education circulating among teachers, professors and politicians, which led to the Casati law, the basis for the Italian educational system until 1923. We have to remark that in 1861 the number of illiterates in Piedmont and Liguria diminished to 54,2%, contrary to the national average of 78%.

Educational journals contributed to make aware the audience of the lacks of the educational publishing industry: they stressed the need for better textbooks and presented themselves as a sort of schoolbooks' supplement. They also contributed to provide immediately usable materials for teachers, who were mostly unprepared, due to the lack of teacher training schools in all the territory.

They focused on the reevaluation of the scientific studies and the technical ones, traditionally subordinated to the humanistic ones. As regards mathematics teaching for elementary schools, journals contributed to promote the decimal metric system, which was newly introduced and so few textbooks were updated. They provided theoretical and practical articles on the geometry teaching too, thus promoting a discipline often overlooked in the elementary schools before 1848. In the following years more challenging topics were introduced, like geometric progressions and the shadows theory.

The elementary educational journals from Turin influenced the birth of similar journals in all of Italy, which contributed to make uniform the education and to compensate the still rampant illiteracy and the differences between territories. But after the Italian Unification their role in mathematics teaching was restricted to advertise for textbooks and to weekly provide arithmetic exercises, and also geometric problems and algebraic equations for secondary and technical schools.

The experience of the *RUC* tried to bridge the gap of scientific journals for secondary teaching, which were not present until the 1870s in the rest of Italy.

The journal attempted to be on the crossroad between teaching and research and had as a collaborator one of the biggest mathematicians of that period in Turin, Genocchi. As regards mathematics teaching the *RUC* was innovative, because it proposed advanced mathematics secondary programs, which included the combinatorial analysis and analytic geometry, and because it presented research issues, like the multiple integers. Due to its short life, the *RUC* didn't really play a role in the local and national mathematics teaching of the following years.

The context was renewed after the involvement of Italian mathematicians towards the teaching's problems. Mathematical journals for teaching, especially for secondary school, spread across Italy. Their advanced purposes and the presence of mathematicians as collaborators linked them to the mathematical journals for research.

Acknowledgment. I wish to extend heartfelt thanks to Clara Silvia Roero and the anonymous referees for the assistance in the revision of this paper and for their helpful advice.

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The production of textbooks in mathematics in Sweden, 1930-1980

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Abstract

This paper presents a bibliometric study on the production of mathematics textbooks in Sweden in the period of 1930–1980. The analysis concerns grades 1-9. The main source is a database of mathematics textbooks. Official reports on the Swedish textbook market comprise a second source.

Introduction

The study presented here is a bibliometric analysis of the production of textbooks during the period of 1930-1980. Apart from presenting the results of the analysis, changes to textbook production are viewed in relation to other changes in the Swedish school system. The purpose of the latter is to illuminate how the conditions for reforming mathematics in Swedish schools changed in this period. I also discuss possible causes of why textbook production changed. My aim here is not to be comprehensive, i.e. there may be more causes not considered in this paper.

The reason for restricting the focus to the period after 1930 is methodological: the database used for the analysis is not reliable prior to 1930. The reason for excluding the period after 1980 is related to format – the paper would be too long.

But 1930-1980 is an interesting period in which the Swedish school system underwent important changes. Moreover, during this period, the New Math reform was prepared and implemented. For further details, see the background section below.

The study presented is part of a research project entitled: *The development of school mathematics and reforms of the Swedish school system in the twentieth century: a comparative and historical study of changes of contents, methods and institutional conditions*. The project aims to examine how Swedish school mathematics changed in primary school (grades 1-9) in the twentieth century, especially during major general school reforms.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Contribution to previous research

Research on the history of Swedish mathematics education in the twentieth century largely concerns the content of various types of educational texts, such as textbooks, teachers' journals, teaching literature, syllabi and other policy documents; see for instance Lundin (2008) and Prytz (2007; 2009; 2012; 2015). Sociological perspectives are also occasionally applied; see for instance Lundin (2008) and Prytz (2009).

There are also two studies by Kilborn (1977) and Prytz & Karlberg (2016) on the preparations of the Swedish New Math reform of the 1960s.

None of the studies mentioned above consider the production of textbooks. In that respect, this study adds new knowledge to previous research. In the final section, I will discuss how this new knowledge may offer new insights into the conditions for initiating, preparing and implementing changes to mathematics in Swedish schools.

From an international perspective, the relevance of this study concerns its methodology: to my knowledge, no similar bibliometric analysis has ever been done on mathematics textbooks.

Method

The work to analyse the production of textbooks involved a database of mathematics textbooks, primarily from around 1900. The construction of the database is described in Prytz (2016) and I only provide an outline of it in this paper.

The database consists of bibliographic data regarding mathematics textbooks. The data have been collected from two sources: the joint catalogue of the Swedish academic and research libraries (LIBRIS) and lists of approved textbooks issued by the national textbook review board. Note that LIBRIS includes the catalogues of the National Library, to which publishing companies have been obliged to send a copy of all publications since 1661.

To enhance the reliability of the database, especially for books published before 1976, the database was checked against lists of approved textbooks issued by the national textbook review board during the period of 1930–1973. LIBRIS was created in 1976 and the categorization of textbooks is less reliable before that year. The missing data were entered into the database.

The best way to analyse the production of textbooks would be to count all editions of all textbooks published. This, however, is not possible, since LIBRIS does not include all editions of all textbooks. Instead, I have chosen to map the production of completely new textbooks, i.e. first editions. LIBRIS includes all first editions of books.

Accordingly, the statistics presented in this paper concern the influx of new textbooks. When searching to determine when new textbooks were published, the basic principle was to filter out the earliest publishing date of all textbooks in the database.

There is however a methodological error associated with this method. In most cases, the earliest publishing date coincides with the first edition. Most exceptions are for cases in which data are collected only from the lists of approved textbooks, i.e., books not found in LIBRIS. The earliest list regarding one school type (*Realskolan*) is from the late 1920s; for the second school type (*Folkskolan*), the first list is from the late 1930s. Consequently, there is an overrepresentation of new textbook publications in the 1930s. This means that first publications that were actually issued in earlier decades are included in the 1930s in my statistics.

Another problem concerns the process of counting textbooks. The basic problem is that textbook formats have changed over time. For instance, in the 1970s, it was common for a textbook series in mathematics to comprise numerous booklets, rather than a book with multiple chapters, which was common in the 1960s and earlier. In LIBRIS, each booklet is registered as a book. A consequence of this format change is that the number of published textbooks in the 1970s appears to have been enormous. To avoid this problem, series of textbooks, rather than single textbooks, have been counted.

However, this does not solve the problem completely. *Realskolan* had no series; each topic (arithmetic, algebra and geometry) had its own textbooks and they could be used for more than one school year. But in practice, this problem is insignificant, because so few new textbooks were produced for *Realskolan* after 1930.

So, even though we lack a perfect common measure, it is possible to accurately discern the increase of new textbooks in the 1950s.

Apart from using the database and bibliometric analysis, I have used official reports on textbook production as source material and I have gathered information about the Swedish textbook market.

Background

In the beginning of the investigated period of 1930-1980, the Swedish school system (1-9) comprised several school types. This paper concerns the two largest: *Folkskolan* and *Realskolan*.

Folkskolan was the larger of these two. When *Folkskolan* was introduced in 1842, it comprised grades 1 to 6. However, throughout its existence, it has included two parts: *Småskolan* (direct translation: *Little School*) constituted the first two grades, while *Folkskolan* constituted the next four grades (Larsson & Westerberg, p. 106).

Textbooks and textbooks series in mathematics were often dedicated to only one of the two school types.

A law passed in 1936 prescribed that all students should attend school for at least seven years. However, the regulation was implemented gradually and was not completed until 1950. But in some (primarily urban) parts of Sweden, *Folkskolan* was extended to eight or nine grades (SCB 1974, pp. 23-24). The portion above grade 6 was called *Folkskolans överbyggnad* (direct translation: *Folkskolan's superstructure*). Many mathematics textbooks were dedicated to only this part of *Folkskolan*.

A partly parallel school type to *Folkskolan* was *Realskolan* – a lower secondary school. When it was introduced in 1905, students entered in grade 4. This was later changed to grade 5 or 7; the students could choose when. In contrast to *Folkskolan*, the syllabus of *Realskolan* was more theoretical; students were supposed to become prepared for further theoretical studies in *Gymnasium* (9-12, upper secondary school) or more advanced vocational educations. Initially, *Realskolan* had only one programme. This changed in 1933, when a practical programme was introduced (Larsson & Westerberg, pp. 126-129).

Before 1905, the predecessors to *Realskolan* and *Gymnasium* were more integrated, belonging to the so-called *Läroverket*. The origins of *Läroverket* date back to medieval cathedral schools. However, even after 1905, *Realskolan* and *Gymnasium* continued to be part of *Läroverket*. Here we should note that there were no alternatives to *Realskolan* within *Läroverket*.

From a sociological perspective, students and teachers alike at *Folkskolan* and *Läroverket* were generally recruited from different socioeconomic classes. *Folkskolan* was a school for the lower class, while *Läroverket* was a school for the upper-middle class and beyond.

When considering the mathematics courses and textbooks of *Folkskolan* and *Realskolan*, the greatest differences can be seen in grades 6-9.[1] The geometry courses of *Realskolan* included proofs, and the textbooks were designed according to the axiomatic-deductive method (Prytz 2007, pp. 125-161). The *Folkskolan* courses did not include these elements. The time at which algebra was introduced and the scope of the topic also differed: algebra was introduced in grade 6 in *Realskolan* and grade 7 in *Folkskolan*. If we consider the syllabi of grade 9, we find formulations about systems of equations and quadratic equations in the *Realskolan* syllabus, but not in the *Folkskolan* syllabus. The latter contains more formulations about simple equations and practical applications.

A reform launched in 1962 that would replace *Folkskolan* and *Realskolan* with a single, mandatory nine-year school: *Grundskolan*. This change was implemented over a ten-year period. But already in the 1950s, preparations for *Grundskolan* had

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[1] In both *Folkskolan* and *Realskolan* during the period investigated, mathematics was taught in all classes and courses were obligatory for all students. Both schools were open for both boys and girls and they took the same courses in mathematics.

begun (Larsson & Westerberg, p. 113). A kind of experimental school type called *Enhetsskolan* (*Unitary School*) was formed, covering grades 1-9. Organization and new types of teaching were tested in this school.

Mathematics education did not change radically in conjunction with the introduction of *Grundskolan* in 1962. A simple characterization is that the former mathematics syllabi of *Folkskolan* and *Realskolan* were integrated into *Grundskolan*. In grades 1-6, all students took the same mathematics courses, but in grade 7, they had to choose between a basic and an advanced course. The former was similar to the *Folkskolan* courses, while the latter was similar to the *Realskolan* courses. A major change, though, was that geometry was given a less prominent place in the advanced *Grundskolan* courses than in *Realskolan*.

The major change to the *Grundskolan* mathematics syllabus arrived in 1969, when *Grundskolan* received a new curriculum. In connection to this reform, New Math was introduced in grades 1-9.

More details on the syllabi in mathematics mentioned above are given in Prytz (2015).

In the period of 1930-1960, the Swedish school system (1-9) expanded in terms of the number of students, as illustrated in Tables 1-5.

Table 1. Number of students in *Folkskolan* (Source: SCB 1974)

Year	*1930*	*1940*	*1950*	*1960*
Number of students	672,823	548,792	612,158	843,110

Table 2. Number of students in *Folkskolan*, grade 8 and above (Source: SCB 1974)

Year	*1941*	*1950*	*1959*
Number of students	3,253	6,529	48,494

Table 3. Number of students in *Realskolan* (Source: SCB 1977)

Year	*1930*	*1940*	*1950*	*1959*
Number of students	61,551	69,891	106,295	150,236

In addition to these school types, there was *Enhetsskolan*, which had relatively few students: it comprised 20,000 students at its highest point in 1961.

Due to the gradual implementation of *Grundskolan*, official statistics on the number of students in grades 1-9 in the 1960s are quite complicated. But if we only consider *Grundskolan* from 1967 to 1972, when the old school types had fewer students, we have the following figures:

Table 4. Number of students in *Grundskolan*, 1967–1972 (Source: SCB 1973)

Year	*1967*	*1968*	*1969*	*1970*	*1971*	*1972*
	872,873	903,885	933,060	954,038	977,194	989,147

From 1973 and onward, when all the old school types were phased out, we have better figures regarding the number of students.

Table 5. Number of students in *Grundskolan*, 1973–1979 (Source: SCB 1980)

Year	*1973*	*1974*	*1975*	*1976*	*1977*	*1978*	*1979*
	1,000,934	1,016,014	1,026,847	1,033,086	1,037,910	1,043,043	1,043,333

For the period of 1940–1960, in Tables 1-3 we can observe an increase in the number of students in *Folkskolan* and *Realskolan*, in contrast to the period of 1960–1979, when the number of students was stable; see Tables 1-5.

In relation to the New Math reform, the big increase of students in grades 1-9, particularly in grades 7-9, took place before the reform. Recall that the reform was prepared in the 1960s and implemented in the 1970s. Thus, during the implementation of New Math, the school system was not under pressure from quickly increasing numbers of students.

The conditions for producing new textbooks appear to have been good in the period of 1940-1960: an increasing number of students meant increasing demand for textbooks. But more students also meant more new teachers who may have been more prone to testing new textbooks. However, the statistics regarding the influx of new textbooks for the period of 1930-1980 show quite the opposite to be true.

Influx of new textbooks, 1930-1980

In Diagrams 1-13, vertical bars represent the number of new textbooks each year in different types of schools during various periods. The broken line represents the sliding average for each year and the preceding four years.

Diagrams 1-3 concern the numbers for *Folkskolan* in the period of 1930-1962.

Diagram 1. *Folkskolan* (1-2) Numbers of new series in Arithmetic

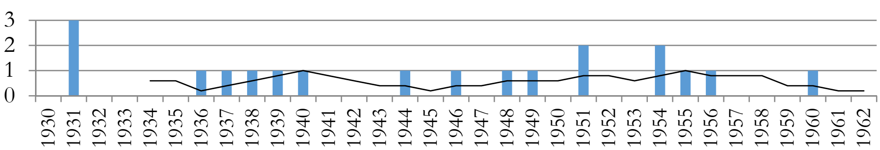


Diagram 2. *Folkskolan* (3-6). Numbers of new series in Arithmetic

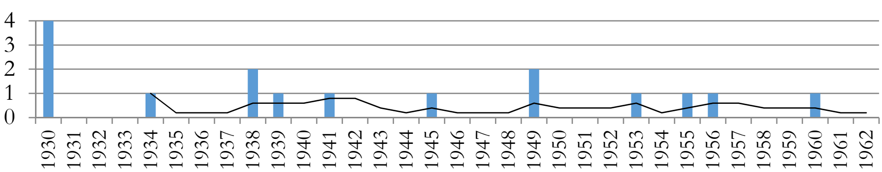
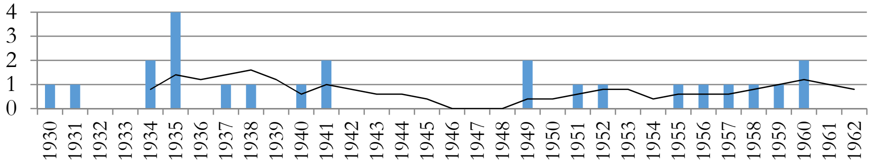


Diagram 3. *Folkskolan* (7-9). Numbers of new series in Arithmetic



The relatively steady influx of new textbooks may be due to few changes made to the syllabus. In the period of 1930-1962, the *Folkskolan* syllabus changed only once, in 1955. Apart from that, the syllabus of 1919 was in effect throughout the period. The slight increase from the late 1940s in grades 1-2 and 7-9 (if we consider the sliding average) may be due to the increase in students. Here, we should also note that mandatory schools, i.e. *Folkskolan* and later *Grundskolan* as well, were required from 1946 and onwards to provide students with free textbooks (SB 1961, p. 176). This might also explain the increases in grades 1-2 and 7-9. Nonetheless, we do not see a similar increase in grades 3-6, which indicates that these factors were not decisive for the production of new textbooks.

The influx of new textbooks for *Realskolan* differed from *Folkskolan*. In general, far fewer new textbooks were produced for *Realskolan*.

Diagram 4. *Realskolan*, Geometry, Number of new textbooks

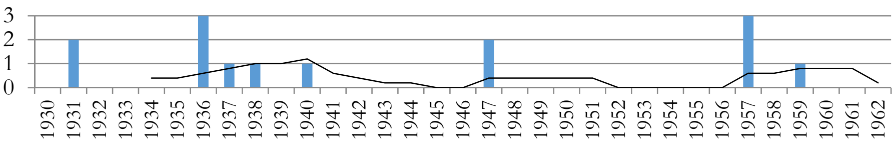


Diagram 5. *Realskolan* Algebra, Number of new textbooks

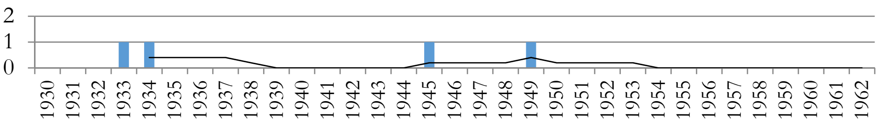


Diagram 6. *Realskolan* Arithmetic, Number of new textbooks

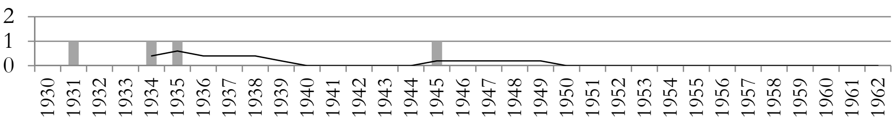
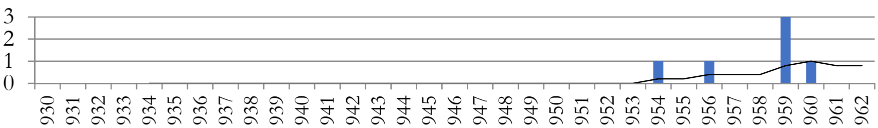


Diagram 7. *Realskolan* Mathematics, Number of new textbooks



The fact that the schools within *Realskolan* were not required to provide free textbooks might explain the low level of influx of new textbooks.

Regarding Diagram 6 and the low level of new textbooks in Arithmetic, the possibility that textbooks intended for *Folkskolan* were also used in *Realskolan* must be taken into consideration, and might also explain the low influx of new textbooks. Arithmetic was a major topic in *Realskolan* (4-6) and also a major topic in *Folkskolan*.

Algebra, however, was not a topic in *Folkskolan* in grades 3-6 and only a minor topic in grade 7. In this case, we can rule out the possibility of sharing textbooks with *Folkskolan*.

As Table 3 shows, the number of students in *Realskolan* increased from 1930 to 1960. However, this does not seem to have caused an increase in the number of new textbooks, especially not in algebra. In geometry, we can see an increase in new textbooks in the second half of the 1930s, but not later, when the rate of increase in students really took off.

Not until the 1950s can we discern an increase in the influx of new textbooks for *Realskolan*. This is probably related to the new syllabus introduced in 1955. Before that, *Realskolan* had not had a new syllabus since 1933.

In the 1950s, the format of the *Realskolan* textbooks had also changed. Rather than separate books for each topic, the books included the word "mathematics" in the title and covered several topics. Here, we should note that three of the six textbooks were intended for the practical programme. Two of the books were analogous to books intended for the theoretical programme and had the same authors. However, separate books in geometry were still published, which I cannot explain.

In the 1950s, some schools were converted to a school type called *Enhetsskolan*, which is described in the previous section. Naturally, textbooks were also produced for *Enhetsskolan*.

Diagram 8. *Enhetsskolan* (1-3), number of new series

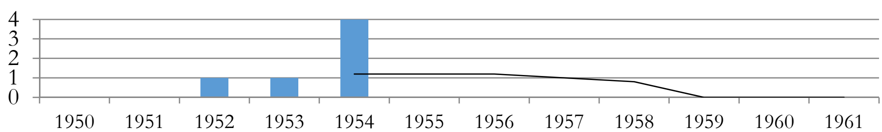


Diagram 9. *Enhetsskolan* (4-6), number of new series

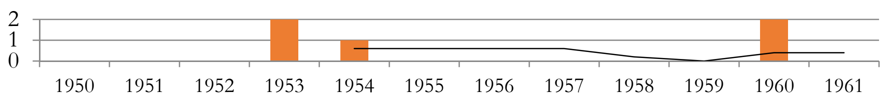
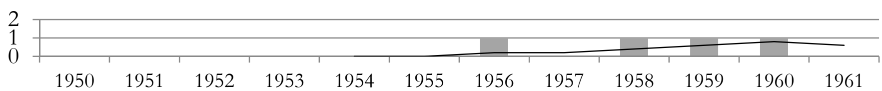


Diagram 10. *Enhetsskolan* (7-9), number of new series



In comparison to the production of textbooks for the other school types, quite a lot of new textbooks were produced for *Enhetsskolan* in the 1950s.

In 1962, *Grundskolan* was introduced, along with a new syllabus. But a new syllabus was launched already in 1969: the New Math syllabus. It was followed by another new syllabus in 1980. If we compare the influx of new textbooks in *Grundskolan* with the influx of new textbooks in *Folkskolan*, *Realskolan* and *Enhetsskolan,* a striking difference can be seen: far more new textbooks, or series of textbooks, were being produced.

In Diagrams 11-13, the sliding average is mainly between 1 and 2. Thus, compared with Diagrams 1-3 regarding *Folkskolan*, the influx of new textbooks was generally twice as big. The differences between *Grundskolan* and *Realskolan* are of course even greater, considering the very small influx of textbooks in *Realskolan*. Note that students had access to free textbooks in *Folkskolan* since 1946, as well as in *Grundskolan*, so that cannot explain the differences between those two school types.

Diagram 11. *Grundskolan* (1-3). Number of new series.

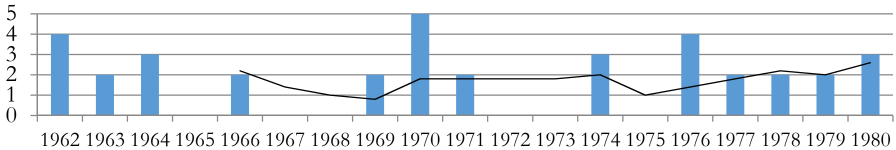


Diagram 12. *Grundskolan* (4-6). Number of new series.

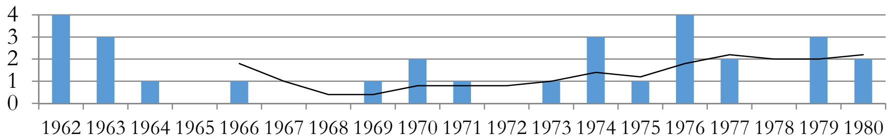
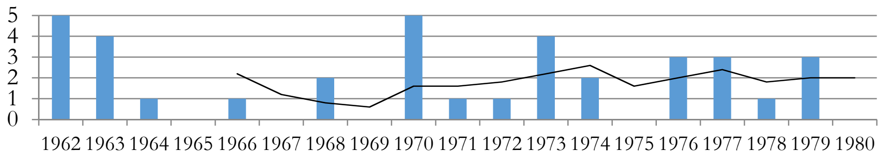


Diagram 13. *Grundskolan* (7-9). Number of new series.



To further illustrate the picture of the situation in the 1960s and 70s, we can also consider the influx of new authors in the period of 1962–1980.

Several of the first authors of the *Grundskolan* textbooks had never published textbooks before. Table 6 below shows the aggregated number of new textbook authors in the periods of 1962-1968 and 1969-1979. These authors had not published when the preceding syllabus was in effect.

Table 6. New authors *Grundskolan* (1-9), 1962-1979

Gr	Period	Number of authors	Number of new authors	Share new authors
1-3	1962-1968	28	13	53.6%
1-3	1969-1979	48	40	83.3%
4-6	1962-1968	28	14	50.0%
4-6	1969-1979	42	37	88.1%
7-9	1962-1968	37	20	54.1%
7-9	1969-1979	38	25	65.8%

In Table 6, we can see that the influx of new authors during the period of 1969–1979 was greater in absolute as well as in relative numbers.

If we compare Diagrams 1-13 regarding the influx of new textbooks and Tables 1-5 regarding numbers of students, the influx of new textbooks does not seem to be related to changes in the number of students. If that were the case, we should have observed the greatest influx of new textbooks before 1960 rather than after. The number of students increased much more slowly after 1960.

However, the increase in the number of new textbooks as well as new authors from the 1950s and onwards coincides with another increase: the schools' spending on teaching materials. According to an extensive state investigation on teaching materials, schools' spending on teaching materials – mainly textbooks – went up 10 times from 1950 to 1970 (LU 1971, p. 53). This increase in spending on teaching materials was part of a general trend. After World War II, the Swedish welfare state expanded significantly (Larsson & Westberg 2011, p. 40), which meant that education in general received much more funding.

In the state investigation mentioned above, several interrelated causes of the increased spending on textbooks were identified: the price per book increased; technical quality was higher; there were more students and free textbooks, and there was greater differentiation in terms of more course programmes, the latter of which required specially designed textbooks (LU 1971, pp. 53, 179, 200). Here we should observe that the price per book did not increase 10 times, according to the report. The report noted that the price per book doubled from 1960 to 1971 (LU 1971, pp. 53).

Still, none of these factors can explain why so many new textbooks in mathematics were published compared with earlier periods. However, the schools' increased spending per se might have been a cause: if the publishing companies knew that in general, schools were willing to spend significant money on textbooks, they may have perceived the risk of a loss on producing a new textbook as low. It is important to observe that the textbook industry seems to have been profitable around 1970. According to the investigation mentioned above, basically every new textbook was a success (LU 1971, p. 190). I do not consider this potential cause to be a triggering factor, but rather a condition that made it possible to produce more new mathematics textbooks.

Finally, if we only consider Diagrams 11-13 and the influx of new textbooks between the syllabi of 1962, 1969 and 1980, we can discern two different patterns.

In both cases, several new textbooks appeared when a new syllabus took effect. But in the 60s, high numbers were followed by a decrease and a low level, with some minor exceptions, until the next syllabus. One of the exceptions occurs in Diagram 13 in 1968. One of those series is an experimental series developed in a project run by the National Board of Education. The second series included books on sets and logic and should instead be counted together with the series published in 1969, when the New Math syllabus was issued. In contrast to the 1960s, the influx of new textbooks increased by the mid-1970s and remained at a relatively high level until the next syllabus was issued in 1980.

The increase in the mid-1970s coincided with a fundamental change in policy for the national textbook review in 1974: the review, whose approval was necessary for publication until then, was no longer mandatory, except for textbooks in civics, history, geography and religion (Johnsson Harrie 2009, pp. 12-13). In this case, I think it is reasonable to discuss a possible triggering factor regarding the increase of new textbooks after the mid-1970s.

Intensified marketing, 1960-1980

Diagrams 1-13 above show a higher influx of new textbooks in the period of 1960-1980 than the period of 1930-1960. In this section, I propose that this change coincided with intensified marketing to teachers by the publishing companies.[2]

Initially, new textbooks probably entailed higher costs for the publishing companies, at least compared with publishing new editions of older books. Furthermore, it may have involved even higher costs if the companies contracted more new authors, which Table 6 indicates, as new authors may need more support and advice. According to one of the state investigations concerning the textbook market, the basic costs of a new textbook were covered in 3-4 years in the 1970s (UL 1980, p. 99).

My point is that the publication of more new textbooks in mathematics in a shorter time frame probably went hand in hand with increased competition and more marketing; the companies want to make profits on products that have initially been nothing but a loss. Here, we should notice that the publishing companies were forced to cover the costs of producing a new textbook within a shorter time frame than before 1950 (LU 1971, p. 53); this meant that fewer editions of a textbook was supposed to cover the production costs. This was a consequence of shorter intervals between new syllabi, issued in 1955, 1962 and 1969. Thus, there were probably more people than before 1970 – both in publishing and authors – seeking to influence teachers' views on teaching mathematics.

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[2] During the period of 1930-1980, all publishing companies were private.

Indeed, the publishing companies did increase their spending on marketing in general in the 1970s. According to a report issued in 1970, the textbook market was price-insensitive and competing by lowering prices was not an option. Rather, the primary means of competition were product development and marketing (LU 1971, p. 201). From 1970 to 1976, marketing costs increased by 171 percent in the major companies and by 151 percent in the smaller companies. The share for marketing of the total costs of textbook production increased from 7.6 to 11.9 percent in the major companies and from 7.8 to 8.3 percent in the smaller companies (SPK 1978, p. 107). This was also the share of the companies' total costs for textbooks that increased the most (SPK 1978, p. 96).

Marketing was done via different channels: advertisements in teacher journals; sending brochures and test copies to schools; exhibitions and so-called schoolbook consultants (LU 1971, pp. 67-68; UL 1980, pp. 65-66). The consultants functioned as links between editors and teachers; they were specially trained to inform teachers about teaching materials and often had backgrounds as teachers. They also gathered information from the teachers about their use of the textbooks. In 1980, about 70 people were fully employed as consultants, mainly in the major companies (UL 1980, p. 65).

According to the state investigations, companies were dominant in the dissemination of information about textbooks, at least compared to the state. In a survey from the early 1970s of 7 municipalities and about 100 school representatives who were responsible for teaching materials, respondents answered that the companies were the most important source of textbook information by far (LU 1971, p. 70). The same survey also indicated that information from the authorities had a small impact on the schools' choices of textbooks.

In 1974, however, a state institute for teaching material information was established (*Statens Institut för Läromedelsinformation*, SIL). But there are reasons to question its success in disseminating textbook information. According to one of the state investigations, the SIL's reports were seldom requisitioned by teachers (UL 1980, pp. 59-61, 71). Moreover, the SIL had fewer resources than the publishing companies. In 1976, their budget for teaching material information was about SEK 2 million. The same year the major publishing companies spent SEK 36.6 million on marketing alone (UL 1980, p. 103).

Alas, we do not have the same detailed information about the publishing industry and its economy in the early 1960s and earlier. However, the companies' marketing is briefly mentioned in an official report on textbooks from 1960. The channels for marketing were basically the same as in the 1970s and marketing activities were considered high (*SB 1961, pp. 118-119, 142*). It is difficult to say what "high" means, but an investigation in the late 1960s analysed the causes of the cost increase of textbooks. Marketing was not identified as a cause (LU 1971, p. 53), indicating that costs of marketing were lower in the early 1960s than in the 1970s. Together with the fact that fewer new mathematics textbooks were published in the 1960s than in the

1970s (see Diagrams 1-13), this indicates that marketing of mathematics textbooks was more intensive in the 1970s than in the 1960s.

It is important to note that the official reports mentioned above mainly concern the textbook market overall, and not each individual school subject. However, since mathematics was one of the major school subjects and the reports do not indicate its market was any different, it is unlikely that the mathematics textbook market was in fact different.

Conclusions and further research

The bibliometric analysis presented in this paper shows that the influx of new textbooks in mathematics (1-9) was greater in the period of 1960-1980 than in the period of 1930-1960. Different causes of this difference have been considered and the likeliest is the schools' increased spending on textbooks. This cause is considered a condition that made a high influx possible and not a triggering cause.

Regarding the period of 1930-1960, the analysis also shows that the influx of new textbooks in *Realskolan* – lower secondary school – was very low in the period of 1930-1950, even though the number of students increased in the same period. This might indicate that the textbooks were good, or at least that teachers thought they were good. However, the results of the national exams in grade 9 suggest otherwise. From the early 1930s, we can observe a clear negative trend in the results until 1950 (Prytz 2007, p. 165). This requires further research. In that situation, why did authors and publishing companies not try or fail to produce new textbooks?

Regarding the period of 1960-1980, the analysis shows that the influx of new textbooks was greater in the 1970s, but also distributed differently in that period. Not surprisingly, we can observe peaks right after the introductions of new syllabi in 1962 and 1969. However, in 1975 the influx of new textbooks took off again and remained at a high level until a new syllabus was issued in 1980. A likely triggering cause of this increase is that the textbook review became voluntary in 1974. Before that, the review was mandatory.

I also propose that the publishing houses intensified marketing to teachers, especially in the 1970s, as production of new mathematics textbooks increased. This conjecture is supported by official reports, which show that the publishing companies spent increasingly more resources on marketing in general in the 1970s, and the official reports give no reason to believe that mathematics was an exception.

Thus, when the state made the textbook review voluntary and lessened the control of the publishing companies, not only did the companies give teachers more choices; they also intensified their efforts to influence teachers.

These changes to textbook production, spending on textbooks, control mechanisms and marketing are interesting in relation to the reforms to mathematics in Swedish schools that took place in the 1960s and 1970s, such as the New Math reform, which was planned in the 1960s and implemented in the 1970s. The results presented in this paper suggest that the conditions for textbook production changed from the time the reform was planned to the time of its implementation. Here, it is important to notice that the New Math reform relied heavily on textbooks. Developing new textbooks comprised a significant part of preparing for the reform (Prytz 2016, pp. 73-74).

Further research is necessary to deepen our understanding of these aspects of the New Math reform. For instance, the publishing companies could be researched in greater depth. Which companies dominated the market and what were their publication strategies? In addition, authors' backgrounds could be analysed. It would also be interesting to study how the content of the textbooks changed when the textbook review became voluntary in 1974.

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New conceptions of mathematics and research into learning and teaching: Curriculum projects for primary and secondary schools in the UK (1960-1979)

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Abstract

This paper is an investigation into the gradual change from entrenched beliefs and research practices in the UK mathematics education community to one of quite differently emerging interests in children's learning and the process of 'teaching-learning' showing how the two activities were inextricably linked.

I describe ways in which new ideas came into the literature for teachers, particularly from the journals of the Mathematical Association (MA) and the Association of Teachers of Mathematics (ATM) and the publication of significant government documents. Of significance were the curriculum projects organised by groups of Secondary school teachers, and the Nuffield Foundation for Primary Schools, that encouraged various local government and individual school-based projects later funded by the Schools Council, for both Primary and Secondary schools.

Of particular interest were the re-investigations into the work of Piaget by Margaret Donaldson, the work of the English psychologist, Richard Skemp, and the discussions on the widening of the curriculum. These ideas formed a background to the development of new ideas and ways of presenting mathematics for both Primary and Secondary schools in this period.

Introduction

This paper follows my earlier text (Rogers, 2015) on the development of a new paradigm in mathematics educational thought in the period after the Royaumont Seminar (OEEC, 1961a; OEEC, 1961b) and discusses the political and social background here in the UK, by looking at individuals, curriculum projects, research trends, and teaching materials that appeared in the period up to the end of 1979. This account must necessarily be selective, and to some extent personal, since I have lived through and been influenced by many of these events, socially and intellectually. I leave it to colleagues to correct any major omissions.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

Political and economic conditions 1950-1979[1]

1945-1960

The 1944 Education Act planned for 'social reconstruction' and established three stages of public education: primary, secondary, and further education to be organized and provided by Local Education Authorities (LEAs). It said nothing about the content or organisation of the curriculum, and made schools responsible for their own management. The system maintained a class-based tripartite system, with grammar schools for the most able, secondary modern schools for the majority, and secondary technical schools for those with a vocational aptitude. Selection of pupils was regulated by tests of English, arithmetic and an 'objective' intelligence test to be taken at age 11[2]. Those who passed the tests went to grammar schools, and the others went to secondary modern or technical schools. Selection at 'eleven plus' (11+), and the continued existence of large classes[3] through to the 1950s, forced primary schools to continue with teaching with emphasis on basic literacy and numeracy, while pupils were trained to pass the tests.

In 1945 the new Labour government continued with this system, and comprehensive secondary education as a general policy was ignored. Teachers were left to their own devices, with one professional organization for mathematics teachers; the Mathematical Association (MA), originally set up in the late 19th century, that was mainly concerned with the grammar and private schools whose pupils aspired to university[4]. A new organization, the Association of Teachers for Teaching Aids in Mathematics (ATAM) was set up in 1952[5].

A new General Certificate of Education (GCE) was introduced in 1951 designed for the top 25 % of the 'ability range'. This examination was taken at age 16 (Ordinary Level) and 18 (Advanced Level), by grammar school pupils and those in independent[6] schools. However, pupils could leave school at age 15, so many pupils in the secondary modern schools had no reason to take the exam, neither were they taught sufficient to do so. Not until September 1972, was the school leaving age was raised to 16.

Teacher Training Colleges had been set up in the 19th and early 20th century. Some universities had also established teacher-training departments that now served as centres for secondary training, and by the early 1950s the government had encouraged LEAs to open 76 new training colleges. However, the provision

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[1] For further details please consult *Education in England: the history of our schools* at http://www.educationengland.org.uk/history/glossary.html

[2] Pupils could be between 10.5 and 11.5 years of age, creating a maturity and ability gap.

[3] In 1945, I was one of a class of 40 pupils.

[4] For a history of the Mathematical Association up to the 1940s see (Price, 1981).

[5] For more details see (Rogers, 2015, pp. 6-11).

[6] Independent schools are private schools and the old Grammar schools. Public schools in England are the well-known Private schools, mostly established in the 19th century.

of specialist mathematics teachers for secondary modern schools was very poor, and while some universities had set up secondary training courses, there was neither encouragement nor compulsion for these to be a qualification to teach. Mathematics graduates went directly into the private or grammar schools, and secondary modern pupils suffered regimes of commercial and 'domestic' arithmetic. Those who attended secondary technical schools fared better, with some practical mathematics and technical drawing. Arithmetic and Mensuration[7] was taught to prospective primary teachers.

Since 1943, the curriculum for the Teacher Training Colleges had been overseen by the Association for Teachers in Colleges and Departments of Education (ATCDE), whose membership was drawn from the colleges and university education departments. However, in 1949, a plea for 'Non-Certificate' mathematics was made from the Teacher Training Colleges, because trainee teachers could have difficulty in passing the new GCE (Williams, 1949). In the 1950s, the shortage of mathematics teachers was serious, problems in some of the new comprehensive schools were brewing (MA, 1950) and teachers in both the Colleges and University Departments produced a report (ATCDE, 1956) recognising shortages and organizational problems, with interventions from both the MA and ATM (Hope, 1956). Proposals for a three-year course were debated (Hope, 1957), while government recognised the serious lack of mathematicians in the training colleges[8] and responded with two reports, (HMSO 1957a, 1957b) but not until September 1960, was the teacher-training course extended to three years. There was a significant intervention in September 1967 when the Mathematics Section of the ATCDE discussed radical changes in the curriculum introducing 'modern' topics, and debated the pedagogy, and methodology of mathematics in training teachers (ATCDE, 1967). The ATCDE Mathematics Section of contributed regularly to debates on the curriculum on behalf of the Colleges of Education.

Conservative governments were in power from 1955 until 1964. The unfairness of the selective system, and awareness of the need for a more child-centred style of education – especially in primary schools – had been growing for some time and the LEAs began to experiment with different types of provision of education. Fortunately, their freedom allowed the newly-founded ATAM to provide assistance and proposals for the mathematics curriculum, particularly in the secondary modern schools.

The 1950s did see the development of some 'experimental' comprehensive schools, but comprehensive reform in England was uneven. In many cases it perpetuated selective teaching within the schools where some comprehensives organised 'grammar streams' for the most able, and publishers started producing graded

[7] Mensuration (Latin, *mensurare* to measure) meaning measuring of all kinds, including memorizing formulae for standard geometrical objects.

[8] At this time only 44% of students has passed the O level exam.

course books for classes of different 'abilities'. In September 1960, the two-year teacher-training certificate was extended to three years, more staff were needed for the Training Colleges, and this expansion opened opportunities for many school teachers, who left senior posts in the schools.

1960-1970

In the 1964 election the Labour Party promised that secondary education would be reorganised on comprehensive lines. LEAs were free to choose their secondary system, and due to pressure from parents, some areas chose to retain their grammar schools. During this period, the proportion of children attending comprehensive schools rose to thirty per cent of the total school population, but the government failed to establish a national comprehensive system and selective areas survived. The Schools Council was established in 1964 to disseminate ideas about curriculum reform in England and Wales, and was dominated by teachers' representatives, so teachers had a leading influence on curriculum change, and government funding became available for new curriculum projects. Finally, after much pressure from education experts and the unions, the four-year Bachelor of Education (B.Ed.) course for training primary and secondary teachers, was introduced in 1965, making teaching an all-graduate profession.

The concerns about the number of pupils in secondary schools incapable of coping with the demands of the GCE, led to the introduction of the Certificate of Secondary Education (CSE) in 1965. However, the CSE increased the pressure on secondary schools to separate students into 'academic' and 'non-academic' streams. Within comprehensives, GCE and CSE students were placed in different teaching groups, while secondary modern students capable of CSE entry were separated from those who were not[9]. However, one positive result was the rise of school-centred curriculum development, and a new system of moderation which gave teachers a considerable amount of independence in content, teaching, and assessment methods in mathematics. There had been a move towards more informal, child-centred education in primary schools with an emphasis on individualisation and 'learning by discovery', and the abolition of selection in many LEAs freed the curriculum and encouraged innovation in schools, enabled by increased professionalisation and autonomy of teachers.

The 'Plowden Report'[10] was produced by the Central Advisory Council for Education (CACE) at a time of great change. Selection and 'streaming' were being abandoned. Comprehensive schools were officially established, and teacher-led

[9] A very few secondary modern pupils were entered for the GCE, calling into question many issues about assessment and 'ability'.

[10] Plowden was a major government report and established new attitudes, ideas, and organizations in primary schools. See http://www.educationengland.org.uk/documents/plowden/index.html

curriculum innovation actively encouraged. The report showed that individual differences between children of the same age were so great that any class, must be treated as a body of children needing individual attention. The main tasks of the primary school were to strengthen children's intrinsic interest in learning and lead them to learn for themselves. Flexibility in the curriculum, learning by discovery, and the importance of evaluating children's progress in different ways, were important.

The Labour government set up the Open University (OU) in 1969. This egalitarian project was intended to provide university education for working people, but became popular among those who had the time to benefit from it, and was overtaken by the middle classes. It provided innovative pedagogy and research, and allowed students to choose from a modular system where mathematics in-service courses for teachers were popular.

1970-1979

Another Conservative government in 1970 encouraged more comprehensive schools, and there were more children in comprehensives than selective schools. In 1974 the Labour government returned, and in November 1976 Prime Minister Callaghan gave his *Ruskin College Speech* in which he called for a 'Great Debate' about the nature and purposes of education, which would allow employers, trades unions, and parents, as well as teachers and administrators, to make their views known[11]. Underlying this speech was the feeling that the educational system was out of touch with the need for Britain to survive economically through the efficiency of its industry and commerce. There were government interventions, funding cuts, and questions about the accountability of teachers, so in 1978 the Department of Education and Science (DES) set up the Assessment of Performance Units (APU)[12] to provide information about the performance of pupils from 11 to 18 in state schools in science, mathematics and English language. The survey continued until 1982. It became clear that while the LEAs were responsible for what went on in the schools, they relinquished their authority to the schools and the teachers. However, there were examples of breakdown in communication and lack of clarity about responsibility, and in the introduction to the initial report (APU, 1976) two essential concerns were raised: while a pupil's performance should be the concern of every educational institution, there were worries about any use that might be made of the findings of monitoring.

The worrying economic climate prompted views presented in five 'Black Papers' written by right-wing educationalists and politicians, and supported by the

[11] Callaghan talked about 'entering the secret garden of the curriculum', which heralded the political controls that were to come.

[12] For details see: https://www.stem.org.uk/elibrary/collection/3618/assessment-of-performance-unit-mathematics-reports-for-teachers

right-wing press. Criticisms had begun in 1969, focusing on the 'progressive' style of education being developed in the primary schools as the main cause, not only of student unrest in the universities, but of other unwelcome social events.[13] These papers attacked the concept of comprehensive education, egalitarianism, and new teaching methods. They deplored the lack of discipline in schools and blamed the comprehensive movement for preventing 'academic' students from obtaining good examination results. Responding to the criticisms, guidelines for LEAs were published in *Mathematics 5-11: A handbook of suggestions* (DES 1979); which, although progressive in tone, its list of aims and objectives produced a firmer direction of the curriculum towards more uniform public education.

Unfortunately the raising of the principle of a 'national system of education, locally administered' would be taken up by Margaret Thatcher's Conservative administration from 1979. They began to change the education system, and abolished the Schools Council in 1984. The school examination system was completely revised in 1985, and first administered in 1986.

The unstable political system with the government changing regularly from 'red' to 'blue' responding to the vagaries of economics and political ideology, with confusions about whether LEAs or teachers were ultimately responsible for children's education, together with differences about the types of school, was not conducive to a reasoned, liberal, debate about schooling or the curriculum. Because there were many divisions within the system, the politicians eventually imposed their will upon the national education system of England and Wales.

Background on educational psychology and philosophy

Developments in mathematics education cannot be separated from political and economic circumstances and we can see that while political ideologies and economic situations have influenced education policies and events, educators flourished from the freedoms remaining in the UK system, and managed to put into practice concepts and principles that have laid the groundwork for the developments in thinking and conditions for implementation, that we see today.

New ideas in psychology

Nathan Isaacs was interested in philosophy and psychology, so primary teachers learnt about Piaget through his short publications (Isaacs, 1960; 1965). He warned about taking the stages of development too literally, but was ignored by many commentators. Although Piaget's *Child's Conception of Number* had already been

[13] These concerns of the Right Wing had been influenced by the events in Paris, in May 1968.

translated (by Gattegno and F.M. Hodgson in 1952), the relevance of this work on early cognitive theory was slow to impact on education. Caleb Gattegno had begun to work on the implications of Piaget's work almost twenty years earlier (Gattegno, 1947; 1959; 1954a) but this was unlikely to reach many primary teachers (MA 1956), or any who were teaching mathematics in Secondary Modern schools. Jerome Bruner (1955) had recently been published, but it took time for his message to be heard outside psychology, and Behaviourism, with its principles of reinforcement and punishment applied in education, emphasised observable behavior[14] and rejected using internal processes or structures to account for learning.

In complete contrast, *Richard Skemp* continued his Piagetian psycho-logical analysis into the field of mathematics, and developed the idea that differences between arithmetic and mathematics are related to the difference between sensori-motor intelligence and reflective intelligence (Skemp, 1961; 1962a). Skemp's earliest writing for teachers introduced the complexity of a 'concept' and the idea of a 'schema'[15] (Skemp, 1962b; 1962c; 1963). His most significant publication for teachers was to be his article in *Mathematics Teaching* on 'Relational and Instrumental Understanding' (Skemp, 1976). He contrasted *instrumental understanding* (knowing how to do something from habit or memory) with *relational understanding* (knowing not just how but why, and the rationale for the procedure) that endows the power to reconstruct, relate, apply and transfer knowledge, rather than the learner remaining dependent on memorising arbitrary instructions. This gave teachers a clearer basis for their pedagogy, and the idea that intelligent self-correction required the ability to become aware of relationships between purely mental events, prompted the distinction between 'products' and 'processes' and laid the groundwork for more interesting and useful developments in cognitive psychology.

The structure of the curriculum was discussed and the Nuffield Project (see below) offered no prescription of best content, but initiated primary teachers into the major concepts of new mathematics, while the MA's (1968) analysis of 12 secondary curriculum development projects, found there was overall agreement in content, but not in order of topics. Bloom's (1950) hierarchy of educational objectives was seen more in the UK as suggesting intellectual abilities rather than teaching objectives, and so rather than questioning the order of topics, views shifted to the need to see the curriculum as whole, and emphasised the individual child's learning.

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[14] From training fighter pilots to react to profiles of enemy aircraft, to pigeons gaining food for 'good' behavior, and 'rote' learning; many American educational publications were using these principles which had some influence on secondary schools at the time.

[15] The idea of a schema appeared in early cognitive psychology in the 1950s with Frederick Bartlett (UK) and was being used by gestalt psychologists. Piaget's definition was 'a cohesive, repeatable action sequence possessing component actions that are tightly interconnected and governed by a core meaning' in 1923.

The idea of using *concrete materials* to support children's learning, originated by Emma Castelnuovo[16] and was espoused in early ATAM (Hope, 1963) and MA materials. As early as 1946 the Visual Aids Sub-Committee of the MA organized an exhibition of materials (MA1947). After some discussion, the matter seemed to rest until Gattegno[17] (1954b)[18] discusses using films in teaching geometry. The debate about 'teaching aids' and 'concrete' and 'structured' materials was the principal rationale for the foundation of ATAM.[19]

Zoltan Dienes' work on structured material (1960, 1963, 1975) contributed to presenting the underlying theory of mathematical topics. He believed that knowledge and abilities were organised around experience as well as abstractions, and provided *structured materials* for pupils, representing certain concepts showing how mathematical structures could be taught to children from an early stage. He first described his theory of 'six stages' of learning mathematics in *Building up Mathematics* in 1960. Dienes' cognitive development was interpreted in modified Piagetian stages. His book was useful when many teachers needed some physical representation of ideas found in the 'new mathematics', and his demonstration of how a number system is structured in the same way in any base, with the Multibase Blocks, helped teachers to understand his *principle of variability*. In *The power of mathematics* (1964) Dienes described the relation between the structure of the task and the structure of an individual's thinking as changing the variables while keeping the concept intact. Later, he worked on some structural versions of games, and the algebra of dance activities (Dienes, 1973).

I described in some detail the research work of Gattegno who was a prolific educational author and original thinker, together with other early members of ATM in (Rogers, 2015, pp. 351-357). The continued ATM commitment to research and curriculum development is clear in its earliest statement (ATM 1955) and in many subsequent issues of their journal, *Mathematics Teaching*. That commitment was reinforced when Vygotski's *Thought and Language* appeared in translation in 1962. The genetic roots of thought and speech, and the development of scientific concepts brought radically new ideas into cognitive theory, which began to turn researchers to social contexts of learning mathematics. A few years later, another Russian translation brought new ideas to the classroom with Krutetskii's view of the abilities of children (1976). Over the next twelve years, the contribution of ATM literature from 1964 to 1976, and its influence on the mathematics education community has already been pointed out in (Rogers, 2015). According to Howson (1978, p. 186) ATM had changed the approach of many subsequent textbooks.

[16] See (Furinghetti & Menghini, 2014).

[17] Gattegno had discovered the Cuisenaire Rods and already promoted their use in schools by 1954.

[18] This was two years after the foundation of ATAM in 1952, where the debate had been continuing for some time (see Fletcher & Harris in MT 3, 1956).

[19] See (*Mathematics Teaching*, 1955, *1*) editorial, the articles on models, and the use of films.

Margaret Donaldson was Professor of Developmental Psychology and Linguistics at Edinburgh University. Her earlier work (1959) concerned the prediction of 'ability' and a review of intelligence testing where she questioned whether the observer could really become aware of another's thought processes[20]. In her book, *Children's minds* (1978), she put forward the view that by the age of three most normal children, given the right kind of 'embedded' instructions, are capable of evincing 'rationality', in the form of deductive inference[21]. She showed that: children are not as egocentric as Piaget claimed; nor are they limited in ability to reason deductively; a child's language-learning skills are not isolated from the rest of their mental growth; and when children interpret what we say to them, their interpretation is influenced by knowledge of the language, an assessment of what we intend, and the manner in which they would represent the physical situation to themselves, if they were not present. Her results showed that most people, from young children to adults, have little difficulty in reasoning in familiar contexts, but struggle when thinking moves into disembedded thought. Her views reinforced the opinions of many teachers who had direct experience of children's thinking, their creativity, and ability to develop abstract thinking. Her work revealed new possibilities of interpreting and supporting children's mathematical development.

The *investigation of the teaching process* became important; the role of the individual teacher, and their teaching methods, could be varied, and investigated. The discussion of the role of the teacher in *Mathematics Teaching* (1958. *8*. 3-4) has an account of giving a lesson, but also discusses the pupils, the onlookers, and the aims of the lesson, calling into question many of the hidden assumptions about 'who learns what' in such a context. This appears to be the first expression in the literature of a fundamental question: How does learning take place? How do children make their own mathematical experience? We may provide a situation that we think is rich in mathematics, we direct attention to the potentialities of the situation, and we hope some mathematics appears. If thereby pupils engage in perception and action, progress is made, but *this is decided by the pupils, not the teacher*[22]. By focussing awareness more on the pupil, this became the *Learner centred approach*. Wheeler's (1970) 'scientific teacher' considered the presentation, with structured materials, of different visual aspects, the sequence of ideas within a lesson, and helping children develop as autonomous individuals. New questions challenged the quality of teacher-pupil interaction in classroom situations, realising that the material presented in texts had to be understood by pupils (Shuard & Rothery, 1975).

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20 Note that this was a serious critique of the whole idea of selection at 11plus.

21 Many interpreters of Piaget's results had seen the 'ages and stages' as a set of barriers to the development of children's thinking, against all the accumulating evidence, rather than thinking about changing the 'laboratory conditions' of the experiments.

22 That pupils may not learn what we teach them was still evident later see (Denvir & Brown, 1986).

Some curriculum development projects

As problems in the schools worsened and teachers struggled to find solutions, opportunities for curriculum development arose. A discussion on 'Mathematics in a Comprehensive School' appeared in the *Mathematical Gazette* (1950) where secondary teachers related the difficulties of teaching pupils who had not passed the '11+' examination. Comprehensive schools should deal with pupils of all abilities, but few pupils could manage the mathematics required to pass the GCE examination; so not only were there inherent difficulties in teaching, schools did not have enough qualified staff, and lacked decent text books. Teachers managed to devise some ways to address these problems, but the situation was dire. This situation prompted the creation of the ATAM, and in the 1950s colleagues organized discussions with teachers which developed into regular local meetings and an annual national conference.

The first large-scale secondary projects began in the school year 1961-62. In principle, schools could ask their examining boards to be examined at 'Ordinary' or 'Advanced' level on their own syllabus. Several groups of teachers set out to devise 'modernising' schemes of work in the wake of publications from the OEEC (1961a,b) by proposing new mathematics Ordinary level examinations. These were The *School Mathematics Project* (SMP), started by Bryan Thwaites, at Southampton University, the *Midlands Mathematics Experimen*t (MME), organized by Cyril Hope,[23] at Worcester College of Education, and the *Contemporary School Mathematics* project (CSM) run by Geoffrey Matthews at St. Dunstan's school.[24] The texts were written by practicing teachers; no government money was involved, and financial support for teachers to take time out for writing, was limited. SMP and MME ran short courses and provided teachers guides. Project writers addressed teachers' meetings organised by LEAs, and the materials were piloted and could be taught to typical pupils. Since texts were to replace the standard O-Level examinations, they addressed new topics for the more able pupils. MME set out to cover a wider ability range and designed its books to cater for the introduction of the CSE in 1965. By 1968, it had already published paperback copies of its texts and announced the first copies of its hard-back texts for CSE and O-level covering the first two years of the secondary school course. It received a grant from the Schools Council enabling the project to fund a small team of teachers to consolidate the work for the 300 schools and 10,000 pupils already involved (C.B., 1968). But the ambitions for the project overwhelmed the energy of the organisers, and it finally became absorbed in the changing examination system. Richard Skemp's Psychology and

[23] Cyril Hope was an ATM member, and had attended the Royaumont Seminar (OEEC, 1961b, p. 307).

[24] Each of these projects produced text books, and samples of these can be found on the STEM website.

Mathematics Project for secondary schools, was based on the principles of concept-building and schematic learning, but his books 'Understanding Mathematics' did not appeal either to pupils or teachers. The only one of these projects that managed to continue was SMP, they obtained some private funding, leading to a publishing contract with Cambridge University Press. It became the largest single secondary school publishing venture in the 1970s and is still running today[25].

The *Abbey Wood Mathematics Project*, devised by David Fielker[26] was quite different. It was supported by teachers, by ATM, and the local community. The Certificate of Secondary Education (CSE) was introduced in 1965, and in the same year Fielker had already applied to his examining board for a programme for Mode 3 examinations[27] based at his school. It became popular, and other schools devised similar curriculum models for Mode 3. The CSE examinations continued until the government revised the whole examination system in 1985.

Secondary Mathematics Individualised Learning Experiment (SMILE), was developed as a series of practical activities for students by teachers in the 1970s. It became a scheme based on a network of activity cards and assessments. The cards were organised so that each student followed their own path through the work, and their progress was supervised and recorded by a teacher. The scheme was self-contained, covering the whole of the secondary school syllabus, with a variety of different activities, including assessment cards. This project was used by teachers, particularly in the Inner London Education Authority, (ILEA) from the early 1970s through to 1988 when the Conservative government abolished the ILEA.

The ATCDE was involved with the changes in the school curriculum, since they were concerned with preparing teachers for schools that had decided to join a project, or to change their own curriculum. After the extension of the teacher-training course to three years, there was insufficient mathematical content – particularly for secondary school teachers. After a number of meetings in 1965/66, they published *Teaching mathematics: Main courses in Colleges of Education* in 1967. This put forward the rationale for the changes in the curricula, discussing the growth and nature of mathematics, mathematics for students, how teaching staff should adjust to changes and update their own knowledge, implementation of methods of assessment, and mathematics courses for the new B.Ed. degree, which had been introduced in 1965.

Later, other curriculum projects became more successful; teachers collaborated with teacher-trainers and universities, or involved industry, in projects like *Mathemat-*

[25] However, even in 1971 there were serious criticisms, see (Sturgess, 1971).

[26] Fielker had a certain 'charisma' and was well-known outside his local area. Rationale and examples of materials for this project can be found in ATM (1968, pp. 65-70).

[27] Mode 3 examinations included radical changes in curriculum content, introducing more 'modern' topics, and the use of course-work to be monitored by the teachers and moderated by external experts.

ics for the Majority, and the *Schools Council Sixth Form Project*, but since they were innovative in content and in assessment methods, these too suffered from lack of implementation, insufficient professional development, and bureaucratic complications. Both the ATM Advanced level and Ordinary level courses failed because validating authorities could not reconcile curriculum proposals that included 'investigations'[28] and new assessment methods that clashed with long held beliefs (ATM 1968) (Bell 1968). While projects may have 'failed' in commercial terms, much good came out of them; teachers gained experience in analysing their teaching and began to pass this on to others, and researchers began to understand the complexities and pitfalls of looking at classrooms and evaluating projects (Bishop, 1972). Another outcome of this activity was the establishment of the *Mathematical Association Diploma* for training specialist mathematics teachers in upper Primary and lower Secondary schools, where a course book *Starting Points* (Banwell, Saunders and Tahta, 1972) was written by members of ATM.

In 1956, Bloom's *Taxonomy of Educational Objectives* appeared; giving a view of the cognitive stages involved in being able to understand a subject area fully, so that a learner can not only to apply the learning, but develop and communicate new ideas in that area. The original levels of the cognitive taxonomy were ordered as: Knowledge, Comprehension, Application, Analysis, Synthesis, and Evaluation.[29] While these categories were intended to describe 'levels' of cognitive development[30], and were used by many to plan teaching, and analyse understanding, it became clear that there were many other aspects involved in mathematical *learning*. Descriptors used by project writers varied considerably, and Avital and Shettleworth (1968) replaced the five levels of Bloom taxonomy with 'recognition or recall', 'algorithmic thinking or generalisation' and 'open search'. Some writers tried to fit their categories to specific mathematical activities; while useful in certain contexts, category choices always brought complications (Watson, 1976, pp. 45-46.)

Investigations and problem solving

An article in ATM (1968; 13-33) explores the nature of 'mathematical activity' and the meaning of 'achievement' and 'ability' in mathematics, offering different levels of 'intellectual activity in mathematics': knowing, translating, manipulating, choosing, analyzing, synthesizing, and evaluating. All these headings have examples of contexts and levels of difficulty. There are other discussions in this publication,

[28] For 'investigations' see ATM 1968 (pp. 68-70).

[29] The taxonomies for the Affective and Psychomotor Domains were published in 1964 and 1972.

[30] 'Levels of development' became a doctrine around which many mathematics projects were organized. The worst example being the UK National Mathematics Curriculum. See pages 31-35 http://www.fft.org.uk/FFT/media/fft/Events/Dave_T_Descriptions.pdf

for example what to do about longer-term projects, completed out of class, all generated by the ideas in the 'Problems Document' (ATM, 1966) and from the writing group (ATM, 1967). There is no doubt that these texts generated substantial discussion of assessment that continues today.

As a significant part of new curricula, *investigations* were unusual problems that do not have an obvious solution algorithm, and are hopefully, at (or just above) the student's 'level'. The student may need to learn some new mathematics, or a particular technique, or get involved in a completely new area of mathematics, in order to make a start on the problem. This was the concern about *mathematical activity,* and it was this activity, quite different from routine problems, that teachers wanted to encourage. Suitable examples were sought outside the 'normal' school curriculum, or the problems found in Polya's (1965) Mathematical Discovery collection. The original model was an account of R.L. Moore's method (MAA, 2009).[31] Clearly, many people had difficulties in understanding the point of the investigation activity, and raised objections and difficulties with assessment. There had to be a sense of trust between teacher and student. A full account of the methodology and contexts, with many examples can be found in the ATM *Problems Document* (ATM 1966 22-45) and *Examinations and Assessment* (ATM 1968 68-70).

A useful example of specific objectives appeared in Leone Burton's research project supported by the Social Science Research Council, and organised from the Southbank Polytechnic in the 1970s. It involved some thirty London teachers and 850 students aged between nine and thirteen. The project produced thirty mathematical problems representing of a wide range of aspects of mathematics, and encouraged pupils to 'think mathematically' in their efforts. The *Problem-Solving Process* was developed that consisted of a guide through the four phases of activity: Entry, Attack, Review, and Extension, mirroring what mathematicians do when solving problems (see Polya, 1945). The value of these was the way in which a reader could work through the structured procedures for each of these phases. *Thinking things through* was published in 1984, and the 'Activities for Students' contains thirty problem-solving activities. In 'training' the problem-solving process, the student would be able to develop their own strategies that could be applied to new situations. At the same time, Leone Burton collaborated with John Mason and Kaye Stacey at the Open University, and they were all involved in writing materials for the course *Developing Mathematical thinking* (EM235)(1982-1991). As a result of this collaboration, *Thinking mathematically* appeared (Mason, Burton, and Stacey 1982, 2010), an outstanding book for all who are involved with teaching or learning mathematics, now in its second edition.

[31] R.L. Moore (1882-1969) was an American Topologist whose revolutionary teaching methods (no textbooks, no lectures, no conferring) prompted the rise of 'inquiry-based learning' projects in universities.

The Nuffield Mathematics Project[32] (1964-1973), was led by Geoffrey Matthews[33] who was appointed to Chelsea College Centre for Science Education. The age range, 5-13 covered all primary years and transition into secondary school.[34] The first Teachers Guides, published in 1967, used a *conceptual* progression based on Piagetian research. These explained the underlying mathematics with ideas for different activities, all illustrated with pupils' work. Expressly written for teachers, they introduced the rationale for the project, explaining the need for change, with ideas for class organization and a new approach to evaluation. The modern structural ideas of mathematics were introduced: sets, relations, number bases, and properties of operations, to help form a conceptual basis for calculation, together with properties of plane transformations and spatial situations. There were also 'Checking Up' guides that provided interviews using everyday objects to review children's understanding of conservation, equivalence and invariance. An important innovation of the Nuffield Project was the creation of *mathematics centres* where teachers could meet with advisers; these evolved into professional development centres to be found in most Local Authority areas. The NFER published a survey of the techniques of primary teachers (Williams, 1971a; 1971b) and the charismatic Edith Biggs[35] published her *Curriculum Bulletin No.1*, a popular practical guide for primary teachers. This project was widespread throughout the UK, inspiring many teachers, involving them in classroom research; this also provided the basis for the attainment tests found in the *Assessment of Performance Units* (APU) and the *Concepts in Secondary Mathematics and Science* (CSMS) investigations in the 1970s and 1980s

Emerging research centres and themes

The National Foundation for Educational Research (NFER) was founded in 1946 as a centre for educational development in England and Wales. Its early work was influenced by the views of Cyril Burt[36] on intelligence, arithmetic, and language

[32] This project was funded by the Nuffield Foundation, a Charitable Trust, founded in 1943 to improve social well-being by funding research and innovation in education and social policy.

[33] Geoffrey Matthews had led the CSM project, and later became the first Professor of Mathematics Education in England.

[34] At the time this project was set up, there were many 'Middle Schools' in England, for pupils aged 9 to 13, the result of the freedom of the LEAs to organise their provision of education.

[35] Edith Biggs was a staff inspector for primary schools, one of Her Majesty's Inspectors, an experienced primary teacher and the first inspector to have her name on a published report. Hitherto, all HMI reports had been published anonymously.

[36] "By the term 'intelligence', the psychologist understands inborn, all-round intellectual ability…... Fortunately, it can be *measured with accuracy and ease*." (Burt 1933:28-9)(italics, mine). Burt's research has been severely criticised as manufactured and unreliable see Gillie, O. (1976, October 24).

testing for the 11+ selection. By the late 1960s it had produced a series of research publications, including research into curriculum development projects and secondary and comprehensive school examinations. NFER carries out comparative surveys such as TIMSS, PISA, etc. and is the UK's independent national educational research and assessment organisation.

From the outset, ATAM organised meetings for teachers, in liaison with Institutes of Education and teachers' unions to test the value of teaching aids and methods by demonstration and discussion. Exhibitions and seminars were organised in different parts of the country, and individual members of ATAM addressed groups of teachers on a range of subjects (Rogers, 2015: 352). In 1963, ATAM published a Special Issue on Teaching aids in Mathematics (Hope 1963).

From 1945, University Departments of Education (UDEs) ran courses for secondary school teachers, but mathematics graduates could work in schools without undergoing education training, The Postgraduate Certificate in Education (PGCE) for graduates wanting to teach in Primary and Secondary schools was not compulsory until 1965.

The LEAs were responsible for funding and administration of schools, and supporting teachers, and in the 1960s began to set up *teachers' centres* to improve teaching and promote good practice[37]; instituting a regime of mathematics advisors and inspectors.[38] Since many primary teachers were insecure in mathematics, these centres organised meetings on aspects of the curriculum, reviews of apparatus, and in-service professional development[39]. Many members of ATM and the MA found themselves invited to these activities and University Departments of Education became involved, bringing 'researchers' into closer contact with 'teachers', emphasising communication of research meaningfully. Students from colleges and the university departments would all be undergoing their practical teaching in schools within the same LEA.

From 1964, the Schools Council provided funds and encouraged innovation, and local authorities developed their own policy centre publishing a mathematics curriculum support programme for their schools.

Chelsea College in South West London was an example of a UDE where courses retrained those who wanted to teach mathematics. Matthews, who became the leader of the Nuffield Project (abive), insisted on setting up teachers' centres to support teachers implementing the new programme. This model of support for

[37] For an account of typical activities at a Mathematics Centre, see Stone (1972) who was Warden of Stapleford Maths Centre and Senior Advisory Teacher for Mathematics, Cambridgeshire.

[38] Advisors were usually specialist subject teachers who could work with teachers and schools, helping with specific problems, and were often responsible for the day-to-day running of the teachers' centres. Inspectors were officials who could assist, not censure, teachers and schools in their areas. Thus, new areas of professional expertise were becoming defined.

[39] The establishment of teachers' centres was energetically supported by Geoffrey Matthews when he began the Nuffield Primary Mathematics project.

teachers and encouragement of research was adapted by a number of other UDEs, and national professional organisations like the ATCDE became involved. By the late 1970s there were a number of other local and national organisations to support mathematics teachers[40].

The Bachelor of Education (B.Ed.) degree in Mathematics included Education Studies, and practical teaching in schools. This was offered by a number of Universities and Colleges, thereby increasing the need for more highly qualified staff and communication between teachers and researchers. By 1968 extending the three-year teachers' certificate to four years involved 21 universities with Institutes of Education who had agreed to offer B.Ed. degrees.

Underlying philosophical and methodological themes were appearing amongst educators, aware of social theory, discussing views on the nature of mathematics, linking these ideas to how pupils might learn mathematics. Polya's heuristic, (1945, 1962) was well known, and in 1965, copies of *Proofs and refutations* (Lakatos 1963/64) began to circulate among mathematics educators. This work offered a philosophical justification for the methodology of the 'experimental approach', and the historical-social context for evaluation of the mathematics produced. His account of developing mathematical knowledge suggested that through conjectures and proofs, the mathematical community could trace the evolution of ideas through a *social-historical dialectic*.

Mathematics Education; ICME and International Study Groups

From the 1960s there was a growing movement to have Mathematics Education and Science Education recognised as separate academic disciplines (Furinghetti, Matos, & Menghini, 2014) and new international journals were established[41]. After the ATM had taken schoolchildren to their workshop at ICME in Lyon 1969, the UK hosted ICME 2 at Exeter in 1972 (Howson, 1973). Many people from the mathematical, education, and teaching community had the experience of meeting with colleagues from many parts of the world for the first time. As a result, new ideas, ambitions, and relationships were formed that influenced attitudes in the UK and encouraged the formation of English research groups. The new British Society for the History of Mathematics (BSHM) organised a *Discussion Group* on History and Pedagogy of Mathematics (HPM) at ICME 2, while other meetings at

[40] For example, the Association of Teachers and Lecturers (ATL) and the National Association of Mathematics Advisers (NAMA).

[41] For example, *Educational Studies in Mathematics* (ESM 1968), the *Journal for Research in Mathematics Education* (JRME 1970), and the *International Journal for Mathematical Education in Science and Technology* (IJMEST 1975).

the conference encouraged the formation of the Psychology of Mathematics Education group.[42] From 1972 there was a Psychology of Mathematics Education Workshop (PMEW) holding occasional meetings at Chelsea College, London. After ICME 3 in 1976, Richard Skemp started the British Society for the Learning of Mathematics (BSLM) that began regular meetings at Warwick University. Two years later in 1978, the groups combined, and continued as the British Society for the Psychology of Learning Mathematics BSPLM[43].

Mathematics, society, and curricula

This is the title of a book by Brian Griffiths and Geoffrey Howson published in 1974 that brought together many aspects apparent in the education system of the early 1970s, and heralded the varying contexts and developments of the system during the succeeding years. Some of the early curriculum projects were coming to an end; CSM and MME were being absorbed into the general education system and the professional development function of the Nuffield project was being taken over by the national network of teachers' centres. SMP continued to extend itself, becoming a standard part of school life. A second phase of curriculum development with new projects was beginning; some of the excesses of the 'new mathematics' needed to be tempered, and among the variety of curricula and project apparatus, a return to more traditional views were beginning to emerge.

The Schools Council set up *The mathematics curriculum: A critical review,* at the University of Nottingham in 1973, aiming to help teachers perform their own critical appraisal of syllabuses and teaching apparatus for secondary pupils. The books covered Geometry, Graphs to Calculus and Algebra (Anderson, 1978), Number, and Counting and Configurations, together with surveys of other areas. It was thought that, "teachers who faced a daunting array of mathematical literature and novel classroom material would welcome a basis for constructive and critical discussion of the content of the school mathematics curriculum." (Anderson, 1978, Preface v)

While the Schools Council was trying to encourage some rationalisation, considerable debate continued; on the nature of society, of children, of mathematics

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[42] The final affiliation of the International Study Group on the History and Pedagogy of Mathematics (HPM) and the International Study Group on the Psychology of Mathematics (PME) with ICMI was announced at ICME 3 Karlsruhe in 1976). For further information about the evolution of international research groups see: (Hodgson, Rogers, Lerman, & Lim-Teo, 2013).

[43] The first meeting of BSPLM was held at North London Polytechnic on 4th November 1978. Further meetings were held at Keele University in January 1979, and Warwick University where Richard Skemp was Professor in the Psychology Department. From that time, the tradition of termly meetings of the BSPLM were established at different institutions, hosted by members.

as a practical as well as theoretical science, on the possibilities of realistic mathematical modelling in the schools, and the effects of technology and the media. There was also the struggle to recognise *Mathematics Education* as a legitimate and separate academic subject. This was particularly the case in the colleges, where mathematics graduates were teaching mathematics, as well as training teachers; similar confusion arose with primary teachers recruited to train students, who found themselves involved in changes in mathematics teaching. Mathematics Education itself, and the people involved, were seeking an identity in both the social and academic sense.

Interest in theoretical issues was expanding; cognitive psychology was becoming less prescriptive and the use of theoretical models of thinking became more common. The alternative interpretations of Bruner, Vygotski, and others put more emphasis on social contexts, and Donaldson's (1978) reinterpretation of children's intelligence was challenging standard dogma. More philosophical aspects appeared on the agenda and Piaget's (1972) epistemological work raised deep issues about the nature and origins of mathematics in human thought.

The role of the teacher (Wheeler, 1970) and interactions between pupils and teachers were being investigated raising questions whether teacher-pupil exchanges (Flanders, 1961), could really reveal either person's knowledge or understanding. Other classroom issues like the role of language, reading, and the illustrations in pupils' books began to emerge (Shuard & Rothery, 1975), and sociology began to take a greater part in the determination of educational values, where Bernstein's (1971) theory of language both reflects and shapes assumptions of a certain social group and, how that group is perceived.

Changes in text books

The presentation of materials to children underwent considerable change; new topics appeared in the primary and secondary curriculum, but after the innovative presentation of mathematical ideas to teachers in the Nuffield project, in many cases, texts for primary pupils became 'work cards', which were produced by individual teachers for their own classes, or commercially produced cards of variable quality. The most complex work card system for secondary pupils was SMILE, mentioned above. Again, the text-books varied a great deal; much of the material produced for primary schools were collections of materials allowing teachers to copy the pages, or poor quality disposable books for pupils' written work. For secondary schools, some books hardly ever changed (particularly for the 'low ability' pupils) they continued to be published as if neither the 'new mathematics' nor the new pedagogy had happened. On the other hand, some projects did produce interesting and useful texts trying to pay attention to the Schools Council recommendations, and textbooks and projects written for the Scottish schools (Rogers, 2014, pp. 23-54) were much more successful. There was also a proliferation of texts for autodidacts: many of them were translations of short Russian problem-solving books, and other innovative introductions to new mathematics published in England or the USA.

Films and television

The serious consideration of filmstrips, films and film loops as aids to teaching and learning appeared early (Fletcher & Harris, 1956) and became a regular subject of discussion where mathematics, particularly geometry, was thought about in purely visual terms (Fletcher & Birtwistle, 1961). The basic principles had been established early (Gattegno, 1954b) where teachers would advocate this approach as supporting a *teaching method*, involving a new geometrical curriculum and creating a substantial course on *transformation geometry* (Brookes, 1967), (Bell, 1971), parts of which appeared in many of the new text books.

From 1969, television presentation of mathematics programmes from the Open University began, sending out lectures and seminar materials for students and teachers, which were also available for the general public to watch. The BBC and Independent television companies were already broadcasting 'new mathematics' programmes for secondary schools. These were intended to supplement the mathematics curriculum, but portable equipment and administrative arrangements were difficult until pre-recording became possible.

Conclusion

The coming years were to see even greater changes in all these aspects of mathematics education as political domination of the system increased, and the efforts of individuals and organisations to retain some influence and independence became much more difficult. After 1979, political ideologies became more dominant, and the evidence from educational research was regularly ignored. While influences of continental philosophy on research became more apparent and nuanced in educational theory, international comparisons became the principal political measure by which 'progress' was judged.

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Mathematics teaching in the process of decolonisation

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Abstract

Research into the history of the teaching and learning of mathematics has so far essentially focused on developed countries. There is little research done, however, investigating the development of mathematics teaching in countries that were former colonies of some imperialist power. There were evident differences in the educational policy of the various colonizing powers and likewise different policies how to establish education in their colonies and how to deal with educational practices and structures already encountered in them. Pyenson has systematized the policies of the major colonial powers along three axes: the functionary axis, the research axis and the mercantilist axis (Pyenson 1989a). After independence, the approaches to mathematics teaching were not necessarily independent, but in many cases were strongly influenced by the former colonial power or international organisations such as the International Monetary Fund.

The paper will at first study differing approaches within the administration of British India in the 19th century, to address the issue of interaction between establishing a colonial educational system and differing models of mathematics education within the homeland. Regarding the colonial period in the 20th century and the post-colonial period, the focus will be mainly on Africa where the practices of a number of colonial powers coexisted: France, Great Britain, Portugal and Spain. Textbooks will serve as a clue for analysing the issue of cultural identity, underlying the elaboration of genuinely "independent" mathematics curricula.

Introduction

Research into the history of the teaching and learning of mathematics has so far essentially focused on developed countries. The main focus has been on countries in Europe, North America and some countries in Latin America, in particular, Brazil. Little research, however, has been done on investigating the development of mathematics teaching in countries that were former colonies of some imperialist power.

Clearly, such research is confronted with even greater methodological challenges, having to consider a broader interdisciplinary conceptual field. Besides deepening assessment of political history and of cultural history, more areas of sociological research have to be considered. Particularly relevant is the area of post-colonial studies.

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

To give an idea of the extension of tis research area, I mention a pertinent publication: a voluminous reader with 84 chapters: *The Post-Colonial Studies Reader* (Ashcroft et al., 1995).

As is evident, too, studying the history of mathematics teaching and learning in colonial and post-colonial times is a field full with mined regions: such landmines concern political, cultural and ideological issues. To give an example, I refer to the paper by Alan Bishop, which was reprinted in this reader: *Western mathematics. The secret weapon of cultural imperialism* (Bishop, 1990).

On the other hand, there is a remarkable fact regarding the decolonized countries, which has not yet been reflected and commented upon: it is well documented to what degree mathematics as a discipline of teaching, say in secondary schools in Europe, used to be questioned during centuries and what a long and difficult process it was to become transformed from a marginal status to the status of "Mathematics for All" (see Schubring 2014). Regarding the former colonies, upon becoming independent, there has be no such questioning at all. Well to the contrary: upon establishing proper educational systems after independence, it went by itself that mathematics had to be a major discipline, it was in fact conceived of as "Mathematics for All" (see Schubring, 2015).

Particularly pertinent for mathematics teaching is the process of decolonization in the transition form colonial times to the period of independence. There one has to be aware that there were evident differences in the educational policy of the various colonizing powers and likewise different policies on how best to establish education in their colonies and to deal with educational practices and structures already in existence. Besides political history and sociology in general, it is, in particular, the sociology of science, which lends analyses and categories for understanding the development of mathematics teaching in the colonies during Imperialism. The most productive contributions in this sense have been elaborated by Lewis Pyenson, with his trilogy on cultural imperialism (Pyenson, 1985; 1989b; 1993).

Pyenson has synthesised and systematized the policies of the major colonial powers along three orthogonal axes: the functionary axis, the research axis and the mercantilist axis, in his important (1989a) paper.

Regarding the union of academic, military, religious and business interests when sending scientists outwards to the colonies of the metropolis:

- functionary axis: a scientist in foreign parts remained entirely subordinated to metropolitan directives (even depressing, if not distinguishing the desire to prosecute original research) – tight union;
- research axis: research ethic remained paramount – loose union;
- mercantilist axis: scientists would have to serve business interests: subordinated to solve one or another problem in technology, in applications.

These different axes reveal at the same time the role attributed by the imperialist powers to education of the colonised population.

British-India

A first revealing case is presented by British India during the 19th century, since there a first period of valorising the traditional Hindu culture became brutally replaced by a crude Orientalism, denigrating the original culture and imposing British culture.

The first period there is tied to the development of the East India Company, chartered in 1600, a trading company, which began to colonize India in the interest of Great Britain. Expanding ever more there, it controlled extensive parts of India since the second half of the 18th century. The EIC became ever more under control of the British government, and in 1858 it was dissolved and British India became governed directly by the British Crown.

Originally without any intention to be active for education, the EIC's policy began to change by the end of the 18th century. From the 1780s, English civilians had begun to found schools, in Calcutta and Benares. The EIC itself initiated establishing education, from 1805. European missionaries and East India Company officials had very different motivations for establishing educational institutions in the country but by the middle of the nineteenth century two prevailed, that of the civilising mission and the attempt first by East Indian Company and then British administrators of the colonial regime to solve the problem of governability. How to prepare a modern educated colonial citizen who would administer the colony on modern lines (Raina 2014, p. 377)? Schools and colleges were established for a variety of motives but before the 1830s company officials were very careful not to encroach upon existing systems of education (ibid.). Thanks to the scholarly work of the early generations of British Orientalists both the antiquity and deep knowledge of astronomy and mathematics on the subcontinent was highlighted within the Sanskritic and Indo-Persianate universes (ibid.). The colleges founded were based on this assessment of the knowledge of the "indigenous". They were named "Oriental colleges", expressing that teaching occurred in the native languages.

It was in the wake of this true "Orientalism" that British scientists studied Sanskrit mathematical texts and published the first editions of this mathematical tradition, notably by Friedrich August Rosen (Univ. of London) and Colebrooke (1817). These orientalists had suggested to the first generation of educators in British India that Indian mathematics (arithmetic and algebra) was grounded on the same principles than those of Europe (see Raina, 2014, p. 378).

From 1835, however, a drastic rupture occurred, supposing now the Indians as an uncultured population who had to be educated in terms of British culture. This change was brought about by Thomas Babington Macaulay (1800-1859), a British historian and Whig politician. By his well-known "Macaulyan Minute" of 1835 – also known as "Bentinck's Resolution" –, he convinced the Parliament and the Governor-General of India to substitute teaching in native languages by teaching in English:

> We have to educate a people who cannot at present be educated by means of their mother-tongue. We must teach them some foreign language. The claims of our own language it is hardly necessary to recapitulate. It stands pre-eminent even among the languages of the West. It abounds with works of imagination not inferior to the noblest which Greece has bequeathed to us (Macaulay, 1835, p. 12).

While aiming at a class of anglicised Indians, he denounced the value of the traditional cultures in India, depreciating Arabic and Sanskrit. He called the local languages "poor and rude" and thought it not to be exaggerated "to say that all the historical information which has been collected from all the books written in the Sanskrit language is less valuable than what may be found in the most paltry abridgments used at preparatory schools in England". Works on science he referred to as, among others, "astronomy which would move laughter in girls at an English boarding school" (Macaulay, 1835, p. 8 and p. 13). Macaulay wanted to constitute an elite class of Indians, and „form a class who may be interpreters between us and the millions whom we govern". It should be left to this class "to refine the vernacular dialects of the country, to enrich those dialects with terms of science borrowed from the Western nomenclature" (ibid., 34).

In the first decades of EIC organizing teaching in British India, teachers sent from England had mainly been trained at special colleges of the EIC or at military colleges; there, the textbooks used were more modern than the adaptations of Euclid used at the universities (see Aggarwal, 2006, pp. 12 sqq.). But when the British Crown took over the educational system, teachers sent to India now mainly came from the universities – thus, the teaching of mathematics became much more like the traditional English system venerating and extolling Euclid.

Post-colonial period

I am now turning to Africa, the main victim of European colonialism and Imperialism from the 19th century. In the Congo Conference of 1885 in Berlin, the European powers thought they were entitled to divide Africa among themselves. I will use primarily mathematics schoolbooks for revealing cultural patterns characteristic for post-colonialism of the respective country. Different from syllabi, textbooks visualise, in particular by its illustrations, how education is dominantly understood in cultural terms and especially how mathematical activities are represented by persons (boys and girls) and by houses, buildings and the landscape.

North Africa: Islamic Maghreb

Tunisia

At first, I will discuss a country of Arab culture: Tunisia, where there is the adapted study by Mahdi Abdeljaouab (2014), which covers three periods: the period of Islamic civilisation, being formally a part of the Ottoman Empire, but factually half independent, governed by the Bey of Tunis: It follows the period from 1881, when France transformed Tunisia into a Protectorate, and lastly the period of independence, from 1856 on.

Tunisia, like India, presents therefore a case of a culture characterised by a proper development of mathematical culture: Islamic mathematics. The pre-colonial period is characterised, as in general in Islamic civilisation by an absence of state organisations of schooling. Basic schooling is mainly organised by the Ulemas, the Islamic religious scholars and imams of the mosques, the most traditional forces of the society, blocking any modernisation. Mathematics for the majority was restricted to basic arithmetic, necessary for calculating heritages according the Qu'ran. But from about the 1830s the governors tried to introduce modernisation: firstly by introducing military education by founding military academies, based mainly on teaching Western mathematics and contracting teachers from European countries. Reform minded personalities tried to create college-like secondary schools, introducing a strong role for mathematics teaching. In reforming in 1875 the most prestigious Tunisian institution, the mosque Al-Zaytuna in Tunis, the leading person Khayr al-Din defined a list of ten mathematics textbooks all of good quality, but all by Islamic mathematicians from the 13th to the 15th century, including Nasir-al-Din's extended Euclid commentary (Abdeljaouad, 2014, p. 410)

The French Protectorate established a strict separation between French Schools, for the French settlers and functionaries, with the identical syllabus as in France, and Arab schools, which should not lead to French Higher Education. Still, this part became better organized and state controlled. Mathematics teaching improved there, too, but had to stay in a level, which would not cause resistance by the Ulemas. A decree of 1933 valid for the secondary schools of Muslim education alerted: Students had to

> open their minds by being trained in those mathematics that are not opposed to the *sharīʿa* and do not disturb learning the basic religious sciences"(quoted from: Abdeljaouad, 2014, p. 412).

Progress was achieved at the primary level: modernised Qur'an schools (with a broader and more demanding curriculum) became paralleled by Franco-Arabic primary schools – with the same syllabus as the French schools but with Arabic language courses.

With the independence in 1956, the school system became unified and mathematics teaching was reinforced at all levels of the educational system. One heritage of the colonial system should prove, however, to turn out highly problematic for the learning of mathematics: mathematics was taught in Arabic during the six years of primary schools, but in French for the three years of intermediate and the three years of secondary level (*collège* and *lycée*).

The problem became more visible in 1988 when it was decided to teach mathematics in Arabic also in the *collège* (intermediate), but to maintain the mathematical symbols in the *collège* the same as in the *lycée*, i.e. in French, what were called in this discussion also the "international" symbols.[1]

What is problematic with this? One has to be aware here of a specific cultural feature of Islamic civilisation: it had received the numbers, which we use to call Arab numbers, from the Indians and these numbers – and formulas using them – have to be read from the left to the right; thus contrary to the direction of reading texts written in Arabic, i.e. from the right to the left.

Thus, since the beginning of symbolisations of mathematical operations, occurring in the Maghreb, symbolisation had to be read from the left to the right, contrary to the direction of reading the text itself:

While decolonizing, the three Maghreb countries had applied, however, diverse manners of dealing with this opposition:

		Algérie	**Maroc**	**Tunisie**
Langue d'enseignement des maths	Ecole primaire	Arabe	Arabe	Arabe
	Collège	Arabe	Arabe	Arabe
	Lycée	Arabe	Arabe	Français
Ecriture symbolique	Ecole primaire	Droite à gauche	Gauche à droite	Droite à gauche
	Collège	Droite à gauche	Gauche à droite	Gauche à droite
	Lycée	Droite à gauche	Gauche à droite	Gauche à droite

Fig. 1. (Abdeljaouad, 2004, p. 37)

Abdeljaouad has studied this problematic as *bilaterality*, i.e. that one has to read in the same sentence some parts from the right to the left and other parts from the left to the right, and in particular that one has to read mathematical notations in the opposed direction than the text in Arabic.
An example to illustrate this:

[1] One has to be aware that in almost all former colonies the language of the former colonial power was maintained at least to a certain degree, as an official language and in teaching. This was partly due to the cultural ties of the now governing elites who had been educated in schools practising the values and the language of the metropolis; a considerable part had even studied and obtained degrees at universities of the metropolis. The other reason, especially in Sub-Saharan Africa, was the multitude of local African languages – so that the colonial language was maintained as lingua franca. The only exception is the Bantu-language Swahili, which was spoken by Africans at the East African coast and which disseminated ever more; it became official language, for instance, in Tanzania, Kenya and Uganda.

مساحة المربع EFGH تساوي $49m^2$

The area of the square EFGH is equal to 49m².

Fig. 2. (Abdeljaouad, 2004, p. 38)

The 1988 school reform meant for mathematics an abrupt change from the sixth grade to the seventh grade:

- algebraic signs were no longer given in Arabic letters, but in Latin letters;
- symbolic expressions have to be read in the opposite direction, from the left to the right;
- the syntactic rules for the use of the signs of operation (+, -, etc.) imply reading the expressions from the left to the right;
- the abbreviations for indicating the units for measuring quantities obtain to two different registers: the arabised international units change to their international form:

	6ème année de base	7ème année de base
Litre	10 ل	10 l
Mètre	78 م	78 m
Kilogramme	123 كغ	123 kg

Fig. 3. Differing writings and readings of metric units. (Abdeljaouad 2004, p. 48)

This split structure has been maintained since; there is the twofold change in the Tunisian mathematics schoolbooks: from the primary school to the *collège*, and from the *collège* to the *lycée,* as is shown by the schoolbook series of 2001.

- In the 6th grade, the text is in Arab, and the formulas have to be read in the same direction – from the right to the left:

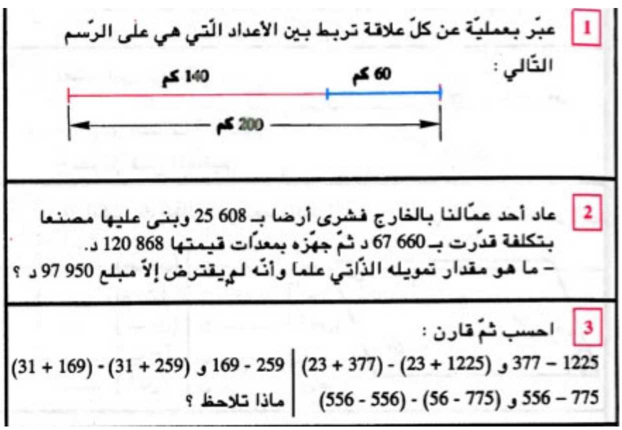


Fig. 4. Ar-riyathiyat 6, p. 7

- In the 8th grade, now in secondary education, the text is in Arab, but the formulas have to be read from the left to the right:

لنعتبر العددين الكسريّين : $x = a + \frac{1}{2}$ و $y = b - \frac{5}{3}$

نريد مقارنة x و y .

بما أنّ معرفة علامة الفرق بين عددين صحيحين نسبيّين تُمكّنُ من المقارنة بينهما. فإذا علمت أنّ هذه النّتيجة تبقى صحيحة بالنّسبة إلى الأعداد الكسريّة النّسبيّة قارن بين العددين x و y .

لنتذكّر : مهما يكن العددان الكسريّان النّسبيّان x و y فإنّ :
$(x - y \geq 0)$ يعني $(x \geq y)$

Fig. 5. Kitab Ar-riyathiyat 8, p. 104

- In the 10th grade, finally, the text is in French, the signs are the “international” ones and all reading is in the same direction:

I– Limites

Activité

Déterminer les limites suivantes

a) $\lim\limits_{n \to 0} \frac{x^2 - x}{x}$, **b)** $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$ **c)** $\lim\limits_{x \to 1} \frac{3}{x}$

Dans ce qui suit, on rappelle quelques résultats relatifs aux limites étudiés en 6eme année.

Fig. 6. *Mathématiques 10*, p. 6

Spanish Morocco

I wanted to study Spanish Morocco for having another case of an Imperialist country occupying a country of Islamic culture, and thus with a mathematical tradition. France and Spain had decided in 1912 to divide Morocco among themselves and to govern it as their colonies.

Although there is sufficient literature on the educational system in the Spanish Protectorate (see González, 2015), so far no research has been undertaken on the mathematics teaching there.[2] I have to restrict myself hence to some structural remarks.

Spain invested enormous energies in establishing a **colonial** school system. It consisted of three types of schools:

[2] I am thanking Luis Rico (Granada) for his cooperation in this research.

- Spanish schools, for the settlers, and for some Moroccans near to the colonial government (administration, etc.). Teachers came from Spain, and instruction was given in Spanish,
- Hispano-Arab schools, for Moroccans, teaching given in Spanish and teachers also from Spain
- Hispano-Hebrew schools, for the Jews in some of the urban centres.

Besides this official system, there began to arise, in the 1920s, the so-called nationalist schools, founded by the nationalist movement and giving instruction in Arabic.

Eventually, there were the traditional schools, here – besides the evident Qur'an primary schools – also Talmud schools; all these were not government controlled but by the respective communities, the Ulemas and the rabbi.

The Franco-Regime brought about a change in this colonial system. To recompense the Moroccans who had supported Franco's putsch, his government united the Hispano-Arab schools with the emerging nationalist schools. Instruction there was given in Arabic, and Spanish teachers were substituted by Moroccan ones. For the teaching in these schools, *concours* (competitions) were organized for the elaboration of new textbooks, from 1940. In the second stage of these *concours*, in 1943, a *concours* occurred also for mathematics (González, 2012, p. 130). Unfortunately, I did not yet find its results.

Sub-Saharan Africa

Basically, the countries of Sub-Saharan Africa can be regarded as having not established a proper mathematical culture in pre-colonial times. Mathematical practices used to be transmitted within the families or the ethnic groups. The imperialist powers practiced in these parts of Africa basically the same educational policy as we already remarked for the Maghreb: segregating schools for their settlers and some indigenous elite groups and almost no schooling for the colonized people – letting basic education to be organized by missionaries. My focus of analysis was on Great Britain, France, Portugal and as a special case South Africa. For Belgium (Belgic Congo) I had no sources available.

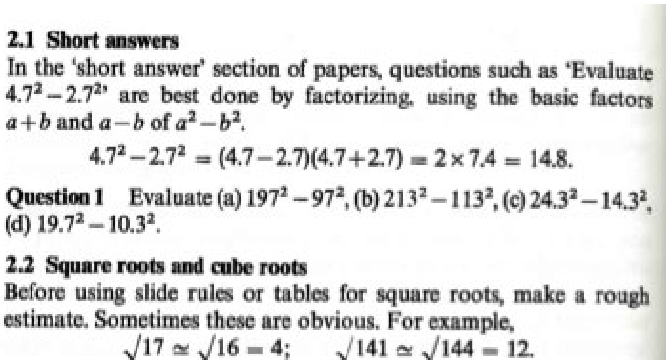
Former British colonies, East Africa

Kenya became independent in 1963, and Uganda in 1962. For an extended period, both countries maintained the British system of examinations and established only early in the 1970s own examination boards. Geoffrey Howson observed about curriculum cooperation work in former British colonies, in particular in Africa:

> In going round the countries, first of all you had to discount the fact that in whatever former colony you went to, you would find a handful of schools which were modelled on the English grammar school and which could match any English school. Often these schools were run by expatriates, used the English external examinations and sent their students on to English universities. But that was just the tip, the very tip of an enormous iceberg, and what was underneath was pretty disastrous in most countries. In fact, this is what increasingly worried me in that CREDO's efforts were perhaps directed too much at this tip (Howson, 2008, p. 55).

Actually, the former British colonies in East Africa stand out in their mathematics textbooks after independence by showing the least degree of adapting to a proper cultural identity.

As a first example, I will mention the book: *Modern Mathematics for East Africa* (1973). As a matter of fact, there is nothing specific in the book, which would legitimate the title "for East Africa". Even more, it does not really look "modern"!



2.1 Short answers

In the 'short answer' section of papers, questions such as 'Evaluate $4.7^2 - 2.7^2$' are best done by factorizing, using the basic factors $a+b$ and $a-b$ of a^2-b^2.

$$4.7^2 - 2.7^2 = (4.7-2.7)(4.7+2.7) = 2 \times 7.4 = 14.8.$$

Question 1 Evaluate (a) $197^2 - 97^2$, (b) $213^2 - 113^2$, (c) $24.3^2 - 14.3^2$, (d) $19.7^2 - 10.3^2$.

2.2 Square roots and cube roots

Before using slide rules or tables for square roots, make a rough estimate. Sometimes these are obvious. For example,

$$\sqrt{17} \simeq \sqrt{16} = 4; \qquad \sqrt{141} \simeq \sqrt{144} = 12.$$

Fig. 7. Jones & Morris 1973, p. 30

It might be that this book served with its solution recipes for passing examinations.

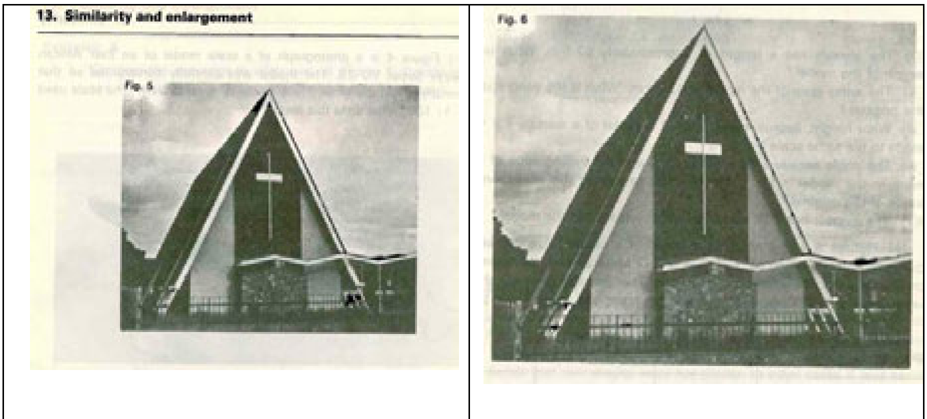
The next example is from a textbook series for Kenya and Uganda of the 1970s: *School Mathematics of East Africa*. The preface emphasises that this is a version considerably revised of the SMP schoolbook series:

> The series had its roots in the books of the School Mathematics Project of the United Kingdom, but these were extensively re-written, and issued in draft form, with examples, questions and language more appropriate for the local situation, by men and women teaching in East Africa (School Mathematics of East Africa, Book 1, 1969, p. iii).

Actually, the book gives no names of authors. The book cover gives a certain geographic reference by showing the contour of entire Africa. Besides this, it is shown a punched tape, apparently for alluding to then modern use of computers as a global symbol for mathematics.

Fig. 8. From the cover of *School Mathematics of East Africa*, book 1

I am showing in Fig. 9 and Fig. 10 two illustrations from this book. The first one regards the topic of similarity; it assumes that Christianity should be the dominant cultural core of these countries. Actually, there were, besides African religions, strong communities of Islamic and of Indian religions in these countries. Surely, schooling had been organised especially by missionaries of the various Christian religions; many later political leaders had been educated at such schools. The same is true also for Mozambique.



Are the prints similar? The number in the last column shows how much bigger one print is than the other. We call this number the *scale factor*. It is the same no matter which corresponding lengths are compared.

Fig. 9. School Mathematics of East Africa, book 2, p. 210

The second one (Fig. 10), to accompany the introduction of perspective, assumes as evident that the cultural heritage of Europe makes sense for the students in East Africa. These schoolbooks were continuing more or less the cultural visions of the former colonial power.

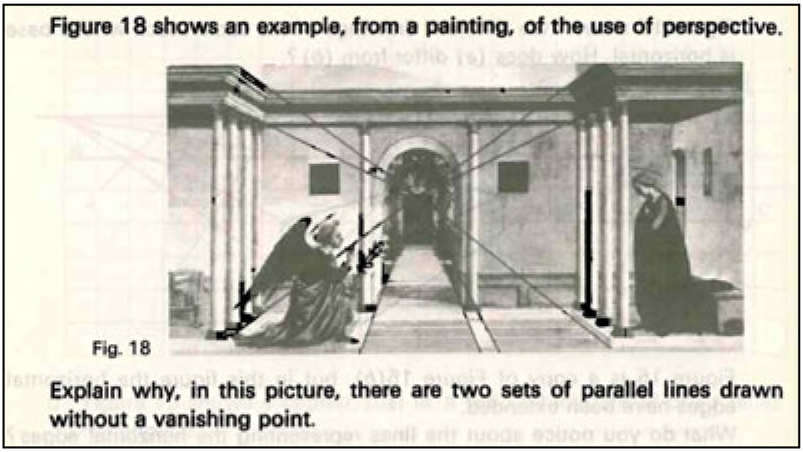


Fig. 10. School Mathematics of East Africa, book 2, p. 162

Remarkably enough, former colonies in West Africa (Ghana, Nigeria) were quite soon active for representing cultural awareness in schoolbooks. See how the same topic of similarity became visualised in a schoolbook for West Africa in 1985:[3]

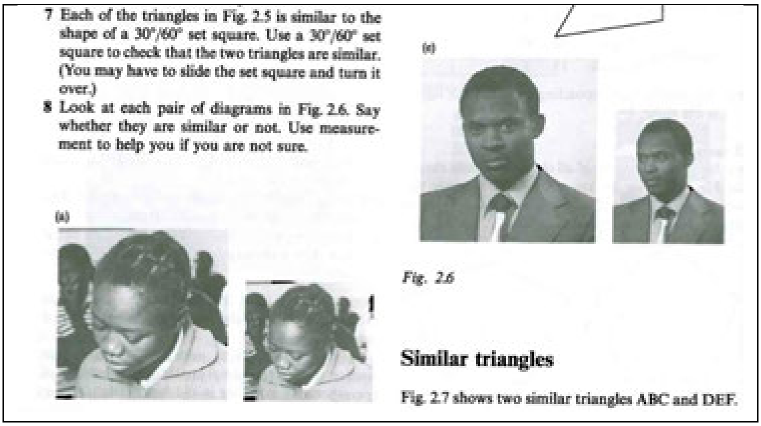


Fig. 11. *General Mathematics* 2, p. 10

[3] This textbook was published in 1997 in a revised version for Kenya, by the Zambia branch of the English publisher Longman. "This revised edition of the "New General Mathematics Series" has been written specifically to cover the needs of the 8-4-4 syllabus for the Kenya Certificate of Secondary Education." One continued, hence, in Kenya to construct syllabi in dependence of the final examination.

Former French colony: Niger

A radically different enculturation has been realised in the former French colony Republic of Niger, which had become independent in 1958 There, schoolbooks – at least since the 1980s – were elaborated according to consciousness for proper African cultural values – thus related to the conception of *Négritude* – by the INDRAP, the *Institut National de Documentation, de Recherche et d'Animation Pédagogique* at Niamey. The mathematics books were elaborated by teams of teachers from Niger and with the conceptual contribution of Ghislain Spaak, a member of the ONG Eirene. Spaak (died in 2011), who originated from Belgium but lived in Niger since the early 1980s. I know from many conversations and cooperation with him how difficult it was for INDRAP to resist ever new pressure from France to adopt again their mathematics schoolbooks; later on, this pressure was taken over by the World Bank, which wants (in the context of aid for development) the Ministry to suppress the innovative developments and to return to the traditional French textbook series. However, there one basically succeeded in continuing to develop the schoolbook series for the Ministry of Education. Already the book covers reveal a radically different spirit: an African teacher working hard at night preparing the next lessons (Fig. 12). Likewise, the books for the students show a vivid new style, bound to local culture (Fig. 13).

Fig. 12. Book cover of: Mahamane et al. 1992

Fig. 13. Book cover of: Ibo et al. 1995

One of the key characteristics of these schoolbooks from Niger is to illustrate the mathematical texts with figures showing African children and teachers interacting:

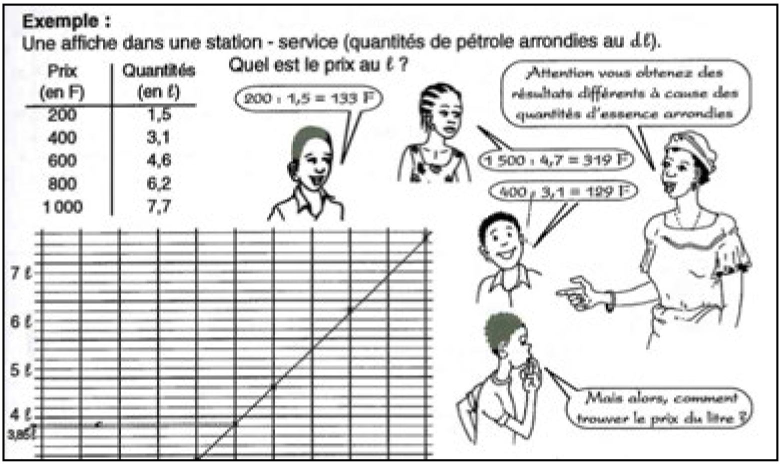


Fig. 14. *Mathématiques. CM 2. Livre de l'élève.* 1995, p. 31

Likewise, buildings and landscape illustrations are taken from the proper region:

Fig. 15. Mathématiques. CM 2. Livre de l'élève. 1995, p. 32

Former Portuguese colony: Mozambique

A parallel development of schoolbooks was realised in Mozambique, an ex-Portuguese colony, after independence in 1975. The independence meant a radical rupture: since almost all Portuguese administrators and settlers flew the for them imminent danger of socialism, there was no continuity with the former cultural system of Portuguese colonisation. The practice of elaborating textbooks based in African culture – which had started even earlier in areas liberated from Portuguese rule – was continued and extended. The mathematics textbooks in Mozambique, *Eu vou à Escola: Matemática* [I am going to school: mathematics], for primary grades, and *Eu gusto* *de matemática* [I like Mathematics] and *Matemática? Claro!* [Mathematics? Sure!], for secondary grades,[4] turned out to be milestones for presenting mathematical concepts in their sociocultural contexts, partly already related to the then-emerging research in ethnomathematics in Mozambique. Paulus Gerdes commented on the difficulties in elaborating the proper textbooks:

> The programmes were developed in a hurry in 1975, in an attempt to respond to the political and socio-economic changes in the country. The mathematics curriculum for primary school was simplified (e.g., the arithmetic programme was reduced so the pupils learned only to operate well with natural numbers and to handle linear measures and money), in order to cope with the difficulties that stem from the weak knowledge of the Portuguese language, the medium of instruction and communication. Already in 1975 the first teachers' manuals had been produced. But the first national mathematics textbooks and exercise books for pupils of the first grade will leave the presses only in 1981 (who could write them before?). At the other end of the educational spectrum, professors of the Eduardo Mondlane University have now produced the mathematics textbooks for the tenth and eleventh grades. (Gerdes 1981, p. 461)

The first textbook series, from the late 1970s document the new approaches taking cultural identity as key pattern. At first, covers from these Mozambican schoolbooks, too (Fig.s 16 and 17), and then an example of contextualising mathematical concepts (Fig. 18).

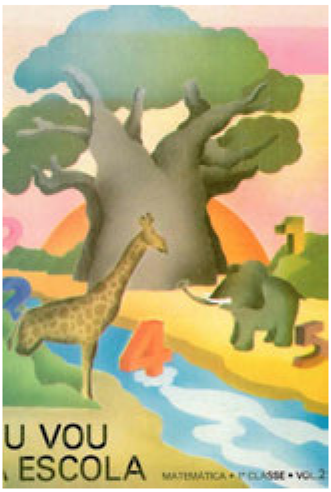


Fig. 16. Cover of *Eu vou à Escola.* 1a classe, vol. 2. 1983

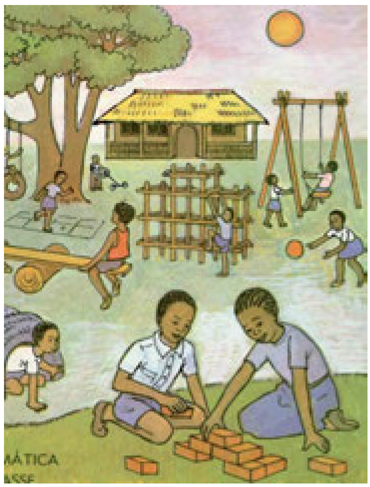


Fig. 17. Cover of *Eu gusto de Matemática.* Classe 2. 1984

[4] The consolidated versions date from 1983 (primary) and 1988 (secondary).

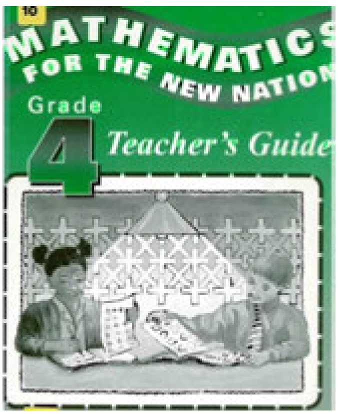
Fig. 18. *Eu gosto de Matemática*. Classe 2. Caderno de Exercícios, vol. I. 1984, p. 11

Later, however, due to pressures by the World Bank, textbooks from Portugal had to be introduced.[5]

World Bank pressure resulted eventually in its policy on education for developing countries to require that any publisher in the world should have the possibility to concur in the production and publication of schoolbooks.

South Africa

South Africa, as the colony of the Cape, had been disputed between Dutch settlers and Great Britain; after cruel wars, eventually Britain imposed itself as colonial power, but the colony became formally independent in 1926 – continuing as a member of the Commonwealth. As is well-known, the history of South Africa is characterised and darkened by the Apartheid – introduced at the impact of a Presbyterian Church of Dutch origin. As one of the consequences of Apartheid, mathematics was taught for the black population at a decisively reduced level. The end of Apartheid, marked by the election of Nelson Mandela as president, abolished this racist reduced curriculum and established equality (Khuzwayo, 1997). The new textbook series from 1999 on, *Mathematics for the New Nation*, realises this



[5] Communication by Paulus Gerdes (Maputo).

equality in a remarkable manner, as shown here by a few examples. Teachers as well as students use to be represented by white and by African persons (Fig.s 19,[6] 20 and 21).

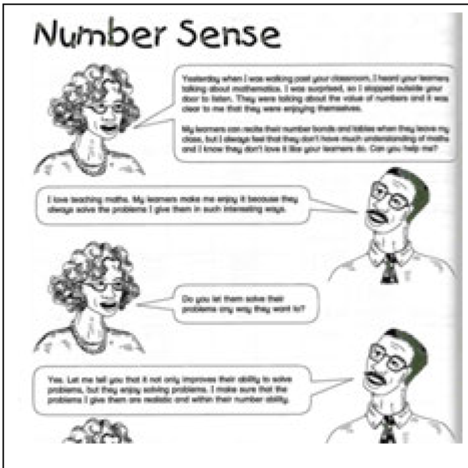


Fig. 20. Westaway et al., 1999, p. xii

Fig. 21. Spanneberg et al., 1999, p. 7

Resuming and outlook

This paper is a first study about mathematics teaching in colonised and in post-colonial countries. Surely, only a rough approach for studying the related changes in teaching mathematics could be given, mainly restricting to the visualisations of dominant cultural patterns in teaching mathematics, as evidenced by schoolbooks. Clearly, a deeper analysis would afford sociological research about continuities in the education system, for instance, regarding maintained examination systems and their impact upon the vision of school mathematics. Likewise, a sociological analysis of the education having shaped the new political generations of the decolonised countries would broaden the approach. I am hoping to instigate more research on this research question.

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[6] Fig. 19. Cover of Bouwer et al., *Mathematics for the New Nation.* Grade 4, Teacher's Guide. 1999.

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Johan Wansink and his role in Dutch mathematics education

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Abstract

Johan Wansink (1894-1985) was a Dutch mathematics teacher, author of textbooks and author of almost one hundred articles on mathematics education, a teacher trainer and the author of the first book on mathematics teaching in The Netherlands in almost 100 years: Didactische oriëntatie. [Didactical orientation]

Wansink made a remarkable career. He never attended a secondary school, but nevertheless he earned in 1931 his PhD in mathematics and he became an honorary member of both the Dutch Association of Mathematics Teachers as well as the Dutch Mathematical Society. Studying his life and career is highly revealing for the Dutch educational system in general and mathematics teaching in particular for a large part of the 20th century.

He was chairman of the board of the Dutch Mathematics Teachers Association and chief editor of its magazine Euclides. He was also the chairman of the curriculum committee that in the fifties of the last century succeeded in breaching the deadlock that had held Dutch mathematics teaching in its grip for decades. By achieving this, he made it possible that Hans Freudenthal "for the first time had indirectly made headway in the mathematics education with his ideas, without his name or even that of the WW associated with it" as Freudenthal's biographer wrote. Wansink and Freudenthal held each other in high esteem, although Wansink certainly was not in complete agreement with all of Freudenthal's ideas. Their most important difference of opinion concerned, as Wansink formulated it, "the role of the teacher" in the teaching of mathematics.

His life and career[1]

Johan Wansink was born in 1894 in Aalten, a small town in a rural region in the east of The Netherlands, close to the German border. He attended three different elementary schools, due to several moves by his parents, and he stayed on elementary school until he was 14. In those years, in remote and rural areas, there was hardly any form of secondary education available. In many local elementary

[1] This chapter relies heavily on (Goffree, 1985).

Bjarnadóttir, K., Furinghetti, F., Menghini, M., Prytz, J., & Schubring, G. (Eds.) (2017). *"Dig where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education.* Rome: Edizioni Nuova Cultura.

schools, the schoolmaster gave extended elementary education for gifted children, for instance to prepare them for the entrance exams of the *Normal Schools* to become teachers in elementary schools themselves. Young Johan was prepared by the headmaster for these exams and in 1908, at fourteen years of age; he was admitted to the State Trainings Institute at Deventer, a medium-sized provincial town about fifty kilometers away from where he lived. To attend the lessons, he was obliged to live there in lodgings.

In 1912, he got his teaching certificate for elementary schools. He could then become a teacher on an elementary school, but he was more ambitious: he wanted to become a teacher on an extended elementary school. Wansink himself had received individual extended elementary education while he was still on the elementary school, but more and more independent-extended elementary schools had been established. That was a school type that legally belonged to elementary education, but in practice functioned as a lower level type of secondary education. These schools became very popular in the first half of the 20th century and played a crucial role in the emancipation of the lower classes. They were mostly attended by two groups of pupils: those who had not the capacities to go to a HBS (the Dutch variant of the German *Realschule*) or a gymnasium, but also by gifted children of lower classes, whose parents thought that such an extended elementary school was already quite ambitious for their children.

When one had obtained a teaching certificate for elementary schools, one could teach some topics, such as Dutch language, history and geography in extended elementary schools, but for the majority of the topics taught there, such as foreign languages, mathematics or bookkeeping, one had to obtain extra certificates. While he was still a pupil of the *Normal School,* young Wansink had already obtained such certificates for gymnastics and mathematics.

When he had finished the *Normal School,* he took an unusual step: he signed a voluntary contract for the army. He had good reasons to do so. He knew that he was obliged to join the army as a conscript, and he thought it was better to do so right away after school on a voluntary basis which would give him a better position in the army. He planned to use his spare time in the army to obtain one or more extra teaching certificates.

His contract with the army expired in 1913 and he had hardly started his teaching career when World War I broke out and Wansink was mobilized again. In 1917 the schoolboard of an extended elementary school in Schoonhoven, a small town more in the western part of the Netherlands, succeeded in getting him out of the army and employed him as a teacher. Altogether he had been in the army for four and a half year, but he had used his spare time there: he obtained extra teaching certificates for French, English and bookkeeping and the certificate for becoming a headmaster of an elementary school.

Wansink had not attended university; he had not event attended any form of formal secondary education that would allow him to go to the university. But in

The Netherlands, there was and still is, a possibility to become a teacher on a secondary school preparing for university, without having university education yourself. This route had its origins in the first half of the 19th century, when mathematics was introduced as a topic on the grammar schools. It was the intention of the government that the lessons on this subject should be given by the regular teachers of the grammar schools, who were of course specialized in Latin and Greek. Many of them felt incompetent to teach mathematics, or simply refused to do so. As a solution, elementary schoolteachers were employed to give these lessons on grammar schools, next to their normal jobs on their elementary schools.

It was intended as a temporary measure, but when the mathematics curriculum was expanded and it became clear that specialized teachers for mathematics became indispensable, the government started to organize special exams for these teachers, to be sure that they had enough knowledge of mathematics. It led to a system with formal programs and exams, not only for mathematics, but for the majority of the school topics. Of course, the demands for these advanced certificates were much higher than the demands for the certificates for extended elementary education. It was not before the second half of the 19th century that university graduates made out a substantial part of the mathematics teachers at the gymnasia and the HBS.

Wansink, while working at the extended elementary school in Schoonhoven, followed the courses that gave training for these exams and he acquired such an advanced certificate to teach mathematics. He became a mathematics teacher at the HBS in Arnhem, the main city in the region where he came from. Eventually, he became deputy headmaster and in his last years, headmaster of the school.

Although this was not required for his job as a HBS-teacher, he wanted to study mathematics on a university level. At first glance, that seemed almost impossible, since he did not have the required diplomas for this. But the ministry of education could give persons of at least thirty years of age a special permit to study at the university when they had shown exceptional talent for their field of study. Wansink received this permit because of the excellent results obtained for his teaching certificates. In 1924 he started his studies – combined with his full time job as a teacher – in Utrecht. In 1929 he got his masters, in 1931 his PhD in mathematics.

He retired in 1959, but that was certainly not the end of his career. In the fifties, after more than a century of stagnation, the Dutch government took at last some steps for the professional education of future teachers on secondary level, including mathematics teachers. Wansink, who not only had published some excellent textbooks, but also many articles on didactics, became, during the first five years of retirement, a teacher trainer at several institutes, including the Technical University in Delft. As he told himself: that was not an easy job, because he had to invent and create courses for the Dutch situation from scratch. The result was impressive: it resulted in the publication of a three-volume work *Didactical orientation*, the first book of its kind in The Netherlands in almost a hundred years. (Wansink, 1971)

When was became seventy years old, he did resign from all official functions. But he stayed active as the author of several interesting articles, for instance about the history of mathematics education in The Netherlands. His last article was about the Dutch schoolbook market in the period 1800-1940, published in 1978 when he was 84. (Wansink, 1978) He died in 1985 in Arnhem, the city where he had lived for more than 60 years.

In the Board of the Mathematics Teachers Association and Editor in Chief of *Euclides*

Wansink was elected to the board of the Mathematics Teachers Association in 1949. That Association was established in 1925. It was an association for teachers of mathematics, mechanics and cosmography at the HBS, the Dutch variant of the German *Realschule*. Mechanics and cosmography were topics taught in the HBS, almost always by a teacher of mathematics. The name of the Association was usually abbreviated as *Wimecos*, referring to Wiskunde (mathematics), mechanics and cosmography. For teachers in a gymnasium there was a separate Association for all gymnasium teachers together, and the mathematics teachers on the gymnasia formed, together with the science teachers, a working group within that Association. Those who taught mathematics on extended elementary schools or on vocational schools were not accepted as members of these Associations.

In 1949 the track record of *Wimecos* was not very impressive. It had shortly after the war between 200 and 300 members, and its main activity was a yearly conference, usually attended by two or three dozen of its members. The board's orientation was mainly conservative and defensive. The Association did not have its own magazine. There was a magazine for mathematics teachers, *Euclides,* founded in in the same year 1925 by Piet Wijdenes, but that magazine was owned by its publisher Noordhoff.

Wijdenes, a former mathematics teacher and a highly successful textbook author who could live by his publications, was the dominant editor in chief. He was not only a conservative man by nature, but he had also his personal reasons why he was opposed to any modernization of the curriculum: it could make his textbooks outdated and therefore threaten his income. From 1939 on, the Association had, together with the group of gymnasium mathematics teachers, the right to publish its official announcements in *Euclides*, and membership of the Association included subscription to the magazine, but the Association had nothing to say about the editorial policy of the magazine. Wijdenes was still the dominating figure and he personally selected his only co-editor, taking care that this person shared his own ideas.

When Wansink joined the board in 1949, he made it clear at once that he wanted change. In the first meeting of the board that he attended, he raised two issues (Archives Wimecos/NVvW). At first he raised the problem of the magazine. "What exactly *is* the relationship of our Association with the magazine?" he asked, and he continued with saying that since the co-editor of the journal at that time, J.H. Schogt, had just resigned, this was the moment for the Association to have appointed his own representative as co-editor. The board agreed, and started discussions with Wijdenes about his matter. That was not easy: Wijdenes tried to avoid by all means interference of the Association with what he considered as his "own" magazine, but in the end Wansink had his way and two representatives, one of the Association of the HBS teachers and one of the group of gymnasium mathematics teachers, were appointed as co-editor.

After a few years Wijdenes, due to his age, finally stepped down and in 1956, Wansink himself became editor in chief, where he stayed until 1968. In that same year, 1956, the editorial board of the magazine became independent of its owner and publisher, the Noordhoff Company, and was only responsible to both mathematics teacher Associations/groups, which also appointed the editors. During his time as the editor, Wansink successfully transformed *Euclides* into a magazine that was much more open minded and provided an interesting platform for discussion.

In that same board meeting of 1949, Johan Wansink also touched on another tricky business. He proposed to invite the *Wiskunde Werkgroep* [Mathematics Working Group] to write an article about its activities and to have this published in *Euclides.* The minutes of that meeting say only that "the other members of the board were not in favor of this without having reservations", but we might assume that the discussions were high.

This Mathematics Working Group was founded in 1936, as a subdivision of the Dutch branch of the *New Education Fellowship,* now the *World Education Fellowship.* The group soon became the center point for discussions about modernization of the mathematics curriculum and mathematics teaching in The Netherlands. It was a loose organization; membership was open for anybody who was interested. Some members were mathematics teachers, but there were also members from outside, like psychologists and educationalists. Before the war, it had 20-30 members, but soon after the war, more than one hundred. The group was founded and before the war dominated by Tatyana Ehrenfest-Afanassjewa, the widow of the physicist Paul Ehrenfest, the successor of Lorentz in Leiden. She was born in Kiev and educated in St. Petersburg as a mathematics teacher. She was not only an excellent mathematician who also had studied in Göttingen with Klein and Hilbert, but she had progressive ideas about mathematics teaching, which were very unusual in The Netherlands in those years. Soon after the war Hans Freudenthal became a member and chairman of the group. For the traditionalists in the Association, this working group was dominated by *dangerous amateurs,* as the chairman of the Association once said, meaning people like Freudenthal who knew nothing about the practice of teaching.

So Wansink's proposal was not very welcome, but as a compromise it was decided that the board should ask the working group for a summary of its activities, and the then board would publish an account in *Euclides* – so they could keep things in their own hand.

Wansink was also a member of the working group; he had joined the group shortly before the war. That did not mean that he was a great innovator. He once described himself on mathematics teaching as an essentially cautious man, not a *daring devil,* and he added that within the working group he was considered conservative, in the teachers Association however as progressive. But he was very open minded, interested in the opinions of others and in new developments. He did want to get the Association out of its isolated position. It took some time, but in 1953 there appeared indeed an article about the working group in *Euclides*. It was written by Wansink, a solution typical for Wansink's ability to maneuver in difficult situations: by writing the article himself both the working group and the board of the Association, combined in one person, had their say in the article.

The *Wimecos* curriculum proposals

In 1950, the group of mathematics teachers in the gymnasia, a group that was a bit less conservative than those on the HBS, formulated proposals to modernize mathematics teaching in the gymnasia. This had no immediate consequences for the HBS, but, if put into practice, it would make mathematics teaching on the HBS still more old fashioned. A few years later, the mathematics working group, now with Freudenthal as its chair, published a lengthy report with proposals for new curricula for gymnasium and HBS. (*Euclides*, 28, 1953, pp 206-226). The working group had tried to write a combined program for both school types, but that had turned out to be a bridge too far. Both programs had a thorough trimming of exuberant algebraic techniques in common and incorporated calculus in the exam program. In 1953, the report of the working group was sent to the HBS Mathematics Teachers Association.

There could not have been a worse-chosen moment. In 1952 Freudenthal had caused an outrage by launching, out of the blue, a fierce attack on the teaching of theoretical mechanics on the HBS. This subject, in fact a 19th century relict, was always taught by mathematics teachers. It was taught in a deductive and theoretical way. In the twenties and thirties, the physicists had claimed the subject, arguing that mechanics was part of physics and that it should be taught in a much more experimental way by the physics teacher. But the mathematics teachers had successfully opposed to that idea, and since then, it had been quiet on that front.

But in 1952, a son of Freudenthal attended the HBS and in this way Freudenthal's attention was drawn to this subject. In a lecture for the mathematics working group, he launched a fierce attack on it, and his conclusion was: Do away with this trashy science! (Freudenthal 1953, p. 42) The lecture was published and quoted in the press, which of course outraged the mathematics and mechanics teachers Association, especially its chairman.

It brought Wansink in an awkward position. He was a member of the working group that was chaired by Freudenthal, but he himself liked teaching mechanics and had even written a textbook for it – the only textbook by the way that Freudenthal did not completely disapprove of. In the board of the Association, he tried to calm down the chairman and in the end he succeeded in persuading him not to launch a fierce personal attack on Freudenthal on the next annual general meeting of the Association, as was his intention.

That was the climate when the proposal of Freudenthal's working group landed on the table of the board. "What shall we do with it? Shall we discuss it, or shall we put it aside?" asked the chairman according to the minutes of the meeting. Most likely he wanted to do the last, but things turned out differently.

Someone – the minutes don't mention a name, but I guess that it was Wansink – argued that it should not be wise to sit back, it was better to take the initiative themselves. He pointed out that the pre-war proposals for modernization were never fully implemented and drew attention to developments in other countries that should not be ignored. Of course, nobody could feel offended by these arguments and they turned the negative attitude into a more positive one. In the same meeting it was agreed that the board would try to appoint a committee that should draw up proposals for a new HBS mathematics curriculum.

Indeed, in the next board meeting such a committee was founded, with Wansink as its chairman. It consisted mainly of HBS teachers, but counted also representatives of the group of gymnasium teachers. The mathematics working group of Freudenthal was not officially represented; this had been unacceptable for the conservative members of the board. But three members of the committee, Wansink, Bunt and Vredenduin, were also members of the working group, so 'it's say' in the committee was unofficially guaranteed.

The work of Wansink's committee was highly successful. It succeeded in producing a joint curriculum for gymnasium and HBS together, containing calculus in the exam program. To reach consent on a joint program, Wansink had to swallow a bitter pill: descriptive geometry, which he liked very much, disappeared from the HBS curriculum.

Descriptive geometry had been on the HBS-curriculum from the start in 1863. It was not taught on the gymnasia, on these schools analytical geometry was taught. Freudenthal was a strong opponent of the teaching of descriptive geometry. When he gave in 1948 a lecture on a conference for the Mathematics Working Group, he was asked in the discussion afterwards about his opinion on descriptive geometry. "Worthless!" declared Freudenthal, adding that it should not be taught on secondary education. Wansink, who was also present, immediately opposed Freudenthal. (Freudenthal 1948, p. 121). In 1952, in his lecture on the teaching of mechanics, Freudenthal characterized descriptive geometry as "a subject, isolated from all other parts of mathematics, from science and from engineering". (Freudenthal 1953, p. 24) On another occasion he declared that the only reason to uphold the subject at the HBS was "because solid geometry had expanded to such an extent that a lot of pupils would no longer pass the geometry exam, if this very easy part of the curriculum 'descriptive geometry', was to disappear". (quoted in la Bastide –van Gemert, 2015, pp 151-152)[2]

When the Mathematics Working Group proposed a new mathematics curriculum for HBS and gymnasium, the position of descriptive geometry had been one of the major obstacles for a combined program for HBS and gymnasium. Wansink had been a member both of the subcommittee for analytical geometry and of that of descriptive geometry, and no doubt he used his influence to uphold descriptive geometry on the HBS-curriculum. (Wansink, 1953)

But now, some years later, Freudenthal had his way. Wansink, now chairman of the Wimecos-committee, realized that the only possibility to achieve a combined program for HBS and gymnasium, which would greatly enhance the chances of its acceptance by the government, was to accept the disappearance of descriptive geometry. He took his responsibility and so, after almost hundred years, descriptive geometry disappeared from the HBS.

The new program was one of moderate modernity, but compared with the old situation, a great step forward (Archives Wimecos/NVvW and personal archives Johan Wansink). Not only were the results satisfactory, but the process towards it had been inspiring. A compromise, even cooperation, between conservatives and reformers had been possible. The proposals in their various stages were discussed by large numbers of mathematics teachers, alternatives were formulated and commented on, and in the end meetings were held by the HBS and the gymnasium mathematics teachers Associations where the proposals were approved by great majorities. The government adopted the proposals and implemented them in 1958. Never before had the community of mathematics teachers been involved so strongly in the process of making a new curriculum and, as I regret to say, it never was that much involved in the sixty years that followed.

[2] The results for solid geometry and descriptive geometry were combined into one mark, the results for descriptive geometry were usually better than those for solid geometry.

No doubt, the original proposals of Freudenthal's working group had had a strong influence on the final result. But that was not said openly, because Freudenthal was still very much disputed in circles of mathematics teachers. In this case, Freudenthal behaved very tactful. He praised the proposals of the Wansink committee and never claimed any credit for himself or for the working group. Decades later, Pierre van Hiele, also a member of the working group, was still indignant about this lack of recognition for the contribution of the working group, but Freudenthal didn't mind. (Goffree, 1985, p. 123)

The Wansink-curriculum was also important in another way. The author of Freudenthal's recent biography formulates it like this:

> Thus for the first time Freudenthal had indirectly made headway in the mathematics education with his ideas, without his name or even that of the WW [WiskundeWerkgroep or Mathematics Working Group] associated with it. (la Bastide-van Gemert, 2015, p. 155)

That this had been possible was for a large part due to Johan Wansink.

Wansink and Freudenthal

What can be said about Wansink's own didactical ideas and beliefs? Wansink was not, like Hans Freudental or Pierre van Hiele, an original thinker or researcher in mathematics education, and he did not develop his own theory or philosophy in this field.

Nevertheless, it is not difficult to see what his deepest conviction about mathematics teaching was. That was, that the role of the teacher is crucial in the teaching process. He formulated this point of view very clearly in an article he published in 1951, with the title (in translation) *The textbook and the teacher in mathematics education.* (Wansink, 1951) One of his statements in this article is: "In my opinion the teacher is more important than the textbook" (p. 123), and he discusses his reasons for this point of view and explores its consequences in a way that still makes interesting reading today.

That does not mean that he was in favor of traditional lecturing with the teacher dominating the learning process, or that he was against modern methods, such as laboratory methods, exploring or guided reinvention. In the same article Wansink discusses the ways by which a textbook can further self-activity of the learner, but he stresses that the role of the teacher in all these activities is crucial. He also argues that the value of the activities in the classroom should be judged by the *results* obtained by them, not by ideologies. In different situations, in different classes, the teacher should be able to choose different strategies and methods of teaching; he strongly opposes the idea that there is only one way which the best is.

It is clear that on these points he did not agree with Freudenthal. In 1975, Wansink wrote a long review of Freudenthal's *Mathematics as an educational task*. (Wansink, 1975) Of course, he admires the book greatly, but he nevertheless formulates some very sharp criticisms; firstly concerning Freudenthal's almost exclusively preference for his method of *guided reinvention,* and secondly about the way Freudenthal sees the role of the teacher.

The principle of *guided reinvention* is central in *Mathematics as an educational task*. According to that principle the children should be not be taught some ready-made mathematics, but they should be presented with meaningful mathematical activities that enables them to develop the relevant mathematical theory behind these activities themselves. The teacher should be a guide for the pupil to help him in his voyage of (re)invention, hence the term *guided reinvention.*

For Freudenthal, this method was not just one among other possible ways to teach mathematics. Although he writes that " It is believed that knowledge and ability acquired by reinvention are better understood and more easily preserved than if acquired in a less active way" (Freudenthal 1973, p. 118), in the end that is not what really matters. Sixty pages earlier he wrote: "(…) I entirely dismiss the question whether people learn better by actively building up the subject than by passive reception of ready-made matter – in fact much depends on the subject matter (…). For the moment I am not interested in the didactical consequences." His argument there is that any authoritarian method, whether it produces good results in teaching or not, is "simply impossible because it does not agree with the character of modern society" (p. 58) At the end of chapter six, exclusively devoted to reinvention, his conclusion leaves no doubt: "Re-invention, that is a didactic principle on research level, should be the principle of all mathematical education (…). (p. 130)

Wansink did not appreciate this vision. As he already wrote in his article of 1951, for Wansink the results obtained by the teaching process were decisive. His reaction on Freudenthal's remark that he was not interested in the didactical consequences was: "But I am interested in them" (Wansink 1975, p. 411). He suggests in his review that for instance sometimes a structured dialogue between teacher and pupils with a more dominating role of the teacher might produce better results than the method of guided reinvention. That possibility was also left open by Freudenthal, but Wansink declared that, unlike Freudenthal, he preferred in that case what he called this dialectic method. Moreover, although Wansink did consider the method of guide reinvention to have great value, he did not believe that it was possible to apply integrally in the teaching of mathematics.

The method of guided reinvention asks for a new role of the teacher; less dominant and more in the background. In the chapter "Tradition and education" of *Mathematics as an educational task* however, it becomes clear that for Freudenthal this was not just an inevitable side effect. His picture of current teaching practices and teacher behavior is not very flattering. (pp 60-61) He regrets that on numerous surveys the students on his university answered that the quality they appreciated most in their mathematics teacher at school was his ability to *explain*, a preference according to Freudenthal induced by a love of ease in the students (p. 57).

Not surprisingly, Wansink does not agree. In his opinion the reproaches Freudenthal makes to teachers and textbook authors are not always fair, and he thinks that Freudenthal's view on the current situation of mathematics education is not always correct. He especially deplores Freudenthal's low opinion of the ability to *explain*, according to Wansink an essential capacity for any good teacher. (Wansink 1975, pp 410-411)

Freudenthal goes even a step further in his reasons why the role of the teacher should change. In the future, he says, "to reach ever broader social layers", education will be a mass product, and a personal relation between a teacher and his pupils will become impossible. "Large parts of mathematics are already suitable for programmed implementation in the learning process. It is an unnecessary luxury to teach such parts in personal intercourse". (Freudenthal, 1973, p. 62) That is even the case for teaching and learning by guided reinvention, for "The major question is now how active and even creative learning can be programmed". (ibidem)

While this prospect for Freudenthal did not seem to be problematic at all – he calls the longing for a personalized university "romantic"- it clearly was a gloomy perspective for Wansink. He has his doubts whether the programming of learning and teaching will really be feasible, but more important: "The teacher acts as a model for his pupils in so many aspects, also concerning his creative achievements, that his replacement by impersonal programming is for me at the moment rather unattractive". Wansink's fundamental disagreement with Freudenthal is summarized in another remark on the same page of his review: "The significance of the personality of the teacher is, in my opinion, done no justice in this book". (p. 411).

It was not a new issue of discussion between them. Almost a year before, on occasion of Wansink's 80th birthday, Freudenthal wrote a small article in *Euclides* to congratulate him and he recalls their first discussion shortly after the war, in which discussion Wansink defended the "point of view of the teacher" against Freudenthal. (Freudenthal, 1974) But it seems that Freudenthal did not realize how important this perspective was for Wansink, for in the same article Freudenthal suggested that this type of discussions was very outdated, by comparing them with the discussions about the teaching of descriptive geometry.

The discussion about the significance of the role of the teacher gained however new importance recently. In a report by the Dutch Royal Academy of Sciences on the results of arithmetic teaching, published in 2009, the crucial role of the teacher is confirmed. (Lenstra, 2009) The report was written on account of the heated discussions about the results of mathematics teaching based on Freudenthal's most important legacy: *realistic mathematics education.* Freudenthal consequently rejected any kind of large scale evaluation by means of statistical analysis of results of mathematics education, which made him vulnerable to the reproach that there was not any proof that his approach did give better results than more traditional teaching methods. It was, according to some critics, just the other way around. The conclusion of the report was that there was in fact no scientific proof for the effectivity of *realistic* versus *traditional* teaching methods; it is the role of the teacher which is much more decisive. The report would no doubt have confirmed Wansink in his opinion.

But these differences in points of view did not threaten the friendly relationship between Wansink and Freudenthal. Wansink was an amiable man, a man of the middle, with an open mind and always trying to bring people together. When the Dutch Mathematical Society celebrated its 200th anniversary in 1978, Freudenthal was the chair of the Society, and in this quality he awarded Wansink with the honorary membership of the Society. As Freudenthal wrote, it was for both one of the highlights of the celebration. (Freudenthal, 1986)

The world of Dutch mathematical education owes a lot to Wansink. The Association of Dutch Mathematics Teachers was very lucky to have a man like Wansink as a member of its board, as its chairman and as editor of its magazine, in a time when a profound change in Dutch math teaching was both desirable and inevitable.

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Marxism and mathematics.
Paul Libois and intuitive geometry in Belgium

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Abstract

In 1948 the mathematics curriculum at Belgian secondary schools was reformed. One of the new features of the curriculum was a course of intuitive geometry to be taught in the first year. The course introduced students to geometry through the observation, manipulation, and measurement of real objects. It was considered an alternative to the traditional Euclidean approach of deductive geometry. The central advocate of intuitive geometry in Belgium was the mathematician Paul Libois. For Libois, intuitive geometry was closely connected to his epistemological conception of geometry, considering geometry as a part of physics. His views also bear a clear parallel to his political position as a prominent Marxist Communist. The emphasis on the real world as the origin of abstract concepts, the equality among teachers and students in the learning process and the dialectical nature of the history of mathematics all refer to a vague but unmistakable Marxist inspiration. Libois' ideas were not generally accepted in Belgium, and were superseded towards the end of the 1950s by the Bourbaki school in the New Math Reform. However, Libois' influence was still clearly discernible at the Dubrovnik conference of 1960, which, at least temporarily, could halt the advance of the Bourbaki school.

Introduction

In the late summer of 1960 a small group of mathematicians came together in the cities of Zagreb and Dubrovnik in Yugoslavia, to work out a new curriculum for school mathematics in Europe. Their meeting was a follow up of the expert Seminar, organized the previous year at Royaumont by the Organisation for European Economic Co-operation (De Bock & Vanpaemel, 2015a). At Royaumont, a general agreement had been reached on the abolishment of the traditional axiomatic and deductive approach in teaching geometry, which was still in use in many countries. There had been less consensus, however, on what exactly had to be put in place of this so-called 'Euclidean' approach. To some, the reform of mathematics education had to focus on a greater attention to mathematical structures in the style of Bourbaki, to others the reform needed to concentrate on new methods of teaching. It was the task of the Dubrovnik group to come up with a practical solution.

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The outcome of the Dubrovnik conference, as it was commonly called, was an ambiguous compromise. It was recommended that in the final years of secondary school education (16-18 years), geometry would be replaced by the study of algebraic structures, while in the early years (12-15 years) the emphasis would be put on a more intuitive approach (Barbin & Menghini, 2014). This attention to intuitive geometry was largely due to the intervention of Paul Libois (1901-1991), a Belgian professor of geometry at the Université libre de Bruxelles. Actually, at the time he was one of the few voices among the group of experts preparing the reform, still advocating an intuition-based teaching of geometry (another important voice being Emma Castelnuovo – not present in Dubrovnik) (OEEC, 1961). In this paper, we will look into the background of Libois' interest in intuitive geometry and his understanding of the central role of intuition in mathematics. This will also lead us into a discussion of the broader issues involved in the reform of mathematics. As Christopher Phillips (2014) noted in his recent analysis of the New Math movement in the US: "the debate about the curriculum […] was also a debate about the underlying identity of the subject itself." The debate about intuitive geometry was indeed much more than a debate about classroom practices. As the most articulated alternative to the Bourbaki inspired definition of the New Math, the advocacy of intuitive geometry was the expression of a widely held, but rarely explicitly voiced, opposition against the direction in which the New Math reform was going.

Intuitive geometry

There is no single definition of intuitive geometry with regard to school mathematics. In general terms, it was considered a reversal of the axiomatic presentation followed by Euclid. As Caleb Gattegno (1955) observed,

> the use of the term 'intuitive geometry' implies nowadays that the aim remains, as before, knowledge of theorems and geometrical facts, but that the presentation of the material will start with wholes which will be analysed and not with definitions and axioms, with actions that will gradually be formalised and whose validity will be extended, rather than with general statements universally valid from the start (at least as far as the teacher's mind is concerned). (p. 351)

Marta Menghini (2010) also considers intuitive geometry as an alternative to the rational-deductive approach to geometry based on Euclid. According to her, it is characterized by the use of visualizations, perceptions, concrete materials and mental images in the generation of knowledge. Intuitive geometry attracted a considerable amount of attention at the beginning of the twentieth century, not only in the context of school mathematics but also as part of a methodological 'introspective' reflection on the nature of mathematical knowledge. After the Second World War, a new wave of interest for intuitive geometry emerged, but now more exclusively in the field of didactics. In Italy, Emma Castelnuovo (1913-2014) published a textbook for secondary schools *Geometria Intuitiva* (1948), in which she worked out a complete intuitive approach for the teaching of geometry. She put the essence of intuitive geometry in the active involvement of both teacher and student in the gradual construction of mathematical knowledge. With reference to the 1741 geometry text of the French mathematician Alexis-Claude Clairaut (1713-1765), she also suggested that the intuitive approach to geometry actually retraced the historical genesis of geometrical knowledge. Whereas Euclid's Elements were the polished pronunciation of this already acquired knowledge, the discovery of geometrical truths which preceded the writing of the Elements was based on an intuitive method. Intuitive geometry was therefore best suited to teach geometry to beginning students.

In Belgium intuitive geometry was made obligatory in the first year of secondary schools (12-year olds) in 1948. The introduction was part of a more general reform of secondary school education in Belgium and the outcome of several years of deliberation (Bosteels, 1950). It was argued that the secondary schools (12-18 years) did not prepare their students enough for them to be able to enter university, but at the same time it was conceded that not all students entering secondary school would later proceed towards higher education. The student had to learn skills that would help him in his daily life. The schools should therefore provide an instruction aimed at developing critical thinking. Bookish learning and a too great emphasis on memorization should be avoided. In particular during the first years of secondary school education, intuition and practical skills were seen as important tools in generating a foundation for abstract understanding.

These considerations were of direct impact on the program for mathematics. In the first year much effort had to be done to situate arithmetical and geometrical learning in the real world. For arithmetic, the teacher should not only take his examples from books, but primarily from economics (calculation of discount, interest, mixing of products, etc.). Geometry teaching had to be practical and visual. Students had to learn how to manipulate and construct geometrical shapes, both in a plane and in three-dimensional space. In accordance with the skills learned in arithmetic, surfaces and volumes would be measured and calculated. Intuitive geometry would continue in the second year, but would be gradually replaced by deductive reasoning. In the third year, geometry would be taught exclusively in a deductive manner.

The course of intuitive geometry was meant to be a bridge between the elements of geometrical practice the students had learned in primary school (6-12 years), and the course of 'rational' geometry, which started in the second year of secondary school. Intuitive geometry aimed at making the students familiar with simple geometrical objects (rectangle, triangle, cube, cylinder), on which they would make practical measurements. Much emphasis was put on the active manipulation of real objects, to develop the students' sense of observation. Students would e.g. cut a triangle out of a piece of paper, and then cut the triangle in three parts. When the pieces were assembled again, with the internal angles of the original triangles next to each other, the students would be astonished to find that the pieces would align themselves in a straight line, no matter the shape of the original triangle. This would lead to an intuitive understanding of the fact that the sum of the angles of a triangle is always equal to two right angles. Students were also challenged to verify this fact by making triangles of unusual shapes. Similar examples were developed in the textbooks appearing to support the new program (Bosteels & Horwart, 1948; Horwart & Bosteels, 1948; Guion, 1949; Berwart, 1951; Devillers, Jaumain, Jeronnez & Ronveaux, 1951; Bockstaele, 1953; Sijsmans, 1954; Bilo, 1956).

Paul Libois between mathematics and Marxism

The most important advocate of intuitive geometry in Belgium was Paul Libois, professor of algebraic geometry at the Université libre de Bruxelles (Bruffaerts, 1994; Bingen & Gotovitch, 2012; Gotovitch, 2014). Libois, the only child of an army colonel, studied at Brussels university, where after graduation "with the highest distinction", he became an assistant to Adolphe Mineur. In 1937 he was promoted to the chair of geometry, but his research would gradually broaden towards mathematical physics. A gifted mathematician, Libois is usually best remembered for being the mentor of Jacques Tits, winner of the Abel prize in 2008.

Fig. 1. Paul Libois, 1927 (Université libre de Bruxelles, Archives)

Libois contributed to the 1948 reform, but his interest in school mathematics went back to the early 1930s (Jeronnez, 1959), when he became involved in the *École de* *l'Ermitage*, founded by Ovide Decroly (1871-1932) in 1907 (Depaepe, Simon & Van Gorp, 2003; Van Gorp, 2005). The Decroly pedagogy was quite influential in Belgium before World War II. The *École* followed the child-centered pedagogy of its founder, putting much emphasis on societal involvement, interdisciplinarity and active learning processes based on real life situations (Fonteyne 1934; L. Libois, 1971). Although the school was situated in an urban center, it attracted much support from the leftist intellectual circles in the capital, in particular with connections to the Université libre. In 1936, Libois married Lucie Fonteyne, a teacher at the school and later its director.

Another influence on Libois may have been his contacts with Italian mathematicians. In 1927 Libois went to Rome and worked for several months with Federigo Enriques (1871-1946) and Guido Castelnuovo (1865-1952). Enriques was critical of the deductive teaching of mathematics, and worked out an alternative based on an intuitive approach. According to Menghini (1998) the influence of Enriques on Libois was profound. Libois made a second trip to Rome in 1934, and he also participated in the Settimana della Scuola di Storia delle Scienze (Rome, 15-22 April 1935) organized by Enriques (Giacardi, 2012).

A third influence on Libois' views on mathematics teaching derived from his Marxist ideology. Already in 1929, Libois visited the Soviet Union. He was favorably impressed by the communist system, and in 1932 became a member of the Belgian Communist Party (Gotovitch, 2014). Very soon, it brought him into conflict with his university, but he managed to be able to continue his career. As a result of the conflict, which was widely discussed in the press, Libois became a symbol of the intellectual faction of the Communist Party, although his real influence was small. Belgian communists were in general rather suspicious of intellectuals. José Gotovitch (2014) characterized his role at the time as the 'vitrine intellectuelle' of the Party. The war, however, offered opportunities to assume a more responsible role, as many of the Party leaders had been arrested or had to go into hiding. Libois became active in the intellectual and ideological education of communist cadres, and one of the main intellectual forces behind the communist clandestine press. After the war, Libois emerged as an important leading figure, who often intervened in doctrinal matters. In 1946 he was elected to the Belgian Senate where he seated until 1950. As a senator, he campaigned against the deliveries of Belgian uranium (from the Congo) to the United States, asking for an open and transparent policy with regard to nuclear research. At the same time, he defended the Soviet (erroneous) point of view during the Lysenko-affair, which caused some scientists to leave the Party (Schandevyl, 2003). In 1954, Libois' was himself accused by the Political Bureau of the Party of 'sectarism' and stripped of his responsibilities. He left the Party in 1956, but always stayed loyal to his Marxist point of view.

In his political career, Libois has been portrayed as being a "rigid fanatic, cold, with an aristocratic contempt for those who could not master the doctrinal texts with the same ease as himself." (Gotovitch, 2014, p. 24). To some, he lacked charisma and maintained an "unconditional loyalty to Stalin. One of the communist students remembered him as being 'cultivated and with some charm. We were afraid of him, so we didn't dispute his ideas. He brought us, with an icy tone, the Truth. For him we were only little boys, we didn't object.' " (Ibid., p. 12)

Libois' connection to the Communist Party is crucial in understanding the network of people with whom he collaborated, or whose ideas he adopted. Libois' involvement in the reform of the mathematics curriculum cannot be fully understood without considering his leading role among leftist intellectuals both at the Université libre de Bruxelles and at the Decroly School. One example of this is his contribution to the *Comité d'Initiative pour la Rénovation de l'Enseignement en Belgique* (CIREB), created in January 1945 by a group of professors of the Decroly School and the Université libre (CIREB, 1945). There are, however, very few sources to reconstruct these networks.

With regard to Libois' scientific inspirations, his frequent references to the work of the French physicist and communist Paul Langevin (1872-1946), also in relation to Langevin's reform of French education, should be further investigated. Although Libois' involvement in Marxism and communist politics does not seem to have interfered in any meaningful way with his scientific activities, a general Marxist outlook did pervade his fundamental conceptions of science. At several occasions, he lectured on a Marxist interpretation of modern science. As he understood it, "dialectical materialism is the current scientific method, the modern synthesis of the experimental method and the rational method." (quoted in Gotovitch, 2014, p. 14) In 1947, he participated in a Marxist conference in Milan, where he applied the dialectical philosophy to the four-dimensional geometry of space-time (Cornu, 1948). Such instances, however, did – as far as we know – rarely appear in print and were not explicitly repeated in papers written for a scientific audience. Interestingly, at least with respect to school mathematics, the ideas of Libois did not reflect the pedagogical developments of that time in Soviet schools (Karp, 2009).

Libois and the teaching of geometry

Libois' approach to the teaching of geometry can be characterized by three specific features:

- The nature of geometry as being a part of physics
- The central role of the student in learning about reality
- The importance of the history of mathematics.

A theme spanning all of these features was the process of abstraction, or the dialectic between abstract and concrete. If mathematics was the art of the abstract, it was only so because of a process starting from the concrete. Students could only learn to look at the mathematical structures behind the real world, by first studying the real world. The more abstract the subject matter became in higher years of secondary school, the more the need for a concrete experience of real-world objects.

> It seems to me that if the teaching of geometry needs to become more abstract, this implies that at the same the source of abstraction becomes larger. Abstraction doesn't fall from the skies. […] Abstract knowledge which is not obtained by a process of abstraction, is not abstract but verbal. Abstraction does not distance [the student] from the concrete, from the nature of things, but it penetrates deeper into its nature. […] The teaching of geometry has to become more abstract by starting from a foundation–a physical foundation in particular–ever larger, and taking into consideration a larger number of objects, and more multilateral relations. […] If we want the teaching of geometry to become more abstract, in parallel with the development of the student, we should beware that it becomes at the same time more largely and profoundly concrete. (Libois, 1955-6, p. 34)

The dialectical process of abstraction and intuitive apprehension was grounded in Libois' epistemological understanding of mathematics as being a part of physics. Libois referred to (among others) Riemann, Poincaré, Langevin, Einstein and Enriques. His position is perhaps best summarized in the words of the French Marxist school psychologist Henri Wallon (1879-1962), quoted by Libois: "Physics creates for itself a geometry according to its needs." (Wallon, 1937, quoted in Libois, 1951, p. 28). Libois maintained indeed that geometry had been developed in response to or as the result of the study of physical objects or phenomena, and not the other way around. Geometrical spaces represent the different states of a physical object or a physical phenomenon. Libois discussed examples of different geometrical spaces, such as the Euclidean space (x, y, z), the Galilean space (x, y, z, t), the space of General Relativity (x, y, z, t) or other spaces (x, y, z, t, p, T) where p and T represented pressure and temperature.

> The space of Euclid […] was obtained through abstraction starting from (essentially) the consideration of solid bodies, imagined independently from time, and fixed with respect to an immovable body (the Earth). […] The space-time of Galileo […] was obtained through abstraction starting from the consideration of solid bodies moving relatively to each other without any of these movements having any particular characteristics, or privileges. The space-time of Einstein and Minkowski (special relativity) is obtained through the abstraction starting from the consideration of optical, electrical and magnetic phenomena. It is the mathematical expression of the ether of Faraday-Maxwell-Langevin. (Libois, 1951, p. 46)

Libois' belief in a physical foundation for all geometrical reasoning suggested to him a similar pedagogical process starting from the naive observations of 'real' physical objects towards increasing abstraction. Here the influence of Decroly's pedagogy shines through. The foundation of Decroly's pedagogy is the child's own spontaneous activity in observing real nature in all its complexity, a process Decroly called 'globalization' (Decroly, 1928). Gradually, the child would continue to investigate, to control and to experiment. The classroom becomes a 'workshop' or a 'laboratory', where the role of the teacher was not to teach but to collaborate with his students in a joint project. The school activities were also expanded "with frequent visits to factories, farms, building sites and laboratories." (Fonteyne, 1934).

Libois also approached geometry as incorporated in the world of industry and permeating modern life. To Libois, a child entering the classroom for the first time has already acquired an understanding of the world surrounding him. Children, he argued, would sooner be acquainted with the 'more advanced concept' of kilometer per hour, than with the 'simple' decigram. In the same vein, a child knows many geometrical shapes, and does so in a much richer variety than the geometrical objects studied in a traditional geometry course. Many objects which the children observe on a daily basis, can't even be given a proper name ("What is the shape of a dinner plate?") (Libois, 1947). Then what if a child would ask if it is possible to multiply kilograms and meters? "Some teachers," Libois explained, "would respond by asking: can you multiply apples and pears, or cows with horses?" But this is wrong, he argued, and it misses the opportunity to show the abstract nature of arithmetical operations. Using considerations of proportionality, one can indeed arrive at formulas in which kg and m are multiplied. In another example Libois (1951) stated–against Piaget–that a child's perception of space is more varied than psychologists usually assume. To understand the mathematical experiences of the child, it is necessary, he argued, to take into account the space-time group of the cyclist and the car driver, or of the tennis or football player.

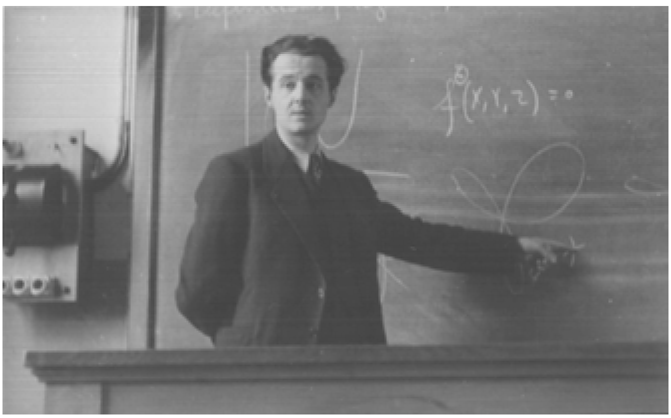

Fig. 2. Paul Libois, 1939 (*Mélanges Paul Libois*, s.d.)

Libois conceded that the child's notions were imprecise and not explicit. Teaching geometry meant that these concepts had to be enriched and specified, in a terminology that was convenient for the young child. More generally, Libois conceived of a series of mental operations: comparison, classification, generaliza-tion, abstraction and concretization, the study of relations, and finally unification, which included the integration of mathematics into physics (Libois, 1958-9).

The empirical basis of geometrical knowledge was not only confined to the learning situation of the young student. It was, according to Libois, an essential part of the history of mathematics. The evolution of mathematics demonstrated the close link between rational thought and the experience of the physical world. To prove his point, Libois sketched a schematic analysis of the notion of space in history, starting with the Euclidian notion of physical space as homogeneous and isotropic. In the sixteenth century, this Euclidian space became numerical, through its contact with algebra and mechanics. It was the beginning of the re-integration of geometry in physics. This would be fully realized by the work of Riemann in 1854. The rise of new technologies, focusing on optical properties in architecture, the construction of sundials or the practice of stone cutting, and the introduction of the notion of infinity, urged a "projectivization" of space. Finally, in the work of Galileo and Relativity Theory, space had become relative, although, according to Libois, the geometrical consequences of this discovery were not yet visible (Libois, 1951). Geometry was essentially a field in development, an observation he liked to link with similar ideas promoted by Federigo Enriques.

From intuitive geometry to the use of concrete materials

The introduction of intuitive geometry in the Belgian school curriculum was not entirely to the liking of Libois. The course on intuitive geometry was confined to the first year of secondary school, which actually led to a further partitioning of the subject of geometry. In 1955, he complained about the obsolescence of the current geometry curriculum, which still presented itself as a number of unrelated topics: intuitive geometry, plane geometry, solid geometry, complements of geometry, plane trigonometry, spherical trigonometry, descriptive geometry, analytical geometry and geometrical drawing. All of this should be brought together in a unified conception of geometry (Libois, 1955-6).

But his ideas did not find a large following in Belgium. Although he was a gifted teacher, his writings lack the rhetoric to make a strong impression on mathematicians and teachers. Libois did not write many papers and his ideas remained too much within his own sphere of interest: the Riemannian conception of geometry as part of physics, demonstrated by a sweeping argument from the historical evolution of the notion of space-time. These papers were rather different in nature from the more usual discussions of classroom activities or the presentation of course material, as they appeared in publications destined for teachers. Even among his friends, Libois' writings were not considered to be very clear and practical. Emma Castelnuovo found his long letters "very dense and very enigmatic", as "in the process of abstraction the jumps towards idealization were too difficult for us" (Castelnuovo, 1981).

His most loyal supporter and active proselytizer, Louis Jeronnez (1905-1981) called Libois "one of the most abstract minded people he had ever known" (Jeronnez, 1955-6). Jeronnez studied mathematics at the Université libre de Bruxelles, and became *préfet des études* (director) of the Athénée Royal de Binche. He was the author of several textbooks on geometry, as well as editor of the Belgian journal for mathematics teachers *Mathematica & Paedagogia* (De Bock & Vanpaemel, 2015b) and a founding member of CIEAEM (Bernet & Jaquet, 1998).

Jeronnez was well aware of the divided reactions of Belgian teachers to the course of intuitive geometry. In 1955 he wrote about the "confusion that reigned in the mind of many teachers." Many considered intuitive geometry a "makeshift job" (*bricolage*) and it became "the subject of many caricatures". To some observers it seemed as if "the beautiful order of Euclid's geometry was lost forever for the younger generations, and that the skills to fold and cut paper had taken the place of the ability to build a rigorous demonstration" (Jeronnez, 1955-6). Critical voices were also heard at the first conference of the newly founded Belgian Society of Mathematics Teachers held on 24 January 1954 in Namur. Several speakers deplored the lack of rigor and logic in the intuitive approach and complained that students after completing the course of intuitive geometry were still not able to think along rational lines. Others pointed to the danger of 'bricolage' replacing geometrical arguments. One speaker raised the problem of the general declining level of mathematical knowledge that was observed in the later years of secondary schools, and asked whether this might be a result of the tendency to 'simplify' the instruction of geometry in the early years (Namur, 1953-4). At the third conference of the Society, held in November 1955, it was pointed out that, although by now most teachers would accept concrete and constructed objects as didactical tools in their classes, they still did not agree on the moment when these objects had to be put aside to make room for rational and deductive reasoning, or when the individual characteristics discovered by observation had to be replaced by abstractly defined mathematical structures. It was also doubted whether physics could ever be used as a means to clarify geometrical understanding (Verstraete, 1956).

Fig. 3. Paul Libois at the École Decroly, early 1960s (Centre d'études decrolyennes)

Jeronnez took up the defense of intuitive geometry. He conceded that the introduction of intuitive geometry had created many problems for teachers. Authors of textbooks had not always well understood what intuitive geometry actually was. "Demonstrations were replaced by folding, cutting and pasting. […] Reasoning was held in horror." (Jeronnez, 1959) But taking his own teaching experience as an example, he explained how an intuitive approach could be used in every single year of secondary school. Against the reproach that intuitive geometry would only benefit weaker students, while the stronger ones were unnecessarily held back, he pointed out that this approach could and did make students better, from mediocre to good, and from good to excellent. In this way, this approach would provide more physicists and mathematicians to the nation. "Science and industry are in ever more urgent need of them. Isn't it our duty to take heed of this social problem?" (Jeronnez, 1955-6) He also added that, notwithstanding his long experience and careful investigations, the intuitive methods were neither unique nor definitive, and could still be improved upon.

Libois did not participate in these polemical debates. His primarily reflexive papers did not address the real or imagined opposition against intuitive geometry. He did, however, promote the intuitive approach by organizing, since 1951, student exhibitions at the Université libre on mathematical themes, such as "quadrics" (1951), "shadows, light and geometry" (1952), "geometry of transformations" (1953), "symmetry" (1954), "the sphere" (1955),… Students showed mathematical models, cubes, pyramids, polyhedrons, hyperbolic parabo-loids,… constructed with cardboard, wood, iron, plexiglass, strings. In November 1955, the Belgian Society of Mathematics Teachers expanded Libois' initiative into a national Exposition at the Royal Athenaeum of Berchem (Antwerp). It was probably the culminating point of intuitive geometry in Belgium. Although Libois' exhibitions continued until 1971 and attracted the admiration of among others

Emma Castelnuovo, the emphasis on mathematical objects created by the students was gradually replaced by the consideration of new, specially designed 'concrete' objects which still could capture some of the intuitive approach, but were further removed from reality, in order to force the student to use abstract, logical reasoning without recourse to measurement or manipulation. Such objects were the Cuisenaire rods, the geoplan or electrical circuits–and later, the formal graphs of the Venn-diagrams.

Conclusion

Libois' commitment to the teaching of intuitive geometry was clearly not only the result of didactical considerations. Although he was from an early age in his professional career involved in the pedagogical reform movement of Ovide Decroly, his views on the teaching of mathematics were firmly rooted in the epistemological notion that the investigation of the physical world is the source of all geometrical reasoning. His account of the history of geometry, in which advances in mechanics were the true 'causes' of conceptual revolutions in geometry, was in full agreement with his didactical approach to geometrical abstraction through increasing manipulation of 'real' objects. To Libois, the defense of intuitive geometry was not merely a debate on didactical principles. It was foremost a debate on the intellectual foundations of geometry.

Behind all of his writings on intuitive geometry, a Marxist ideological inspiration cannot be concealed. In papers published for a general audience, Libois did not address Marxist themes, but too many resemblances to Marxist parlance pop up in his writings to be ignored. For Libois, the application of dialectical modes of thought (in research and teaching) formed the basis of science. This was evident in the interactive relationship between concrete objects and abstract ideas, in the equal roles allotted to teachers and students. In particular, Libois echoed the communist interpretation of Isaac Newton as an idealist philosopher, whose physical views, "based not on physical but ideological arguments" (Libois, 1951, p. 36), constituted a regression in the history of science. On the other hand, the fairly abstract group theory was only the inevitable–if belated–elaboration of the principle of relativity, originally formulated in an empirical and primitive way by Galileo. True science could always be understood to be produced by a dialectical process between empirical knowledge and theoretical formulation. As dialectical reasoning was a fundamental characteristic of science, it should also be used in teaching. The Marxist model of dialectical reasoning was the ultimate foundation of Libois' views on mathematics, on the teaching of geometry and on the importance of an intuitive approach starting from real-world observations.

By the end of the 1950's, it became clear that another conception of modern mathematics education was gaining ground in Belgium. Intuitive geometry had to face the criticism of Bourbaki inspired reformers, such as Willy Servais (1913-1979), Frédérique Lenger (1921-2005) and Georges Papy (1920-2011). Although these reformers also promoted the active involvement of students, in particular through the manipulation of concrete models, they put more emphasis on introducing from the start an abstract mathematical language (in particular the language of sets). The intuitive part of mathematics was reduced to a simple didactical technique, which in practice left little to the initiative of the students. In a few years, Libois' views were completely discarded from the center of debates in favor of the proposals made by his colleague at the Université, Georges Papy.

Libois' influence continued to be noticeable for many more years in Italy. Since 1949, Libois collaborated with Emma Castelnuovo, who regularly came to Brussels to visit the exhibitions at the university and at the École Decroly. Drawing on the work of Libois, she introduced similar student exhibitions in Italy and remained in close contact with Libois. However, in the 1960s Castelnuovo had to admit that the dominant Belgian school was now headed by Papy, and during her stay in Brussels she visited the École Berkendael where Papy taught to 12-year old students (Castelnuovo, 1965). More mathematicians were 'defecting' to the Papy side: Frédérique Lenger who in the early 1950s was Libois' assistant and who also taught at the Decroly school, invited Papy to assist her in her own experiments and later even married Papy. Libois did not participate in the Centre belge de Pédagogie de la Mathématique (Belgian Center for Mathematics Pedagogy) created by Papy in 1961, nor did we find any trace of Libois in the annual teachers' conferences at Arlon, where the New Math reform in Belgium was prepared before it became effective in 1968.

Papy advocated a completely different approach to mathematics education. Children should be exposed as early as possible to clear logical reasoning and to the fundamental mathematical structures. When teaching to 15-year old school children, he found them already oriented in a direction completely opposite to the development of a real mathematical mindset. All the previous work had to be undone. Papy concluded that the course of intuitive geometry had simply been a waste of time and energy (Papy, 1966).

In this respect, the suggestions made by Libois (Libois, 1960, 1963) to the Dubrovnik group of experts formed the swan song of his work during the preceding decades. The Dubrovnik proposal was still positively received at the Decroly School (Ministère, 1962), but, in general, most attention went to the Mathématique Moderne of Papy. In conclusion, one of his ideas proposed in Dubrovnik can be mentioned here, as it captures very well his view of the aims of the intuitive approach to teaching young students. When the initial stage of experimenting and discovering is finished (12-15 years), Libois urged the teachers to introduce their students to the history of mathematics and to explain to them under which circumstances major steps forward had been taken. Students should learn how the Elements of Euclid were actually abstracted parts of the study of nature: the study of solid bodies, the study of whole numbers and the study of proportions. A further elaboration on this point would then enable the students to make a connection between the mathematics of Greek Antiquity and the mathematics of their own time. "Thus," Libois wrote in the preparatory document for the Dubrovnik meeting, "mathematics will cease to appear a dead science, but it will be seen as the result of the centuries' old effort of humanity, as a science which has never ceased to live, and which also today is in full development." (Libois, 1960, p. 6) He was certainly right on that point.

Note. *All translations were made by the authors.*

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Olivier string models and the teaching of descriptive geometry

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Abstract

Théodore Olivier (1793-1853), a disciple of Gaspard Monge, designed, from 1830 onwards, a set of dynamic string models that depict ruled surfaces by tensioned silk threads, thereby developing an original methodology for the study of this kind of surfaces, a main topic in the 19th century French textbooks of descriptive geometry.

Monge was an upholder and one of the most prominent agents in the renewal of the educational system that follows the French Revolution. He established descriptive geometry as a cornerstone of the whole curricula *in the teaching of engineers and practitioners, a model that would influence the organisation of many polytechnic, industrial and military schools in France and also in other countries throughout the 19th century. Théodore Olivier inherited this perspective and stressed the pedagogical importance of tangible models and devices, when pursuing the comprehension of abstract geometrical objects and tools. He conceived the* omnibus*, an apparatus for the physical representation of the process that leads to the* épure *of any object, and developed a large collection of ruled surface models. The spread of the French educational system promoted the diffusion of these models into many schools in several countries.*

Introduction

The birth of descriptive geometry

Gaspard Monge (1746-1818) formalized descriptive geometry, or the double orthogonal projection, while at Mézières, and possibly since the years he was a *Répétiteur* of mathematics (1766-1768), see (Arago, 1846; Loria, 1921). However, this new formal tool was not disclosed outside the school until 1795, after the French Revolution and the subsequent struggles that made clear the need of a strong commitment with education, especially in what concerned the industrial activities, for France to ensure its independence. Monge, who had a very active role on the resolution of military problems at these very turbulent years, also contributed to the organisation of courses for engineers, factory workers and craftsmen, and to

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the foundation of the École Centrale des Travaux Publics (1794, one year later renamed École Polytechnique) and the École Normale (January 20 to May 19, 1795), see (Arago, 1846; Dhombres, 1992). In these two institutions Monge presented, almost at the same time (beginning, respectively, in January 10 and 20, 1795), a course on descriptive geometry, see (Dhombres, 1992).

The identity and structure of the École Centrale des Travaux Publics at its foundation, and the role given there to descriptive geometry, were deeply rooted on the ideas of Gaspard Monge. The curriculum (1796) presented at the end of the first year was, schematically:

1.mathematiques:
- 1.1 analyse
- […]
- 1.2 description graphique des objects
  - **1.2.1 géométrie descriptive**
    - **1.2.1.1 stéréotomie (règles générales et méthodes de la géométrie descriptive […] applications à la coupe des pierres, a la charpenterie, aux ombres des corps, […])**
    - **1.2.1.2 travaux civils […]**
    - **1.2.1.3 fortifications**
  - 1.2.2 art du dessin
  - […]

2.physique

[…]

The connections between descriptive geometry and several applications were structural features of the envisaged education — in the first years, more than an abstract language of representation, it was taken as an efficient tool for solving many constructive problems. And, in fact, descriptive geometry was widespread as the common language of all engineers, due to the prominent role of Monge and of the two referred schools.

The school of Monge, however, was very soon replaced by the school of Laplace, in the École Polytechnique, whereas its spirit remained alive in the Conservatoire des Arts et Métiers (Grison, 1999) or, later, in the École Centrale des Arts et Manufactures.

Théodore Olivier

The École Centrale des Arts et Manufactures was founded, in 1829, by Théodore Olivier (together with Baptiste Dumas and Eugène Péclet, and the financial support of Alphonse Lavallée), as a result of this discontentment with the changes in the École Polytechnique that no longer focused on the practice and the applications but only on the theoretical approach, and was much concerned in producing algebraists rather than engineers, see (Olivier, 1851, pp. VII-XII).

At the École Centrale des Arts et Manufactures, Olivier took the responsibility of teaching descriptive geometry and mechanics, assuming the direction of the studies in 1832. His guidelines were to shape a school of engineering that taught theory always in connection with its applications to industry, and where mathematical topics were limited to descriptive geometry.

Graduated by the École Polytechnique de Paris (1810-1814) and official by the school of artillery of Metz (1815-1818), Théodore Olivier devoted his life to descriptive geometry — teaching, writing textbooks and creating many models.

Before his long career at the École Centrale des Arts et Manufactures, that lasted to the end of his life, he was a Professor of *géométrie descriptive et de ses applications aux divers services militaires* at the Royal School of Marienberg, Sweden (1821-1826). He also was *Répétiteur* at the École Polytechnique de Paris (1829-1844) and Professor of *géométrie descriptive* at the Conservatoire des Arts et Métiers, in Paris (1839-1853), where he became administrator a few months before his death, see (Peligot, 1853, p. 504).

In the theoretical field, besides his titles on descriptive geometry, like his famous *Cours de Géometrie Descriptive* (1st edition: 1843-1844), he also devoted a particular interest to the study of gear surfaces, which was considered one application of descriptive geometry, making several wood models.[1]

But, eventually, it was in the didactic field that his contribution to the development of descriptive geometry was most remarkable.

The use of models for teaching descriptive geometry

The practical purpose of the discipline always guided the conduct of Olivier. The first lines of the preamble of the *Mémoires de géométrie descriptive* (1851) are revealing in this regard:

> A science is not really useful to men unless it leads to terrestrial applications; (…)
>
> Those, among scholars, which deal exclusively with science from the theoretical point of view, are sublime dreamers who populate the world of ideas with fortunate ideas. But do not forget that an idea cannot lead to anything useful on this earth unless it is contained in a material body,

[1] These models are often referred as Olivier models, see (Hervé, p. 297), while the ones treated on this paper are known as Olivier string models.

> unless it becomes practical, and therefore applicable to terrestrial things.[2] (p. III)

Consequently, in order to feed the intelligence and imagination, he advocated the use of concrete models in the classroom as a departing point for the development of abstract thinking, models that, in turn, simulated recurrent construction issues in the manufacture of machines, buildings, bridges, and other artefacts.

The dialectics between imagination and visualisation, between the world of ideas and the contingent reality, that Olivier pursued for the apprenticeship of descriptive geometry, earned him strong criticism at the time, even if only from those who considered that the immateriality of the geometric elements should not be maculated by any attempt to turn them concrete, although that is precisely what drawing does. Anyway, according to Olivier's own words the common sense might have prevailed:

> For several years the use of models was even outlawed, and so to speak officially and by the scientific authority, in teaching elementary geometry and descriptive geometry: we taught using only chalk, the blackboard and a sponge. To show a model to a student was a forbidden thing, as we seemed to doubt his high intelligence. Also, assuming this principle, we came to teach physics making no experience; chalk, blackboard and sponge were the only things needed for the teacher, whatever the science he would teach, physics, astronomy, geodesy, descriptive geometry, mechanics, etc.
>
> Luckily the common sense has finally done justice to this strange utopia. (Olivier, 1852, p. X)

The omnibus

When Olivier started, in 1829, his activity at the École Centrale des Arts et Manufactures, he designed a pedagogical tool for teaching descriptive geometry – the *omnibus* – which was intended to cover the early stages of apprenticeship of the discipline, namely the representation of the point, the line and the plane – the 1st part of the *Cours de Géométrie descriptive* (1852) –, but also the study of ruled surfaces – integral matter of the 2nd part (1853).

> To provide the teacher of descriptive geometry with the means to make as many figures as he wishes and with a single instrument, is thus a useful thing.
>
> Useful in the interest of education, useful considering budget restrictions.

[2] This and the following texts by Olivier are translated by the authors.

> These concerns of utility and economy led me, in 1829, to build an instrument, which I named *omnibus*, that was able to display all the problems related to the point, to the right line and to the plane, in a word, capable of showing the three dimensional configuration of all geometric system composed of straight lines. (1852, p. X)

The *omnibus* is composed of two articulated planes made of cork. These faces simulate the projection planes, and a set of small spheres and needles allows to locate points and projecting lines in space, as well as their projections in the orthogonal planes. Removing the spatial elements the vertical plane can be open back down on the horizontal so as to obtain the *épure,* the two-dimensional representation. Simple and cheap, the *omnibus* payed tribute to its generous name, and met a remarkable diffusion. That was the case of Portugal where an *omnibus* box was produced, from 1895, for high schools and industrial schools, together with the official textbooks on descriptive geometry (Abreu, 1916).

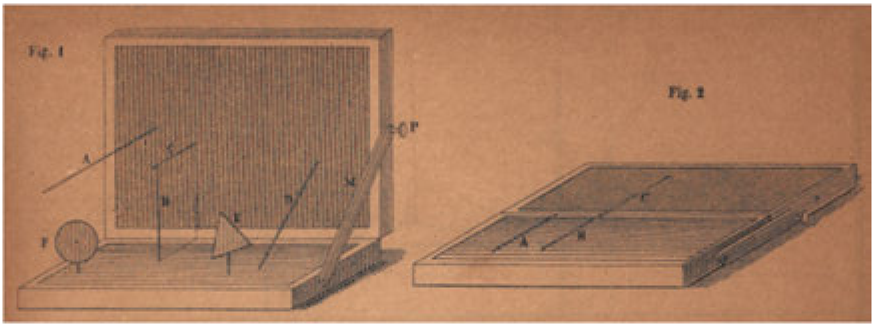

Fig. 1. The *omnibus* box in a Portuguese secondary grade textbook. (de Abreu, 1916)

It should be emphasized the versatility of the *omnibus*, an instrument able to spatially display all problems relating to the point, the line and the plane (which Barbin, 2015, identifies with the *élémentation* followed by Olivier), and all the geometric systems that are composed of straight lines, that is, ruled surfaces. And the reason is simple: if the instrument lets us easily represent a line segment in space and double projection, the same is true for all of the generatrices forming a ruled surface. Or, conversely, since a ruled surface can be represented by a set of generatrices, it is easy to design each of them in the *omnibus*, and to display it in space and double projection.

Therefore, with this instrument it is possible to show: developable surfaces and their edge of regression, cones, cylinders, paraboloids, hyperboloids, conoids, actually, any ruled surface.

The Olivier String Models

Théodore Olivier developed, since 1830 and until the end of his life, a collection of almost fifty string models of ruled surfaces for the teaching of descriptive geometry. These models were intended to complement and support his classes on the subject but were also widespread outside the classroom, in the Conservatoire des Arts et Métiers, to which Olivier offered duplicates of his models, see (Morin, 1851; Morin, 1855). This was a much celebrated collection by the second half of the 19th century, and a witness of Olivier's commitment to the practical approach in the teaching of descriptive geometry:

> …When we want to talk to students about the properties of a surface, the first thing to do is to put before them the model of this surface, so they clearly see what we want to say. (Olivier, 1852, p. X)

The Olivier models depict surfaces, mostly ruled surfaces, by tensioned silk threads. These threads materialise some of the rulings, or straight generatrices, that suggest the surface when taken together, and are supported by perforated metal pieces that correspond to directrices of the ruled surface. Hyperbolic paraboloids, hyperboloids, cylinders, cones, conoids and other surfaces are illustrated, along with some of their properties, including several tangency and intersection situations through models that display more than one surface at a time.

The use of string models to assist in the teaching of descriptive geometry was not a new idea. The catalogues of the collections of the Conservatoire des Arts et Métiers, see (Morin, 1851, p. 17), mention the static string models built by Monge for the École Polytechnique, and then reproduced by Brocchi, the school curator. Beyond these fixed models, the son of Hachette, Ingénieur des Ponts et Chaussées, also invented a string model of a non-ruled surface that was included in the Olivier collection, see (Sakarovitch, 1998, p. 315).[3]

The dynamic nature of the Olivier string models is one of their major features – some of the metal components are movable parts and allow the transformation of the depicted surfaces. The silk threads remain in tension during movement, either because their lengths do not change, or due to the use of weights, which are placed on their ends and hidden inside a wooden box. Movement, in fact, opens many possibilities, in particular the generation of a family of surfaces, the highlight of the invariants of geometrical transformations, as projective invariants, the intersection or tangency of surfaces.

[3] The information on the model with inventory number 04454-0000- of the Conservatoire des Arts et Métiers confirms this authorship.

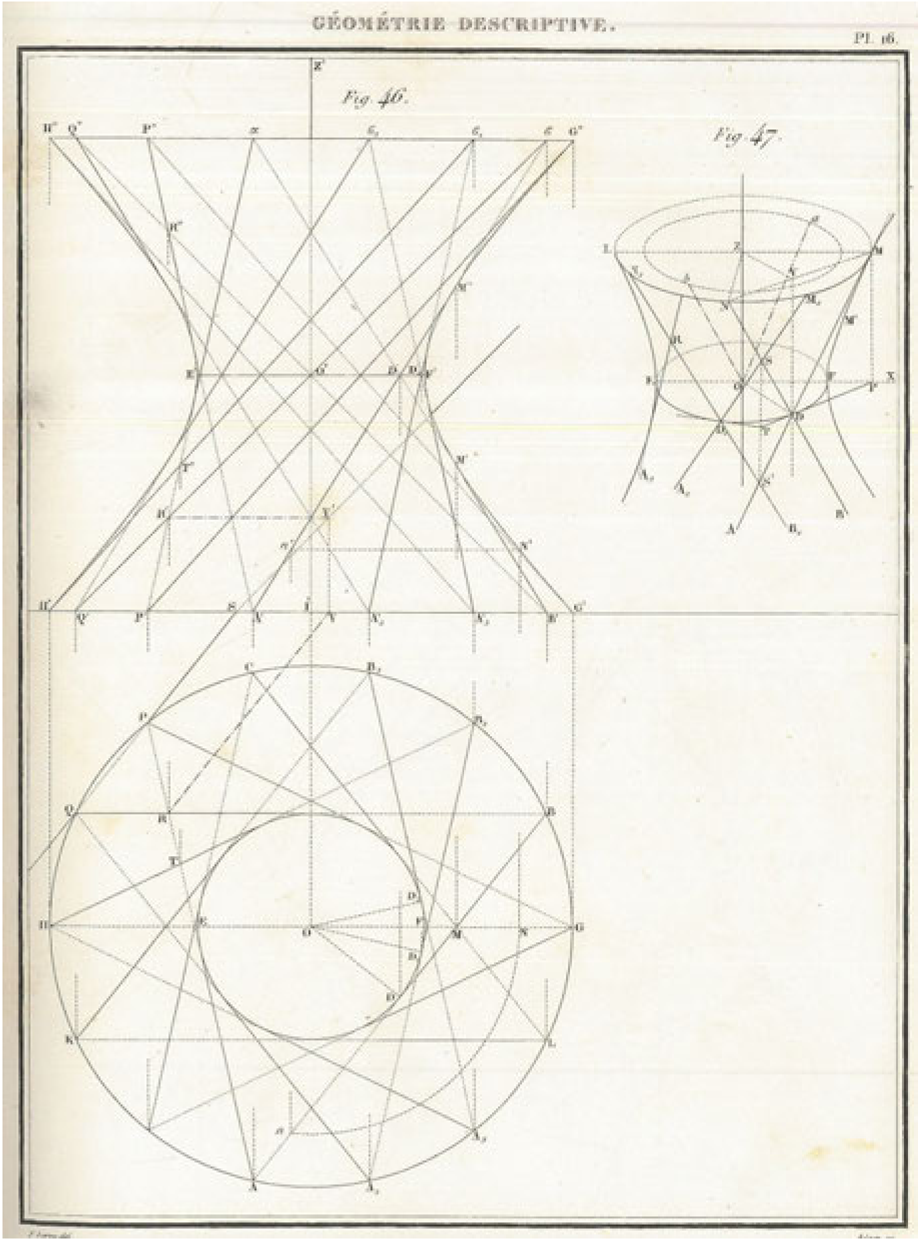

Fig. 2. Double orthogonal projection of a hyperboloid of revolution represented as a doubly ruled surface (Leroy, 1862, II, Pl. 16)

Fig. 3. Olivier model of a hyperboloid. Made by Secretan, c. 1868
(Museum of ISEP, MPL321OBJ)

Among his predecessors – namely Monge and Leroy (1834) – who devoted attention to string models to illustrate what Hachette named ruled surfaces (*surfaces réglées*), we should highlight the work of Joseph Adhémar, where we find the concept of surface transformation as addressed in Olivier's models. That is the case of the *biais passé* that he presents in a dynamic way defying the reader to appreciate the transformation of its form, departing form a rectangular plan to a parallelogram whose diagonal becomes orthogonal to the wall limits (Xavier & Pinho, 2016, p. 359). More than the dynamic concept of surface, expressed by the idea of a moving generatrix, and already acquired before Monge's systematization of descriptive geometry, see (Frézier, 1768, Tome II, Liv. IV, pp. 3-11), we recognize, in Olivier's models as well as in some Adhémar's descriptions, the concepts of transformation and invariants, a keystone in projective geometry.

The models have a remarkable educational interest and can indeed be a starting point, and a permanent reference, to understand the properties of the surfaces. In Fig. 2 and Fig. 3 we confront the double orthogonal projection of a hyperboloid of revolution and the Olivier model of the same surface. The model may be seen as a useful tool for the construction of the rigorous two-dimensional representation of the surface, but surpasses largely its two-dimensional counterpart, allowing an intuitive exploration of the surface, either in its own or embedded in a set of similar surfaces.


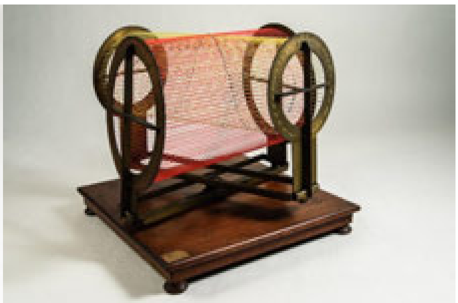

Fig. 4, 5. Olivier model of the intersection of two cylinders. The respective intersection curve is emphasized through rings marking the points of intersection of the coplanar generatrices (Museum of ISEP, MPL307OBJ)

Fig. 6, 7. The model shown in fig. 3 has a movable upper ring. Turning the upper ring leads to the depiction of two different hyperboloids: the inner hyperboloid (lighter strings) becomes more and more closed, until a cone, and the outer hyperboloid becomes more open, until a cylinder (Museum of ISEP MPL321OBJ)

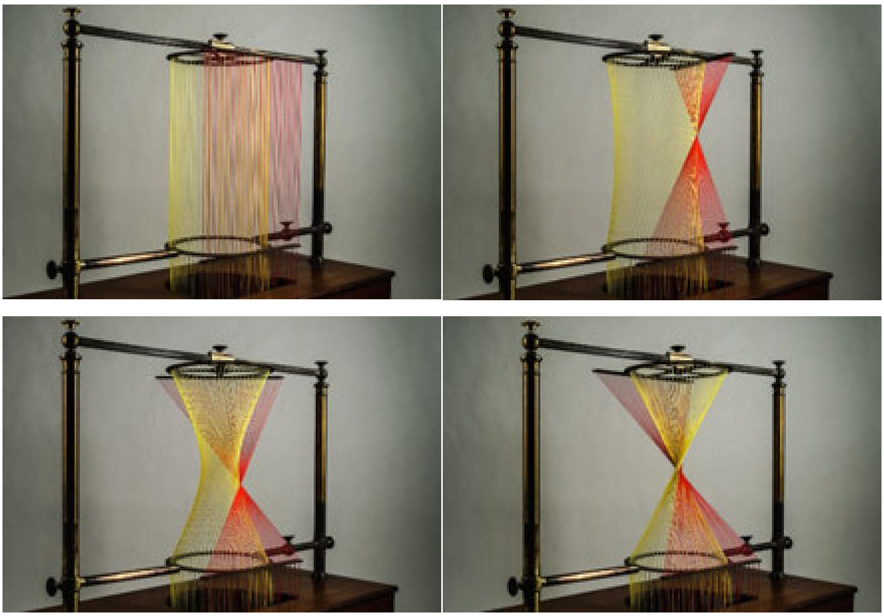

Fig. 8, 9, 10, 11. Olivier model of two tangent surfaces. Looking at the lighter threads, we see the transformation from a cylinder into a cone, through different hyperboloids in the intermediate states. And the simultaneously transformation, see the darker threads, from the plane that is tangent to the cylinder, to the plane tangent to the cone, through the hyperbolic paraboloids that are tangent to the hyperboloids (Museum of ISEP, MPL308OBJ)

Fig. 12, 13. Idem. We can also move the lower brass bar, so that the cone, cylinder and hyperboloids may be straight or oblique (Museum of ISEP, MPL308OBJ)

Leroy (1834), in the footsteps of Hachette (Hachette, 1817, pp. 11-12), had already pointed to the applicability of ruled surfaces to the arts.

> All surfaces that can be generated by the motion of a straight line, are generally referred to as RULED SURFACES, because you can obviously define them on a solid body by a ruler, which makes its use in the arts very frequent. (p. 258)

And, in fact, the special attention given to ruled surfaces comes, first-hand, from their applicability to the construction world, being associated with practical and effective resolution of stereotomy complex problems of stone and wood.

So, when Hachette proposed to identify them as ruled surfaces he was just taking in consideration this tradition, well represented by Frézier in his monumental work on stonecutting (Frézier, 1768). Later on, in 1828, it will be published the first textbook dedicated to ruled surfaces by Gabriel Gascheau.

In fact, many ruled surfaces are known for their application in particular architectural elements built in stone, such as the arrière-voussurre de Marseille, de Montpellier, de Saint-Antoine, the biais passé or the vis Saint-Gilles carré. Some Olivier models reproduce precisely these last two surfaces and are referenced with such designations.

Although the models of ruled surfaces designed by Théodore Olivier might be inscribed in a well-established tradition of studying ruled surfaces, they represent a substantial innovation by tackling this subject so systematically, with illustrations of the main problems and topics of study, and by allowing the mobility of the surface directrices and generatrices. The pursuit of completeness, the movable elements, the wise choice of materials and the perfection of execution makes this a unique collection and a remarkable work. Moreover, notice that the dynamic approach of the Olivier models is an avant-garde attempt of a parametric representation, presently a common procedure in digital drawing and fabrication.

Some collections of Olivier models

From 1830 onwards, Théodore Olivier conceived the models of ruled surfaces and instructed the firm Pixii, Père et Fils to produce them, departing from his drawings and under his close supervision.[4] At least two collections[5] were produced within this collaboration, one of which was deposited, by Olivier's own initiative, at the Conservatoire National des Arts et Métiers, between 1849 and 1853. Presently it is composed by 47 models that, according to the catalogues of the CNAM (Morin, 1851; Morin, 1855), are: 10 paraboloids, 6 hyperboloids, 10 conoids, 6 special ruled surfaces (warped and developable), 12 intersections of ruled surfaces (warped and developable), and 3 models to solve particular problems.

The original collection of around fifty models that Olivier kept for himself, was sold by his widow, in 1855, to William M. Gillespie, a professor of civil engineering, between 1845 and 1868, at the Union College in New York. Having studied in Europe, after graduating from Columbia, Gillespie knew personally Théodore Olivier and his wife, and was aware of the existence of this collection, which, after Gillespie's death, was acquired by the College.

The firm of constructers of scientific instruments was soon renamed Fabre et C.ie, succ.r de Pixii, and proceeded with the production of Olivier models. One of the collections from this period was purchased by Harvard University, circa 1856,[6] from which 4 models still exist.

With the new designation of Fabre de Lagrange, the firm produced several collections for various institutions, among which we find, in the United States, the West Point Academy, which acquired 26 models. Arthur Hardy, graduate teacher by West Point Academy in 1869, wrote this curious comment about the use of these models in the classroom:

> In descriptive geometry, the Academy had a magnificent collection of models, but they were shown us after the study was finished – in other words, mental discipline was the object – practical helps and ends were secondary. (Shell-Gellasch, 2003, p. 78)

In Portugal, the descriptive geometry professor Motta Pegado, from Escola Politécnica de Lisboa (Polytechnic School of Lisbon), had at his disposal a collection of 49 models, also made by Fabre de Lagrange. Today there are no more than 20 of these models, dated from 1861, and we know for sure that they were actually used, according to the exigent curriculum that contains an important section dedicated to the study of ruled surfaces, see (Palaré, pp. 159-160).

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[4] There is even a reference that some models were *made in part by his own hands*, see (Stone, 1969, p. 2).

[5] There is a reference to a third collection that was sent to Madrid, see (Stone, 1969, p. 12).

[6] See the webpages of the Historical Collection of Scientific Instruments of Harvard University.

In 1872, C.W. Merrifield, in *A catalogue of a collection of models of ruled surfaces constructed by M. Fabre de Lagrange*, described the 45 models kept in the South Kensington Museum, 14 of which photographed. In 1973 this collection was split in two, one part remaining in London's Science Museum, and 28 models being donated to the Canada Science and Technology Museum in Ottawa.

Other companies involved in the production of scientific instruments also built Olivier models, as is the case of Secretan, in Paris, who provided the Instituto Industrial do Porto (Industrial Institute of Porto) with 30 models (which are kept in the Museum of ISEP). Here the responsible for the acquisition was the director Gustavo Adolfo Gonçalves e Souza, who was also a professor of descriptive geometry, both at the Institute and at Academia Politécnica do Porto (Polytechnic Academy of Porto).

Gustavo's statement (Souza, 1872) confirms a strong conviction on the extraordinary didactic value of the string models to the practical feature he wanted to give to descriptive geometry, in the steps of Théodore Olivier, along with their common faith on the irreplaceable role of the discipline.

> For the study of the different sections of drawing, this Institute has already a quite good and large collection of plaster models of ornaments, vases, body sections and statues, as well as **a very good collection of models of descriptive geometry figures,** which we can compare to the best of this kind that exist in foreign countries, acquisition **that greatly assists the study of descriptive geometry, which today has become indispensable to all artists**.[7]

Acknowledgment. We would like to thank the valuable support and tireless help of Patrícia Costa, Head of Documentation and Culture, Museum of ISEP. We also thank Raquel Almeida, Museum of ISEP, José Francisco Rodrigues, CMAF-UL, Marta Lourenço, MNHNC, Cyrille Foasso, CNAM, Pedro Varela, DFL/CEAU/FAUP, and Jorge Silva, photographer, who took the pictures of the Museum of ISEP models. Eliana Manuel Pinho has a post-doctoral grant from FCT (SFRH/BDP/61266/2009). The authors were partially supported by CEAU, which is funded by FCT (Portugal) with national (MEC) and European structural funds (FEDER), under the partnership agreement PT2020.

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[7] Text translated from the Portuguese and highlighted by the authors.

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The function of a preface: Contextual information and didactical foundation described in the preface and introduction of a textbook in arithmetic from 1825

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Bernt Michael Holmboe (1795-1850) was one of the most influential persons in the development of school mathematics in the first half of the 19th century in Norway, and he wrote the textbooks in mathematics that were predominantly used in the learned schools in Norway between 1825 and 1860. The theme for this short paper is the preface and introduction in the first edition of his textbook in arithmetic from 1825 (Holmboe, 1825).

Holmboe starts his preface by stating that there are no subjects in school where the students complain more, than mathematics. One of the first tasks for a teacher in mathematics should be to give the students practice in the use of mathematical signs. The teacher should give the students practice in saying a statement in words when the student sees it in mathematical signs, and vice versa. An examples is (Holmboe, 1825, p. IV):

> When a student sees the statement "$(a + b) - c = (a - c) + b$", he should be able to say that "[i]nstead of subtracting a number from the sum of two other numbers, one may subtract it from one of the two adding numbers, and then add the other one".

Holmboe was clearly influenced by the Norwegian philosopher and politician Niels Treschow (1751-1833), who was professor in philosophy and the first rector at the new university in Norway at the time when Holmboe was a student. Treschow published several books, among them a textbook in Common Logic (Treschow, 1813), clearly influenced by Immanuel Kant. Treschow has a rigorous classification of statements – in *direct* and *indirect* statements, and in *analytic* and *synthetic* statements – based on the relation between subject and predicate in the

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statements, and on the nature of the statements (Treschow, 1813, pp. 161-162). Exactly the same classification is described in the introduction chapter of Holmboe's textbook (Holmboe, 1825, pp. 1-3). A *statement* is a connection of two concepts. The first thought of, in the connected concepts in a statement, is called *subject*, and the second is called *predicate*. Statements are called *direct* if one perceives the subject's connection to the predicate without regarding other statements, and it is therefore not necessary to prove or demonstrate the statements. Correspondingly are statements called *indirect* if one uses other statements when considering the connection between subject and predicate, and it becomes necessary to prove or demonstrate the statements. In the latter case, the indirect statement will then be a consequence of the statements used to elucidate it. The presentation of the steps of reasoning used to elucidate the subject's connection to the predicate in an indirect statement is called a *proof*.

The statements are divided in *synthetic* or *practical* statements, expressing that a connection between concepts shall be made, and *analytic* or *theoretic* statements, expressing a connection between concepts that already exists. Both *synthetic* and *analytic* statements may be *direct* as well as *indirect*, and Holmboe is therefore introducing four types of statements:

- **Direct synthetic statements** which he calls *Postulates*. A postulate expresses that two concepts shall be connected.
- **Direct analytic statements** which he calls *Fundamental Statements* or *Axioms*. An axiom expresses the relations between two connected concepts.
- **Indirect synthetic statements** which he calls *Problems*. A problem expresses that two concepts shall be connected using already existing connections.
- **Indirect analytical statements** which he calls *Theorems*. A theorem expresses a connection between two concepts that is proven to be a consequence of preceding connections.

There are in Holmboe's textbook 10 direct statements called *Fundamental Statements*, there are 27 indirect synthetic statements, or problems, and there are 30 theorems and 163 corollaries.

The preface of Holmboe (1825) may seem complicated and difficult for a textbook meant for young students. Holmboe clearly states that his textbook is not meant for self-study, but requires a skilled teacher.

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A case study on the teaching of mathematics in the Italian Renaissance: Niccolò Tartaglia and his *General Trattato*

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Niccolò Tartaglia (1449-1557) is well known in the history of mathematics thanks to the discovery of the solving algorithm of the third degree equation and also for the controversies about the authorship of this result, at first against Girolamo Cardano and then against his pupil Ludovico Ferrari.

Maybe less known, but certainly no less important, it is his activity as Abacus Master (*maestro d'abaco*), attested in Verona since from 1529.

Shortly after Tartaglia moved to Venice where, in addition to his teaching activity, he held public readings on Euclid's *Elements*, that earned him some fame.

In Venice, Tartaglia started to print his books; the last treatise, partly posthumous, was the *General Trattato di numeri et misure*. Divided in six *Parti* (Parts), printed between 1556 and 1560, it was a real encyclopedia in which matters and methods of mercantile mathematics coexist with mathematical humanism, represented for example by the translation in vernacular of the First Book of the Archimedes' *Sfera e cilindro*.

Even if the *General Trattato* can't be considered as a textbook, in it we find many observations which help to partially retrace Tartaglia's ideas on the teaching of mathematics.

To bring his readers – practitioners like artisans, merchants, architects, soldiers and so on – to a mathematics more speculative than the one they were used to, Tartaglia relied on a language rich in metaphors and similes taken from daily life. For example, when he explained the general meaning of 'measuring a surface' he evoked the figure of a shoemaker who 'measures' a piece of leather (the surface) placing upon it the model of a sole (the unit of measure) several times until its very end, so to see how many shoes he was able to make.

Definitions are similarly conceived and they are, when possible, anchored to the real world, so that they will be of some practical usefulness: this approach is easily successful with the geometric definitions of genetic type.

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The particular attention that Tartaglia paid to the language is also the fruit of his activity as both a translator and a teacher. It is in fact important to remember that he was the first to publish Euclid's *Elements* in a current language, that is the Italian vernacular.

The meeting point between Tartaglia-the-translator and Tartaglia-the-abacus-Master, gives rise to surprisingly modern reflections about possible obstacles aroused by the use of the common language in the learning of mathematics.

The attention that Tartaglia paid to the use of mathematical language in the process of learning, it is just one of the features of modernity that clearly emerges from the reading of his works. Another aspect of high interest concerns his approach to problem solving. We can briefly take an example into account.

Facing a typical problem of surveying, as that one to determine the area of a triangle whose sides' length is known, Tartaglia presented different strategies of resolution, that are represented by different formulas to be chosen in relation to the concrete context in which we need to apply them.

A first approach, for example, is to determinate the height of this triangle using the propositions 12 or 13 of Books II of Euclid's *Elements* depending on whether the triangle is obtuse-angle or acute-angle. As an alternative, Tartaglia suggested the use of so-called "Heron's formula", of which he also provided the proof – an unusual mathematical 'object' in a practical geometry treatise – aimed "to satisfy speculative people".

There is also another interesting expedient that Tartaglia used to focus his reader's attention on the resolution procedures. The *escamotage* is to consider every example with the same numerical data, in other words to consider the triangle of 13, 14, 15 sides length. These numbers allow to make simple calculations and to not to deflect the attention from the comprehension of the resolution procedure. Only after that the procedure has been completely internalized, more complex calculation can be introduced.

To conclude, even if the *General Trattato* could not be considered a teaching handbook, the examples we can find highlight some interesting ideas of the Master Niccolò da Brescia on teaching and learning mathematics. A purely mnemonic learning, in other words a learning not subordinated to the comprehension of the processes, it was liable to fade out in a short time without leaving any trace in the learners' mind. It is for this reason that Tartaglia offered to his readers also the possibility to explore the causes that are behind the rules.

Maybe the most significant pedagogical effort is the attempt to educate to abstract mathematical reasoning a public mainly interested in "useful" results, immediately employable in everyday life. Tartaglia addressed his work to these readers using a strongly evocative language and an approach rich in metaphors in which the readers can easily recognize the surrounding world and so can clearly perceive the mathematical laws hidden in their daily life. A message that after so many centuries has not yet lost its efficacy.

Contributors

This chapter contains short biographies of the authors of the papers collected in this volume and the editors of the volume. The biographies are ordered alphabetically by family names.

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He has been invited to many international conferences as plenary speaker, has participated to several international research programs in Mathematics Education and is still active in some of them. He has been supervisor of more than 100 graduate dissertations and of a dozen of PhD dissertations in Mathematics Education.

Evelyne Barbin is full professor of Epistemology and History of sciences at the University of Nantes (France), researcher in the Laboratory of Mathematics Jean Leray and member of the Institute on Research on Mathematical Education (IREM) of Nantes. Her researches concern history of mathematics in 17th and 19th centuries and relations between history and teaching of mathematics. She is a member of the HPM Group and Chair of HPM from 2008 to 2012. As President of the National Committee IREM «Epistemology and History of Mathematic », she edited many books, organized 20 National Colloquia, 8 Interdisciplinary Summer Universities and the first European Summer University «Epistemology and History of Mathematics in Mathematics Education» in 1993. Recent edited books: *Les ouvrages de mathématiques entre recherche, enseignement et culture*, Presses Universitaires de Limoges, 2013; *Les constructions mathématiques dans l'histoire. Avec des instruments et des gestes*, Paris, Ellipses, 2014.

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She has organized the celebrations of the Centenary of the journal *L'Enseignement Mathématique* and of ICMI and was one of the editors of the proceedings. She has developed the website on the history of the first hundred years of ICMI. She wrote the chapter "From mathematics and education to mathematics education" in the *Third International Handbook of Mathematics Education* and the chapter on the history of the international cooperation in mathematics education in the *Handbook on the history of mathematics education*. She is a coauthor of the ICME-13 Topical Survey *History of mathematics teaching and learning*.

In 2000-2004 she chaired HPM (the International Study Group on the relations between History and Pedagogy of Mathematics affiliated to ICMI).

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Livia Giacardi, born in S. Vittoria d'Alba on 22 November 1952, received her degree in mathematics from the University of Turin in 1974, where she is full professor of History of Mathematics. She is the author of over 200 essays, critical editions, books and CD-Roms on the history of mathematics, with a particular focus on the scientific traditions in Turin and Piedmont, the Italian School of algebraic geometry, and the history of mathematics education. She has held various positions in professional associations: Secretary of the Italian Society for the History of Mathematics (SISM, 2000-2008) and board member (ongoing); the Scientific Committee of the Italian Mathematical Union (2003-06, 2009-2015); board member of the Italian Commission for Mathematics Teaching (2003-2012); board member of the INDAM Committee for the History of Mathematics (2014-2015). She has organized national and international seminars and conferences, including "Homage to Corrado Segre (1863-1924)", Turin 2013, and ICHME4, Turin 2015, as well as several exhibitions. She is a member of the Scientific Committee for the *Edizione Nazionale delle Opere* of Roger Joseph Boscovich. In 2008 she was a member of the IPC for the International Symposium celebrating the First Century of the ICMI (1908-2008).

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She held a Regular Lecture on the historical development of practical geometry at ICME 12 in Seoul; she was co-author of the chapter on "History of teaching geometry" in the *Handbook on the history of mathematics education*, and the chapter "From mathematics and education to mathematics education" in the *Third international handbook of mathematics education*.

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Index

Finito di stampare nel mese di luglio 2017
con tecnologia *print on demand*
presso il Centro Stampa "*Nuova Cultura*"
p.le Aldo Moro, 5 - 00185 Roma
www.nuovacultura.it

Per ordini: ordini@nuovacultura.it

[Int_9788868128630_17x24bn_BM05]