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The New Methods in Arithmetic

Thorndike

THE NEW
METHODS IN ARITHMETIC

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By

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THE PREFACE

In the *Psychology of Arithmetic* the writer has presented the applications of recent dynamic psychology and experimental education to the teaching of arithmetic, in form suitable to students who approach the topics as part of a general systematic study of education in elementary schools.

The present volume deals with somewhat the same material, but from the point of view of the working teacher or student in a normal school seeking direct help in understanding the newer methods and using them under ordinary conditions of classroom instruction. No knowledge of psychology is assumed as a requirement for profitable study of this book. Discussions of the general psychological basis of the new methods and of the evidence in their favor are here omitted or much simplified. The treatment is constructive throughout. The practical consequences are treated more specifically and with abundant detailed illustration and application.

In order to aid the teacher still further in putting the new principles of teaching into active operation, each chapter is accompanied by exercises, which are even more detailed and concrete in nature than the text.

The choice of textbook material for illustrations of current practice in the text and for various uses in the exercises requires a word of explanation. As a matter of scientific care and of convenience to the student almost all of this material is taken from the same textbook. Scientifically this is almost necessary; for a procedure that is correct in one total teaching plan might be weak or even wrong in a different total teaching plan. Each detail of method ought to be judged with reference to its setting. All the details presented here are parts of one same teaching plan or textbook—all belong in the same setting.

Practically it seems unwise to require constant consultation of more than one textbook series. After the facts are clearly in mind as they work out in one textbook or total teaching plan, the student may study them in others so far as he has time and facilities.

The textbooks chosen are the *Thorndike Arithmetics*, with which the author is best acquainted and which were written with the definite purpose of applying "the principles discovered by the psychology of learning, by experimental education, and by the observation of successful school practice to the teaching of arithmetic."

One other feature of this volume needs explanation. It may seem that the older methods are not given a fair defense. This is, in a sense, true. But it must be remembered that the older methods are those by which the readers of this book have been taught, which they understand and are accustomed to, and which their unconscious tendencies will strongly favor. A certain advocacy of the newer methods by the author is thus necessary to achieve a real impartiality. In fact, even very vigorous advocacy will hardly suffice to balance the prepossession in favor of the methods by which we learned and which have become a part of us. If the newer methods as presented in this volume win assent and confidence, it will be on merit.

TEACHERS COLLEGE, COLUMBIA UNIVERSITY

THE NEW METHODS IN ARITHMETIC

CHAPTER I

REALITY

The older methods taught arithmetic for arithmetic's sake, regardless of the needs of life. The newer methods emphasize the processes which life will require and the problems which life will offer.

INDISCRIMINATE VERSUS USEFUL COMPUTATION

The old idea was that arithmetic should teach the children to add, subtract, multiply, and divide any numbers. Pupils subtracted ninths from fifteenths and multiplied $\frac{5}{54}$ by $\frac{9}{50}$ in school, though they never would be required to do so all their lives thereafter.

The work shown below illustrates the sort of computation which textbooks and teachers used to assign, but which the newer methods seek to replace by training which can be directly beneficial in the real world:

Reduce to integral or mixed numbers:

$$\frac{35}{15} \qquad \frac{48}{21} \qquad \frac{198}{14} \qquad \frac{2134}{67} \qquad \frac{413}{413} \qquad \frac{6125}{3175}$$

Simplify:

$$\frac{3}{4} \text{ of } \frac{8}{9} \text{ of } \frac{3}{5} \text{ of } \frac{15}{22} \qquad\qquad \frac{7}{8} \text{ of } \frac{15}{18} \text{ of } \frac{4}{5} \text{ of } \frac{1}{36}$$

Reduce to lowest terms:

$$\frac{357}{527} \qquad \frac{264}{312} \qquad \frac{492}{779} \qquad \frac{418}{874} \qquad \frac{854}{1789} \qquad \frac{77}{847} \qquad \frac{18}{243}$$

Square:

$$\frac{2}{3} \qquad \frac{4}{5} \qquad \frac{5}{7} \qquad \frac{6}{9} \qquad \frac{10}{11} \qquad \frac{12}{13} \qquad \frac{15}{16} \qquad \frac{19}{20} \qquad \frac{17}{18} \qquad \frac{41}{53}$$

Subtract:

$$\begin{array}{ccccc} 6\frac{2}{7} & 8\frac{5}{11} & 8\frac{4}{13} & 5\frac{1}{4} & 7\frac{1}{8} \\ \underline{3\frac{1}{14}} & \underline{5\frac{1}{7}} & \underline{3\frac{7}{13}} & \underline{2\frac{11}{14}} & \underline{2\frac{1}{7}} \end{array}$$

Multiply:

$$60\times\frac{11}{28} \qquad 63\times\frac{2}{27} \qquad 65\times\frac{3}{13} \qquad 432\tfrac{2}{7}\times 42\tfrac{1}{12}$$

Much more than nine-tenths of the arithmetical calculations of the real world are with numbers under a hundred, so the newer methods emphasize facility and absolute accuracy with small numbers. Such work as

Add	Subtract	Multiply	Divide
46793	68750	7295	$217\vert\overline{436905}$
128516	$\underline{31925}$	$\underline{6152}$	
91380			
20769			
8465			
$\underline{73600}$			

would be given only a few times to show that it could be done by the same methods already learned for smaller numbers. The main practice with addition and subtraction of fractions is restricted to such as will occur in connection with fractions of a yard, pound, dozen, inch, and the like in the life of the household, store, shop, and trade. The pupil may learn to add fifths to fifths, because such additions may be needed with stop-watch measurements, but he will not be taught to add fifths to thirds, because not one pupil in ten thousand will ever be required to do so.

INTEREST FOR UNUSUAL TIMES

The difference is well illustrated by the case of interest. It has been customary to teach pupils to compute interest for any length of time. More effort in fact was spent with such times as 2 yr. 6 mo. 9 da. than with times of 30, 45, 60, and 90 days, 6 months, and 1 year, all put together. Practically all the interest that the pupil will ever have to compute, however, will be for these usual periods. Mortgages require annual or semi-annual interest payments; almost all bank loans are made for fixed periods and then renewed.* Even the informal personal loan without security is usually made for a fixed period and with stipulated dates of interest payment. Those who do have to compute interest for unusual times do so usually by means of interest tables.

So the newer methods devote attention to the arithmetic that a person actually borrowing or lending money will really need to know, and to the general significance of interest for thrift and business credit operations.

The older methods gave indiscriminate training in finding the rate of interest, or the time, or the principal, the other three being given. So we had:

What is the rate percent when the interest:
- *a.* of $240 for 1 year 9 months is $29.40?
- *b.* of $475 for 3 years 4 months is $95.00?

In what time will:
- *a.* $400 produce 62.06\frac{2}{3}$ at 7 percent?
- *b.* $998 produce $185.145 at 5 percent?

What sum of money will produce:
- *a.* $33.75 interest in 2 years 3 months at 6 percent?
- *b.* $50.32 interest in 5 months 27 days at 8 percent?

These problems are obviously of very trifling or no importance and likely to mislead. In the real situations the interest rate would appear on the note or mortgage and the time would be fixed by circumstances; and, in planning to secure a certain yield, the plan would count on the interest being paid at regular

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*With the exception, of course, of loans to stock brokers, though these may well be regarded as loans made for one day and renewed. The interest on them would always be computed by the aid of tables.

intervals and reinvested; nobody would in any case make a plan about how long it would take to obtain 62.06\frac{2}{3}$, or how much he must invest to receive $50.32 in 5 months and 27 days!

GENUINE PROBLEMS

The older methods permitted the teacher to set any problem that was a problem, regardless of whether it would ever occur as a real problem in a real world. The following are samples of problems accepted as satisfactory by textbooks and teachers twenty years ago:

> Alice has $\frac{3}{8}$ of a dollar, Bertha $\frac{11}{16}$, Mary $\frac{3}{25}$, and Nan $\frac{3}{4}$. How much have they together?
>
> Mollie's mother gave her 40 apples to divide among her playmates. She gave each one $2\frac{2}{9}$ apples apiece. How many playmates had she?
>
> There are 9 nuts in a pint. How many pints in a pile of 6,789,582 nuts?
>
> Mrs. Smith is $\frac{3}{4}$ as old as Mr. Smith, who is 48 years old. Their daughter Alice is $\frac{4}{9}$ as old as her mother. How old is Alice?
>
> Suppose a pie to be exactly round and $10\frac{1}{2}$ miles in diameter. If it were cut into 6 equal pieces, how long would the curved edge of each piece be?

Such problems as the above could occur in real life only in an insane asylum.

> There are ten columns of spelling words in Susie's lesson and 32 words in a column. How many words are in her lesson?

This was perhaps not unreal, since a school that would give such problems might also assign 320 words for a single spelling lesson!

Consider this clever way of finding the thickness of a board:

> A nail 5 inches long is driven through a board so that it projects 2.419 inches on one side and 1.706 on the other. How thick is the board?

Consider the thoughtfulness of this horse in eating exactly 16 ounces of hay:

> Just after a ton of hay was weighed in market a horse ate 1 lb. of it. What was the ratio of what he ate to what was left?

Consider the perfectly fantastic and futile nature of this problem for a problem's sake:

> A man 6 feet high weighs 175 pounds. How tall is his wife, who weighs 125 pounds and is of similar build?

The newer methods set a higher standard in the selection and construction of problems, requiring not only that they give the pupil an opportunity to think and to apply arithmetical knowledge, but also that they teach him to think and to apply arithmetic to situations such as life may offer, in useful and reasonable ways, and so to esteem arithmetic not only as a good game for the mind, but also as a substantial helper in life's work.

In particular, the newer methods reject problems which would not occur in reality because the answer has to be known to frame the problem. For example: "I spent three-eighths of my money for a gun and one-half of it for a tent. I had $12 left. How much had I at first?" Or, "Mr. Jones sold a cottage for $1500, which was 25 percent more than he paid for it. How much did he pay for it?" To give the practice required, the newer methods would seek some genuine situation. For example, they might replace the second problem by: "A dealer sells automobiles at 25 percent advance over what he pays. What does he pay for an auto that he sells for $1500?"

The newer methods also avoid problems which, though real, would not be solved in the way the problem requires. For example: "A farmer bought 160 peach trees, which he set out in rows 24 in a row. How many rows were there and how many trees did he have left?" would usually be solved by simply counting the 6 rows and the 16 trees. Moreover, the farmer would probably set out the 16 trees as an incomplete row. In fact he would probably not have bought 160 trees, but 150. The newer methods would amend the problem so as to remove these elements of improbability. For example: "A farmer has 150 peach trees. He plans to set them out 24 in a row. He figures out how many full rows can be made. Then he picks out the more sickly looking trees so as not to use them in these full rows. How many sickly trees does he pick out?"

"At 3 cents apiece what will be the cost of 4 dozen oranges?" calls for 4×12×3, but the price per dozen would probably be less than 12 times 3. The newer methods would replace this by a genuine problem or amend it to: "A boy is paid 3 cents per box for picking berries. How much is he paid for picking 4 dozen boxes?"

ARITHMETIC FOR ARITHMETIC'S SAKE AND ARITHMETIC FOR LIFE

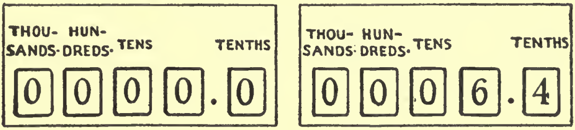
In general, everywhere, the newer methods try to teach, not merely arithmetic, but arithmetic as a help for life. They seek to find just where and how each feature of arithmetic should serve boys and girls while they are in school and after they leave school, and to teach it in such a way that it will serve them. They ascertain the facts of reality with which each arithmetical fact or principle needs to be connected and help the pupil to make the connection.

Thus, since the multiplications 2 times 2, 3 times 2, ... to 10 times 2 are needed in life, and the divisions 4÷2, 6÷2 . . . . are needed in life early and often in connection with quarts and pints and the cost of postage stamps, these tables are taught in those connections. Similarly the tables 2 times 3, 3 times 3, ... 10 times 3 and the corresponding divisions are taught in connection with feet and yards, while quarts and gallons go with the tables of "times 4" and "how many 4's in." This procedure not only makes desirable connections with reality, but also makes the multiplication and division facts themselves more intelligible. The time spent in learning that there are two pints in a quart, three feet in a yard, four quarts in a gallon, seven days in a week, that a nickel equals five cents and that a dime equals ten cents, is more than saved by the comprehensibility and interest which thereby accrue to drills on the multiplication and division facts.

The older custom of teaching the facts about feet, yards, pints, quarts, gallons, etc., off by themselves in a chapter on "Denominate Numbers," with the multiplication and division tables in other chapters by themselves, wasted a chance to make arithmetic serve life and made both topics harder to learn.

Knowledge of decimal fractions is connected by the newer methods with the records in tenths of a mile on bicycles and automobiles, with railroad-distance tables in hundredths of a mile, with rainfall records in thousandths of an inch, and with standard butter-fat records in ten-thousandths of a pound, as shown below and on pages 8 and 9 following.

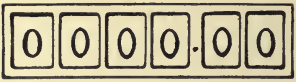
Measuring Distance with a Cyclometer



The left-hand picture shows Fred's cyclometer when he bought it. The right-hand picture shows Fred's cyclometer after he had put it on his bicycle and ridden 6.4 mi. (or $6\frac{4}{10}$ mi.).

1. How will the cyclometer look after he rides 2.3 mi. ($2\frac{3}{10}$ miles) more?
2. How does Fred tell how far he goes in one day or one trip?
3. If the cyclometer reads 0071.2 at 9 A.M. and 0084.9 at 11 A.M., how many miles has the bicycle covered in the two hours?

Hundredths



This is a special cyclometer that shows thousands, hundreds, tens, ones, tenths, and hundredths of a mile. Alice's father had one which he put on her bicycle.

Find the length of each of these trips from the amounts the cyclometer showed at the start and at the finish of each trip.

	At the Start	At the Finish
Trip 1.	0000.00	0011.46
Trip 2.	0011.46	0016.89
Trip 3.	0016.89	0050.03
Trip 4.	0050.03	0067.20
Trip 5.	0067.20	0078.50

A Railroad Table of Times and Distances

Miles			Hr.	Min.
0	Lv.	New York	**5**	**34**
7.10	"	High Bridge	**5**	**52**
8.06	"	Morris Heights	**5**	**55**
8.73	"	University Heights	**5**	**57**
9.64	"	Marble Hill	**6**	**00**
12.24	"	Riverdale	**6**	**08**
13.68	"	Ludlow	**6**	**11**
14.49	"	Yonkers	**6**	**18**
15.58	"	Glenwood	**6**	**21**
17.19	"	Greystone	**6**	**24**
18.75	"	Hastings	**6**	**30**
20.00	"	Dobbs Ferry	**6**	**35**
21.03	"	Ardsley	**6**	**37**
21.98	"	Irvington	**6**	**42**
24.52	"	Tarrytown	**6**	**48**

1. Read this time table.

2. Which station is about 22 miles from New York?

3. Which station is almost exactly 14½ miles from New York?

4. Which station is exactly 18¾ miles from New York?

5. Which station is almost exactly twice as far from New York as Riverdale is?

Measuring Rainfall

Rainfall per Week
(cu. in. per sq. in. of area)

June	1– 7	1.056
	8–14	1.103
	15–21	1.040
	22–28	.960
	29–July 5	.915
July	6–12	.782
	13–19	.790
	20–26	.670
	27–Aug. 2	.503
Aug.	3– 9	.512
	10–16	.240
	17–23	.215
	24–30	.811

1. In which weeks was the rainfall 1 or more?

2. Which week of August had the largest rainfall for that month?

3. Which was the dryest week of the summer? (Dryest means with the least rainfall.)

4. Which week was the next to the dryest?

5. In which weeks was the rainfall between .800 and 1.000?

6. Look down the table and estimate whether the average rainfall for one week was about .5, or about .6, or about .7, or about .8, or about .9.

Dairy Records

Record of Star Elsie

	Pounds of Milk	Butter-Fat per Pound of Milk
Jan.	**1742**	**.0461**
Feb.	**1690**	**.0485**
Mar.	**1574**	**.0504**
Apr.	**1226**	**.0490**
May	**1202**	**.0466**
June	**1251**	**.0481**

Read this record of the milk given by the cow Star Elsie. The first column tells the number of pounds of milk given by Star Elsie each month. The second column tells what fraction of a pound of butter-fat each pound of milk contained.

1. Read the first line, saying, "In January this cow gave 1742 pounds of milk. There were 461 ten thousandths of a pound of butter-fat per pound of milk." Read the other lines in the same way.

2. How many pounds of butter-fat did the cow produce in Jan.? **3.** In Feb.? **4.** In Mar.? **5.** In Apr.? **6.** In May? **7.** In June?

It will be instructive for the reader to compare these with treatments of decimal numbers of fifteen or twenty years ago. These latter will be found to use decimal numbers as they would never be used in life, or as they would be used only by a few scientific and statistical experts.

Consider two cases: (*a*) the teaching of Roman numerals, and (*b*) the teaching of the multiplication of one common fraction or mixed number by another. The older methods were content to give a general account of Roman numerals with miscellaneous exercises applying them. These exercises were at times fantastic, such as: "How much are CXVI and XIX?" "Subtract CCXIV from MCII." "Eliza found XIV eggs one week and XVI eggs the next week. How many did she find in all?" The problem for the teacher is, according to the newer methods, to teach such Roman numerals as life requires in such connections as life requires. This the reader will probably solve somewhat as follows:

> The meanings of I to XII should be taught, for those numbers are still used fairly often on clocks and watches; XIII to XXX may be taught, for these are used somewhat in numbering chapters; I am in doubt about XXXI to C, since very few books to which the ordinary person will ever wish to refer by chapter number will have over thirty chapters, and if they have, he can easily learn the system when he needs it from the book itself. It is true that diagrams and statistical tables are sometimes numbered with Roman numerals and run to these higher numbers. Few elementary-school graduates, however, will ever need to do more than copy such headings. Are dates sometimes printed in Roman numerals? I think so, but such cannot be very frequent, for I cannot think of ever having seen one so printed. I don't think this is a frequent enough use to justify spending time on Roman numerals in the elementary school. There is one, however, that I nearly forgot. M is used for 1000 in certain trades, as the lumber trades. On the whole, I would teach I to XXX in the lower grades, C for 100, D for 500, and M for 1000 in the upper grades. The others I would leave for the pupil to learn in

life when and where he needed them. All that the elementary-school pupil needs to know is how to *understand* these. He will never need to add, subtract, multiply, or divide with them. I would teach I to XXX so as to show the system, not by mere rote. I would connect these with their most important use, the clock face. I would teach that M means 1000 in connection with board measure.

My readers may differ somewhat about the minor uses of multiplications like $2\frac{3}{4} \times 1\frac{1}{2}$, but there will be agreement that the uses deserving most attention are finding the areas of such rectangles as rugs, and the capacity of boxes, whose dimensions come in uneven multiples of a yard and foot. There is thus good reason for teaching the method of computing volume from length, width, and height soon after the multiplication of fractions is taught, and not months or even years later, as used to be done.

It is instructive to consider some measures of the frequency of certain sorts of work in good modern practice in the teaching of arithmetic. For example: How often will a problem be given that is just "made up" to fit a certain arithmetical process? What percent of the common fractions used will be other than halves, thirds, fourths, fifths, eighths, twelfths, or sixteenths? What percent of the multipliers will be of over three figures? It will be found that such unreal problems occur almost never—only when there is some very special gain from their use—that such fractions do not make up one-fiftieth of the total occurrences of common fractions, and that four- and five-figure multipliers will not have a frequency of 1 percent.

The good textbook and the good teacher scrutinize every task they assign to make sure that it fits the pupil for life. They seek to find, for every arithmetical principle and fact, the real affairs to which it applies and with which it should be connected in the pupil's mind.

EXERCISES

1. Replace each of these problems by one which involves the same arithmetical principles, but is such as might really occur:

a. A workman saves $3\frac{5}{7}$ dollars a week. How much will he save in a year?

b. Forty apples were divided among a lot of boys, giving each boy $\frac{4}{5}$ of an apple. How many boys were there?

c. In a schoolhouse there are nine rooms; in each room there are 48 pupils; if each pupil has 9 cents, how much have they in all?

d. Find the perimeter of an envelope which is 5 inches by $3\frac{1}{4}$ inches.

e. $\frac{7}{8}$ of the total product of writing paper in 1900 was 100,000 tons. What was the total product?

2. What are some real situations that require the use of "in the proportions 2, 3, 5," "in the proportions 1 and 4," "2 parts of *a* to 4 parts of *b* to 5 parts of *c*," and the like?

3. How is learning to understand the calendar for a month used in Book I, page 36?* Think of other uses that might be made of it.

4. How is the calendar for a year used in Book I, page 103?

5. Note what features of cooking and domestic science are used in connection with arithmetic and how they are used in Book III, pages 49, 74–77, 130, 184–186, 189–194, 197, 258, and 259.

6. Pair these arithmetical processes and real facts so as to make the best use of all. State your pairing in the form:

a 2 *c* 7
b 1 *d* etc.

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*Reference in the Exercises of this volume are always to the *Thorndike Arithmetics*, unless otherwise stated. I, II, and III refer to Books One, Two, and Three, respectively.

a. Addition of integers
b. Subtraction of integers
c. Multiplication of integers
d. Multiplication of United States money
e. Division of integers
f. Division of United States money
g. Meanings of numbers 40–60
h. Meanings of numbers 60–100
i. $\frac{1}{2}$ of 80 and $\frac{1}{4}$ of 80
j. $\frac{1}{2}$ of 60 and $\frac{1}{4}$ of 60
k. $\frac{1}{2}$ of 16 and $\frac{1}{4}$ of 16
l. Adding halves and thirds
m. Adding halves and fourths
n. Adding fifths
o. Discount

1. Children's ages
2. Children's heights
3. Children's weights
4. Bean-bag scores
5. Planning for a party
6. Cost of a present
7. Cost of suits for a ball team
8. Cost of second-hand books
9. Cost of candy
10. Sales of cream
11. Sales of cloth
12. Ounces and pounds
13. The clock-face
14. Athletic records
15. Rate of travel

CHAPTER II

INTEREST

THE INTERESTS IN MENTAL ACTIVITY AND ACHIEVEMENT

Arithmetic makes a very strong appeal to two potent interests — the interest in mental activity and the interest in achievement. Many children like arithmetic in the same way and for much the same reasons that they like puzzles, riddles, checkers, chess, and other intellectual games. Almost all children like to have their tasks definite so that they can know what they have to do and when it is done, and enjoy the sense of action, achievement, and mastery.

Unless it is very badly taught, arithmetic is one of the best intellectual games that the elementary school has to offer; and its tasks are definite so that the pupil can know rather clearly what he has to do, how much of it he has done, and how well he has done it. The newer methods increase the strength of these two appeals, making arithmetic a more attractive game for young intellects and giving the interests in achievement and mastery greater stimulus and fuller play.

First, they free the work of arithmetic from irrelevant, useless difficulties and strains.

Consider the language used by the textbook and teacher in explaining procedures, stating problems, and the like. In the first fifty pages of eight standard textbooks of about 1900 there were found such words as: *absentees*, *account*, *Adele*, *admitted*, *Agnes*, *agreed*, *Albany*, *Allen*, *allowed*, *alternate*, *Andrew*, *Arkansas*, *arrived*, *assembly*, *baking powder*, *balance*, *barley*, *beggar*, *Bertie*, *Bessie*, *bin*, *Boston*, *bouquet*, *bronze*, *buckwheat*, *Byron*.

Over half of the pupils in the last half of Grade 2 or the first months of Grade 3 simply could not read these words.

They were thwarted in their arithmetical thinking as truly, though not as much, as if the problem had been stated in Greek. The game of thought was spoiled by the intrusion of irrelevant linguistic difficulties.

In these first pages for beginners there were used over fifty different proper names, including *Byron*, *Charlotte*, *Denver*, *Graham*, *Horace Mann*, *Lula*, *Morton*, and *Oakland*. What, the newer methods ask, has ability to read these rare personal appellations to do with learning arithmetic in Grade 2 or 3? Why risk losing interest in the problem and its solution when Tom, Dick, Mary, a boy, and a girl are just as good arithmetically as Horace Mann?

Consider these problems in each of which arithmetical difficulty is almost *nil*, but where the language is, for a little child, a veritable puzzle:

1. What sum should you obtain by putting together 8 cents, 4 cents, 7 cents, and 6 cents? Did you find this result by adding or multiplying?
2. How many times must you empty a peck measure to fill a basket holding 64 quarts of beans?
3. If a boy commits to memory 3 pages of history in one day, in how many days will he commit to memory 9 pages?
4. If Dick had 4 rabbits, how many times could he give away 2 rabbits to his companions?
5. If a croquet player drove a ball through 2 arches at each stroke, through how many arches will he drive it by 3 strokes?
6. If mamma cut the pie into 4 pieces and gave each person a piece, how many persons did she have for dinner if she used 4 whole pies for desert?

Imagine a thoughtless teacher sarcastically asking a child, "Can't you do 3 and 2?" "Don't you know how many two 4's are?" after the child's failure with these.

The newer methods insist that the textbook and the teacher should preserve the pupil's healthful interest in arithmetical thinking by preserving it from being wasted and balked by useless difficulties of vocabulary or construction.

Consider the matter of copying the numbers which are to be added or subtracted or multiplied. The eye strain involved in copying numbers is, minute for minute, many times greater than the strain from reading. If a pupil has much of it to do, the monotonous task tends to make him lapse into error occasionally, even though he is faithfully doing his best. Then a task that was right arithmetically is scored wrong, and he is disheartened. The time required to copy the numbers is for many pupils, with much of the work of the elementary school, more than the time required to do the arithmetical work itself. The purely clerical work of copying is destructive of the joy in thinking.

Consequently the newer methods recommend that, so far as possible, the pupils do only so much copying of numbers as is desirable to train them in ease and accuracy of copying, and proper formation and arrangement of numbers. More than that is likely to involve waste. To put the matter very emphatically, *a pupil should not be made to copy all the numbers that he computes with, any more than he should be made to copy all the stories that he is to read.* Just as his chief task with words is to read them, so his chief task with numbers is to compute with them.

In the textbook much of the work for computation should be arranged so that the pupil may lay a sheet of paper below a row (or above, in the case of division) or beside a column of tasks and write only the answers. He then folds his paper and does the same for a second row or column. Subject to the need for training in copying and arrangement noted above, almost all the written work in addition and subtraction, and in multiplication and division by a one-figure number, may be so arranged. In the case of multiplication by a two-figure multiplier, the partial products and the answer may be written.

Much of the work that has been written on the blackboard to be copied may better be given out on mimeographed or printed sheets, the pupil doing his work on the sheet itself. Not only is much time saved and the pupils' interest much increased; the efficiency of the teacher's supervision is much increased also, since each pupil's paper has the same work in the same place. A few samples of such sheets are shown on the following pages (18–21), taken from the author's *Exercises in Arithmetic*. Since the publication of these *Exercises* in 1909, numerous sets of practice sheets of the same general pattern have been published. The admitted usefulness of such, if the tasks are well chosen and well graded, is solid evidence of the profit that comes from reducing the amount of copying the numbers in the school.

When it is necessary to put matter on the blackboard for pupils to copy, it should obviously be clearly written and well spaced. Children should also be taught to lighten their own labors by making legible figures and spacing them properly. The commonest error is to write them too closely together and to make fractions too small.

Besides freeing arithmetical work from useless difficulties and strains, it is possible to feed the interest in achievement and mastery by helping the pupil to define his goal, know his successes and faults, and measure his progress. Instead of being told vaguely to learn a certain topic, the pupil is instructed to "Do the work of this page. Do it again, keeping a record of how many minutes you spent. Practice until you can get all the answers right in 12 minutes." Instead of being taught merely to do the computation, he is taught a means of checking his work so that he can be sure of 100 percent accuracy if he desires. The time spent in such checkings is in no degree wasted. Multiplying 427 by 358 to check the product of 358 multiplied by 427 is as good practice in multiplying as any. Multiplying 58 by 24 and adding 17 to the product is as good

Subtract. Check your results by adding.

A. 1.	2.	3.	4.	5.
812	592	933	642	759
378	429	181	476	587

B. 8.	9.	10.	11.	12.
765	546	495	327	283
365	238	195	87	126

C.

$5.25	$86.00	$1.50	$37.62	$3.75
1.75	56.32	.64	19.74	1.25

Find the products. Check your results by multiplying.

1.	Check here.	2.	Check here.
232	24	312	26
24	232	26	312

4.		5.	
425	21	246	35
21	425	35	246

Write the missing numbers:

A.

$\frac{1}{2}$ of 6 =

$\frac{1}{2}$ of 10 =

$\frac{1}{2}$ of 8 =

$\frac{1}{3}$ of 12 =

$\frac{1}{3}$ of 15 =

$\frac{1}{4}$ of 8 =

$\frac{1}{4}$ of 40 =

$\frac{1}{5}$ of 40 =

$\frac{1}{6}$ of 18 =

$\frac{1}{8}$ of 56 =

B.

$\frac{1}{3}$ of 27 =

$\frac{1}{2}$ of 18 =

$\frac{1}{3}$ of 18 =

$\frac{1}{6}$ of 12 =

$\frac{1}{2}$ of 16 =

$\frac{1}{2}$ of 14 =

$\frac{1}{9}$ of 18 =

$\frac{1}{4}$ of 36 =

$\frac{1}{4}$ of 32 =

$\frac{1}{7}$ of 35 =

C.

$\frac{1}{5}$ of 35 =

$\frac{1}{5}$ of 30 =

$\frac{1}{6}$ of 30 =

$\frac{1}{3}$ of 21 =

$\frac{1}{8}$ of 32 =

$\frac{1}{8}$ of 16 =

$\frac{1}{8}$ of 48 =

$\frac{1}{6}$ of 48 =

$\frac{1}{6}$ of 60 =

$\frac{1}{4}$ of 28 =

F.

$\frac{2}{3}$ of 9 =

$\frac{3}{4}$ of 16 =

$\frac{2}{5}$ of 20 =

$\frac{4}{5}$ of 20

$\frac{3}{5}$ of 20

$\frac{2}{3}$ of 15

$\frac{3}{4}$ of 20 =

$\frac{3}{4}$ of 8 =

$\frac{3}{4}$ of 12 =

Write the whole numbers or mixed numbers which these fractions equal.

A

$\frac{5}{4}$ =

$\frac{7}{4}$ =

$\frac{8}{4}$ =

$\frac{11}{4}$ =

$\frac{9}{4}$ =

B.

$\frac{4}{3}$ =

$\frac{5}{3}$ =

$\frac{6}{3}$ =

$\frac{3}{3}$ =

$\frac{7}{3}$ =

C.

$\frac{15}{8}$ =

$\frac{16}{8}$ =

$\frac{8}{8}$ =

$\frac{11}{6}$ =

$\frac{12}{6}$ =

D.

$\frac{10}{8} = 1\frac{}{8}$ or $1\frac{}{4}$

$\frac{14}{8} = 1\frac{}{8}$ or $1\frac{}{4}$

$\frac{12}{8} = 1\frac{}{8}$ or $1\frac{}{4}$ or $1\frac{}{2}$

$\frac{10}{6} = 1\frac{}{6}$ or $1\frac{}{3}$

$\frac{9}{6} = 1\frac{}{3}$ or $1\frac{}{2}$

A. Write the missing figures. An × means that there is not any single figure that is right.

$\frac{1}{2} =$	$\frac{1}{2}$	$\frac{\times}{3}$	$\frac{}{4}$	$\frac{\times}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$
$\frac{1}{3}$	$\frac{\times}{2}$	$\frac{1}{3}$	$\frac{\times}{4}$	$\frac{\times}{5}$	$\frac{}{6}$	$\frac{\times}{8}$	$\frac{\times}{10}$	$\frac{}{12}$
$\frac{1}{4}$	$\frac{\times}{2}$	$\frac{\times}{3}$	$\frac{1}{4}$	$\frac{\times}{5}$	$\frac{\times}{6}$	$\frac{}{8}$	$\frac{\times}{10}$	$\frac{}{12}$
$\frac{1}{5}$	$\frac{\times}{2}$	$\frac{\times}{3}$	$\frac{\times}{4}$	$\frac{1}{5}$	$\frac{\times}{6}$	$\frac{\times}{8}$	$\frac{}{10}$	$\frac{\times}{12}$
$\frac{1}{6}$	$\frac{\times}{2}$	$\frac{\times}{3}$	$\frac{\times}{4}$	$\frac{\times}{5}$	$\frac{1}{6}$	$\frac{\times}{8}$	$\frac{\times}{10}$	$\frac{}{12}$
$\frac{1}{8}$	$\frac{\times}{2}$	$\frac{\times}{3}$	$\frac{\times}{4}$	$\frac{\times}{5}$	$\frac{\times}{6}$	$\frac{1}{8}$	$\frac{\times}{10}$	$\frac{\times}{12}$

B. Write the missing figures when you can. Write × when there is not any single figure that is right.

$\frac{2}{3} =$	$\frac{}{2}$	$\frac{}{3}$	$\frac{}{4}$	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$
$\frac{3}{4} =$	$\frac{}{2}$	$\frac{}{3}$	$\frac{}{4}$	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$
$\frac{2}{5} =$	$\frac{}{2}$	$\frac{}{3}$	$\frac{}{4}$	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$
$\frac{5}{6} =$	$\frac{}{2}$	$\frac{}{3}$	$\frac{}{4}$	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$
$\frac{3}{8} =$	$\frac{}{2}$	$\frac{}{3}$	$\frac{}{4}$	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{10}$	$\frac{}{12}$

Addition.

A. 37.846
1.02
8.109
70.61

31.1
20.988
71.37
52.63

9.246
18.09
44.17
45.763

B. 21.405
48.19
4.01
77.024

52.417
19.8
90.8
41.753

48.1
37.87
41.907
15.963

C. 1.09
8.64
1.6143
5.7086

3.275
9.01
5.98
8.1093

4.0125
1.5907
4.10
8.671

Subtraction.

A. 10.
8.481

47.18
36.297

9.
8.809

B. 32.
13.409

2.
1.5017

40.36
6.675

C. .92412
.62

.2547
.13225

50.
44.636

practice if used as a check on $58|\overline{1409}$ as if done independently. Instead of an ambiguous mark on some hastily constructed examination, a precise statement of how accurately and rapidly he can do specified tasks and of how hard tasks he can do with substantially 100 percent efficiency is made to the pupil. He can compare his present achievement with his achievement a week, or month, or year ago. The standard tests of Courtis, Woody, and others are providing better means of doing this year by year.

Of special importance for the teacher are the tests of the "ladder" type graded in difficulty from easy to hard, or from simpler to more elaborate operations. If four or five tasks at each step are given, as shown in the samples on pages 23 and 24, these show a pupil clearly just where his weak spots are.

The addition ladder has six steps. The first and second require only understanding of the elements of the method and surety with the combinations with sums to nine. The third requires ability to handle zeros in column addition. Step 4 requires knowledge of the combinations to 9+9. Step 5 requires knowledge of carrying, but with no cases where 0 is to be written in units place in the sum. Step 6 has such cases and also has zeros and empty spaces in the columns.

The multiplication ladder, for use in Grade 5, leaves out the very easiest abilities. In Steps 10 and 11 it reaches very difficult tasks in multiplication with decimals. Its main purpose is to show in general how hard tasks a pupil can master, but it will detect to some extent special weaknesses, as in placing the decimal point (Step 3), the "zero difficulties" (Steps 4 and 5), multiplying with fractions by easy cancelling (Step 6), multiplying with fractions requiring selection of methods and the use of aliquot parts or elaborate work (Steps 7, 8, and 9).

An Addition Ladder

Begin with Step 1 and see how far up you can climb without making a mistake.

Step 6.			25	17	16
	14	48	7	6	10
	9	19	19	30	9
	20	15	6	18	17
	27	34	13	15	8
	—	—	—	—	—

Step 5.				16	27
	16			28	19
	27	38	19	17	15
	19	49	37	26	28
	49	65	23	18	24
	—	—	—	—	—

Step 4.		8	6	8	6
	7	7	9	3	9
	5	9	6	7	8
	8	6	8	7	9
	7	6	9	5	4
	—	—	—	—	—

Step 3.	10	40	30	30	30
	21	30	10	10	12
	20	25	20	20	27
	14	12	20	13	30
	—	—	—	—	—

Step 2.	34	43	22	12	5
	2	5	31	43	62
	51	41	6	23	21
	—	—	—	—	—

Step 1.	21	21	12	54	32
	32	51	25	12	14
	15	24	21	33	42
	—	—	—	—	—

A Multiplication Ladder

Here is a multiplication ladder. Begin at the bottom, climb to the top. Find the products. Express common fractions or mixed numbers in your results in lowest terms.

Step 11. *a.* $.65 \times 104.7$ mi. *b.* $.625 \times \$10.50$
c. $.0325 \times \$103.25$ *d.* $3\frac{5}{8} \times 4.6$ *e.* $.0426 \times 10904$

Step 10. *a.* $90.04 \times \$925.00$ *b.* $.035 \times \$103.50$
c. $.75 \times \$1.20$ *d.* $.15 \times 39.37$ *e.* $.06 \times \$5$

Step 9. *a.* $12 \times \$\frac{5}{6}$ *b.* $24 \times 16\frac{2}{3}$¢ *c.* $36 \times 12\frac{1}{2}$¢
d. $8 \times 87\frac{1}{2}$¢ *e.* $\frac{3}{4} \times 1\frac{2}{3}$

Step 8. *a.* $9 \times 1\frac{1}{8}$ *b.* $5\frac{1}{2} \times 3\frac{1}{2}$ *c.* $25\frac{1}{2} \times \$120$
d. $16\frac{3}{4} \times \$500$ *e.* $7\frac{1}{2} \times 11\frac{3}{4}$

Step 7. *a.* $3\frac{1}{2} \times \$1.50$ *b.* $7\frac{1}{4} \times \$1.25$ *c.* $5\frac{3}{8} \times \$1.00$
d. $4\frac{3}{4} \times 144$ *e.* $2\frac{1}{3} \times \$1.00$

Step 6. *a.* $\frac{3}{4} \times 10$ *b.* $\frac{2}{3} \times 8$ *c.* $\frac{7}{8} \times 5$ *d.* $15 \times \frac{3}{5}$
e. $\frac{3}{4} \times \frac{3}{4}$

Step 5. *a.* $\begin{array}{r}3.07\\ \underline{60}\end{array}$ *b.* $\begin{array}{r}57.5\\ \underline{40}\end{array}$ *c.* $\begin{array}{r}6.14\\ \underline{5.03}\end{array}$ *d.* $\begin{array}{r}530\\ \underline{4.6}\end{array}$ *e.* $\begin{array}{r}30.9\\ \underline{40.7}\end{array}$

Step 4. *a.* $\begin{array}{r}605\\ \underline{20}\end{array}$ *b.* $\begin{array}{r}225\\ \underline{20}\end{array}$ *c.* $\begin{array}{r}214\\ \underline{102}\end{array}$ *d.* $\begin{array}{r}850\\ \underline{27}\end{array}$ *e.* $\begin{array}{r}908\\ \underline{506}\end{array}$

Step 3. *a.* $\begin{array}{r}9.3\\ \underline{2.1}\end{array}$ *b.* $\begin{array}{r}\$2.47\\ \underline{16}\end{array}$ *c.* $\begin{array}{r}74\\ \underline{.32}\end{array}$ *d.* $\begin{array}{r}1.24\\ \underline{1.7}\end{array}$ *e.* $\begin{array}{r}3.18\\ \underline{5}\end{array}$

Step 2. *a.* $\begin{array}{r}43\\ \underline{15}\end{array}$ *b.* $\begin{array}{r}27\\ \underline{29}\end{array}$ *c.* $\begin{array}{r}52\\ \underline{38}\end{array}$ *d.* $\begin{array}{r}75\\ \underline{17}\end{array}$ *e.* $\begin{array}{r}84\\ \underline{46}\end{array}$

Step 1. *a.* $\begin{array}{r}62\\ \underline{7}\end{array}$ *b.* $\begin{array}{r}94\\ \underline{8}\end{array}$ *c.* $\begin{array}{r}73\\ \underline{6}\end{array}$ *d.* $\begin{array}{r}85\\ \underline{9}\end{array}$ *e.* $\begin{array}{r}48\\ \underline{5}\end{array}$

Finally it should be noted that all the improvements in teaching to be described in this volume will have a beneficial effect upon the interest in thinking and achievement in so far as they help the pupil to learn more easily and to learn matters better worth knowing. In spite of all their faults, boys and girls on the whole prefer to learn rather than be ignorant, and to learn what is useful rather than what is useless!

OTHER INTERESTS

Besides the interests in arithmetic as a game where you use your mind, win results, and show your strength and skill, there are many others to which appeal may be made. Other things being equal, work will be more interesting to children in proportion as there is physical action, variety, sociability, a chance to win, a practical gain, a connection with something or somebody that one cares for, and, most of all, perhaps, a significance for some aim or purpose that is a ruling factor in one's life at the time.

There is opportunity for much care and skill in choosing and arranging and presenting arithmetical facts and principles so that these interests will work for rather than against learning, and so that interest will come to be in the arithmetic, not simply a sugar-coating over it or a draught to swallow after it. It is easy to go too far. It would be folly to try to turn arithmetic into a mixture of gymnastics and parlor games, or to expect to find in all of a class of thirty fifth-grade children at the same time any very strong ruling purpose that demanded knowledge of decimal fractions.

The older methods were neither careful nor skillful. They made up problems not only with disregard of child life and interest, but often with disregard of vital interests at any age.

Problem after problem was of the level of interest of these (for pupils of Grade 3):

1. A fly has 6 legs. How many legs have 9 flies?

2. A box has 8 corners. How many corners have 3 boxes together?
3. Ernest has 64 buttons. How many rows of 8 buttons each can be made?
4. John Smith deposited in the First National Bank $23.72 and a week later $16,952. How much did he deposit in all?
5. In 1890 St. Louis had 460,357 inhabitants, Boston had 447,720, Baltimore 432,095, and San Francisco 297,990. How many had these four cities together?
6. Milton was born in 1608 and died in 1674. How many years did he live?
7. President Lincoln's first inaugural address contained 3500 words. His second inaugural address contained 580 words. How many more words did the first contain than the second?

A standard textbook of 1893, excellent for its time, advises the teacher, when she needs to add variety to any topic, to use in addition to the exercises printed for that topic similar ones, but using:

Animals	Dog, puppy, cat, kitten, rabbit, cow, calf, pig, horse, colt, sheep, lamb, goat, kid, fox, mouse, squirrel, monkey.
Birds	Robin, sparrow, swallow, canary, parrot, crow, bluebird, kingbird, hawk, owl, jay, loon, swan, pigeon.
Clothes	Hat, cap, bonnet, coat, vest, dress, socks, boots, shoes, collar, cuffs, slippers, rubbers, mittens, gloves.
Flowers	Rose, pink, daisy, pansy, lily, geranium, violet, poppy.
Fowls	Hen, chicken, turkey, duck, goose, gosling.
Fruits	Apple, pear, quince, orange, lemon, peach, grape, fig.
Garden	Peas, beans, corn, potatoes, carrots, parsnips.
House	Room, door, window, chair, table, picture, carpet, cup, plate, saucer, fork, knife, spoon, pitcher, clock.
Insects	Fly, spider, bee, hornet, butterfly, beetle, cricket.
School	Desk, slate, pencil, pen, book, paper, chair.
Smallwares	Buttons, pins, needles, spools of thread.
Store	Tea, coffee, sugar, starch, soap, candles, matches, eggs, axe, rake, pail, spade, hoe, saw, nails.

Toy-store	Doll, top, ball, whip, basket, marbles, whistle.
Tradesmen	Baker, butcher, grocer, milkman, blacksmith.
Trees	Apple, oak, cherry, plum, ash, birch, beech.
Vehicles	Train, car, coach, hack, buggy, wagon, gig, sleigh, sled, barge, bus.

Such was the concept of interest through variety of the older methods!

When urged to use genuine vital problems which children would care about solving, in place of the puzzles about boats and streams and currents and men digging wells and about where the hands of a watch would be, the older methods brought forth nothing better than statistics about Wisconsin's cheese, or New York's water supply, or the growth in the production of steel rails, or still dryer details of factory procedure.

There were efforts to utilize the interests of childhood to give motive to the work of arithmetic, but they were too often clumsy, as in such problems as:

1. A class uses 8 pads of paper in its arithmetic work during one week; how many pads will be required during the entire term of 20 weeks?
2. One boy throws a hammer $18\frac{1}{4}$ ft.; another throws it $13\frac{7}{12}$ ft. How much farther does the first boy throw than the second?
3. One baseball team wins 68 games. Another team wins $\frac{1}{4}$ as many. How many games are won by the second team?
4. Five boys form a basket-ball team. Their average weight is $118\frac{3}{4}$ lb. Find the total weight of all.
5. Nine-tenths of a class were promoted; 4 pupils were not promoted. How many were in the class? How many were promoted?

These five have the semblance of utilizing the interests of childhood, but it is only a semblance. Who, for example, cares by how much he is beaten when he is beaten by 30 or 40 per cent? Who cares about the total weight of a team when its average weight is already known?

6. The cover of a box is made of three pieces of wood. The pieces are $4\frac{5}{8}$ in. wide, $3\frac{1}{4}$ in. wide, and $6\frac{5}{6}$ in. wide, respectively. Find the width of the cover.
7. An oblong baseball field contains 25,000 sq. ft. The length is $41\frac{2}{3}$ yds. What is the width?

These two *sound* like shop work and athletics, but it is only sound. There is really no appeal to the constructive or athletic interests.

8. A reader costs $0.50. What will be the cost of all the readers used by this class?

The time taken to count all the readers would suffice for a dozen good problems.

The newer methods demand that the textbook and teacher should at least:

Consider childish life and affairs in school and out and try to use them when they will be of real help.

Seek a vital, engaging problem as an introduction to each new process, if there is such.

Apply each process to matters to which children then or later may be reasonably expected to care to apply it, when such applications are just as instructive as remote and artificial applications.

Use arithmetical games, races, matches, and the like as means of drill and motives for drill in preparation, when such games, races, and the like are just as instructive as mere drill for drill's sake.

Associate arithmetical work with humor, sociability, variety, and action when this can be done at no loss to order, system, and workmanship.

The pages that follow (pages 29–34) illustrate efforts in these directions.

Christmas Presents

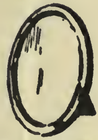
For Father

INK WELL
15¢

FISH LINE
18¢

SUSPENDERS
25¢

For Mother



PICTURE FRAME
15¢

SUGAR BOWL
17¢

TEAPOT
38¢

For a Boy

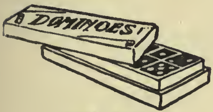
HAMMER
9¢

WHISTLE
13¢

BATTERY
14¢

KNIFE
24¢

For a Girl



DOMINOES
9¢

RIBBON
19¢

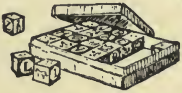
DOLL'S SLIPPERS
22¢

CANDY
25¢

For a Baby

TRUMPET
8¢

RUBBER BALL
15¢



BOX OF BLOCKS
17¢

WAGON
25¢

Christmas Presents

Try to think out for yourself the way to find the right sums.* If you need help, study page 40. Page 40 will show you a quick way to find the sums and have them all right.

1. Choose three presents, one for father, one for mother, and one for baby. Write the cost of each and add to find the total cost. *Total cost* means the cost for all three together.
2. Choose three presents for yourself. Find the total cost.
3. Choose three presents for a girl. Do not spend more than 60 cents. What is the total cost of the three you chose?
4. Choose three presents for a boy. Do not spend over 40 cents. What is the total cost of those you chose?
5. Find the total cost if you buy a fish line for father, a sugar bowl for mother, and dominoes for sister.
6. Find the total cost if you buy suspenders for father, a picture frame for mother, and a box of blocks for baby.
7. How much is the total cost of the two cheapest presents for a girl?
8. How much is the total cost of the three most expensive presents for a girl?
9. What is the total cost of all four presents for a boy?
10. What is the total cost of all four presents for a girl?
11. What is the total cost of all four presents for a baby?

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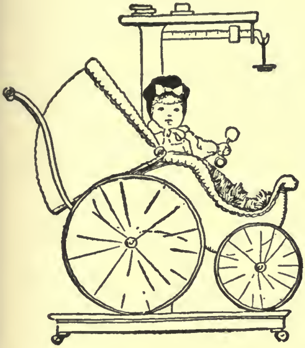
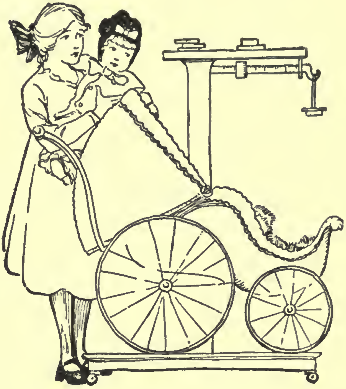
* To THE TEACHER.—Only a few of the most gifted pupils will invent "carrying" for themselves, but it is well for all the children to face this problem and feel a need for its solution before learning the solution.

Weighing the Baby





The baby and the baby carriage weigh $38\frac{1}{8}$ lb. The baby carriage without the baby weighs $14\frac{1}{2}$ lb.

How much does the baby weigh?

$38\frac{1}{8}$ *Think "$\frac{1}{2} = \frac{4}{8}$, $1\frac{1}{8} = \frac{9}{8}$."*
$\underline{14\frac{1}{2}}$ *Think "$\frac{4}{8}$ and $\frac{5}{8} = \frac{9}{8}$."*
$23\frac{5}{8}$ *Write $\frac{5}{8}$. Increase the 4 of 14 to 5.*

Check your result by adding $23\frac{5}{8}$ and $14\frac{1}{2}$.

Nell's baby sister weighed $7\frac{3}{8}$ lb. when it was born and $9\frac{1}{4}$ lb. when it was a month old. How much did it gain in the first month?

$9\frac{1}{4}$ *Think "$1\frac{1}{4} = \frac{10}{8}$."*
$7\frac{3}{8}$ *Think "$\frac{3}{8}$ and ... $= \frac{10}{8}$."*
Write $\frac{7}{8}$. Increase the 7 to 8.
Check your result by adding.

This table of numbers tells what Nell's baby sister Mary weighed every two months from the time she was born till she was a year old.

Weight of Mary Adams

When born	**$7\frac{3}{8}$ lb.**
2 months old	**$11\frac{1}{4}$ lb.**
4 months old	**$14\frac{1}{8}$ lb.**
6 months old	**$15\frac{3}{4}$ lb.**
8 months old	**$17\frac{5}{8}$ lb.**
10 months old	**$19\frac{1}{2}$ lb.**
12 months old	**$21\frac{3}{8}$ lb.**

1. How much did the Adams baby gain in the first two months?
2. How much did the Adams baby gain in the second two months?
3. In the third two months?
4. In the fourth two months?
5. From the time it was 8 months old till it was 10 months old?
6. In the last two months?
7. From the time it was born till it was 6 months old?

This table of numbers tells how much Alice Stern's baby brother Alfred weighed.

Weight of Alfred Stern

At 0 months	**$7\frac{7}{8}$ lb.**
At 2 months	**$9\frac{3}{4}$ lb.**
At 4 months	**$11\frac{5}{8}$ lb.**
At 6 months	**$13\frac{1}{4}$ lb.**
At 8 months	**$16\frac{5}{8}$ lb.**
At 10 months	**$19\frac{1}{4}$ lb.**
At 12 months	**$23\frac{1}{8}$ lb.**

Mrs. Stern keeps account of how much the baby gains every two months and writes it in a table like this.

Gain from 0 to 2 months =
Gain from 2 to 4 months =
Gain from 4 to 6 months =
Gain from 6 to 8 months =
Gain from 8 to 10 months =
Gain from 10 to 12 months =

The 5th-grade children had a FRACTIONS DASH. The teacher put 10 problems on the blackboard in a column like the one at the left of this page, and covered them with a chart. When she uncovered them, each boy and girl raced to write the correct answers as quickly as he or she could. The best record was 39 sec., by a girl. Practice with the exercises at the right of the page. Then try to beat the record. Your record does not count unless all answers are correct and are expressed in lowest terms.

FRACTIONS DASH

Add $8\frac{1}{8}$
$3\frac{1}{2}$

Add $2\frac{3}{4}$
$3\frac{3}{4}$

Add $2\frac{3}{4}$
$3\frac{7}{12}$

Subt. 7
$\frac{1}{2}$

Subt. $8\frac{1}{4}$
$2\frac{7}{8}$

$\frac{5}{2} \times \frac{8}{10}$

$17 \times \frac{1}{4}$

$\frac{1}{3} \times \frac{5}{8}$

$1\frac{1}{2} \times 2\frac{1}{4}$

$1\frac{1}{2} \times 50$

Material for Practice for Fractions Dash

Addition

A.	$5\frac{5}{16}$	$9\frac{1}{3}$	$9\frac{5}{8}$	$8\frac{3}{4}$	$7\frac{7}{12}$	$1\frac{2}{3}$
	$9\frac{1}{4}$	$2\frac{1}{6}$	$9\frac{3}{4}$	$3\frac{2}{3}$	$8\frac{1}{2}$	$1\frac{1}{4}$
B.	$9\frac{3}{8}$	$2\frac{1}{6}$	$9\frac{7}{8}$	$9\frac{1}{2}$	$7\frac{3}{4}$	$5\frac{1}{4}$
	$8\frac{1}{2}$	$2\frac{5}{12}$	$3\frac{1}{4}$	$5\frac{3}{16}$	$4\frac{5}{8}$	$4\frac{3}{4}$
C.	$9\frac{3}{8}$	$9\frac{3}{4}$	$8\frac{1}{3}$	$7\frac{7}{12}$	$9\frac{3}{8}$	$8\frac{2}{3}$
	$7\frac{7}{8}$	$4\frac{3}{4}$	$4\frac{2}{3}$	$9\frac{11}{12}$	$8\frac{5}{8}$	$8\frac{2}{3}$

Subtraction

A.	8	7	5	6	9	4
	$2\frac{1}{2}$	$2\frac{2}{3}$	$3\frac{5}{8}$	$3\frac{7}{12}$	$5\frac{9}{16}$	$1\frac{3}{4}$
B.	$9\frac{1}{8}$	$6\frac{1}{3}$	$7\frac{3}{8}$	$21\frac{2}{3}$	$3\frac{1}{3}$	$8\frac{3}{4}$
	$4\frac{1}{2}$	$1\frac{1}{6}$	$5\frac{7}{8}$	$11\frac{4}{5}$	$1\frac{3}{4}$	$7\frac{11}{16}$

1. $\frac{2}{3} \times \frac{3}{4}$
2. $\frac{1}{2} \times \frac{2}{3}$
3. $\frac{3}{8} \times \frac{3}{4}$
4. $\frac{5}{6} \times \frac{1}{2}$
5. $15 \times \frac{2}{3}$
6. $15 \times \frac{3}{5}$
7. $15 \times \frac{1}{2}$
8. $5 \times \frac{3}{8}$
9. $\frac{1}{2} \times \frac{1}{3}$
10. $\frac{1}{4} \times \frac{1}{3}$
11. $\frac{3}{4} \times \frac{3}{4}$
12. $1\frac{1}{4} \times 1\frac{1}{2}$
13. $2\frac{1}{2} \times 40$
14. $3\frac{1}{3} \times \frac{2}{5}$
15. $\frac{5}{8} \times 1\frac{1}{2}$
16. $10\frac{1}{2} \times 8\frac{1}{3}$

A Second-Hand Party

The boys and girls in the 6th grade of the Irving School had a *Second-Hand Party.* Each pupil brought one article that he wanted to sell and marked it with a tag telling the price he paid for it and the price he would sell it for. First they figured what percent the selling price was of the cost of the article when new, and wrote the percent on the tag. They arranged the articles around the room in a line, beginning with the article that was marked at the lowest percent of the original cost and ending with the article that was marked with the highest percent of the original cost. Then the children who wanted to buy any of the articles did so.

Here are some of the articles and prices. Figure out for each article what percent the selling price is of the cost when new.

		Selling Price Second Hand	Cost When New
1.	Book	30¢	$1.75
2.	Skates	25¢	.98
3.	Game	15¢	.49
4.	Racket	40¢	3.25
5.	Picture	17¢	.25
6.	Toy	9¢	.25
7.	Doll	15¢	1.25
8.	Sled	25¢	1.75

Find quotients only to the nearest thousandth, giving results to the nearest tenth of a percent, as shown below for No. 1.

$$\begin{array}{r} 17.1\% \\ .171+ \\ 1.75\overline{).30000} \\ 175 \\ \hline 1250 \\ 1225 \\ \hline 250 \\ 175 \\ \hline 75 \end{array} \quad \text{or} \quad \begin{array}{r} 17.1\% \\ .171+ \\ 175\overline{)30.000} \\ 17\;5 \\ \hline 12\;50 \\ 12\;25 \\ \hline 250 \\ 175 \\ \hline 75 \end{array}$$

Ask your teacher to let you have a *Second-Hand Party* when you have learned to find percentages quickly and without mistakes.

EXERCISES

1. Replace each of these problems by one which gives the same arithmetical training but is more interesting, or freer from irrelevant difficulties of language, or both:

a. If a dealer buys three barrels of sugar containing respectively 310.7 lb., 314.6 lb., and 312.5 lb., how many pounds does he buy in all?

b. If the height of the upright piece in a sun dial is $\frac{5}{32}$ of the diameter of the dial and the diameter of the dial is 12 in., what is the height of the upright piece?

c. How far will a commercial salesman travel in 8 days at an average of $52\frac{1}{2}$ miles a day?

d. Measure the cover of your arithmetic and draw a plan of it on the scale of $\frac{1}{4}$.

e. Automobiles help to pay for making roads. In a certain year there were 2,900,000 motor vehicles of all kinds in this country, and they paid on the average $10 a year for registration and licenses. How much did they pay to the states?

f. The area of British India is 1,004,616 square miles and the population 150,767,851. How many inhabitants are there to a square mile?

g. How many persons die in a year in a city of 190,000 inhabitants if the annual death rate is 10 per thousand?

h. If a man steps on the average $2\frac{3}{4}$ feet, how many steps will he take in walking a mile (5280 ft.)?

i. An attorney collected a debt of $324.50 and charged 10 percent for his services. What was his commission?

j. The Pyramids of Egypt are supposed to have been built 337 years before the founding of Carthage; Carthage to have been founded 49 years before the destruction of Troy, and Troy to have been destroyed 431 years before Rome was founded; Carthage was destroyed 607 years after the founding of Rome, and 146 years before the Christian era. How many years before the Christian era were the Pyramids of Egypt built?

2. What are the means taken to infuse interest in learning how to add and subtract with 0? (Book I, pages 26–29.)

3. In reviews of multiplication by a one-place number, subtraction, and knowledge of the tables of measures and short division? (Book I, pages 136, 137.)

4. In understanding what shares of stock are? (Book III, pages 153, 154.)

5. In circular measure? (Book III, pages 111, 112, 113.)

6. Examine Book I, page 214, with reference to the spacing, the size of the fractions, and the motives used to appeal to the interest in achievement.

7. Examine Book III, page 31. Would this type be suitable in Grade 3 or 4? Would it be suitable if printed in regular paragraph form?

8. Examine Book II, pages 130 and 131. Was Alice an average pupil or a superior pupil?* Suppose that you were to have your pupils use this form of test repeatedly in Grade 5 as training in quick reaction, adaptability, knowledge of fundamentals, and combining two steps in one. How would you allow for individual differences, so that the quick, adaptable, and gifted pupils would not be bored, and the slow and dull pupils would not be disheartened?

9. Examine Book III, page 6. What would be the result if the instructions were: "Practice with these until you can do all without a mistake in 4 minutes"?

In such practice to reach a standard accuracy and speed, care should be taken to make the speed standard a reasonable one for children at that stage of development.

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* See also Book I, pages 87, 166, 204, for other facts about Alice.

CHAPTER III

THEORY AND EXPLANATIONS

DEDUCTIVE REASONING

The older methods explained the various rules and procedures in arithmetic from "carrying" in addition of integers to the placing of the decimal point in division by a decimal, if they explained them at all, deductively as necessary consequences of fundamental axioms and of the nature of our system of numbers in which a digit signifies so many ones, or tens, or hundreds, or tenths, etc., according to the place which it occupies; the number written above the line in a fraction signifies the number of parts, and the number below the line the ratio of unity to the size of one of these parts.

Experience showed, however, that the pupils did not learn much from these deductive explanations, so that year by year less and less stress has been put upon them. No competent textbook maker and no expert teacher would now give to pupils such explanations as those shown below, though such were highly esteemed in the days of our fathers:

I

Divide 3465 dollars equally among 15 men.

Solution. When the divisor exceeds 12, as in this example, instead of performing all the operations in the mind, it becomes necessary to write down a part of the process, as in example 3 preceding.

We find that 15 is not contained in 3 (thousands), therefore, there will be no thousands in the quotient. We next take 34 (hundreds) as a partial dividend, and find that 15 is contained in it 2 (hundred) times; that is, we have 2 hundred dollars for each of 15 men, which requires in all 15×2 (hundreds) = 30 hundreds. Subtracting 30 (hundreds) from 34 (hundreds), there are 4 (hundreds) remaining, to which we bring down the 6 (tens), and have 46 (tens) for a second partial dividend. In this, 15 is contained 3 (tens) times, which gives each man 3 tens (30) dollars more, and requires for all 15×3 (tens) $= 45$ tens of dollars. Subtracting this, and bringing down the 5 (units), we have 15 (units) for a third partial dividend, in which the divisor, 15, is contained once, which gives to each man 1 (unit) dollar. Hence, each man received 2 hundred dollars, 3 ten dollars, and 1 dollar, that is 231 dollars.

	thous.	hund.	tens	units		hund.	tens	units
15)	3	4	6	5		(2	3	1
	3	0	hundreds.					
		4	6	tens.				
		4	5					
			1	5	units.			
			1	5				

By this process, the dividend is separated into parts, each of which contains a divisor a certain number of times. Thus, in the first part, 30 (hundreds), the divisor is contained 2 (hundred) times. In the second part, 45 (tens), the divisor is contained 3 (ten) times; and in the third part, 15 (units), it is contained 1 (unit) times. It is easily seen, that the several parts together are equal to the given dividend; and, that the several partial quotients make up the entire quotient.

Divisor	Parts	Quotients
15	3000	200
	450	30
	15	1
	3465	231

II

To Divide a Fraction by a Fraction.

How many pounds of tea can be bought for $\frac{11}{12}$ of a dollar, at $\frac{2}{3}$ of a dollar a pound?

OPERATION

First step, $\frac{11}{12} \times 3 = \frac{33}{12}$

Second step, $\frac{33}{12} \div 2 = \frac{33}{24} = 1\frac{3}{8}$

Whole work, $\frac{11}{12} \div \frac{2}{3} = \frac{11}{\cancel{12}_4} \times \frac{\cancel{3}}{2} = \frac{11}{8} = 1\frac{3}{8}$, Ans.

ANALYSIS. As many pounds as $\frac{2}{3}$ of a dollar is contained times in $\frac{11}{12}$ of a dollar. 1 is contained in $\frac{11}{12}$, $\frac{11}{12}$ times, and $\frac{1}{3}$ is contained in $\frac{11}{12}$ 3 times as many times as 1, or 3 times $\frac{11}{12}$, which is $\frac{33}{12}$ times, which is the number of pounds that could be bought at $\frac{1}{3}$ of a dollar per pound; but $\frac{2}{3}$ is contained but $\frac{1}{2}$ as many times as $\frac{1}{3}$, and $\frac{33}{12}$ divided by 2 gives

$\frac{33}{24}$ equal to $1\frac{3}{8}$ times, or the number of pounds that can be bought at $\frac{2}{3}$ of a dollar per pound.

We see in the operation that we have multiplied the dividend by the denominator of the divisor, and divided the result by the numerator of the divisor, which is in accordance with **140** for dividing a fraction. Hence, by inverting the terms of the divisor, the two fractions will stand in such relation to each other that we can multiply together the two upper numbers for the numerator of the quotient, and the two lower numbers for the denominator, as shown in the operation.

III

Division of Fractions is the process of dividing when the divisor or dividend, or both, are fractions.

To Divide a Fraction by a Whole Number.

Ex. 1. Divide $\frac{8}{9}$ by 4. Ans. $\frac{2}{9}$

FIRST OPERATION

$\frac{8}{9} \div 4 = \frac{2}{9}$

We divide the numerator of the fraction by 4 and write the quotient, 2, over the denominator. It is evident this process divides the fraction by 4, since the size of the parts into which the whole number is divided, as denoted by the denominator, remains the same, while the number of parts taken is only $\frac{1}{4}$ as many as before, therefore,

Dividing the numerator of a fraction by any number divides the fraction by that number.

Ex. 2. Divide $\frac{5}{7}$ by 9. Ans. $\frac{5}{63}$

SECOND OPERATION

$\frac{5}{7} \div 9 = \frac{5}{63}$

We multiply the denominator of the fraction by the divisor, 9, and write the product under the numerator, 5.

It is evident this process divides the fraction, since multiplying the denominator by 9 makes the number of parts into which the whole number is divided 9 times as many as before, and consequently each part can have but $\frac{1}{9}$ of its former value. Now, if each part has but $\frac{1}{9}$ of its former value, while only the same number of parts is expressed by the fraction, it is plain the fraction has been divided by 9. Therefore,

Multiplying the denominator of a fraction by any number divides the fraction by that number.

RULE. *Divide the numerator of the fraction by the whole number, when it can be done without a remainder, and write the quotient over the denominator,* Or,

Multiply the denominator of the fraction by the whole number, and write the product under the numerator.

To Divide a Whole Number by a Fraction.

Ex. 1. How many times will 13 contain $\frac{3}{7}$? Ans. $30\frac{1}{3}$

OPERATION

$$13 \div \tfrac{3}{7} = \frac{13 \times 7}{3} = \frac{91}{3} = 30\tfrac{1}{3} \text{ Ans.}$$

13 will contain $\frac{1}{7}$ as many times as there are *sevenths* in 13, equal 91 sevenths. Now, if 13 contains 1 seventh 91 times, it will contain $\frac{3}{7}$ as many times as 91 will contain 3, or $30\frac{1}{3}$.

RULE. *Multiply the whole number by the denominator of the fraction, and divide the product by the numerator.*

To Divide a Mixed Number by a Whole Number.

Ex. 1. Divide $17\frac{3}{8}$ by 6. Ans. $2\frac{43}{48}$

OPERATION

$$\begin{array}{l} 6\underline{)17\frac{3}{8}} \\ \quad 2 \end{array} \quad 5\tfrac{3}{8} = \tfrac{43}{8}; \quad \frac{43}{8 \times 6} = \tfrac{43}{48};$$

$$2 + \tfrac{43}{48} = 2\tfrac{43}{48}$$

Having divided the whole number as in simple division, we have a remainder of $5\frac{3}{8}$ which we reduce to an improper fraction, and divide it by the divisor, as in Art. **159.** Annexing this result to the quotient 2, we obtain $2\frac{43}{48}$ for the answer. That is, we

Divide the integral part of the mixed number; and the remainder, reduced if necessary to a simple fraction, divide as in Art. **159.**

To Divide a Whole Number by a Mixed Number.

Ex. 1. Divide 25 by $4\frac{3}{5}$. Ans. $5\frac{10}{23}$

OPERATION

$$\begin{array}{l|l} 4\frac{3}{5} & \frac{25}{5} \\ 5 & \\ \hline \end{array}$$

$$\begin{array}{r} 23)\ 125\ (5\frac{10}{23} \\ 115 \\ \hline 10 \end{array}$$

We first reduce the divisor and dividend to fifths, and then divide as in whole numbers.

The divisor and dividend were both multiplied by the same number, 5; therefore their relation to each other is the same as before, and the quotient is not changed. (Art. 115, Note.) Hence,

Reduce the divisor and dividend to the same fractional parts as are denoted by the denominator of the fraction in the divisor, and then divide as in whole numbers.

To Divide a Fraction by a Fraction.

Ex. 1. Divide $\frac{7}{8}$ by $\frac{4}{9}$. Ans. $1\frac{31}{32}$

FIRST OPERATION

$$\frac{7}{8}\times 9=\frac{63}{8};\ \frac{63}{8\times 4}=\frac{63}{32}=1\frac{31}{32}$$

SECOND OPERATION

$$\frac{7}{8}\div\frac{4}{9}=\frac{7}{8}\times\frac{9}{4}=\frac{63}{32}=1\frac{31}{32}$$

Since 1 is contained in $\frac{7}{8}$, $\frac{7}{8}$ times, $\frac{1}{9}$ is contained in $\frac{7}{8}$, 9 times $\frac{7}{8}$ times, or $\frac{63}{8}$ times; and $\frac{4}{9}$ is contained in $\frac{7}{8}$, $\frac{1}{4}$ of $\frac{63}{8}$ times, which is $\frac{63}{32}$ or $1\frac{31}{32}$ times.

That is, we have multiplied the denominator of the dividend by the number denoting the numerator of the divisor, and the numerator of the dividend by the number denoting the denominator of the divisor; hence, for convenience, as in the second operation, we can simply invert the terms of the divisor and proceed as in Art. **196**.

The fact that these deductive explanations have dwindled in length and importance is taken by some teachers to mean than no real understanding of rules and processes is desirable — that the pupil should simply learn mechanically to do as he is told. The newer methods assert that there is a third alternative — that, although the deductive explanations as used did not produce rational understanding of the rules and processes, such understanding is obtainable — that the pupil need not be left to a blind, mechanical rote memory of what to do. The newer methods aim to make arithmetic a science that the pupil knows as well as a trade that he can work at skillfully; they aim to secure real understanding of rules and principles. The means they take to attain this are the topic of this chapter.

INDUCTIVE REASONING

There are two sorts of reasons that may be given as answers to such questions as "Why should you 'carry' in addition?" "Why should you write the first digit of a partial product under the figure by which you are multiplying?" or "Why in dividing by $\frac{3}{4}$ do you multiply by $\frac{4}{3}$?" The first sort refers back by a chain of argument to axioms and the general nature of our arithmetical system, and is, of course, a deductive explanation of the sort described above. The second sort is very different, being in essence, "Because I find that doing so always gives the right answer." It refers to some valid verification. It is experimental and inductive.

The newer methods make large use of this second sort of reasoning. The pupil is taught to verify rules and processes.
He verifies the procedure taught him for multiplying 412 by
412
3 by adding 412. He verifies the procedure taught him for
412
dividing 675 by 25 by multiplying 27 by 25. He verifies the rule for adding fractions by objective measurement. He verifies the rule that the number of decimal places in the product equals the sum of the decimal places in the multiplier and of those in the multiplicand by comparing the results when the numbers are expressed as common fractions, checking .25 × .5 by $\frac{1}{4} \times \frac{1}{2}$ and the like. He also checks here in this way:

7.14 It cannot be 2.7132, for 3×7 is more than 20.
3.8 It cannot be 271.32, for 4×8 is not so much as 200.

The newer methods lay more stress on the pupil's surety that the rule or process is right and less stress on his ability to state in words a proof that would satisfy a mathematician. They do not wish him to take rules and processes on faith and follow them mechanically. On the other hand, they do not insist that he should be able to express deductions from the theory of numbers in exact and complete form. They find that requiring a pupil to do so tempts him to mere memorizing of definitions, rules, analyses, and explanations.

For example, suppose that a child has worked $6\div\frac{3}{4}$ by $6\times\frac{4}{3}$ and verified the result by dividing a 6-inch strip into $\frac{3}{4}$-inch lengths, has worked $2\frac{1}{2}\div\frac{5}{8}$ by $\frac{5}{2}\times\frac{8}{5}$ and verified the result by dividing a $2\frac{1}{2}$-inch strip into $\frac{5}{8}$-inch lengths, and similarly for other cases. He has also made out by addition or multiplication tables like

$$\begin{gathered}1\tfrac{2}{3}\div\tfrac{5}{6}=2\\2\tfrac{1}{2}\div\tfrac{5}{6}=3\\3\tfrac{1}{3}\div\tfrac{5}{6}=4\\4\tfrac{1}{6}\div\tfrac{5}{6}=5\end{gathered}$$

and used them to verify the rule. Such a child understands in a certain true and useful way the reasons for "Invert and multiply" or "Multiply by the reciprocal" (supposing him to have been taught the meaning of *reciprocal*). He may not be able to state the deductive proof from the nature of a fraction. Neither can some of my readers, perhaps!

ADAPTABILITY TO THE LEARNER

The newer methods are less concerned with making rules and explanations satisfactory to put in an encyclopedia for mathematicians, and more concerned with making them true guides to the young learner. Attention is given to dynamic truth and exactness such as will not mislead the pupil, as well as to logical and verbal correctness such as fits a dictionary.

For example, the newer methods prefer A to B for children at the beginning of Grade 5.

A

Numbers like 2, 5, 7, 9, 11, 75, 250 are whole numbers.
Numbers like $\frac{3}{8}$, $\frac{1}{5}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{11}{8}$, $\frac{7}{6}$ are fractions.
Numbers like $4\frac{1}{4}$, $2\frac{7}{8}$, $12\frac{3}{4}$, $1\frac{2}{3}$ are mixed numbers.

B

A whole number, or a number expressed without fractions, is called an integer. A number which shows what number of equal parts of a unit is taken is called a fraction. A fraction which has both terms expressed is called a common fraction. A number composed of an integer and a fraction taken together is called a mixed number.

Dynamically—that is, in action—a definition may be regarded as correct if it leads to correct applications; a rule is correct if it leads to correct operations; a process is correctly understood if the pupil can use it to obtain correct answers—and, in all three cases, if the definitions, rules, and explanations *do not hinder the pupil in later work.* Thus it does no harm for a pupil to think of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, etc., first as "numbers smaller than 1," and a little later of fractions as "numbers like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{7}{8}$, $\frac{3}{8}$, $\frac{2}{5}$," without any specific inclusion of or reference to improper fractions. He uses this knowledge to master the addition and subtraction of fractions with the same denominator, and of halves, fourths,and eighths, and of halves and thirds and sixths, and is in no way misled or hampered in his later work when the idea and definition of a fraction is extended. In fact, the pupil in the elementary school should probably never be expected to understand a perfect definition of fraction, such as would include $\frac{b+a}{c}$, $\frac{\frac{b+a}{7}}{8\frac{3}{4}}$, and $\frac{.426-.169}{2.3}$.

THE DEVELOPMENT OF KNOWLEDGE OF THEORY

A common procedure in the older methods was to teach the general theory, rule, and explanation for a certain process, such as the addition of common fractions, or the subtraction of denominate numbers, or division by a decimal, and then give drill on the process until the pupil could use it accurately and quickly. Almost all of the understanding was supposed to come before any of the use; after the pupil could use the process well he was permitted to forget the reasons for it. "First learn why you do so and so; then forget the whys and wherefores."

Such a plan is perhaps defensible. But the newer methods are suspicious of learning only to forget, and in particular consider that the general principles should be the last things to be forgotten. If principles are taught that are really helpful, that really act in learning and retention, and are taught in the right way, it would seem that, even if certain details of how to compute were forgotten, these vital general principles would not be.

The newer methods teach a principle gradually along with the actual practice in the process (and often *after* the process is used) as an explanation of why the process is and must be right. The principle is then better understood and better remembered because it concerns something that the pupil is doing and has been doing. Thus a pupil begins his work in the addition of unlike fractions by meeting exercises like

$$\begin{array}{r} 6 \\ 9\frac{1}{4} \\ \underline{4\frac{1}{2}} \end{array} \qquad\qquad \begin{array}{r} 7 \\ 7\frac{1}{2} \\ \underline{8\frac{3}{4}} \end{array}$$

and being given the very simple principle, "Think of $\frac{1}{2}$ as $\frac{2}{4}$." He later meets additions with $\frac{\ }{8}$s, $\frac{\ }{4}$s, and $\frac{\ }{2}$s, and he learns two more simple principles, namely: "Think of $\frac{1}{2}$ as $\frac{4}{8}$," "Think of $\frac{1}{4}$ as $\frac{2}{8}$ and of $\frac{3}{4}$ as $\frac{6}{8}$." He is thus prepared to understand the more general principle "When you add fractions, express them as fractions having the same denominator."

The newer methods organize minor principles together into a more general understanding of some large topic, and give, after the pupil has had experience of certain operations, a full explanation which he can then understand and value, but which would have been incomprehensible and useless at the beginning. Thus the pages quoted on pages 47–49 following are very suitable at the end of Grade 6 or the beginning of Grade 7.

The newer methods put more confidence in teaching a pupil to understand the theory of arithmetic by what the teacher and textbook have him do than by what they tell him. Telling him is too likely to produce mere rote memory of definitions and rules and explanations, unless the definition or rule or explanation sums up conveniently something he has already seen to be true in his actual work.

So, instead of being told much about the place-value of numbers, the pupil is given from time to time exercises like the following:

$6 \times 9 =$
$6 \times 90 =$
$6 \times 900 =$

$7 \times 8 =$
$7 \times 80 =$
$7 \times 800 =$

$$\begin{array}{r}243\\2\\\hline\end{array}$$

Check by $2 \times 200 =$
$2 \times 40 =$
$2 \times 3 =$ ______
Sum is

$$\begin{array}{r}975\\8\\\hline\end{array}$$

Check by $8 \times 5 =$
$8 \times 70 =$
$8 \times 900 =$ ______
Sum is

Oral Review

Addition means finding the sum of two quantities.

The correct result in addition is the result that would be obtained by accurate counting or measuring.

We obtain the correct result in adding whole numbers or decimal numbers by—

adding ones to ones and counting 10 ones as 1 ten

adding tens to tens and counting 10 tens as 1 hundred

adding hundredths to hundredths and counting 10 hundredths as 1 tenth

adding thousandths to thousandths and counting 10 thousandths as 1 hundredth

1. How do you count 10 tenths in adding?

We obtain the correct result in adding fractions, all having the same number as denominator, by adding the numerators.

2. How do we count $\frac{2}{2}$ or $\frac{3}{3}$ or $\frac{4}{4}$ or $\frac{5}{5}$ or $\frac{8}{8}$?

We obtain the correct result in adding fractions with different numbers as denominators by first expressing them as fractions with the same number as denominator, or by expressing them as decimal numbers.

3. Express $\frac{7}{25}$, $\frac{9}{50}$, and $\frac{11}{20}$ as decimal numbers.

4. Express $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{5}{6}$ as $\frac{}{12}$s.

5. Read, saying the right words or numbers where the dots are:

We obtain the correct result in adding quantities like 4 bu. 2 pk. 3 qt. and 1 bu. 3 pk. 7 qt. by adding qt. to....and counting 8 qt. as....pk. and by adding pk. to....and counting 4 pk. as....bu.

6. Tell how you "carry" with seconds, minutes, pints, inches, feet, and ounces.

7. Let b stand for bushels. Let p stand for pecks. Let q stand for quarts. Helen's father bought 3 bags of nuts.

The first bag contained 2b.+1p.+3q.
The second bag " 1b.+1p.+2q.
The third bag " 2b.+1p.+2q.
How much did all three bags together contain?

Units of Measure

Whatever quantity is called 1 is the unit of measure.

1. Read. Supply the missing words as is shown in the first two lines.
   - *a.* A half mile is $\frac{1}{2}$ if we are using *a mile* as the unit of measure.
   - *b.* A half mile is 160 if we are using *a rod* as the unit of measure.
   - *c.* A half mile is 880 if we are using....as the unit of measure.
   - *d.* A half mile is 2640 if we are using...as the unit of measure.
   - *e.* This square is 2×2 if we use an....as the unit of length.
   - *f.* This square is $\frac{1}{6}\times\frac{1}{6}$ if we use a....as the unit of length.
   - *g.* An hour is 1 if we use an ...as the unit of measure.
   - *h.* An hour is $\frac{1}{24}$ if we use... as the unit of measure.
   - *i.* An hour is 60 if we use... as the unit of measure.

Any quantity is a multiple of some unit.

Thus 9 mi. is 9×1 *mi.,* $10\frac{1}{2}$ *mi. is* $10\frac{1}{2}\times 1$ *mi.,* $3\frac{3}{4}$ *lb. is* $3\frac{3}{4}\times 1$ *lb.*

In using the dimensions of any surface to find its area, express both dimensions as multiples of the same unit. Choose a convenient unit.

2. Supply the missing words:
   - *a.* Length of a rectangle in....×width in....=area in sq. in.
   - *b.* Length of a rectangle in.....×.....=area in sq. ft.
   - *c.* Length of a rectangle in yards×.....=area in.....
   - *d.* Base of a parallelogram in inches×altitude in.....= area in.....
   - *e.* Base of a parallelogram in miles×altitude in.....= area in.....

f. Base of a triangle in feet × ½ of...... = ...sq. ft.

g. Average of two parallel sides of a trapezoid × altitude = area. If dimensions are in inches, area is in.... If dimensions are in feet, area is in....... If dimensions are in miles, area is in........

(*With pencil.*)

3. How many square feet are there in a road 2.4 miles long and 18 ft. wide, counting the road as perfectly straight?

4. How many square yards of material are there in a big flag 5 yards long and 10 ft. wide?

5. What fraction of a square mile is the area of this park?

In using the dimensions of any box or bin or solid to find its capacity or volume, express all dimensions in the same unit.

6. How many cubic feet will a rectangular trough contain that is 10 ft. long, 2 ft. 6 in. wide, and 18 in. deep?

7. A rectangular pile of wood 4 × 4 × 8 ft. equals 1 cord of 4-ft. wood. How many cords of 4-ft. wood are there in a pile 4 ft. wide, 4 ft. high, and 24 yards long?

8. How many cubic yards are excavated in digging a hole 40 ft. by 24 ft. by 8 ft.?

In solving any problem, think what the units of measure mean.

9. The Merchants' Express goes 220 miles in 4 hr. 24 min. The Continental goes at the rate of a mile in 80 seconds. Which goes faster? Prove that your answer is right.

10. Helen can add 100 two-place numbers in 248 seconds. Alice can add the numbers at the rate of 30 a minute. Which girl adds more rapidly? Prove that your answer is right.

SCIENTIFIC VERSUS CONVENTIONAL RULES AND EXPLANATIONS

The newer methods distinguish between those rules and explanations which are true and necessary because of the very nature of our system of numbers and those which are simply convenient, or even simply customary. Samples of the former are:

Subtrahend + difference should = minuend.

Divisor × quotient should = dividend.

"Percent of" means "hundredths times."

"What percent of" means "How many hundredths of."

The number of decimal places in the divisor plus the number of decimal places in the quotient should = the number of decimal places in the dividend.

Rules such as these are always and everywhere necessarily true. Samples of the latter are:

Rules for adding units first, then tens, etc.

Rules for carrying in addition.

Rules for placing the partial products in multiplication.

Rules for reducing fractions to those of the same denominator before adding.

It is not true that we must begin with the units column and "carry" to obtain the correct answer.

$$\begin{array}{r} 88 \\ 56 \\ \underline{97} \\ 220 \\ \underline{21} \\ 241 \end{array}$$

is entirely sound and correct. It simply is not customary, and probably not quite so rapid. We can secure the product of $\begin{array}{r} 475 \\ \underline{261} \end{array}$ in many other ways than by

$$\begin{array}{r} 475 \\ \underline{261} \\ 475 \\ 2850 \\ \underline{950} \end{array}$$

For example,

$$\begin{array}{r} 80000 \\ 24000 \\ 14000 \\ 4200 \\ 1000 \\ 400 \\ 300 \\ 70 \\ \underline{5} \\ 123975 \end{array}$$

is just as sound and correct.

It is not necessary to reduce $\frac{1}{2}$ to $\frac{2}{4}$ in adding $\frac{3}{4}+\frac{1}{2}$. Indeed, most of my readers would not do so, but would at once know the total. If we all knew the subtraction combinations of $\frac{\ }{2}$s, $\frac{\ }{4}$s, and $\frac{\ }{8}$s as well as we know the subtractions to $18-9$, we should do all the exercises below without reducing the halves and fourths to eighths.

$$\begin{array}{cccccccc} \frac{7}{8} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \frac{7}{8} & \frac{7}{8} & \frac{3}{4} & \frac{3}{4} \\ \underline{\frac{1}{2}} & \underline{\frac{5}{8}} & \underline{\frac{3}{8}} & \underline{\frac{1}{8}} & \underline{\frac{3}{4}} & \underline{\frac{1}{4}} & \underline{\frac{1}{2}} & \underline{\frac{3}{8}} \end{array}$$

There is in fact no more absolute necessity for the rule about reducing to fractions of the same denominator than there is for the rule, "In adding numbers from 1 to 10, count on your fingers." One rule is a good one and the other a bad one, not because one is true and the other false, but because one is in general desirable to follow while the other is not.

Some rules of the second sort, which used to be taught as of equal importance with the necessary essential rules, are really not even desirable to follow. Children used to be taught that the way to solve exercises in the multiplication of fractions and in reducing fractions to lowest terms was by finding the greatest common divisor and dividing both terms by it. We would now teach them not to. Children used to be taught in adding fractions always to reduce them to the *least* common denominator. We would now teach them to reduce them to any common denominator that they could most readily use. Children still are sometimes taught that failure to reduce a fractional answer to lowest terms is as unscientific and incorrect as to secure a wrong answer. It is unfair to ask pupils to reason about arithmetic if their teachers are as unreasonable as that!

In general, by substituting proofs by experimental verification for incomprehensible deductive explanations and derivations, by giving children reasons when they need them and in such form that they can use them, by so arranging arithmetic that the pupil's own work reveals the science and logic of arithmetic to him, and by distinguishing essential principles from arbitrary rules made for convenience, the newer methods have reinstated reasoning in the learning of arithmetic.

EXERCISES

1. Compare with the explanations on pages 37–41 the explanations of long division (Book I, pages 175, 176) and division by a fraction (Book II, pages 52, 53). (Note also the verifications by checking on pages 54 and 55.)

2. A boy asked his teacher why he should "carry" in adding. The teacher replied, "Because the value of the figures increases from right to left in a decimal ratio." What is the defect in this explanation?

3. What criticism would you make of a teacher who used just the same explanation of the formulae for computing areas of triangles and parallelograms in Grade 7 as she had already used in Grade 6?
4. Which of the following rules are important parts of the science of arithmetic? Which are not?
   a. Write all the numbers given in a line. Divide by any common prime factor of two or more of the numbers, bringing down the quotients and any number not divisible. Continue the division by a common prime factor of two or more of the numbers until the final quotients are prime numbers.
   b. To divide by any number, you may multiply by its reciprocal.
   c. In finding the volume of any solid express all dimensions in the same unit of measure.
   d. In dividing United States money divide the number as in ordinary division and place a point in the quotient directly over the point in the dividend.
   e. If the dividend is a concrete number and the divisor is an abstract number, the quotient and the dividend are like numbers.
   f. If the divisor and dividend are multiplied or divided by the same number, the quotient will not be changed.
   g. Rule for notation: Begin at the left and write the figures of each period in their proper orders, filling all vacant orders and periods with ciphers.
5. Examine the gradual development of understanding of ratio and proportion and their applications in II, 137,* lower half; II, 230, bottom, 231 and 232; III, 76, 77, 78, 79, 114, 115, and 116. Note especially how the pupil's own activities are arranged to teach him.

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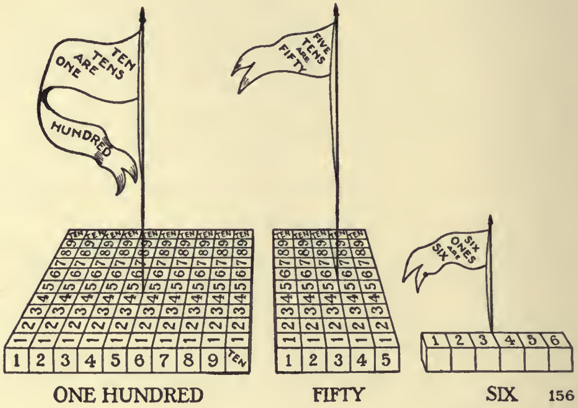
*From here on the books and the pages of the *Thorndike Arithmetics* will be referred to simply by I, II, or III and the appropriate page numbers.

6. Consider these pairs of objective aids to understanding arithmetical procedures. In each case decide which seems the more useful.

A 1

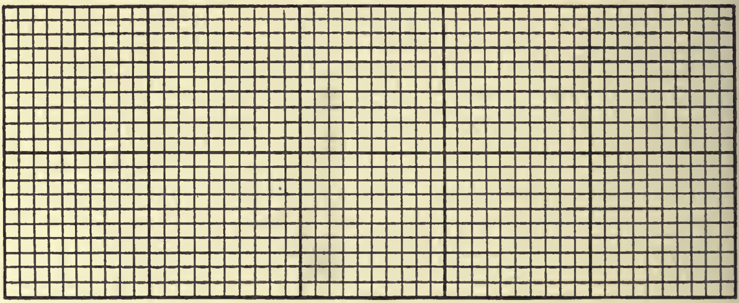
Lines 1, 10, 100, 50, 6, and 156 inches long are drawn on the blackboard.

A 2

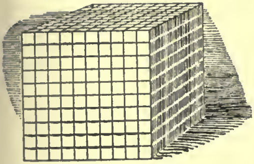


B 1

There are 1000 small squares in this picture.



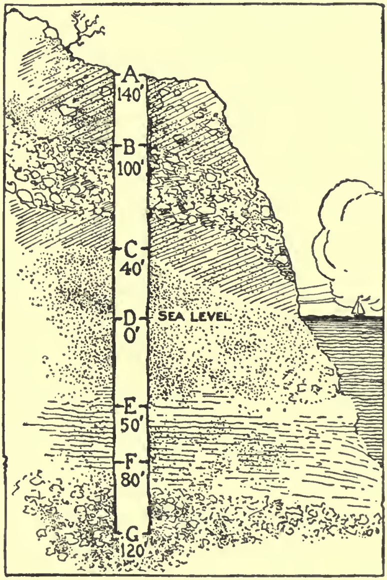
B 2



There are 1000 small cubes in this pile.

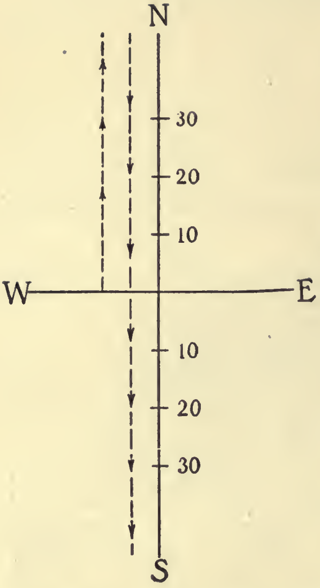
C 1

Negative Numbers



1. How much higher is A than B on this map?
2. How much higher is A than C?
3. Than D?
4. Than E?
5. Than F?
6. How much higher is D than E?
7. Than F?
8. How much higher is E than F?
9. Call distances above sea level $+$; and call distances below sea level $-$. M is $+42$ ft. R is -36 ft. How much higher is M than R?
10. N is $+940$ ft.; S is -60 ft. How much higher is N than S?

C 2



A ship sails north from *O* at 10 miles an hour for four hours. How far north of *O* will it be then?

Another ship sails north from *O* at 10 miles an hour for four hours and then sails south for an hour at the same rate. How far north of *O* will it be then?

Call distance north of *O* plus (+) and call distance south of *O* minus (−).

If a ship sails north from *O* for 10 miles and then sails south for 90 miles, where will it be?

If a ship sails from *O* north for 3 hours at the rate of 15 miles per hour and then sails south for 4 hours at the same rate, what will be its distance from *O*?

CHAPTER IV

HABIT FORMATION AND DRILL

REPETITION VERSUS MOTIVATION

The older methods trusted largely to mere frequency of connection — that is, to mere repetition — to form habits of arithmetical knowledge and skill. Pupils said their tables over and over. They heard and saw $7+9=16$, $6\times8=48$, and the like again and again, hour after hour and day after day. Yet scores of such repetitions did not form the bonds perfectly. A girl who learned to connect the names of the forty-five children in her class with their faces infallibly in a few weeks from casual incidental training did not learn to connect the forty-five addition combinations, $1+1$ to $9+9$, with their answers in the systematic drills of twice that time. A boy who in two months of vacation learned, from a few experiences of each, to know a thousand houses, turns of paths, flowers, fishes, boys, uses of tools, personal peculiarities, slang expressions, swear words, and the like, without effort, seemed utterly incapable of learning his multiplication and division tables in a school year.

Something besides repetition is evidently at work, something which we may call interest or motive or satisfyingness. Those bonds or connections which satisfy some want or craving of the learner are formed from very few repetitions. The psychologist states two laws for the formation of mental connections.

The *Law of Exercise* is that, other things being equal, use strengthens and disuse weakens mental connections.

The *Law of Effect* is that, other things being equal, connections accompanied or followed by satisfying states of affairs are strengthened, whereas connections accompanied or followed by annoying states of affairs are weakened.

The second law, the law of effect, evidently gives the explanation of the enormous variation in the ease of learning matters which, so far as mere amount and complexity go, would be equally easy to learn. If we are to have rapid learning, we must so far as possible get the force of satisfyingness on our side. This the newer methods try to do. They use all the general means of arousing interest in arithmetical work which were described in a previous chapter. They use also particular means adapted to the special forms of arithmetical work which we call habit formation or drill. Let us consider some of these.

The active connection of two things by the person is more potent than the passive hearing or seeing of them in connection. So we have the pupil study part of the table or other facts to be learned, then cover the answers with a card, and give them himself, looking at each to make sure he is right or if he is unable to think of any answer in which he has confidence. This he continues until he can give all correctly and fluently. He thus not only comes to know the facts more quickly, but also to know that he knows them. Cards with the questions on one side and the question and the answer on the other side may be used, especially where it is desirable not to have any help from the printed orders of the facts.

Almost all the drill work of arithmetic consists, not of isolated, unrelated facts, but of parts of a total system, each part of which may help to knowledge of all other parts, if it is learned properly. To be learned properly in this respect means to be learned with such related facts as are already known, ready to connect with the new fact. Thus nobody in his senses would give as the first lesson in multiplication 2×3, 8×5, 14×9, 9×7, 10×40, 6×60, and 4×7. We try to put together in the pupil's mind those things that belong together. Now this principle is capable of wide and ingenious use. If we can have "three 9's = 27," "27 and 9 = how many?"

and "9, 18, 27, 36," as it were awake, alert in the background of the pupil's mind ready to work to help him when he asks himself "four 9's = how many?" the $4 \times 9 = 36$ has a chance of being learned more easily, joining in the system, and helping other bonds in turn. Time spent in understanding facts and thinking about them is almost always saved doubly by the greater ease of memorizing them. Almost all arithmetical knowledge should be treated as an organized interrelated system.*

In some cases the cause of the failure to form the new bonds easily lies far back in the pupil's early training. If, for example, he had no real sense of the meaning of numbers, if he could not tell whether the children in the room made 20 or 40 or 60 or 80, or would as often call a yardstick 15 inches or 55 inches as 36, or would choose 10 cents and 10 cents and 10 cents rather than 70 cents, then obviously the multiplication tables might be to him only a set of series of nonsense syllables, difficult to learn and well-nigh impossible to remember. If a pupil has no real understanding of either common fractions or decimal notation, he cannot readily learn to operate with aliquot parts of a hundred.

If we try to learn all of a game at once, we may learn none of it, and perhaps think it beyond our capacity, or at least take the harmful attitude of expecting to blunder and fail. If we take the same game one feature at a time, putting each new feature into coöperation with the others until we are playing the whole game as it is really played, we succeed.

An operation like column addition for a child in Grade 2, or long division for a child in Grade 4, or division by a decimal for one in Grade 6, means, not the formation of a habit, but the formation and organization of many habits quite comparable to the tasks of an intelligent adult in learning an

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*The exceptions are such matters as 1 ton = 2000 lbs., or circumference = $\frac{22}{7}$ diameter.

elaborate game. Hence the progress of teaching has, during the past forty years, been steadily toward the gradual building up of certain abilities, habit being added to habit and all being gradually integrated together. By thus focusing upon one thing at a time, we can be sure that the pupil knows what he is trying to learn, learns it, and enjoys learning it.

So we find drill exercises to give practice on just selecting the quotient figure in long division, or on the one ability to place the decimal point in division by a decimal, or even on the one fact that "x percent of" means "x hundredths ×," as shown below:

Find the quotients and remainders. Sometimes you may think of a wrong figure for the quotient. Then you must see whether it is too large or too small and change it. But try to think of the right number the first time.

11. $28|\overline{817}$ Are there 3 28's in 81 or only 2?

12. $47|\overline{992}$ Are there 2 47's in 99 or only 1?

13. $27|\overline{538}$ Are there 2 27's in 53 or only 1?

14. $17|\overline{476}$ Are there 3 17's in 47 or only 2?

15. $358|\overline{1062}$ Try 2 as the quotient figure. How do you know 2 is right and not 3?

16. $139|\overline{276}$ Try 1. Why is 2 wrong?

17. $312|\overline{1249}$ Try 4. Why is 3 wrong?

18. $151|\overline{375}$ Shall you try 3 or 2?

19. $123|\overline{375}$ Shall you try 3 or 2?

20. $225|\overline{650}$ Shall you try 3 or 2?

21. $25|\overline{425}$ Shall you try 2 or 1?

22. $15|\overline{470}$ Shall you try 4 or 3?

(*Without pencil.*)

1. The correct quotient for $395\overline{)302175}$ is 765.

a. State the correct quotient for $3.95\overline{)30.2175.}$

b. State the correct quotient for $39.5\overline{)30.2175.}$

c. State the correct quotient for $.395\overline{)30.2175.}$

d. State the correct quotient for $3.95\overline{)3021.75.}$

e. State the correct quotient for $39.5\overline{)302.175.}$

f. State the correct quotient for $395\overline{)3021.75.}$

2. State the correct quotients when the numbers have decimal points as they have here.

A.	B.	C.
736 is right	*628 is right*	*650 is right*
$349\overline{)256864}$	$924\overline{)580272}$	$475\overline{)308750}$
$34.9\overline{)2568.64}$	$9.24\overline{)5802.72}$	$.475\overline{)30.8750}$
$.349\overline{)256.864}$	$9.24\overline{)58.0272}$	$4.75\overline{)308.750}$
$3.49\overline{)2.56864}$	$92.4\overline{)5802.72}$	$47.5\overline{)308750}$
$3.49\overline{)25.6864}$	$.924\overline{)58.0272}$	$475\overline{)30.8750}$
$34.9\overline{)25.6864}$	$.924\overline{)5.80272}$	$4.75\overline{)30.8750}$
$.349\overline{)2.56864}$	$92.4\overline{)5802.72}$	$.475\overline{)3.08750}$
$349\overline{)2568.64}$	$924\overline{)580.272}$	$475\overline{)308.750}$
$349\overline{)256.864}$	$924\overline{)58027.2}$	$47.5\overline{)308.750}$

"Percent of" Means "Hundredths Times"

1. Read, supplying the missing numbers:

a. 5 percent of 30 means $\frac{5}{100}$ of 30, or $.05 \times 30$, or...

b. 6 percent of 30 means $\frac{6}{100}$ of 30, or $.06 \times 30$, or...

c. 12 percent of 50 means $\frac{12}{100}$ of 50, or $.12 \times 50$, or...

d. 95 percent of 100 means $.95 \times 100$ or...

e. 4 percent of 25 means... ×25 or...

f. 8 percent of 120 means... ×120 or...

g. 15 percent of 30 means... ×30 or...

h. 18 percent of 1000 means... ×1000 or...

THE SPECIALIZATION OF HABITS

An arithmetical bond, say between 6×7 and 42, may operate perfectly if just the same conditions are maintained as existed during its formation, but may operate imperfectly or even not at all if these conditions are somewhat altered. The pupil who answers perfectly to $6\times7=$ . . . may thus fail in $\begin{array}{r}378\\ 6\\ \hline\end{array}$ when he has to keep in mind the 4 to be added and is oppressed by the fact that what he gets from 6×7 must have something added to it, only a part of the result written down, and another part kept in mind for later use. Thus ability with $3+9=12$ does not imply ability with $13+9=22$ or $23+9=32$. It even occurs sometimes that a pupil who has no difficulty in adding 5 to a 6 that he sees, may have difficulty in adding $\begin{array}{c}5\\ 2\\ 4\\ \hline\end{array}$ where the 6 is not seen but thought of

Theoretically any change in the accompanying conditions or circumstances may disturb the operation of any bond or mental connection. And in actual experience it is found that changes in accompanying conditions to which the older methods paid no attention do often seriously interfere with the bonds or habits concerned. So the newer methods provide against disturbance from such changed conditions, making it a rule to give such help in adapting the habit to the new circumstances as is feasible. Sometimes much help is needed, as in extending the habit of correct response to the addition combinations to their use in higher decades. Sometimes only a slight direction is needed, as in using $3+3=6$ to answer two 3's = ? or in using $4+4=8$ to answer two 4's = ?

It is important to give just enough special practice in each case; it is still more important to give the right sort of practice. Consider, for example, bridging the gap between ability to use the fundamental multiplication bonds, 1×1 to 9×9, each by itself alone and ability to use them in examples like $\begin{array}{r}729\\ \hline 4\end{array}$, $\begin{array}{r}648\\ \hline 9\end{array}$, etc.

Consider these two methods:

A. Proceeding directly to the latter exercises, but explaining the need of keeping in mind the number that has been "carried," and remembering what is to be done with the product obtained, and giving exercises such as "Multiply each of these numbers by 7 and add 2 to the product: 6, 4, 3, 9, 2, 5, 7, 1, 8. Multiply each by 5 and add 3 to the product," etc.

B. Giving many exercises like the following: $(4\times7)+3$, $(5\times9)+8$, $(6\times8)+4$, before proceeding to exercises with $\begin{array}{r}729\\ \hline 4\end{array}$, $\begin{array}{r}748\\ \hline 9\end{array}$, etc.

Method B is defensible, but is probably not so good as method A, for it includes teaching the use of the parenthesis, which is perhaps as troublesome to many pupils as the new habits themselves. Also the number to be added is, in these exercises, always visible to the eye; whereas in those of method A it is more likely to be held in the mind, as it should be in the real multiplication.

C. Permitting the pupil to write down the number to be carried, so that he does not have to remember it.

This method seems much worse than either A or B. It is permissible, of course, to use this crutch for a few times to make sure that the pupil learns *how to do* such multiplications, but to continue it for a long time just to save him the work of forming the new habit is simply avoiding a difficulty, not conquering it. It may actually make it harder for the pupil to shift to the right habit later. If the use of this crutch is continued with multiplication by two- and three-figure multipliers, there results a strain on the eye and mind in picking out the partial-product figures which are to be added from the "crutch" figures which are to be neglected. Errors result. Also, if pupils are well taught, they will be able to multiply more rapidly without this crutch than with it.

NEGLECTED HABITS

In general, the newer methods pay more attention to habit formation than did the older methods. The latter very often assumed that the pupil would reason out a certain procedure and use it without the need of special practice with it. Actual classroom experience, however, proves that in many cases where such an assumption was made it was true only of the most gifted pupils. It is not safe to assume that the rank and file of a class will form bonds of their own initiative, even in cases that seem to us very easy. They seem easy to us, in fact, partly because we have them already formed.

Thus it is not safe to assume that the formation of the bonds $9+4 \rightarrow 13$ and $6 \times 9 \rightarrow 54$, will insure the formation of $4+9 \rightarrow 13$ and $9 \times 6 \rightarrow 54$. In general, the reverses of the combinations need some separate attention and drill. It is not safe to assume that pupils who can respond to 54 = . . . 9's and 63 = . . . 9's perfectly will be able to respond perfectly to 58 = . . . 9's and . . . remainder, 70 = . . . 9's and . . . remainder. On the contrary, it is much safer to assume that three out of four will not. The newer methods give specific drill on all the divisions with remainders, using such exercises as those shown on pages 65–67.

A pupil who has learned that "a divided by $b=c$" will be very much puzzled when confronted by "$a=b \times$ what?" "How many b's $=a$?" "$a=$ what $\times\, b$?" and "How many times as large as b is a?" That is, each of the important equational and verbal forms in which the division facts may appear needs special drill.

Buying Stationery

Pencils, 2¢ each.
Penholders, 3¢ each.
Erasers, 4¢ each.
Ink, 5¢ a bottle.
Envelopes, 6¢ a package.
Crayons, 7¢ a box.
Pads, 8¢ each.
Notebooks, 9¢ each.

Supply the missing numbers:

A.
For 10¢ you get...pencils.
For 10¢ you get...penholders and...¢ change.
For 10¢ you get...erasers and...¢ change.
For 10¢ you get...bottles of ink.
For 15¢ you get...pencils and...¢ change.
For 15¢ you get...penholders.
For 15¢ you get...erasers and...¢ change.

B.
10=...3's and...remainder.
10=...4's and...remainder.
10=...5's and no remainder.
15=...2's and...remainder.
15=...3's.
15=...4's and...remainder.
15=...5's.
15=...6's and...remainder.
15=...7's and...remainder.

C.
5=...2's and...remainder.
5=...3 and...remainder.
5=...4 and...remainder.
6=...2's.
6=...3's.
6=...4 and...remainder.
6=...5 and...remainder.
7=...2's and...remainder.
7=...3's and...remainder.

Read these lines. Say the right numbers where the dots are. Read "remainder" where you see r.

D.
8=...2's.
8=...3's and...r.
8=...4's.
8=...5 and...r.
3=...6 and...r.

E.
9=...2's and...r.
9=...3's.
9=...4's and...r.
9=...5 and...r.
9=...6 and...r.

F.
11=...2's and...r.
11=...3's and...r.
11=...4's and...r.
11=...5's and...r.
11=...6 and...r.

A Remainder Race

Read these, saying the right numbers where the dots are. Read "remainder" for r.

When you know them all, ask the teacher to have a race, to see how many each child can do correctly in 60 seconds.

A.
12 = . . . 2's.
12 = . . . 3's.
12 = . . . 4's.
12 = . . . 5's and . . . r.
12 = . . . 6's
12 = . . . 7's and . . . r.
12 = . . . 8's and . . . r.
12 = . . . 9's and . . . r.

13 = . . . 2's and . . . r.
13 = . . . 3's and . . . r.
13 = . . . 4's and . . . r.
13 = . . . 5's and . . . r.
13 = . . . 6's and . . . r.
13 = . . . 7's and . . . r.
13 = . . . 8's and . . . r.
13 = . . . 9's and . . . r.

14 = . . . 2's.
14 = . . . 3's and . . . r.
14 = . . . 4's and . . . r.
14 = . . . 5's and . . . r.
14 = . . . 6's and . . . r.
14 = . . . 7's.
14 = . . . 8's and . . . r.
14 = . . . 9's and . . . r.

B.
16 = . . . 2's.
16 = . . . 3's and . . . r.
16 = . . . 4's.
16 = . . . 5's and . . . r.
16 = . . . 6's and . . . r.
16 = . . . 7's and . . . r.
16 = . . . 8's.
16 = . . . 9's and . . . r.

17 = . . . 2's and . . . r.
17 = . . . 3's and . . . r.
17 = . . . 4's and . . . r.
17 = . . . 5's and . . . r.
17 = . . . 6's and . . . r.
17 = . . . 7's and . . . r.
17 = . . . 8's and . . . r.
17 = . . . 9's and . . . r.

18 = . . . 2's.
18 = . . . 3's.
18 = . . . 4's and . . . r.
18 = . . . 5's and . . . r.
18 = . . . 6's.
18 = . . . 7's and . . . r.
18 = . . . 8's and . . . r.
18 = . . . 9's.

C.
19 = . . . 2's and . . . r.
19 = . . . 3's and . . . r.
19 = . . . 4's and . . . r.
19 = . . . 5's and . . . r.
19 = . . . 6's and . . . r.
19 = . . . 7's and . . . r.
19 = . . . 8's and . . . r.
19 = . . . 9's and . . . r.

20 = . . . 2's.
20 = . . . 3's and . . . r.
20 = . . . 4's.
20 = . . . 5's.
20 = . . . 6's and . . . r.
20 = . . . 7's and . . . r.
20 = . . . 8's and . . . r.
20 = . . . 9's and . . . r.

21 = . . . 2's and . . . r.
21 = . . . 3's.
21 = . . . 4's and . . . r.
21 = . . . 5's and . . . r.
21 = . . . 6's and . . . r.
21 = . . . 7's.
21 = . . . 8's and . . . r.
21 = . . . 9's and . . . r.

Quotients and Remainders

State quotient and remainder for each of these:

A.	B.	C.	D.	E.	F.
22 = . . . 3's and . . . r.	$6\overline{)25}$	$9\overline{)28}$	$8\overline{)32}$	$8\overline{)36}$	$9\overline{)40}$
22 = . . . 4's and . . . r.	$7\overline{)25}$	$3\overline{)29}$	$9\overline{)32}$	$9\overline{)36}$	$5\overline{)41}$
22 = . . . 5's and . . . r.	$8\overline{)25}$	$4\overline{)29}$	$4\overline{)33}$	$4\overline{)37}$	$6\overline{)41}$
22 = . . . 6's and . . . r.	$9\overline{)25}$	$5\overline{)29}$	$5\overline{)33}$	$5\overline{)37}$	$7\overline{)41}$
22 = . . . 7's and . . . r.	$3\overline{)26}$	$6\overline{)29}$	$6\overline{)33}$	$6\overline{)37}$	$8\overline{)41}$
22 = . . . 8's and . . . r.	$4\overline{)26}$	$7\overline{)29}$	$7\overline{)33}$	$7\overline{)37}$	$9\overline{)41}$
22 = . . . 9's and . . . r.	$5\overline{)26}$	$8\overline{)29}$	$8\overline{)33}$	$8\overline{)37}$	$5\overline{)42}$
23 = . . . 3's and . . . r.	$6\overline{)26}$	$9\overline{)29}$	$9\overline{)33}$	$9\overline{)37}$	$6\overline{)42}$
23 = . . . 4's and . . . r.	$7\overline{)26}$	$4\overline{)30}$	$4\overline{)34}$	$4\overline{)38}$	$7\overline{)42}$
23 = . . . 5's and . . . r.	$8\overline{)26}$	$5\overline{)30}$	$5\overline{)34}$	$5\overline{)38}$	$8\overline{)42}$
23 = . . . 6's and . . . r.	$9\overline{)26}$	$6\overline{)30}$	$6\overline{)34}$	$6\overline{)38}$	$9\overline{)42}$
23 = . . . 7's and . . . r.	$3\overline{)27}$	$7\overline{)30}$	$7\overline{)34}$	$7\overline{)38}$	$5\overline{)43}$
23 = . . . 8's and . . . r.	$4\overline{)27}$	$8\overline{)30}$	$8\overline{)34}$	$8\overline{)38}$	$6\overline{)43}$
23 = . . . 9's and . . . r.	$5\overline{)27}$	$9\overline{)30}$	$9\overline{)34}$	$9\overline{)38}$	$7\overline{)43}$
24 = . . . 3's.	$6\overline{)27}$	$4\overline{)31}$	$4\overline{)35}$	$4\overline{)39}$	$8\overline{)43}$
24 = . . . 4's.	$7\overline{)27}$	$5\overline{)31}$	$5\overline{)35}$	$5\overline{)39}$	$9\overline{)43}$
24 = . . . 5's and . . . r.	$8\overline{)27}$	$6\overline{)31}$	$6\overline{)35}$	$6\overline{)39}$	$5\overline{)44}$
24 = . . . 6's.	$9\overline{)27}$	$7\overline{)31}$	$7\overline{)35}$	$7\overline{)39}$	$6\overline{)44}$
24 = . . . 7's and . . . r.	$3\overline{)28}$	$8\overline{)31}$	$8\overline{)35}$	$8\overline{)39}$	$7\overline{)44}$
24 = . . . 8's.	$4\overline{)28}$	$9\overline{)31}$	$9\overline{)35}$	$9\overline{)39}$	$8\overline{)44}$
24 = . . . 9's and . . . r.	$5\overline{)28}$	$4\overline{)32}$	$4\overline{)36}$	$5\overline{)40}$	$9\overline{)44}$
25 = . . . 3's and . . . r.	$6\overline{)28}$	$5\overline{)32}$	$5\overline{)36}$	$6\overline{)40}$	$5\overline{)45}$
25 = . . . 4's and . . . r.	$7\overline{)28}$	$6\overline{)32}$	$6\overline{)36}$	$7\overline{)40}$	$6\overline{)45}$
25 = . . . 5's.	$8\overline{)28}$	$7\overline{)32}$	$7\overline{)36}$	$8\overline{)40}$	$7\overline{)45}$

Repeat this page until you can give all the quotients and remainders correctly in 20 minutes or less.

A pupil who has learned to respond correctly to "2 is what part of 4?" "4 is what part of 10?" and "6 is what part of 8?" etc. (and who knows how to divide by a fraction), may still be insecure or even entirely baffled in the case of "$\frac{3}{8}$ is what part of $1\frac{1}{2}$?" "$\frac{3}{4}$ is what part of $2\frac{1}{4}$?" He knows that "what part of" requires a division and how to divide by a fraction, but he may be unable to combine the two facts so as to inaugurate the new habit.

The newer methods, of course, stimulate the pupil to reason out what is to be done so far as he is able. They do not favor mechanical as against rational learning. But they take pains to see that somehow he does actually acquire the new habits, form the new bonds, and have enough practice with them to keep them alive and active.

THE AMOUNT AND DISTRIBUTION OF PRACTICE

By the newer methods, then, proper motivation is secured for the first formation and for continued exercise of arithmetical bonds, each bond is adapted to its various uses, and every bond that needs to be formed is taken care of. Further, the amount of drill on each is made sufficient but not wasteful, and it is distributed throughout the elementary-school course so as to come when it is needed.

The older methods were careless in these respects, as appears from an actual inventory of the amount of practice given and its arrangement, samples of which are shown in the tables on pages 69–74.

From these tables it appears that the older methods were careless about the amount of practice, giving far too little, relatively, to the harder facts. It is surely unwise to give only one-fourth as much practice on $8+8$ as on $2+2$, only one-eighth as much practice on 9×8 as on 2×2, and less than a tenth as much practice on $17-8$ as on $2-2$. They also probably gave too little practice, absolutely, to some of the facts. Surely 60÷7, 60÷8, 60÷9, 61÷7, 61÷8, 61÷9, and the like should occur oftener than once a year!

TABLE 1

AMOUNT OF PRACTICE: ADDITION BONDS IN A RECENT TEXTBOOK (A) OF EXCELLENT REPUTE, BOOKS I AND II, ALL SAVE "SUPPLEMENTARY" AT ENDS OF PARTS 1, 2, 3, AND 4

The table reads 2 + 2 was used 226 times, 12 + 2 was used 74 times, 22 + 2, 32 + 2, 42 + 2, and so on, were used 50 times.

1	2	3	4	5	6	7	8	9	
2	226	154	162	150	97	87	66	45	.....
12	74	53	76	46	51	37	36	33	.....
22, etc.	50	60	68	63	42	50	38	26	.....
3	216	141	127	89	82	54	58	40	.....
13	43	43	60	70	52	30	22	18	.....
23, etc.	15	30	51	50	42	32	29	30	.....
7	85	90	103	103	84	81	61	47	.....
17	33	25	42	32	35	21	29	16	.....
27, etc.	30	23	32	29	24	23	25	28	.....
8	185	112	146	99	75	71	73	61	.....
18	28	35	52	46	28	29	24	14	.....
28, etc.	53	36	34	38	23	36	27	27	.....
9	104	81	112	96	63	74	58	57	.....
19	13	11	31	38	25	14	22	11	....
29, etc.	19	17	27	20	32	32	19	18	.....
2, 12, 22, etc. +	350	267	306	259	190	174	140	104	1790
3, 13, 23, etc.	274	214	238	209	176	116	109	88	1424
7, 17, 27, etc.	148	138	177	164	143	125	115	91	1101
8, 12, 28, etc.	266	183	232	183	126	136	124	102	1352
9, 19, 28, etc.	136	109	170	154	120	120	99	86	994
Totals...	1174	911	1123	969	755	671	587	471	

It is not to be expected that a perfect adjustment of the amount and distribution of drill will be made, for there are many others needs that have to be considered. For example, when some new process is being explained, the numbers used in connection with it should deliberately be made easy to handle so that attention can be focused on the process itself. Also, some bonds, such as 3+3, need much practice very early and necessarily receive an excess of practice later in

TABLE 2

Amount of Practice: Subtraction Bonds in a Recent Textbook (A) of Excellent Repute, Books I and II, All Save "Supplementary" at Ends of Parts 1, 2, 3, and 4

Frequencies of Subtractions of: 1 from 1, 1 or 2 from 2, 1, 2, or 3 from 3, etc.

Minuends	Subtrahends								
	1	2	3	4	5	6	7	8	9
1	372								
2	214	311							
3	136	149	189						
4	146	142	103	205					
5	171	91	92	164	136				
6	80	59	69	71	81	192			
7	106	57	55	67	59	156	80		
8	73	50	50	75	50	62	48	152	
9	71	75	54	74	48	55	55	124	133
10	261	84	63	100	193	83	57	124	91
11		48	31	50	36	41	32	46	35
12			48	77	57	51	35	80	30
13				35	22	40	29	35	28
14					25	37	36	49	32
15						33	19	48	20
16							16	36	26
17								27	20
18									19
otal excluding 1 – 1, 2 – 2, etc.	1258	755	565	613	571	558	327	569	301

TABLE 3

Frequencies of Subtractions Not Included in Table 2

These are cases where the pupil would by reason of his stage of advancement probably operate 35 — 30, 46 — 46, etc., as one bond.

Minuends	Subtrahend									
	1 11 21 etc.	2 12 22 etc.	3 13 23 etc.	4 14 24 etc.	5 15 25 etc.	6 16 26 etc.	7 17 27 etc.	8 1 2 etc.	9 19 29 etc.	10 20 etc.
10, 20, 30, 40, etc.	11	29	16	52	32	51	7	30	22	69
11, 21, 31, 41, etc.	42	14	22	32	12	26	19	52	17	10
12, 22, 32, 42, etc.	47	97	5	13	9	21	11	24	19	17
13, 23, 33, 43, etc.	7	40	7	14	15	13	19	19	22	3
14, 24, 34, 44, etc.	8	28	14	58	13	16	14	26	19	7
15, 25, 35, 45, etc.	21	28	29	54	51	15	21	12	24	8
16, 26, 36, 46, etc.	5	18	12	27	35	69	13	17	19	2
17, 27, 37, 47, etc.	5	9	12	40	32	54	24	12	12	1
18, 28, 38, 48, etc.	2	16	10	23	22	36	18	47	16	0
19, 29, 39, 49, etc.	5	7	7	10	13	28	14	23	16	0
Totals	153	276	134	323	234	329	160	261	186	117

TABLE 4

AMOUNT OF PRACTICE: MULTIPLICATION BONDS, IN ANOTHER RECENT TEXTBOOK (B) OF EXCELLENT REPUTE, BOOKS I AND II

MULTIPLIERS	MULTIPLICANDS										TOTALS
	0	1	2	3	4	5	6	7	8	9	
1........	299	534	472	271	310	293	261	178	195	99	2912
2........	350	644	668	480	458	377	332	238	239	155	3941
3........	280	487	509	388	318	302	247	199	227	152	3109
4........	186	375	398	242	203	265	197	163	159	93	2281
5........	268	359	393	234	263	243	217	192	197	114	2480
6........	180	284	265	199	196	191	148	169	165	106	1923
7........	135	283	277	176	187	158	155	121	145	118	1755
8........	137	272	292	175	192	164	158	157	126	126	1799
9........	71	173	140	122	97	102	101	100	82	110	1098
Totals .	1906	3411	3414	2287	2224	2095	1836	1517	1535	1073	......

TABLE 5

AMOUNT OF PRACTICE: DIVISIONS WITHOUT REMAINDER IN TEXTBOOK B, BOOKS I AND II

DIVIDENDS	DIVISORS								
	2	3	4	5	6	7	8	9	TOTALS
Integral multiples of 2 to 9, in sequence, i.e., 4 ÷ 2 occurred 397 times, 6 ÷ 2 occurred 256 times, 6 ÷ 3, 224 times, 9 ÷ 3, 124 times.	397	224	250	130	93	44	98	23	1259
	256	124	152	79	28	43	61	25	768
	318	123	130	65	50	19	39	19	763
	258	98	86	105	25	24	34	20	650
	198	49	76	27	22	30	33	16	451
	77	54	36	31	28	27	16	9	278
	180	91	50	38	17	13	22	16	427
	69	46	37	24	12	17	16	15	236
Totals.............	1753	809	817	499	275	217	319	142	......

$\frac{3}{4}+\frac{3}{4}$, $\frac{3}{8}+\frac{3}{8}$, etc. Also some bonds, like the products of 5, though easy to form, will occur often because of their frequent use in life.

It should also be noted that some "overlearning" or practice with a bond after it is well enough learned does relatively little harm. If, say, $4\times3=12$ is very well learned, it will take only a second to act so that even three hundred more practices than are needed will mean only a loss of 5 minutes.

TABLE 6

Division Bonds, with and without Remainders, Book B

All work through Grade 6, except estimates of quotient figures in long division

Dividend	**2**			**3**				**4**					**5**						**6**							
Divisor	1	2		1	2	3		1	2	3	4		1	2	3	4	5		1	2	3	4	5	6		
Number of occurrences	41	386		27	189	240		26	397	66	185		23	136	43	53	135		21	256	224	68	43	83		
Dividend	**7**								**8**									**9**								
Divisor	1	2	3	4	5	6	7		1	2	3	4	5	6	7	8		1	2	3	4	5	6	7	8	9
Number of occurrences	23	72	55	38	46	32	54		17	318	30	250	22	28	39	91		19	50	124	49	25	15	18	30	38
Dividend	**10**									**11**									**12**							
Divisor	2	3	4	5	6	7	8	9		2	3	4	5	6	7	8	9		2	3	4	5	6	4	8	9
Number of occurrences	258	38	46	120	19	9	24	24		32	21	16	3	7	11	14	3		198	123	152	29	93	9	16	7
Dividend	**13**									**14**									**15**							
Divisor	2	3	4	5	6	7	8	9		2	3	4	5	6	7	8	9		2	3	4	5	6	7	8	9
Number of occurrences	45	16	15	11	7	4	5	3		77	20	13	5	8	44	8	6		69	98	16	79	8	8	4	6
Dividend	**16**									**17**									**18**							
Divisor	2	3	4	5	6	7	8	9		2	3	4	5	6	7	8	9		2	3	4	5	6	7	8	9
Number of occurrences	180	19	130	14	6	9	98	3		61	9	15	14	6	6	12	3		69	49	13	6	28	7	7	23
Dividend	**19**									**20**								**21**								
Divisor	2	3	4	5	6	7	8	9		3	4	5	6	7	8	9		3	4	5	6	7	8	9		
Number of occurrences	21	6	10	5	3	4	10	4		24	86	65	11	3	23	5		54	12	8	5	43	10	5		
Dividend	**22**								**23**								**24**									
Divisor	3	4	5	6	7	8	9		3	4	5	6	7	8	9		3	4	5	6	7	8	9			
Number of occurrences	17	16	15	8	13	6	15		7	8	11	8	6	3	2		91	76	18	50	5	61	1			

Dividend	25								26								27										
Divisor	3	4	5	6	7	8	9		3	4	5	6	7	8	9		3	4	5	6	7	8	9				
Number of occurrences	11	13	105	5	6	5	3		5	6	3	3	4	6	3		46	8	10	4	2	6	25				
Dividend	28								29								30										
Divisor	3	4	5	6	7	8	9		3	4	5	6	7	8	9		4	5	6	7	8	9					
Number of occurrences	4	36	8	3	19	3	7		6	8	0	5	11	2	3		21	27	25	6	7	13					
Dividend	31							32							33							34					
Divisor	4	5	6	7	8	9		4	5	6	7	8	9		4	5	6	7	8	9		4	5	6	7	8	9
Number of occurrences	4	3	1	1	4	2		50	11	3	6	39	5		8	7	7	2	6	1		8	3	5	2	1	1
Dividend	35							36							37							38					
Divisor	4	5	6	7	8	9		4	5	6	7	8	9		4	5	6	7	8	9		4	5	6	7	8	9
Number of occurrences	10	31	5	24	5	3		37	16	22	2	6	19		12	8	7	5	3	9		7	8	7	1	1	5
Dividend	39							40						41						42							
Divisor	4	5	6	7	8	9		5	6	7	8	9		5	6	7	8	9		5	6	7	8	9			
Number of occurrences	4	3	7	4	3	1		38	9	2	34	2		6	6	3	7	5		7	28	30	10	3			
Dividend	43						44						45						46								
Divisor	5	6	7	8	9		5	6	7	8	9		5	6	7	8	9		5	6	7	8	9				
Number of occurrences	7	5	10	13	3		7	6	4	5	0		24	6	7	10	20		3	3	2	2	2				
Dividend	47						48						49						50					51			
Divisor	5	6	7	8	9		5	6	7	8	9		5	6	7	8	9		6	7	8	9		6	7	8	9
Number of occurrences	6	2	2	0	3		7	17	4	33	2		4	7	27	9	2		4	6	3	8		2	3	1	2

TABLE 6 — *Continued*

DIVISION BONDS, WITH AND WITHOUT REMAINDERS, BOOK B

All work through Grade 6, except estimates of quotient figures in long division

Dividend	52					53					54					55					56					
Divisor	6	7	8	9		6	7	8	9		6	7	8	9		6	7	8	9		6	7	8	9		
Number of occurrences	5	5	5	3		4	3	2	2		12	5	1	16		5	3	4	2		0	13	16	8		
Dividend	57					58					59					60				61						
Divisor	6	7	8	9		6	7	8	9		6	7	8	9		7	8	9		7	8	9				
Number of occurrences	0	3	1	3		2	2	3	1		2	3	0	3		3	9	1		1	2	5				
Dividend	62				63				64				65				66				67					
Divisor	7	8	9		7	8	9		7	8	9		7	8	9		7	8	9		7	8	9			
Number of occurrences	4	6	1		17	5	9		5	22	0		1	10	1		2	1	4		0	1	1			
Dividend	68				69				70			71			72			73			74			75		
Divisor	7	8	9		7	8	9		8	9		8	9		8	9		8	9		8	9		8	9	
Number of occurrences	1	3	2		0	6	1		6	2		1	0		16	10		7	5		3	3		5	3	
Dividend	76			77			78			79			80		81		82		83		84		85			
Divisor	8	9		8	9		8	9		8	9		9		9		9		9		9		9			
Number of occurrences	3	2		3	0		4	1		0	2		4		15		2		4		1		2			
Dividend	86		87		88		89																			
Divisor	9		9		9		9																			
Number of occurrences	0		3		2		7																			

It is the "underlearning" of the hard bonds rather than any overlearning of the easy bonds which is the chief defect in the four cases given.

Finally, it should be repeated that interest and skillful arrangement to help the learner are very much more important than any control of the mere amount of drill.

It is, however, possible to provide reasonably for adequate but not wasteful drill without any sacrifice of interest and skillful arrangement, or of the general excellence of the teaching of arithmetic. The newer methods try to do this.

The best method of distributing the practice with a bond or group of bonds seems to be to give at the time of first learning enough practice to form the bond rather well, and then to give practice in smaller and smaller amounts at longer and longer intervals as shown in Fig. 1. This holds so far as the learning and retention of the bond itself is concerned. Its connection with other arithmetical bonds and use in relation to practical problems are matters worthy of at least equal consideration.

FIGURE 1

The commonest errors of teachers and textbooks are:

(1) To give too much of the practice at the first learning.

(2) To leave too long intervals with no practice.

(3) To leave a group of bonds in too great isolation from others with which they should be connected.

If very much of the practice is concentrated at the time of first learning, not only will there be insufficient review, but the first learning may become so monotonous as to be unthinking and consequently unprofitable. If the interval is too long, not only is the bond itself lost, but there may be various difficulties in the formation of other bonds where its help is needed. If the connections and correlations that are needed are not made, we leave the pupil with his knowledge more or less in separate compartments, unable to combine old knowledge in a new emergency, able to answer questions only when asked just as his teacher asked them, ready to use arithmetic only when the circumstances under which he learned it are reinstated.

The proper distribution of practice for each of all the different abilities to be developed by arithmetic thus becomes a delicate and complicated affair. The individual teacher cannot be expected to attend to it fully. If the textbook or course of study which is her guide does it well, her teaching will be made easy and effective. If the textbook or course of study is careless about this, her teaching will suffer. She can reduce the injury only by omitting excessive practice at certain points and supplying it at others.

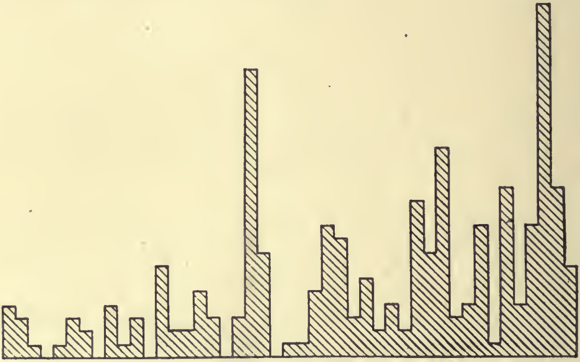

FIGURE 2

Fig. 2 shows the distribution of practice on 5×5 in the first two books of the three-book series E. The diagram thus represents nearly four years of school work, from near the beginning of Grade 3 to the end of Grade 6. Each fifteenth of an inch along the base-line represents ten pages of the textbook in question (beginning with the first treatment of 5×5). Each two-hundred-twenty-fifth of a square inch of the shaded area represents one occurrence of 5×5, assuming that a pupil did all the work offered. That is, in doing all the work of the first ten pages in which 5×5 first appeared, he would have to think 5×5=25 four times; in the next ten pages, three times; in the next ten pages, once; in the next ten pages, not at all; in the next ten pages, once, and so on.

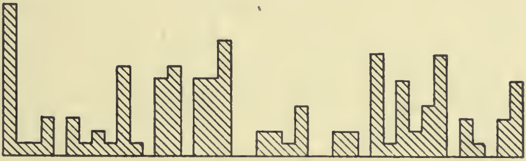

FIGURE 3

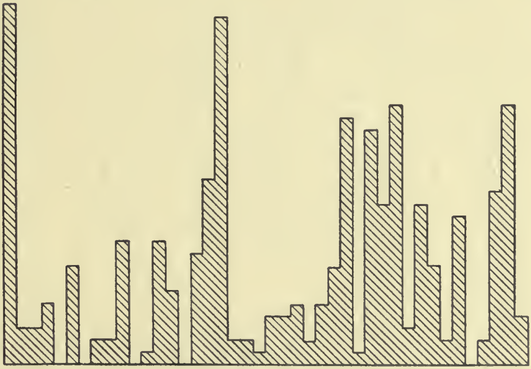

FIGURE 4

Figs. 3, 4, and 5 show in just the same way the distribution of the practice on 7×7, on 6×7 and 7×6 together, and on 81, 82, 83, 84, 85, 86, 87, 88, and 89 (all of these) divided by 9. Any occurrence is counted, whether in a practice drill or a problem, whether in work with integers or with common or decimal fractions or percents.

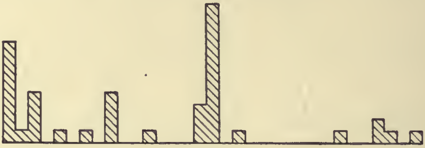

FIGURE 5

These diagrams show no consistent plan for distributing practice, nor is any one of the four a very good plan. In general, the older methods were very careless about it. The newer methods try to distribute practice in the best possible way that is consistent with the other desirable features of the general teaching plan.

EXERCISES

1. In the case of each of the ten following drill lessons, which motives of those listed below are used, beyond the general interest in mental activity and achievement, to add zest to the drill? Use the abbreviation before the motives to save time in writing.

I, 7, upper half; I, 140, 141, Section 19; I, 180; II, 34; II, 49; II, 101; II, 238; III, 31; III, 136; III, 138.

(1)	Phy.	Opportunity for physical activity
(2)	Puz.	The puzzle interest
(3)	Pri.	Pride
(4)	Nov.	Novelty
(5)	Pra.	Practical use in life
(6)	Chi.	Interest in other children and what they do
(7)	Soc.	Sociability and group action
(8)	I. C.	Interest in individual competition
(9)	G. C.	Interest in group competition
(10)	Self D.	Interest in directing one's self

2. In many cases just a slight suggestion of competing, or of a race, or of a game, or of a definite attainable standard to be reached, or of genuine use in life will add to the satisfyingness of success in acquiring or improving an ability. What is the suggestion in each of the following ten cases: I, 8; I, 25; I, 31 and 33, Section 51; I, 48; I, 118, 119; I, 213 or 214; II, 46, lower half; II, 178; II, 221; III, 164, Ex. 5?

3. What criticisms have you on this page of review practice in multiplication? 72 examples, 8 of a 3-place by a 2-place number, 16 of a 3-place by a 3-place number, 22 of a 4-place by a 4-place number, 6 of a 5-place by a 2-place number, 18 of a 4-place by a 3-place number, and 2 of a 5-place by a 3-place number. This is preceded by 168 cases of a 3-place by a 3-place number, from which, however, the teacher is to select only what she thinks wise.

4. What criticisms have you of the following treatment of long division as a review at the beginning of Grade 5? $1\frac{1}{4}$ pages of explanation; 54 examples, 18 of 3-place numbers, 12 of 4-place numbers, 24 of 5-place numbers as dividends, the divisors being 11 or 21 in 20 cases and 31, 41, 51, 61, 71, or 91 in the others; $\frac{2}{3}$ page of further explanation; 18 examples of 4-, 5-, and 6-place numbers divided by 2-place numbers; a page of miscellaneous problems; a page of further explanation and rules; 50 examples, almost all of 6-place numbers divided by 2-, 3-, or 4-place numbers; a page of problems; $\frac{1}{2}$ page of explanation of division of United States money; 18 examples with United States money as dividends (5 to 8 places) with 2-, 3-, and 4-place divisors.

5. What three distinct habits are used in learning the divisions by 6? (I, 73.) What fourth habit is added in the case of the 7's? (I, 78.)

6. What specialization of habit is provided for by Ex. 1 on page 199 of Book II? What further precaution is taken in the same lesson to insure the correct action of the habit?

The table below gives the number of occurrences of multiplications with various multipliers in four textbooks, including all work through Grade 6, except as noted:

X means any digit except 0
XXX thus means a multiplier like 385 or 419
XXO means a multiplier like 380 or 410
XOX means a multiplier like 305 or 409
XX means a multiplier like 47 or 52
XO means a multiplier like 20 or 70
XOO means a multiplier like 700 or 500

Cases of multiplication by 10 are not counted as XO, but listed separately as 10.

FREQUENCY OF OCCURRENCE OF MULTIPLICATIONS WITH DIFFERENT SORTS OF MULTIPLIERS

	XO	XOO	XX	XXO	XXX	XOX	10
A...........	198	55	725	75	155	33	73
B*..........	114	38	287	8	60	55	†
C...........	107	30	478	27	93	42	55
D...........	159	21	377	33	91	53	131

* Book B has also three sets of materials for computation to be used at the teacher's discretion, comprising 9 pages in all, from which by various shifts there may be made up 32 XO, 22 XOO, 113 XX, 64 XXO, 68 XXX, and 10 XOX multiplications. These are, however, not to be considered in answering question 7.

† The 10 cases were not recorded for Book B.

7. Which seems to you to give the most suitable amount of practice with multipliers of the XX and XXX type, assuming that reasonable care and skill are devoted to making the work satisfying?

8. Which seems to give the most suitable amount of practice with multipliers of the XOX type?

9. The diagrams below and on page 82 give the distribution of practice with multipliers of the XOX type in the four books. Which two seem to you the best distributions of the four?

Figure 6

Figure 7

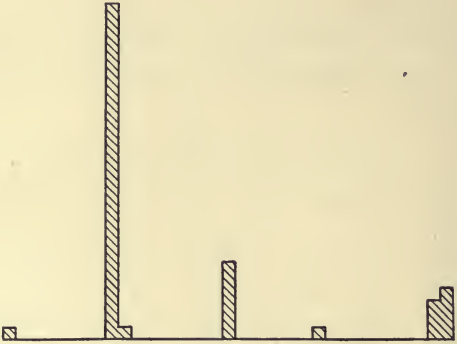

FIGURE 8

FIGURE 9

10. What do you think is the explanation of the single case of this, one of the hardest things in multiplication, in Book C weeks ahead of the time when it is regularly taught?

CHAPTER V

THE ORGANIZATION OF LEARNING

THE OLDER SYSTEM

The older scheme of organization of arithmetical learning was beautiful to look at, but very hard to learn by. The pupil was supposed to learn in order:

To read, write, and understand integers
To add with integers
To subtract with integers
To multiply with integers
To divide with integers
To read, write, and understand United States money
To add with United States money
To subtract with United States money
To multiply with United States money
To divide with United States money
To read, write, and understand fractions
To reduce them to higher and lower terms
To find the least common multiple
To add with common fractions, then with mixed numbers
To subtract with common fractions, then with mixed numbers
To multiply with common fractions, then with mixed numbers
To divide with common fractions, then with mixed numbers
To read, write, and understand decimal numbers
To reduce common fractions to decimals and vice versa
To add with decimal fractions and decimal mixed numbers
To subtract with decimal fractions and decimal mixed numbers

To multiply with decimal fractions and decimal mixed numbers
To divide with decimal fractions and decimal mixed numbers
To understand denominate numbers
To reduce them, "ascending" and "descending"
To add with denominate numbers
To subtract with denominate numbers
To multiply with denominate numbers
To divide with denominate numbers
To read, write, and understand percents
To manipulate the "three cases" of percentage:
    I. Multiplying by a percent
    II. Dividing one number by another and expressing the result as a percent
    III. Dividing a number by a percent to find what number it is that percent of
To understand the uses of percents in computing interest, discounts, insurance premiums, taxes, dividends, yields of bonds, etc.
To understand and compute square root and cube root
To compute the areas of certain surfaces and the volumes of certain solids or the contents of certain receptacles

Noting the difficulties which pupils had in learning the early part of this system, certain teachers long ago attacked it. "Why," they wisely said, "should a young beginner learn about hundreds, thousands, and millions which he cannot easily understand and has no need to understand before he learns that 2 and 3 are 5, or that 6 from 10 leaves 4? Why should he learn to add all integers before he subtracts any?" But they unwisely exaggerated their point into a system that was also very hard to learn by. They organized learning around the numbers, the pupil learning all the addition, subtraction, multiplication, and division he could with 4, then with 5, then with 6, and so on.

Certain teachers, irritated by the obvious defect that there were many very hard things early, and many very easy things late in this old system, wisely sought to remedy it. But they again unwisely overdid their correction by fashioning a new "spiral" system whereby the pupil had just a little of addition, subtraction, multiplication, and division, then a little more of each, then somewhat more of each, and so on. The artificialities and restrictions of this were nearly as troublesome to the learner as the difficulties of the older scheme, and they lost the chief merit of the old system, which was that if you did learn a part of it, that learning often led on to something—it did not leave you hanging at a loose end.

THE PURPOSE OF ORGANIZATION

The newer methods seek first to get beneath surface criticisms to an understanding of what the purpose of a general plan of arrangement of arithmetical work should be, and of the criteria or standards by which such a plan or system should be judged. They find that the main purpose is to help the learner to learn and remember arithmetic and use it in life. Whether or not the system looks well on paper, or is a good inventory of the contents of arithmetic to put in a catalogue of studies, or is a convenient list by which a writer may be sure he has left nothing out, or shows clearly the main topics in arithmetic to a person who already knows it—all these are of relatively trifling consequence.

There has been among educators a ruinous passion for system for system's sake. Spelling books are still in use which teach first all words of one syllable, then all words of two syllables, and so on; or which group together for study all the pairs of words that sound alike but are spelled differently; or all the common abbreviations. Reading books used to begin with

ba	be	bi	bo	bu
da	de	di	do	du
fa	fe	fi	fo	fu, etc.

Courses of study and textbooks in arithmetic have suffered their full share from this passion for system. For example, only a veritable mania for system would have deferred teaching facts like $\frac{1}{2}+\frac{1}{2}=1$, or $\frac{1}{2}$ of $4=2$, easily learned and needed by the young child in school and out, until after the intricacies of long division had been mastered; or have left 12 inches = 1 ft., 3 ft. = 1 yd., 2 pts. = 1 qt., 4 qts. = 1 gal., or 7 days = 1 week, until late in the school course; or have used the computation of interest as an excuse to require innocent children to juggle any three of the quartet, principle, interest, time, and rate, so as to find the absent member.

Against this tendency the newer methods protest that *mere system* in teaching, system for system's sake, is chiefly a scholar's idol. After a pupil has learned arithmetic, it may be worth while for him to spend some time in arranging his knowledge into a "logical" system for contemplation, and even to spend a little time on matters useless for life in general, but of some interest as filling out gaps in the system. In general, however, the system is valuable only in so far as it helps the pupil to learn arithmetic and use it in life.

ORGANIZATION FOR THE LEARNER

Logical beauty and progression of organization, as in the scheme on pages 83 and 84, is largely wasted on the young learner. He cannot appreciate the progression toward what is to come, because he does not know what is to come! The simplicity and balance which we admire as we read through a course of study, he never even sees; for it takes him six years to go through that course of study! By the time he is in

Grade 8 he probably has not the slightest remembrance of whether he learned to add 1 and 3 before he learned to divide 400 by 40, or whether he learned 10×10 before or after $\frac{1}{2} = \frac{2}{4}$. We may mar the symmetry of the organization at no cost to him!

There is not, in fact, very much symmetry or system of the older sort left to the organization after the newer methods have made it over to suit the learners' needs. What was one topic may be scattered over the course. Thus reduction, ascending and descending, of denominate numbers disappears as a topic by itself. Part of this work is put in with the first learning of the multiplications to 90×9 and the divisions to $89 \div 9$. Part of it appears in connection with multiplication by two-place numbers and long division. Part of it is associated with the four operations with compound numbers.

The logical completeness of a topic may be rudely marred by dropping out a part that was put there only to complete the scheme, and is not needed for later arithmetic or for life. Thus the pupil learns to find interest when he knows the principal, time, and rate, but not to find the rate from principal, time, and interest, or the time from principal, rate, and interest.

A topic that is a single unit in the mathematician's system may be broken up into several teaching units. For example, multiplication by three-place multipliers is separated into:

A. Multiplication by multipliers with no zeros, like 465, 289, 372
B. Multiplication by multipliers like 460, 280, 370
C. Multiplication by multipliers like 400, 200, 300
D. Multiplication by multipliers like 405, 209, 302

So also, in long division, cases where there are zeros in the quotient are treated as a separate unit, delayed until the simpler procedure is fully mastered. Teachers are warned not to assign exercises or problems involving 0 in the quotient until special training with this most difficult feature of long division is given.

A general topic that could be learned consecutively with no great difficulty may be interrupted so that there may be inserted a sort of work which enables the abilities so far acquired to be put to use in their proper connections, and enables each new ability within the general topic, when that is resumed, to be put to use as soon as learned. This sort of modification of the older topical system occurs again and again in the newer treatments of arithmetic. For example, as soon as the addition combinations with sums to 9 are well known, the pupil may be taught to use them in column additions like

$$\begin{array}{ccccccc} 3 & 2 & 3 & 2 & 3 & 2 & 2 \\ 1 & 3 & 2 & 1 & 2 & 1 & 2 \\ \underline{5} & \underline{4} & \underline{4} & \underline{3} & \underline{3} & \underline{2} & \underline{4} \end{array}$$

and even in column additions like

$$\begin{array}{ccccc} 23 & 22 & 12 & 21 & 12 \\ 12 & 31 & 52 & 33 & 12 \\ \underline{14} & \underline{33} & \underline{11} & \underline{15} & \underline{65} \end{array}$$

before the addition combinations 5+5, 6+4, 4+6, 7+3, 3+7, etc., to 9+9 are learned.* The learning of the multiplication combinations may be interrupted, after the products of 1, 2, 3, 4, and 5 by the numbers from 1 to 10 (or 1 to 9, in some plans) are learned, by the introduction of multiplication of two- and three-place numbers by a one-place number. The work given is of course restricted to such as requires only the combinations learned; for example:

$$\begin{array}{rrrrrr} 23 & 42 & 51 & 53 & 34 & 25 \\ \underline{2} & \underline{9} & \underline{8} & \underline{5} & \underline{4} & \underline{7} \end{array}$$

$$\begin{array}{rrrrrrr} 254 & 315 & 223 & 513 & 452 & 113 & 345 \\ \underline{6} & \underline{9} & \underline{7} & \underline{5} & \underline{8} & \underline{7} & \underline{3} \end{array}$$

---

*This particular feature of the organization of learning is not necessarily the best, and depends somewhat upon the grade in which such formal work in arithmetic is begun, but it has been adopted by many expert teachers.

This plan secures the early use of the multiplication facts in a real connection in which they are to be used and enables the pupil to put the "times 6's," "times 7's," etc., to real use as fast as they are learned. It also lessens the monotony of oral memory work at this stage, and of written computation at a later stage.

Certain arrangements of work may even be made for variety's sake alone. For example, it is desirable to give, very early in the course, some knowledge of $\frac{1}{2}$ and $\frac{1}{4}$ in very simple cases because of the practical worth of this knowledge. This may be put in where it will do the most good as a change from drills on addition and subtraction. It is not often that variety is the sole or even the chief reason for an arrangement, but it is often a subsidiary reason.

Part of a topic may be taken out of the place where the so-called "logical" systems put it, in order that it may be put where the ability gained will notably help, or be helped by, some other ability. This is one of the reasons for two extensive changes from the older systems. These are (1) teaching the subtraction combinations along with the addition combinations, and (2) teaching each set of division combinations or "tables" along with the corresponding multiplications. By such teaching the pupil is helped to use knowledge he has to gain new knowledge, and also to check his results in the new process. The contrast also helps to emphasize the nature of each process. Another case of this sort is the removal of 11 and 12 times 2, 3, 4, etc., and of 2, 3, 4, etc., times 11 and 12, from the early learning of the multiplication tables. This reduces the memory work of the tables by much more than one-sixth, since these are specially difficult combinations; it also gives very great help in learning the process of "long" multiplication and long division, since cases like $\begin{array}{r}45\\ \underline{11}\end{array}$, $\begin{array}{r}52\\ \underline{12}\end{array}$, and $11|\overline{462}$, $12|\overline{396}$ help to present the essential procedures with a minimum of difficulty in computation. Later the products of 2, 3, 4, etc., times 12 should be learned thoroughly, because of their frequent use in connection with the dozen. The reverses (the products of 12 times 2, 3, 4, etc.) may also be learned, though these are less often used. There seems to be no more need for learning the products of 11 at any stage than for learning the products of 25 or 16. In fact, there is rather less need for them. They were included in the older system just for system's sake. Since 1 to 10 and 12 were needed, 11 was put in to make the plan look better!

Shifts may be made to tie together or "integrate" abilities that otherwise might not get into proper mutual relations. This is one reason for using "$\frac{1}{2}$ of" 2, 4, 6, 8, etc., in early association with the division tables. The problems of finding the cost of fractions of a yard of cloth, pound of meat, fish, butter, cheese, candy, and the like demand facile use of this form of statement of the call for division, using $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, etc. Problems in "sharing" may use it with other numbers. There are also other notable advantages in this procedure. So we have, in the best recent organizations of arithmetical subject matter, certain multiplications of integers by common fractions taught long before the word fraction is used, and over a year before the main and ostensible study of multiplication of an integer by a fraction is begun.

Perhaps the most evident breakdown of the system outlined on pages 83 and 84, from the point of view of the learner's needs, is its lack of reviews to keep alive and healthy the abilities that have been acquired. What, for instance, is to happen to long division during the many months that common fractions are the topic for study? This difficulty was early recognized, but, as usual, love of a system easy to plan and admirable to look at in a table of contents misled author and teachers into neglect of a great opportunity. They simply inserted "Reviews" from time to time, in which the pupil did over again what he had done before. Their reviews were mere repetitions.

The newer methods set a far higher standard for a review than an indiscriminate repeating of the same work in the same way.

First of all, they inquire what abilities will need no reinstatement, being given abundant exercise in the course of the learning of later topics. Obviously, for example, the forty-five addition combinations and very short column addition will be repeatedly exercised in the addition of the partial products in multiplication with integers, United States money, decimals, and percents. If the additions and subtractions of fractions are practiced, as they should be, largely in mixed numbers, there is added practice for addition with columns of moderate length. The addition of decimals contributes further. In general, it is clear that plans for review of any ability should consider all uses of that ability to date. Reviews should not be indiscriminate. Some abilities need little or no special review; the amount of review and the interval after which it should be given differ for each ability; any one set system of reviews must be wrong.

In the second place, the newer methods seek to do, if possible, something better than repeat the same work in the same way. The pupil is always older; he presumably has learned more arithmetic in the meantime; he ought, according to the findings of chapter iii, to be able to understand the reasons for and general theory of the processes better; variety and interest have some claims; perhaps the review can be used to "integrate" old habits, to facilitate new learning, and to show interrelations and new uses. The newer methods realize these facts and seek to make reviews fit the learner's abilities and needs just as skillfully as the first learning did.

It is not possible to illustrate these points properly, since the nature and value of a review can be understood only if the nature and amount of all the previous relevant work are known, and to show this would require many pages. Some idea may be gained, however, of the way the newer theory of reviews works out in practice from the notes given and from the illustrations shown below and on pages 93–95 following.

The first is a review of the multiplication tables, but with a change to suit the way they are used in multiplication of numbers of two or more figures. It is given late in Grade 3.

I

3 9 5 7 2 6 8 1 4

1. Multiply each of these numbers by 6 and add 2 to the product.
2. Then multiply each of them by 7 and add 3 to the product.
3. Then multiply each of them by 8 and add 4 to the product.
4. Then multiply each of them by 9 and add 5 to the product.
5. Then multiply each of them by 5 and add 6 to the product.
6. Then multiply each of them by 4 and add 7 to the product.
7. Then multiply each of them by 3 and add 2 to the product.

The second is a review of the additions especially of 7, 8, and 9, of the meaning of average, and of certain subtractions of fractions. It is given at the end of Grade 4. The old material is here taken up in a new way.

II

1. Helen's exact average for December was $87\frac{1}{3}$. Kate's was $84\frac{1}{2}$. How much higher was Helen's than Kate's?

$87\frac{1}{3}$ How do you think of $\frac{1}{2}$ and $\frac{1}{3}$?
$84\frac{1}{2}$ How do you think of $1\frac{2}{6}$
How do you change the 4?

2. Find the exact average for each girl. Write the answers clearly so that you can see them easily. You will use them in solving problems 3, 4, 5, 6, 7, and 8.

	Alice	**Dora**	**Emma**	**Grace**	**Louise**	**Mary**	**Nell**	**Rebecca**
Reading	91	87	83	81	79	77	76	73
Language	88	78	82	79	73	78	73	75
Arithmetic	89	85	79	75	84	87	89	80
Spelling	90	79	75	80	82	91	68	81
Geography	91	87	83	75	78	85	73	79
Writing	90	88	75	72	93	92	95	78

3. Which girl had the highest average?

4. How much higher was her average than the next highest?

5. How much difference was there between the highest and the lowest girl?

6. Was Emma's average higher or lower than Louise's? How much?

7. How much difference was there between Alice's average and Dora's?

8. How much difference was there between Mary's average and Nell's?

9. Write five other problems about these averages, and solve each of them.

The third is a review of the use of signs, of some of the harder addition, subtraction, multiplication, and division combinations, of multiplication by multiples of 10, of the principle of finding a fraction of a number when the number is a multiple of the denominator of the fraction, and of the principle of addition and subtraction of fractions; arranged also so as to help fix permanently in memory certain facts like $\frac{3}{4}+\frac{3}{4}=1\frac{1}{2}$, $100\div 25=4$, and $\frac{1}{2}$ of $50=25$. This review is deliberately superficial. It is given at the beginning of Grade 5 as a part of a set of reviews which together enable the teacher to detect and the pupils to remedy any fundamental weaknesses left by the work of previous grades. That is, this review is in part a test.

III

(*Without pencil.*)

1. Give as many right answers as you can in 2 minutes:

A.	B.	C.	D.	E.
$19+8=$	$20\times 9=$	$\frac{1}{3}$ of $27=$	$7\times 11=$	$6\times 8\ =$
$16-9=$	$10\times 17=$	$12-\ 9=$	$75-25=$	$36\div 9\ =$
$8\times 7=$	$63\div\ 7=$	$\frac{1}{4}$ of $28=$	$10\times 30=$	$240\div 6\ =$
$54\div 6=$	$3-1\frac{1}{2}=$	$\frac{3}{4}$ of $16=$	$66\div 11=$	$23+9\ =$
$7\times 6=$	$2\frac{1}{2}+6\frac{1}{2}=$	$\frac{3}{4}$ of $36=$	$\frac{1}{4}+\frac{1}{4}\ =$	$\frac{3}{8}$ of $16=$
$72\div 8=$	$81\div\ 9=$	$30\times 12=$	$\frac{3}{4}+\frac{3}{4}\ =$	$\frac{1}{2}$ of $50=$
$32+9=$	$35+\ 8=$	$56\div\ 8=$	$100\div 25=$	$15\frac{1}{2}-5\frac{1}{2}\ =$
$13-8=$	$80\div 20=$	$7\times 50=$	$\frac{3}{4}-\frac{1}{4}\ =$	$\frac{2}{3}$ of $36=$

Practice until you can do all five columns in 2 minutes and have every answer right.

The fourth is a review of some of the essential elements of knowledge of the nature of common fractions, decimal fractions, place value, the use of zero, and the technique of dividing by a fraction. The questions demand thoughtful analysis and would be catch questions if they were given one at a time in circumstances tending to mislead. As given here, they are fair means of making the pupil realize clearly the essential principles on which he has been acting. This review is totally unlike the original learning in its form, and represents not only a review, but also a considerable advance.

IV

(*Without pencil.*)

1. In which of these pairs do the two numbers have the same value, or mean the same amount?

a.	$\frac{3}{4}$	.75	*l.*	\$.001	$\frac{1}{10}$ of a cent
b.	$\frac{3}{4}$	$\frac{4}{3}$	*m.*	$1\frac{5}{8}$	$\frac{18}{5}$
c.	\$10.5	\$10.50	*n.*	$3\frac{1}{4}$	$\frac{4}{3}$
d.	\$10.5	\$10$\frac{1}{2}$	*o.*	86	860
e.	\$10.50	\$105	*p.*	8.6	8.60
f.	1 bu.	32 qt.	*q.*	.45	.450
g.	$1\frac{1}{2}$ bu.	$32\frac{1}{2}$ qt.	*r.*	.45	.045
h.	0146.3 mi.	146.30 mi.	*s.*	.33$\frac{1}{3}$	$\frac{1}{3}$
i.	018.7 mi.	180.7 mi.	*t.*	$\frac{1}{4}$	.25
j.	$66\frac{2}{3}$¢	\$$\frac{2}{3}$	*u.*	$\frac{1}{6}$	$16\frac{1}{3}$
k.	$66\frac{2}{3}$ mi.	$\frac{2}{3}$ mi.	*v.*	.4	$\frac{4}{5}$

2. Examine the pairs of numbers again. When the two numbers of a pair do not have the same value, prove that they do not. Use pencil if you need to.

3. Read each of these *equations* or statements of two things that are equal. If the statement is true, say "True." If the equation is not true, say "False." Then change it to make it true.

a. $\frac{8}{16}=\frac{1}{2}$

b. $.08+.09=.017$

c. \$$\frac{3}{8}$ = 375 cents.

d. $12\times\frac{1}{2}=12\div 2$

e. $9\div\frac{3}{2}=9\times\frac{2}{3}$

f. $7\frac{1}{2}\div 1\frac{1}{2}=\frac{15}{2}\times\frac{2}{3}$

g. $\frac{1}{8}$ of $24=24\div\frac{1}{8}$

h. $6\div\frac{3}{4}=6\times\frac{4}{3}$

i. $100\times .46=46$

j. The reciprocal of $3\frac{1}{2}$ is $\frac{2}{7}$

ORGANIZATION FOR LIFE'S NEEDS

The facts about modern practice in organizing the subject matter of arithmetic so far given and other facts that might have been given have to do with better adaptations to the work of learning. We now turn to the problem of better organization to fit the needs of life.

Life organizes its arithmetical demands, not so much by the nature of the processes as by the situations involved. You are choosing Christmas presents, or arranging a vacation, or saving for a bicycle, or planning a garden, or securing capital to start in business, or cooking in your kitchen. The newer methods seek to organize arithmetical learning around such frequent instructive situations demanding arithmetic, so far as this can be done, with no loss to the learning of the purely arithmetical facts and principles. Thus we have, in Grade 3, after a review of the elements of telling time, the work shown on pages 97 and 98, including all four operations with integers and some very simple uses of fractions.

Some of the situations which are thus used as organizing centers for arithmetical training are suggested by the following titles of lessons or groups of lessons in Grade 4:

1. Vacation Activities
9. School Supplies
14. Playing "How Far"
15. Playing "Saving"
18. Telegrams, Express, and Freight
19. Playing "Cashier"
20. House Plans
21. Drawing to Scale
24. The School Program
45. Weighing
46. Buying Candy
51. School Marks and Averages
53. Keeping Accounts
54. Buying Fruit
58. Henry's Orchard
60. How Lewis Earns Money
61. How Elsie Earns Money
67. At the Fish Market
72. A Christmas Party
74. Earning and Saving
79. At the Butcher Shop
87. Buying in Quantity
98. Report Cards
99. Earning and Saving

How many hours does it take the hour hand to go—

1. From 6 in the morning to 11 in the morning?
2. From 6 in the morning to 3 in the afternoon?
3. From 8 in the morning to noon?
4. From 8 in the morning to 5 in the afternoon?
5. All the way round from 12 noon to 12 midnight?
6. From midnight to noon and then all around again to midnight? From midnight to noon and then again to midnight is 1 day. How many hours equal 1 day?
7. From midnight to 2 o'clock in the afternoon is how many hours?
8. From noon to 6 o'clock in the morning of the next day is how long?
9. On some railroads they call 1 o'clock in the afternoon 13 o'clock. They call 2 o'clock in the afternoon 14 o'clock, and so on to 23 o'clock. What do they call 5 o'clock in the afternoon? What do they call 9 o'clock in the evening?
10. On most railroads they call the hours from midnight to noon 1 A.M., 2 A.M., 3 A.M., etc. They call the afternoon and evening hours from noon to midnight 1 P.M., 2 P.M., 3 P.M., etc. How long does it take the hour hand to go from 5 A.M. to 7 P.M.? From 9 A.M. to 4 P.M.? From 3 A.M. to 7 P.M.?

[Then follows work on $\frac{1}{2}$ of 12, $\frac{1}{4}$ of 12, $\frac{1}{6}$ of 12, and $\frac{1}{3}$ of 12.]

Clock Problems

1. How many minutes does it take the minute hand to go from 2 to 3? From 2 to 4?
2. From 2 to 9? From 12 around to 12 again? From 12 to 1? From 12 to 2? From 12 to 8?
3. What part of an hour is 30 minutes? How many minutes make $\frac{1}{6}$ hr. or one sixth of an hour? What part of an hour is 15 minutes? How many minutes are there in an hour and a half?
4. How many minutes are there in $\frac{3}{4}$ hr. or three quarters of an hour? In half an hour?
5. At 10 minutes past 5, Dick's mother told him, "You must come in in a quarter of an hour." At what time must Dick come in?
6. Another day at 5 minutes past 4 she said, "You may stay just three quarters of an hour." At what time did he have to come in on that day?
7. Another day at quarter of five she said, "You must come in 25 minutes." At what time did he have to come?
8. It was quarter past 4. "You can play till 5 o'clock," said Will's mother. "How long is that?" asked Will. How long was it?
9. How many minutes is it from 9:40 A.M. to 10 A.M.? From 9:40 to 10:20? From 2:50 P.M to 3 P.M.? From 2:50 P.M. to 3:25 P.M.?
10. From 3:48 P.M. or 12 minutes of 4 P.M. to 4:09 P.M. or 9 minutes past 4? From 9:52 or 8 minutes of 10 to 10:07 or 7 minutes past 10?
11. How long is $\frac{3}{4}$ hr. and $\frac{1}{4}$ hr. in all?
12. How long is $\frac{1}{4}$ hr. and $\frac{1}{4}$ hr. in all?

The lessons organized around these life situations make up about one-fourth of the entire work of the grade. If they are chosen wisely and arranged wisely, such situations and activities can be used at no cost to the learning of purely arithmetical matters. Some of them will indeed serve as admirable introductions to new processes. Some will give chiefly needed drill on some one process, but with other processes coming in naturally. Some will require the pupil to use a large part of his repertory.

As a sample of the organization, not of single lessons or small groups of lessons, but of the work of six months or more, by the situations of life, we may examine the following Divisions II and III of the total plan for Grade 7:

I. The General Theory and Technique of Arithmetic.

(Sections 1 to 29 comprise 27 pages, reviewing all the important difficult features of arithmetic up to percents, with emphasis on the general theory.)

II. Owning, Buying, and Selling:

Sections 30–33	Review of percents
34	Fixing prices
35	Property: inventories
36, 37	Protection against loss of property by fire
38	Insurance: rates
39	Insurance: valuation
40	Buying: sales slips, bills, and receipts
41, 42	Buying by mail and telegraph
43	Paying by check or draft
44	Buying: discounts for cash
45	Buying: trade discount
46	Practice in computing discounts
47	Buying for the home
48	Selling: profit and loss
49	Selling: profit per unit of time spent
50	Selling: the risk of loss
51, 52	Some of the expenses of selling

53 Selling on commission
54 Receiving a commission for buying

III. Borrowing and Lending: Interest:

55 Saving money and acquiring property
56 How money increases when interest is added to it
57 The Postal Savings Bank
58 Starting in business
59 Borrowing money to go into business
60 Borrowing money for a short time
61 Borrowing money for a long time
62 The number of days between two dates
63 Interest tables
64 Buying on the installment plan
65 Review

Divisions IV and V include ratio, board measure, circular measure, similar triangles, the use of symbols and equations, and practice in all computations.

As a consequence of organization around such situations taken from life, there is a great reduction in the isolated problems of the older courses of study. Such are not to be discarded entirely, however. The older series of "miscellaneous" problems given in "General Reviews" served a real purpose in demanding that a pupil keep his entire repertory of abilities alive and ready to act. If each by itself is real, well stated, and not beyond the pupil's experience of language or of facts, a set of such isolated problems is useful both as training and as test. Twenty or thirty such can test a wider sampling of abilities than any twenty or thirty problems that are likely to belong properly to any one real situation.

The *order* of topics may be changed to fit life's needs. If, for example, pupils were to leave school in most cases by the end of Grade 5, it would probably be best to delay work in division with decimals until Grade 6, replacing it by a fundamental acquaintance with percents. Division by a decimal is rarely required in life, whereas understanding the meaning of percent and finding a given percent of a number are in very common use. It seems to the writer that the main significance of interest to the many children who leave school before completing Grade 8 or even Grade 7 is in relation to thrift and saving, and that consequently compound interest should be taught early in Grade 7, soon after the bare general meaning of interest is taught, before much drill on simple interest, and long before interest on bank loans. Compound interest has usually been delayed until very late, on the assumption that it is harder than simple interest. This is quite erroneous, the hard feature of interest being the treatment of the time. Computation of compound interest may be long, but it is not at all hard, since the pupil simply multiplies again and again by just the same multiplier and since savings banks do not compute interest on fractions of a dollar.*

If the needs of life are given influence, certain features of the older organization are given much less attention, or even none. Life very seldom demands multiplication with two mixed numbers both large, such as $48\frac{1}{2} \times 213\frac{1}{4}$; or the addition or subtraction of fractions except

$\frac{}{2}$s, with $\frac{}{3}$s, or $\frac{}{4}$s, or $\frac{}{8}$s, or $\frac{}{12}$s,

$\frac{}{3}$s, with $\frac{}{2}$s, or $\frac{}{4}$s, or $\frac{}{12}$s,

$\frac{}{4}$s, with $\frac{}{2}$s, or $\frac{}{3}$s, or $\frac{}{12}$s,

$\frac{}{5}$s, with $\frac{}{5}$s,

$\frac{}{8}$s, with $\frac{}{2}$s, or $\frac{}{4}$s,

*The older methods added a needless burden by their ignorance or neglect of this fact.

or any use of complex fractions. So the treatment of multiplication of one mixed number by another may safely be given as reduction to common fractions with cancellation; the whole topic of least common multiple is best omitted; the general conception of a fraction as any number divided by any other may be left unmentioned. For similar reasons, the rare applications of dividing may be omitted or slurred with no very great loss.

To take one more illustration, life insurance seems an especially undesirable topic for a boy or girl to meditate upon! This last case of life insurance brings us back again to our first beginning, the passion for mere system. Not only is insurance against death a morbid topic for a child of thirteen or fourteen; but it really does not belong logically or arithmetically as an "application of percentage" or as a pair with fire insurance. There is no important connection between the premium rate per $250 or per $1000 of insurance and percents. Insurance of property is for the benefit of the person insured; insurance of life is for the benefit of others. One is a matter of business; the other is usually a matter of love or duty. The passion for filling out a system, with main topics and subtopics, seizes avidly upon the likeness in the word insurance, and puts both in; the premium is *per* something, so life insurance is put as an application of percents!

ARITHMETIC AS SCIENCE AND AS ART

The subject of system and organization in arithmetic is too broad and too intricate to be summed up in any brief way. We may, however, keep the main issues in mind if we think of arithmetic as both a science like anatomy which the pupil is to know, and an art like surgery which he is to practice, or even a game like tennis which he is to play. We wish him, so far as he has the capacity, to know the science of arithmetic well, so that he can, when confronted by a problem, think through the science and get whatever aid it has for the problem's solution—so that he could even, if necessary, write down the main facts of the science for preservation, and so that he can have, as part of his mind's training, knowledge of an orderly, progressive, interrelated set of facts and principles. We also wish him to practice the art of actual arithmetical work well on the street, in the home or factory, when buying, selling, planning, and working. We wish him to play well at the game of responding to the situations of life by the arithmetical thought and action that they need. The newer methods teach the science as well as the older methods, probably much better for the majority of pupils. But their especial care in the matter of organization is to train pupils to play the game well. To learn to play tennis it is not wise to list all the strokes in some such fashion as is shown below and learn them one at a time:

A. FOREHAND STROKES:
- I. Above the waist
  - 1. Very swift
    - *a.* to the right
      - i. with a cut, etc.
      - ii. without a cut
    - *b.* to the left
      - i. with a cut
      - ii. without a cut
  - 2. Swift, etc., as above
  - 3. Slow, etc., as above
- II. Below the waist
  - 1. Very swift, etc., as above
  - 2. Swift, etc., as above
  - 3. Slow, etc., as above

B. BACKHAND STROKES:
- etc., as above

It is surely wiser to learn the easier strokes first, and to learn to make all strokes with a real ball, on a real tennis court, in response to a real opponent's play.

The ideal organization of learning to play tennis would be for the learner to have a teacher who would show him how to make the strokes and so play against him that he would be given just the right amount of practice on the different strokes and combinations of strokes in the conditions when they were appropriate, all being arranged in the order making for most rapid progress and all being integrated into a total ability to play a real game of tennis.

The ideal organization of learning to play arithmetic will be, to some extent at least, a similar series of graded acquisitions and activities fitted always to the learner's status, and leading always to competence in the real game, under real conditions.

EXERCISES

1. Some of the older arithmetics gave $0+0=0$, $0+1=1$, $0+2=2$, etc., very early in addition. Why is it better to delay this until Grade 2 when written column addition is learned?

2. Some teachers used to teach United States money after the general treatment of decimals. What are some advantages of teaching it very early?

3. Is there any reason in the science of arithmetic for teaching children how to keep simple accounts in any one of these places rather than another? Grade 4 late, Grade 5 early, Grade 5 late, Grade 6 early, Grade 6 late, Grade 7 early, Grade 7 late, Grade 8 early, Grade 8 late? What considerations would guide you to put it in one place rather than another?

4. How early could you teach computing the area of parallelograms and triangles, so far as the general science of arithmetic is concerned?

5. What topics would be made easier if the metric system were taught very early, say in Grade 4?

6. Examine the organization of the teaching of the multiplication and division tables, Book I, Divisions III and IV, pp. 49–83. Compare this treatment with learning all the multiplication tables to 12×12, then learning all the division tables to 144÷12, then learning short multiplication.
7. For many reasons it is of the utmost importance to teach pupils to verify addition by objective work, multiplication by addition, division by multiplication, etc. Find pages where such verification serves as a useful form of practice or review.

CHAPTER VI

LEARNING MEANINGS

THE MEANINGS OF NUMBERS

Any word or figure acquires meaning by being connected with some real thing, event, quality, or relation. *Six* is mere nonsense except as it has gone with six real boys, beans, toothpicks, inches, feet, or the like. The connection may be direct, as when we show a line 45 inches long, or have a child lift 45 pounds, or count the children in the room as 45. It may be indirect, as when children who have had 40 and 5 each connected with reality learn 45 as "40 and 5 more."

Other things being equal, direct connection is better. Thus, "The wall of your schoolroom contains about 30,000 square inches. That empty freight car weighs about 35,000 pounds." "Draw lines 40, 50, 60, 70, 80, 90, and 100 inches long." "Hold your hand about 40 inches from the floor. Now hold it 10 inches higher, or 50 inches from the floor. Now hold it 10 inches higher, or 60 inches from the floor. Can you hold it 70 inches from the floor? The tallest boy may stand on my desk and show us 100 inches."

Some features of the meanings of numbers are so taken for granted by us that we may neglect them in teaching. We all know that $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{5}{6}$, and the like are smaller than 1 so well that we may never mention this fact, but $\frac{7}{8}$ may look like a rather large number to a child. It is worth while to make sure that pupils realize that each of these proper fractions means something smaller than 1. To arrange $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$ in order of size (with the aid of a foot rule, if necessary) is an excellent exercise. We all know perfectly well that 10,000 means a very large number, but to a child, seeing the 1 and four 0's and being not very clear about the theory of decimal notation, it may not mean so large a quantity as 987. It is then safer to show him a 10,000 square-inch area and tell him that it is about 10,000 feet to some familiar place 2 miles away.

A number has not one meaning, but several. Thus eight means a certain point or place in the number series, 1, 2, 3, 4, 5, 6, 7, 8, 9, etc., which is 1 beyond 7 and 1 before 9. This we call the *series* meaning. Eight also means the number of single things in a collection of 8 boys, or 8 hats, or 8 beans, or 8 pencils. This we may call the *collection-size* meaning. Eight also means 8 times a certain unit, say a pint, whether isolated as 8 separate pints or combined together in a gallon can. This we may call the *quantity-size* or *ratio* meaning.*

The teacher should not neglect this *quantity-size* or *ratio* meaning. Children should measure as well as count, and should learn to use 3 for 3 inches if 1 is one inch, for 3 feet if 1 is one foot, for 3 yards if 1 is one yard, etc., as well as for 3 clearly separated objects like apples or pieces of chalk. Somewhat later they should use 3 for 3 pairs, or 3 dozen, or 3 hundreds, as well as for 3 ones. Later still three should mean for them 3 times whatever is taken as one.

Knowledge of the meaning of a number may be of varying degrees of exactness and completeness. It does not have to be either zero knowledge or perfect knowledge. The child who knows that a thousand is a great many, that a thousand dollars would be better to have than fifty, and that a thousand pounds would be more than he could lift, has made some progress, though he does not know that a thousand is ten hundreds, and that each hundred is ten tens. It is not necessary or desirable to teach the full and exact meaning of a number all

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*Eight may also be considered to mean a number possessed of certain properties in relation to other numbers. By this view to know eight is to know that it is two 4's, 3 more than 5, 2 less than 10, etc. This knowledge about a number's relations to other numbers is perhaps better considered as knowledge of the relations of numbers than as knowledge of their meanings.

at once, for pupils learn more and more about the meanings of numbers by using them.

Consider, for example, the number 24. The pupil in Grade 1 may be taught to find page 24 in his reader, to count from 1 to 100, and to count to 100 by tens. Later he may be taught that 24 equals 2 tens and 4 ones, that 24 cents equals 2 dimes and 4 cents. Later still he finds that $19+5=24$, $18+6=24$, etc. Later still he learns that eight 3's$=24$, that six 4's$=24$, that 2 dozen$=24$. Later still he has experience with 2400 and 24,000. These and other operations teach him more and more fully what 24 means.

It is rather the rule than the exception that the meaning of numbers is known incompletely and vaguely at first and is filled out and clarified by use of the numbers. Thus, in learning the meanings of common fractions, the pupil may first be taught a few very simple facts about one-half and one-quarter, as commonly used about the home, then be taught to give the division tables in response to $\frac{1}{2}$ of $4=\ldots$, $\frac{1}{2}$ of $6=\ldots$, $\frac{1}{4}$ of $8=\ldots$, $\frac{1}{4}$ of $12=\ldots$, etc. Later he may be taught to recognize $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$ of clearly divisible units like a pie or apple. Then he may be taught to recognize $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, $\frac{1}{8}$ inch, $\frac{1}{2}$ yd., $\frac{1}{4}$ yd., and other easily measured fractions of common measures. He does not at any of these stages know the full meaning of these fractions, but each is a worthy step toward that knowledge; and that knowledge is more easily gained and more useful by being thus developed gradually. It is wasteful to give knowledge of the meanings of numbers too long before it can be put to use. The good teacher will make sure that, at any stage, the pupil knows the meaning well enough to use the number intelligently in those uses which are then necessary, but will be cautious about teaching any more elaborate meaning than that.

In "objectifying" a number—that is, connecting it with realities that show its meaning—we should consider, not only formal, systematic presentations with dots, counters, lines, etc., as shown in Fig. 10, but also such informal and incidental connections as can be made with objects and acts of daily life. The latter are likely to be more interesting and to be themselves more surely understood. We must not let the facts used to explain a number be harder to understand than the number!

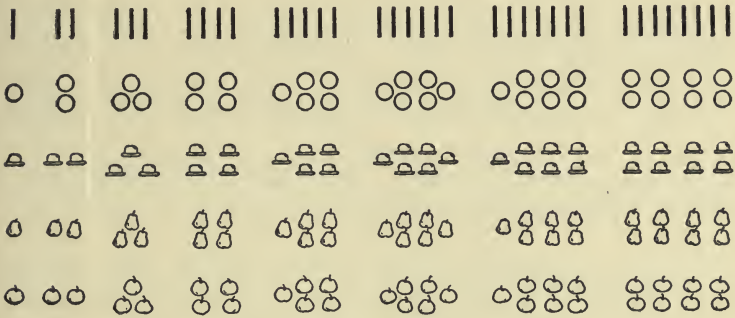
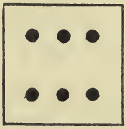


FIGURE 10

Teachers are sometimes careless about teaching the meaning of perhaps the most important number of all, 0, best called zero. 0 is, like all numbers, primarily an adjective meaning *no* or *not any*, and should be so read, not as *naught*. To read it as "nothing" is just as unwise as it would be to read 1 as "one thing" or 4 as "four things," or 5 as "five things." 0 may be objectified or connected with its appropriate reality by a blackboard presentation as shown below, and by hidden subtractions where 0 is the answer, such as: "I put 5 pencils in the long box and 5 pencils in the short box [doing so].



6 dots

0 dots

3 dots

I take 2 pencils out of the long box [doing so]. How many are there left in the long box? I take 5 pencils out of the short box [doing so]. How many pencils are there left in the short box?"

THE MEANINGS OF GROUPS OF NUMBERS

Knowledge of the common meaning of the numbers of a certain sort (such as integers, common fractions, mixed numbers, proper fractions, improper fractions, like fractions, unlike fractions, decimal fractions, decimals, compound numbers) should be built up out of knowledge of enough samples of the single numbers in question. The pupil should learn what *fraction* means, *after* he knows what $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, etc., mean.

As with single numbers, the common meaning of a group of numbers often has several aspects. A proper common fraction is less than 1; it has a numerator to show how many parts are taken, and a denominator to show the size of each part; it represents an uncomputed division.

As with the meanings of single numbers, so with the meanings of groups, it is neither necessary nor desirable that perfect knowledge be given as soon as any knowledge is given. The pupil first learns: "Numbers smaller than 1 are fractions;" later, "Numbers like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, etc., are fractions;" later, after learning much about single fractions and mixed numbers, "Numbers like 2, 5, 7, 9, 11, 11, 250 are whole numbers," "Numbers like $\frac{3}{8}$, $\frac{1}{5}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{11}{8}$, $\frac{7}{6}$ are whole fractions," "Numbers like $4\frac{1}{4}$, $2\frac{7}{8}$, $12\frac{3}{4}$, $1\frac{2}{3}$ are whole mixed numbers;" later still, after experience with decimals, "Numbers like .1, .01, .001, .6, .06, .006, .8, .28, .004 are called decimal fractions or simply decimals. Numbers like 16.24, 9.05, 1.3, 2.7, 4.81 are called decimal mixed numbers, or simply decimals." Common fractions are then distinguished from decimal fractions.

It will be observed that these statements that guide the pupil and sum up his knowledge do not claim to be rigorous and complete definitions. He does not say, "Common fractions are so and so," but "Such and such are common fractions," which is perfectly true, though not a full definition. If the reader will frame a definition for *integer* or *common fraction* which does cover all cases, he will find it to be less helpful for learning, stage by stage, than these summaries of working knowledge. Only rarely does a pupil's early working knowledge exactly tally with a rigorous definition.*

THE MEANINGS OF OPERATIONS, TERMS, AND SIGNS

Adding, subtracting, multiplying, and dividing are usually understood by pupils even with poor teaching. The teacher may reduce difficulties, however, by giving clear cases of objective adding, of objective subtraction (both [1] taking away and learning what is left, and [2] learning what must be added to make the difference between two numbers), objective multiplication, and objective division (both [1] dividing a number into, say, 3's to find how many 3's there are and what the remainder is, and [2] dividing a number into 3 equal parts to find out how large each such part will be).

After enough practice with concrete cases with the issue stated in unmistakable terms, the words *add*, *subtract*, *find the sums*, *find the differences*, *find the remainders*, may be safely taught, and the signs + and — .

The operation of multiplication should be introduced first in the verbal form "Four 5's = . . ., Seven 5's = . . .," and with very clear cases, such as "1 nickel = 5 cents, 2 nickels = . . . cents, 3 nickels = . . . cents, 1 yd. = 3ft., 2 yds. = , . . ft." The word *times* may best be introduced by such a series as, "It costs

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*Cases where it does so tally are: A prime number is any number which is divisible without remainder by no integer except itself and one. A fraction is in lowest terms if the numerator and denominator cannot be both divided by 2 or 3 or 4 or some other whole number without remainder.

5 cents to go to the moving pictures once, it costs 6 times 5 or 30 cents to go to the moving pictures six times, it costs 4 times 5 or 20 cents to go to the moving pictures four times." × may then be at once taught as meaning *times*. The words *multiply* and *multiplication* may best be delayed until many cases have been experienced, including cases with two-place multiplicands such as: "One long trolley car holds 42 men. Four long trolley cars hold . . . men. One short car holds 23 men. Three short cars hold . . . men."

The general word is then defined by the particulars, and by contrast with addition and subtraction. For example:

"You *multiply* when you find the answers to questions like:

How many are nine 3's?
How many are 3×32?
How many are 8×5?
How many are 4×42?

"If you add 3 to 32, you have 35. 35 is the sum.

"If you subtract 3 from 32, you have 29. 29 is the difference or remainder.

"If you multiply 32 by 3, you have 96. 96 is the product."

In general, any new operation or new form of an old operation, such as finding $\frac{1}{2}$ of, $\frac{1}{3}$ of, $\frac{1}{4}$ of, etc., as a new form of division, finding "$\frac{3}{8}$ of" by "dividing by 8 and multiplying by 3," or adding with fractions, should be introduced by concrete problems which show clearly what is the issue, and arouse a reasonable amount of interest in it. The following (pages 113–116) are samples in the case of:

I. The first steps in interpreting scale drawings and computing areas of rectangles (Grade 3).
II. Multiplication with 2-place multipliers.
III. First steps in long division with United States money.

I

Square Feet

1. The teacher will show, on the blackboard, rectangles containing 1 square foot, 2 square feet, 4 square feet, and 10 square feet. Look at them. Then tell how many square feet there will be in a rectangle 3 feet long and 2 feet wide.

Think of the top of the teacher's desk.

Think of the door of your room at home.

Think of the floor of the schoolroom.

2. Which contains about 10 square feet?

3. Which contains about 20 square feet?

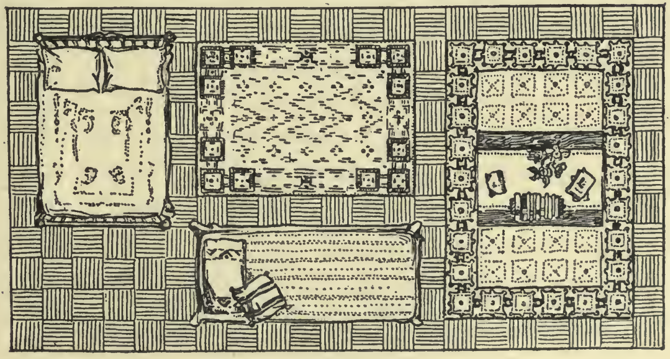
4. Which contains about 500 square feet?

5. Draw on the blackboard a rectangle 4 ft. long and 2 ft. wide. How many square feet does it contain?

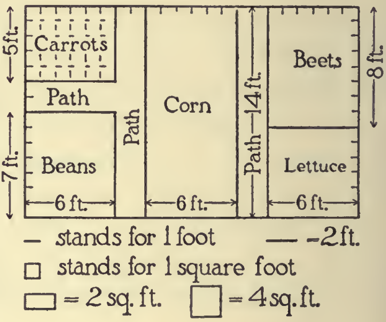
6. Draw a rectangle 3 ft. long and 3 ft. wide. How many square feet does it contain?

Drawing Plans

This is the plan of a room. — stands for 1 foot long. ☐ stands for one square foot.



1. The bed is 6 feet long by 4 feet wide. How many square feet does it cover?
2. The couch is 7 feet by 3 feet. How many square feet does it cover?
3. The big rug is 6 by 9 feet. How many square feet does it cover?
4. The table is 3 by 4 feet. How many square feet does it cover?
5. The little rug is 5×7 feet. How many square feet does it cover?
6. This is the plan of Tom's garden. How long and how wide is the space for carrots?
7. How long and how wide is the space which is planted with beans?



8. How long and how wide is the space planted with corn?
9. How many square feet are planted with carrots? With beans? With beets? With lettuce? With corn? (Use pencil if you need to.)
10. How many square feet are there in the path between the corn and the beets and the lettuce?
11. Draw a plan of a garden. Let one inch stand for four feet. Then ½ inch will stand for how many feet? An inch and a half will stand for how many feet? Two inches will stand for how many feet? Three inches?

Observe in the floor plan on page 113 that the flooring shows square feet, to make the dimensions realistic, but that the bed and couch and rugs do not permit obtaining the area by counting.

II

School Supplies

The Second Grade, Rooms A, B, and C, had these supplies:

3 boxes of pencils, 144 pencils in a box.
6 boxes of chalk, 144 pieces in a box.
3 big boxes of inch cubes, 1728 cubes in a box.
5 boxes of play money, 250 pennies in a box.
72 pads of paper, 96 sheets in a pad.

1. How many pencils did they have in all?
2. How many pieces of chalk did they have in all?
3. How many inch cubes did they have in all?
4. How many play pennies did they have in all?
5. How many sheets of paper did they have in all?

Here is a quick way to find out:

96 *Think "2 6's = 12." Write the 2 under the 2 of 72 in the*
72 *ones column. Remember the 1.*
192 *Think "2 9's = 18. 18 and 1 = 19." Write the 19.*
672 *Think "7 6's = 42." Write the 2 under the 7 of 72 in the*
6912 *tens column. Remember the 4.*
Think "7 9's = 63. 63 and 4 = 67." Write the 67.
Add. Remember that the 672 counts as 6720 in adding.

III

1. The boys and girls of the Welfare Club plan to earn money to buy a victrola. There are 23

boys and girls. They can get a good second-hand victrola for \$5.75. How much must each earn if they divide the cost equally?

Here is the best way to find out:

$$\begin{array}{r} \$.25 \\ 23\overline{)\$5.75} \\ \underline{46} \\ 115 \\ \underline{115} \end{array}$$

Think how many 23's there are in 57. 2 is right.
Write 2 over the 7 of 57. Multiply 23 by 2.
Write 46 under 57 and subtract. Write the 5 of 575 after the 11.
Think how many 23's there are in 115. 5 is right.
Write 5 over the 5 of 575. Multiply 23 by 5.
Write the 115 under the 115 that is there and subtract.
There is no remainder.
Put \$ and the decimal point where they belong.
Each child must earn 25 cents. This is right, for \$.25 multiplied by 23 = \$5.75.

In some cases the operation illustrates itself better than concrete problems requiring it. For example, "A square lot contains 62,500 square feet. How long is it?" is not so good as a straightforward series like that which follows.

Examine this table. Supply the missing numbers in the last five lines. $\sqrt{}$ means "square root of."

The square root of 16 is 4	$4 \times 4 = 16$	$\sqrt{16} = 4$
The square root of 484 is 22	$22 \times 22 = 484$	$\sqrt{484} = 22$
The square root of 25 is 5	$5 \times 5 = 25$	$\sqrt{25} = 5$
The square root of 400 is 20	$20 \times 20 = 400$	$\sqrt{400} = 20$
The square root of 49 is	$\sqrt{49}$ is	
The square root of 81 is	$\sqrt{81}$ is	
The square root of 36 is	$\sqrt{36}$ is	
The square root of 64 is	$\sqrt{64}$ is	
The square root of 100 is	$\sqrt{100}$ is ...	

As with the meanings of numbers, so with the meanings of operations, as soon as the pupil knows enough to make intelligent use of the operation, it is commonly best to let him begin using it, trusting that intelligent use of it will establish and extend and refine his understanding of it. If he has insufficient understanding of it, the fact and the nature of his difficulty will be clearer to him by the errors he makes in its use than from extended verbal discussion of the operation.

THE MEANINGS OF MEASURES, GEOMETRICAL FACTS, AND BUSINESS OPERATIONS AND TERMS

The meanings of inch, foot, quart, gallon, rod, acre, sq. yd., cu. ft., angle, parallel, altitude, base, radius, diameter, discount, interest, insurance, notes, stocks, dividends, bonds, and the like are all to be taught according to the same general principles that we have been considering in the case of numbers and operations. The reader will be able to apply them for himself, and we need note only certain facts which are sometimes misconceived.

The connection with reality of the larger measures like mile, acre, and ton, though not so convenient to secure as is the case with the smaller measures, is well worth the trouble. Even children bred in the country often have very inaccurate appreciation of what a mile or an acre really is. The teacher should find some well-known distances in the neighborhood to approximate 1 mile, 10 miles, ½ mile, and ¼ mile, and some well-known areas to represent ½ acre, 1 acre, 10 acres, or the like. Thus a New York City block is about 4 acres. A ton of coal may be easily seen in most cities, a ton of hay or grain in the country, and ton may be illustrated everywhere roughly by comparison with the weight of the Ford touring car (¾ ton).

Where the actual reality is inaccessible, or too complicated for observation by children, a simplified dummy form of it may be used. Thus we cannot insure property and wait for it to burn down, but we can play insurance as shown on pages 119 and 120.

This game represents no time-cost, since the problems solved are worth solving quite apart from the game. The realities of insurance are probably also very much clearer to children from this game than they would be from seeing premiums paid, a house burn down, the insurance money paid, and so on.

As with other matters, we should not teach pupils everything about one of these insurance facts because we teach them something about it—everything about stocks because we teach "stocks," or everything about bonds because we teach "bonds." Pupils may well learn by playing "bank" how to draw a check, and cash it, but they do not need to go through all the details of opening an account. In many cases the observation or dramatization of the full realities would only mean the expense of much time with more confusion than comprehension. Pupils would not profit so much, in respect to the purposes of arithmetic, by being present at the organization of a stock company or declaration of a dividend, or by visiting the stock exchange, or by reading the text of a railroad bond, as by a quarter of the time spent in a simplified study of the essential facts. Textbooks and teachers who show an ordinary bond to teach children what a bond is either have never read such a bond themselves, or have entirely fantastic ideas of the ability of elementary-school pupils, or have failed to appreciate the fundamental axiom that the purpose of teaching is to help children to learn. The same is true of many of the elaborate bookkeeping devices, tax blanks, and the like which are sometimes shown to pupils. They are real, but mere reality is not enough; it should be instructive reality.

Protection against Loss of Property by Fire

1. Do you pay anything for insurance against loss by fire?
2. Does your father?
3. If you know something useful about life insurance, fire insurance, accident insurance, insurance against theft, or insurance against sickness, be ready to tell it to the class clearly.
4. Play "INSURANCE" in this way:

One pupil is the "*Insurance Company.*" One pupil is "*Fire.*" The property to be insured is the written work of the test printed on page 120. Each pupil does the work of the test and puts his paper in a pile on the teacher's desk. "*Fire*" comes to the desk with his eyes shut and destroys one of the test papers. If that pupil is *not* insured, he has to do the 20 problems all over again after school. If he *is* insured, "*Insurance Company*" has to give him 20 problems all solved to use in place of the test paper. The pupil whose paper is lost gives these 20 problems to the teacher and does not have to do the test problems again. To be insured a pupil has to solve one of the extra problems and give it to "*Insurance Company.*" If you are willing to run the risk of having to do all 20 problems over again, you do not have to do an extra one to buy insurance. If you wish to be insured against the chance that "*Fire*" will happen to destroy your test paper, do one of the extra problems to pay for the insurance. "*Insurance Company*" uses the problems he receives from the other pupils to pay for the losses caused by "*Fire.*"

"*Insurance Company*" writes out for each pupil who pays him a problem an agreement like this:

Policy No.— Premium, 1 problem.
............10 A.M.

The Seventh-Grade Insurance Company agrees to insure........... to the amount of 20 problems against the loss of his test paper by fire within 5 hours from date.

(*Signed*)

....

An insurance agreement like the one at the bottom of the preceding page is called a **Policy.**

The amount that the Insurance Company may have to pay is called the **Face.**

The amount that the pupil who is insured pays is called the **Premium.**

The length of time during which the pupil is insured is called the **Term.**

5. What is the face of this policy?
6. What is the premium?
7. What is the term?
8. Read this description again so that you will know what to do in the game if you are "*Insurance Company*," "*Fire*," or an "*Insured person.*"

Twenty-Problem Test

1. The Davis family plan to save for an automobile. They found that they spent 80 cents a week in going to the motion pictures last year. They decided to spend only half as much. How much will they save in a year by this?

2. Last year they spent for clothes as follows:

Mr. Davis,	$110.50	Helen,	$115.30
Mrs. Davis,	175.25	Arthur,	70.10

They plan to reduce these expenses for clothes—

20% in the case of Mr. Davis
35% in the case of Mrs. Davis
35% in the case of Helen
15% in the case of Arthur

How much will they save in a year if they do so?

3. Mrs. Davis had a maid at $18 a month and paid $1.35 per week for a woman to do the washing. She plans to do her own work; and Helen and Arthur promise to do the washing. Counting the cost of food for the maid as $2.25 per week, how much will they save in a year by doing the housework and washing themselves?

[17 other problems follow in the test and 5 extra problems to be used to pay premiums.]

TESTING KNOWLEDGE OF MEANINGS

In testing whether pupils really know the meanings of numbers, operations, measures, geometrical facts, and facts about business, it is not enough to ask them to give a definition or description. The questions, "What is a fraction?" "What is a cubic yard?" "What is bank discount?" "What is a trapezoid?" are too likely to stimulate mere learning by rote. The ability to respond to them depends too much upon ability to express oneself in language. Questions which require the comparison of meanings, such as, "What are the differences between common fractions and decimal fractions? In what are they alike?" "What is the difference between a rectangle and a parallelogram? What is the difference between a parallelogram and a trapezoid?" are better, since they are free from the first objection, and because, with them, it is easier to distinguish deficiencies of knowledge from deficiencies in expression.

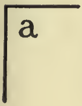
Not only are definitions and descriptions insufficient as tests; they are rarely very good tests. Tests where the knowledge is used in recognizing and in classifying facts and in giving illustrations are in general better. For example:

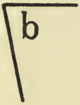
A. Gr. 4 or 5. Write as many fractions as you can in 4 minutes.

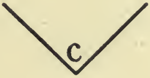
B. Gr. 4 or 5. Which of these fractions mean less than half a pound? Mark them *l*. Which of them mean more than half a pound? Mark them ***m***.

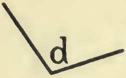
$\frac{3}{4}$ lb.	$\frac{1}{3}$ lb.	$\frac{1}{16}$ lb.	$\frac{5}{16}$ lb.	$\frac{7}{8}$ lb.
$\frac{2}{3}$ lb.	$\frac{3}{8}$ lb.	$\frac{1}{5}$ lb.	$\frac{3}{10}$ lb.	$\frac{5}{8}$ lb.

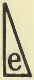
C. Gr. 4 or 5. Which of these angles are right angles?

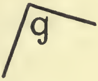








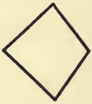




D. Gr. 4 or 5. Which of these are rectangles?


a


b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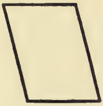
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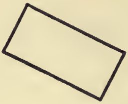

e

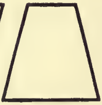

f

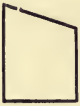

g


h

E. Gr. 5, 6, or 7. Which of these are parallelograms? Mark them *P*. Which are trapezoids? Mark them *T*.


a


b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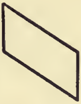

d


e


f


g

F. Gr. 5. Read the four denominators:

$\frac{1}{2}$ $\frac{3}{4}$ $\frac{6}{7}$ $\frac{11}{12}$

Name some fractions in which 5 is the numerator.
Name some fractions in which 5 is the denominator.

G. Gr. 5. *Prime Number.* Read each of these and state whether it is a prime number or not. If it is not, tell how you know that it is not.

10 11 12 13 14 15 16 17 18 19 20
21 22 23 24 25

H. Gr. 5. *Reciprocal.* State the reciprocals of:

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{5}$ $\frac{5}{8}$ $\frac{7}{8}$ 2 3 4 $\frac{3}{2}$ $\frac{11}{8}$
$1\frac{1}{4}$ $3\frac{1}{2}$ $2\frac{3}{4}$

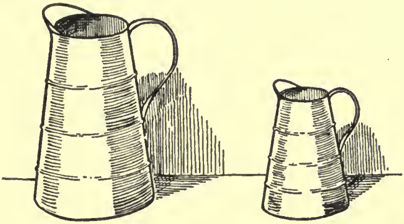
Such work as that shown above reveals clearly to the teacher a pupil's knowledge or lack of knowledge. It is easy to score. The pupils can see clearly that they are right or wrong and cannot hide behind, "I knew it, but I could n't write it."

EXERCISES

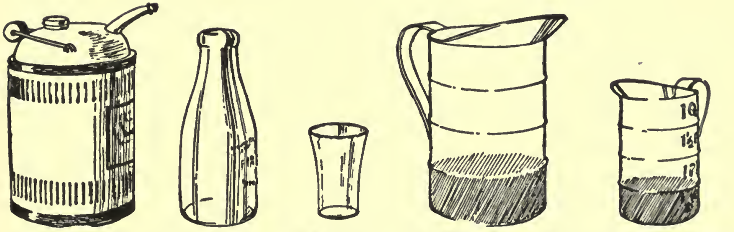
1. Name as many as you can of the things or events in school life in Grade 1 which can be used incidentally to teach the meanings of numbers.

2. Which are the more useful illustrations of gallon and quart, those in *A* or those in *B?* What would be better than either?

A



B



1 gallon **1 quart** **1 half pint or glassful** **This gallon measure has 1 quart of water in it** **This quart measure has 1 glassful or half pint in it**

3. In what respects would play money simply made of different sized squares marked 1¢, 5¢, 10¢, 24¢, and 50¢ be inferior to circular pieces printed to resemble the coins? In what respects would it be superior?

4. What use or uses would you make of each of these in teaching the meanings of numbers, groups of numbers, operations, measures, or geometrical facts?
   The height of pupils
   The top of the teacher's desk
   The floor
   The thickness of a sheet of paper in a book
   A pin
   The clock
   A row of desks
5. List the different things in arithmetic in learning which a foot rule is useful.
6. Examine the teaching of the meaning of pounds and tons (I, 114, Ex. 1–8), pecks and bushels (I, 117, Ex. 1 and 2), and five-place numbers (I, 150, Ex. 1–10). Plan additional illustrations and questions to use in case they are needed.
7. What illustrations are used to teach the meaning of minus quantities and negative numbers (III, 283 and 284)? What other aid is given?
8. Examine the explanation of mortgages (III, 161), If it were important that every pupil should surely understand what a mortgage was, what means would you employ? (Compare the explanations of checks [II, 182, 183] and commission [II, 200]).
9. Would you teach the meaning of "reciprocal" by pictures and illustrations as liquid measure and negative numbers are taught, or by cases of itself as square root is taught? (See II, 52, 53, and 54, if necessary.) Why?
10. Examine the teaching of the meaning of shares of stock (III, 153 and 154). Why is this better than a visit to the stock exchange? In what respects is the certificate on p. 153 better than a regular stock certificate?

CHAPTER VII

SOLVING PROBLEMS

DESIRABLE QUALITIES IN ARITHMETICAL PROBLEMS

Teachers in the past have too often been content to assign any problem that was a problem. They have assumed that the discipline the mind received from trying to discover the solution of any problem which required thinking was so valuable that it did not much matter whether the problem was real or artificial, well or ill stated, common or rare. For this they have had some justification, or at least some excuse; for it is true that solving arithmetical problems is one of the best single tests of intellect that psychologists have yet found; and that a problem may be a good exercise for the intellect even though its data are foreign to, or even contrary to, experience.

However, it seems certain that if we take enough pains and have enough ingenuity, we can find an abundance of problems which will exercise the intellectual powers well and at the same time prepare the pupils more fully and directly to apply arithmetic to the problems they will really encounter in life. So the newer methods, as was noted in chapter i, set a higher standard for problems. A problem should, preferably, (1) deal with a situation which is likely to occur often in reality; (2) in the way in which it should be dealt with; (3) should make the situation neither much harder nor much easier to understand than it would be when really present to the pupil's senses; and (4) should be supported by somewhat the same degree of interest and motive as attach to the problems which the pupil will meet in the actual conduct of his affairs. It is admitted, however, that these standards may have to be somewhat relaxed in order to have problems which can be used conveniently under the conditions of classroom instruction. They are desiderata, not requirements.

SITUATIONS PRESENT TO SENSE, IMAGINED BY THE PUPIL, AND DESCRIBED IN WORDS BY ANOTHER

One important limitation due to the conditions of classroom teaching is that the facts of the problem can so seldom be presented to sense—must so often be described in words. The problems of life are most often questions about situations or facts actually existing before the pupil's eyes, less often questions which the person puts to himself in connection with his past affairs or future plans, and least often questions put to him in words by another. In proportion as we can escape this limitation and actually present the situations, we not only are surer of preparation for life, but also find it easier to teach the pupils how to attain correct solutions.

There are three main elements in problem solving: (1) to know just what the question is, (2) to know what facts you are to use to answer it, (3) to use them in the right relations.

When the actual situation is present, and, so to speak, itself defines the question, there is likely to be almost no difficulty in respect to the first of these three, and relatively little in respect to the others. When pupils actually play store with groceries and money, or lay out their own school garden or baseball field, or decide which side won a game of bean-bag, or who jumped farthest, they are, as a rule, somewhat easily taught to use their intellects effectively. They do not so often multiply two numbers just because one is very large and the other very small, or add the numbers because there happen to be three of them! The real situation helps to make the question clear and protects them against many follies.

When the pupil initiates a question for himself in connection with his own past experiences or plans for the present or future (as in deciding the quantities he will need for a party or how long it will take him to have enough to buy a certain thing, starting with what he has and saving at a certain rate), his thinking may be somewhat less easy to stimulate and guide than when the situation is present to the senses. But it will probably be much more active and ready for correction than in the case of a problem put to him in words by another.

Many of the difficulties of pupils in learning, and of teachers in teaching problem-solving, are due to the use of problems described in words. With imposed tasks in no real setting the pupil is much less likely to know what the question is, or to have any strong interest in obtaining its answer. And these difficulties are, to a certain extent, unprofitable, since in life the question will commonly be his own and come in a real setting that helps to guide him to its answer. Life problems are thus easier than book problems.

Consequently the newer methods try (1) to provide real situations or projects where that is feasible, and (2) to encourage the pupil to identify himself with the person whom the problem represents as acting or planning. If the reality cannot be supplied, and if the sense of personal participation cannot be aroused, they try at least (3) to free the problem from difficulties due (*a*) to its vocabulary and structure or (*b*) to lack of experience by the pupils of the facts described. As samples of the use of situations present to the senses as they would be in reality (or very nearly so) we may take the problems on force and distance and gear ratios (for boys in Grade 8) shown below. As samples of problems planned to attract pupils to think of themselves as personally concerned with the problem we may take the vacation trip problems (beginning of Grade 6) (see pages 129–131) and the problems in starting in business (middle of Grade 7) (see pages 131–132).

Force and Distance

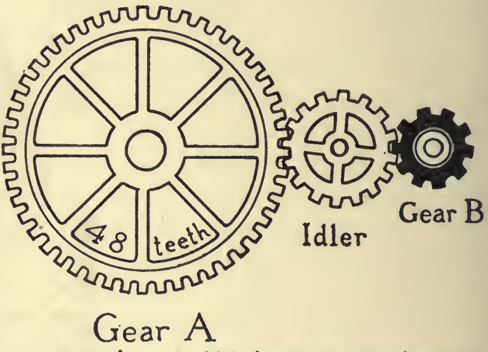
1. If a man pulls rope *A* down 6 ft., how far up will the weight *W* go?
2. How far up will the weight go (*a*) If *A* is pulled down 20 ft.? (*b*) If *A* is pulled down 14 ft.?
3. How far down must rope *A* be pulled (*a*) To hoist weight *W* up 6 ft.? (*b*) To hoist it 8 ft.?

Neglecting friction, a downward force of 1 lb. acting through 1 ft. will raise 2 lb. half a foot, or 10 lb. a tenth of a foot, etc.

4. Neglecting friction, how far must a force of 100 lb. act (*a*) To raise a weight of 500 lb. 2 ft.? (*b*) To raise 500 lb. 3½ ft.? (*c*) To raise 400 lb. 4 ft.?

Gear Ratios

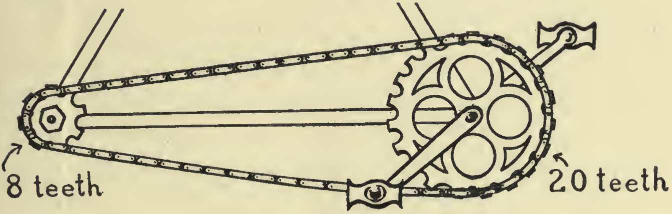
1. How many teeth are there on Gear A? On the idler? On Gear B? Record the numbers to use in Ex. 2, 3, and 7.
2. When Gear A makes one complete revolution, how many times will the idler revolve—2 or 2½ or 3?
3. When the idler makes one complete revolution, how many times will Gear B revolve—one time or 1½ times or 1⅗ times or 1¾ times?



Gear C

4. Gear C has 12 teeth. How many teeth must a gear have that fits into Gear C and revolves once while Gear C revolves 8 times?

5. How many times does the rear wheel of this bicycle revolve for each complete revolution of its sprocket wheel?



6. If the rear wheel is 28 in. in diameter, how far does the bicycle go for each revolution of its sprocket wheel?

7. If Gear A in the picture at the top of the page makes 100 R.P.M. (revolutions per minute), how many R.P.M. does Gear B make?

A Vacation Trip

Mr. and Mrs. Sears, with Ruth and Alec, went on a camping trip for their vacation. They carried a tent, blankets, food, and a little stove in a wagon. Ruth and Alec rode on their bicycles.

1. They drove north for 2 weeks and 2 days and spent 1 week 6 days coming back. How long was the whole trip?

2. Mr. and Mrs. Sears drove 438.9 miles in 25 weekdays. How many miles did they average per day?

3. The children rode more than this, because they went to different places on errands and took some side trips. Ruth's cyclometer read 586.7 at starting and exactly 1175 when they reached home. Alec's read 738.46 at starting and 1341.24 when they returned. How far did each child ride?

4. Mr. Sears arranged an old bicycle-cyclometer on the wagon wheel. They found by measuring that when this cyclometer showed 1 mile, the wagon had really gone 2.09 miles. When the cyclometer showed 2 miles, the wagon had really gone 4.18 miles. How far had the wagon really gone when the cyclometer showed 6.4 miles?

5. What did the cyclometer show when they had gone 438.9 miles?

6. Mr. Sears paid $100 for the horse, $12 for the harness, and $48 for the wagon. At the end of the trip he sold all three for $125. How much did it cost for the use of them during the trip?

7. The tent and stove cost $21.00. If they last 12 years and Mr. Sears uses them each summer for camping, what will be the annual cost for the use of tent and stove? (Annual cost means cost per year.)

8. They bought food to take with them at a cost of $16.82, and spent $21.46 for food on the way. (*a*) What was the total cost for food for the 29 days? (*b*) What was the average cost per day for food?

9. Repairs on the wagon cost $1.45. Oats and hay for the horse cost $9.28. What was the average daily cost for care of the wagon and feed for the horse?

10. The Sears family always try to earn special money for their vacation, beginning Jan. 1, each year. They try to earn an average of 83⅓ cents a day during Jan., Feb., Mar., and April. How much do they try to earn in all during the four months?

11. Mr. Sears tries to earn $62.50 in all, Mrs. Sears tries to earn $25.00, and each of the children tries to earn $6.25. What fraction of the $100 does each member of the family try to earn as his share of the vacation expenses?

12. Ruth earned $7.50. Alec earned $4.50. Did Ruth earn 1½ times as much as Alec, or 1⅔ times as much, or 1¾ times as much?

13. In a previous year Ruth earned $3.00 and Alec earned $7.50. How many times as much did Alec earn as Ruth earned?

14. Write three problems about a vacation trip for the class to solve.

Starting in Business

1. Dick studied in evening school at the Y. M. C. A. and learned how to run an automobile. He worked in a garage for 6 months at $5 a week, 6 months at $7 a week, 6 mo. at $8 a week, and 6 mo. at $9 a week. He saved one third of all that he earned. How much did he save?

2. He borrows enough more money to spend $75 for a garage, $375 for a second-hand automobile, and $10 on robes, tools, etc. (*a*) How much does he borrow? (*b*) At 6% interest per year, how much interest must he pay each year until he pays back what he has borrowed?

3. He plans that the first year he will pay $27.50 for insurance on the garage and auto, $100.00 toward a savings fund for a new auto when this one wears out, and $112.00 for interest and for paying back part of what he borrowed. How much will he have left if he does this, supposing that he receives $920.00 from fares and spends $115.00 for gasoline, oil, supplies, and repairs?

Dick's rates for passengers are:

	1 Mile or Less	Extra for Each ¼ of a Mile over 1 Mi.
For 1 passenger	30¢	5¢
For 2 passengers	40¢	6¢
For 3 or 4 passengers	50¢	7½¢

4. How much does he charge for a trip for 2 passengers when his speedometer reads 110.2 mi. when they get in and 113.7 mi. when they get out?

5. How much does he charge a trip for 3 passengers when his speedometer reads 116.3 mi. when they get in and 118.0 mi. when they get out? (.6 mi. or .7 mi. is counted as ¾ mi.)

6. How much does he charge for a trip for four passengers when his speedometer reads 149.4 mi. when they get in and 151.9 mi. when they get out?

7. For a trip of 12 miles with a single passenger Dick received his regular rate less 30%. How much did the passenger pay him for this trip?

PROBLEMS MADE UNDULY EASY BY VERBAL DESCRIPTION

In three respects the verbally described problems as ordinarily given were easier to solve than corresponding problems actually encountered. The first was that it became the custom to give no numbers in a problem that did not have to be used to obtain the answer. Consequently the pupil knew that he must work them all in somehow, whereas in life the situation may contain many irrelevant numbers which you must neglect. The second was that it became the custom to give (except for facts like 12 in. = 1 ft., or 1 gal. = 4 qts.) in each problem all the numbers needed to solve it. The pupil working Problem *x* on page *y* did not have to look anywhere outside its own two or three lines of print, whereas in life the problem may require him to inspect the price list, remember his mother's instructions, and question the storekeeper. The third was that a certain verbal form was so uniformly associated with a certain procedure (e. g., "bought .... at ....," with multiplying) that the correct response could be given by the sheer force of habit.

These are not desirable customs, and the newer methods vary them. Instead of putting in each problem all the data for that problem and nothing besides, they often set forth certain data followed by a group of problems each of which uses only a few of the data given, as illustrated on pages 134–136. They give problems which require the pupils to obtain data (such as "How many square feet are there in your schoolroom floor?"). They occasionally require the pupil to see and act on the need of recalling for use in one problem a result obtained in a problem preceding it in the group. They are careful to use each verbal form in the varied ways in which it may properly be used in life. Thus they use "bought 30 crullers at 20¢ per dozen" as well as "bought 4 crullers at 2¢ apiece" or "bought 6 dozen crullers at 20¢ per dozen."

[For Grade 2 or early in Grade 3]

The 1 2 3 4 5 6 7 8 9 Cent Store

Things for Boys and Girls at Every Price

1¢ a picture postcard
2¢ a paper doll
3¢ a dozen jackstones
4¢ a top
5¢ a ball
6¢ a pad
7¢ a toy pistol
8¢ dominoes
9¢ a book

How much do you pay for —

1. A ball and a pad?
2. A ball and a toy pistol?

[Followed by eight more problems]

1. Play that you have 8 cents to spend for two things at the *1 2 3 4 5 6 7 8 9 Cent Store.* You can buy a pad and ......, or you can buy a ball and a ......, or you can buy two tops.

What can you buy if —

2. You have 9 cents to spend for two things?

Running Races

In one class at the Lincoln School every boy and girl who was well ran 50 yards as fast as he could. The teacher timed them with a stop watch. Here are some of the times in seconds:

Boys	
Alfred	**$10\frac{1}{5}$**
Arthur	**$8\frac{3}{5}$**
Ben	**$8\frac{2}{5}$**
Charles	**$7\frac{4}{5}$**
Dick	**9**

Girls	
Alice	**$10\frac{2}{5}$**
Clara	**$9\frac{1}{5}$**
Ella	**$11\frac{3}{5}$**
Kate	**$8\frac{4}{5}$**
Helen	**$14\frac{1}{5}$**

1. Which boy ran in the shortest time? Who was next? What was the difference between their times?
2. How much under 10 seconds was Arthur's time? Ben's? Charles's? Dick's? Clara's? Kate's?
3. How much over 10 sec. was Alfred's time? Alice's? Ella's? Helen's?

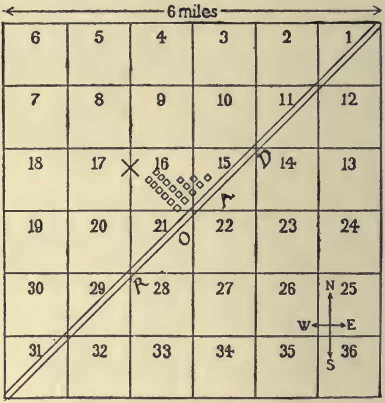
From 12 yr. 0 mo. to 13 yr. 0 mo.
Dick gained $\frac{3}{8}$ in.
Fred gained $\frac{5}{8}$ in.
George gained $1\frac{3}{4}$ in.
Oscar gained $1\frac{1}{8}$ in.
Paul gained $1\frac{1}{4}$ in.
Robert gained $1\frac{1}{2}$ in.
Sam gained $2\frac{1}{2}$ in.

Supply the missing numbers:

4. George gained . . . times as much as Dick.
5. Oscar gained . . . times as much as Dick.
6. Robert gained . . . times as much as Dick.

A Township Map

1. Examine this map. It is a map of a township. The numbers on it are the numbers of the sections into which it is divided. Consider the township as a square and each section as a square.



2. How many square miles are there in the township?

3. How many acres are there in a quarter of a section?

4. What fraction of a section is 80 acres?

5. Find how long it takes each of these trains to go (*a*) from Chicago to Marion; (*b*) from Chicago to Omaha.

	Missouri River Express	San Francisco Limited	Local Express
Lv. Chicago	**6 05**	**9 35**	**10 00**
Dan. Jct.	**8 14**	**11 32**	12 30
Marion	**11 50**	3 10	5 30
Pickering	1 30	....	7 35
Omaha	7 19	10 10	**3 25**

Numbers in heavy type are **P.M.**
Numbers in light type are A.M.

6. Which train takes about 40 per cent longer than the San Francisco Limited?

7. When will the Missouri River Express arrive at Omaha if it loses 1 hr. 16 min. and then makes up ¾ of the loss?

Some teachers would go still farther to counteract the first custom, giving single problems with irrelevant numbers (such as, "I went to 4 stores and bought 9 pads of paper at 7¢ each. Each pad contained 50 sheets. How much did they cost me?" or "I went with 3 friends to a sale, which began at 10:25 and lasted until 1:10. We spent $2.35 each. How much did we spend in all?")

This does not seem wise, however, unless it is done frequently or with warning to the pupils that such problems will be interspersed with the others. It is hard to devise natural irrelevancies; also the feeling that the teacher or textbook is bent on catching you unaware and tripping you up is prejudicial to good work in general. Except for irrelevant numbers as they often appear in reality* and special exercises with

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* An illustration of such natural irrelevancies is the following, where the alleged amounts of discount and the lengths of the pieces in the advertisement should, of course, not be used in planning costs:

Buying Remnants

Remnants at ⅓ to ½ Regular Prices
Lengths from ½ yd. to 5 yd.
Ginghams, 8¢ per yd. **Serges, 16¢ per yd.** **Flannels, 24¢ per yd.**

1. Mrs. Andrews bought two pieces of gingham. One was 2⅜ yd.; the other was 1¾ yd. How many yd. did she buy in all? How much did the gingham cost in all? How much change should she receive from a two-dollar bill?
2. Mrs. Johnson bought three pieces of flannel. One was 1⅛ yd., the second was 1½ yd., the third was 1⅝ yd. How many yd. did she buy in all? How much did the flannel cost? How much change should she receive from a two-dollar bill?

warning given to be alert and not get "caught," it seems better to accept the conventional custom that a single problem shall not contain irrelevant extra numbers. The same end seems better attained by grouping problems. For example, the teacher quoted above might better have done as follows:

Put on the blackboard the story of the forenoon's shopping; then ask:

1. How much did the 9 pads cost?
2. How many sheets of paper did they contain?
3. How long did the sale last?
4. How much did my friends and I spend in all at the sale?

Certain very innocent "catches" (for example, "Train No. 20 goes 30 miles an hour for 6 hours. How far does it go in all?") may, perhaps, be useful to detect pupils who are solving problems just by chance shuffling of the numbers or to shame pupils who are very careless. There are, however, better ways to do this:

THE TECHNIQUE OF SOLVING PROBLEMS

Concerning the technique of solving problems and expressing the solution, the newer methods advocate extreme catholicity.

Until Grade 5 attention may best be given almost exclusively to obtaining the right answer. Only rarely should the pupil be directed to state what he intends to do before doing it, or why he did a certain thing after doing it. Sometimes he may be given problems without numbers (such as, "If you know how many hours an auto went and how far it went, how do you find how fast it went?") which force him to think of the method of solution and to express it in words.

In Grade 5 or Grade 6 he may be taught the following principles and given some practice in using them:

(1) If you know surely how to solve the problem, go ahead and solve it.

(2) If you do not at once see how to solve the problem, consider the question, the facts, and their use, asking yourself:
  What question is asked? What am I to find out?
  What facts are given? From what am I to find it?
  How shall I use these facts? What shall I do with the numbers, and with what I know about them?

(3) Plan what you are going to do, and why, and arrange your work so you will know what you have done.

(4) Check the answer obtained, to see if it is true and reasonable according to what the problem says.

In this connection the pupil may be shown how to put the data in an equation showing what is to be done, and given some practice in so doing. Neither this nor any written form of analysis should be made compulsory, however. Problems without numbers may be assigned occasionally.

In Grade 7 or 8 he may be taught to express the solution of typical and familiar problems in generalized form, using either words or shorter symbols or a mixture. For example:

Let y *= the number of miles a train moves per hour.*
Let d *= the total distance (in miles) the train goes.*
Let t *= the total time (in hours) the train is in motion while going that distance.*

1. What is the value of y (*a*) When $d = 200$ and $t = 5$? (*b*) When $d = 270$ and $t = 9$? (*c*) When $d = 105$ and $t = 3$?

2. Which of these are correct equations?

$y = d \times t$	$y = d + t$	$y = d \div t$	$y = d - t$
$y = \frac{d}{t}$	$y = 2d + 3t$	$y \times d = t$	$t \div y = d$
$d \div y = t$	$\frac{d}{y} = t$	$y \times t = d$	

3. Express this rule, "*The area of a triangle* = (½ *the altitude*) × (*base*)," in an equation, letting

x *= the area of a triangle in square inches.*
a *= the altitude of that triangle in inches.*
b *= the base of that triangle in inches.*

4. Supply the missing words and express this rule in an equation:

Interest = Principal × Rate × Time.

("*Principal*" means the number of dollars borrowed.)

Let i = the number of dollars paid for interest.
Let p = ...
Let r = ...
Let t = ...

5. What is the value of i, when $p = \$200$, $r = 6\%$ or .06, and $t = 3$ years?

Written Practice

Let w = the regular weekly salary of a salesman in dollars.
Let c = the rate of his commission on sales.
Let s = the total sum of his sales for the year in dollars.
Let e = his total earnings for the year in dollars.

1. Write an equation to use in computing e.
2. What is e if $w = 12$, $c = 7\frac{1}{2}\%$, and $s = 8950$?
3. What is e if $w = 15$, $c = 10\frac{1}{2}\%$, and $s = 12{,}740$?
4. What is c if $e = 1156$, $w = 10$, and $s = 10{,}600$?

Such work is valuable, but is very hard for many pupils. It may best be introduced by work where the equations are in words and where one element is to be supplied, as shown on page 141. The pupil in the upper grades may write out a full statement and justification of a solution occasionally either as an exercise in arithmetic, or, more fittingly, in English.

In general, however, statements, oral or written, about what is to be done or what has been done or why it should be done are of very minor importance in comparison with doing it accurately and quickly. Statements, analyses, and explanations by pupils are valuable chiefly in proportion as they help the pupils to solve problems. They have some value also as training in the use of language, but this does not justify their use when they take much more time than they save and interfere with thinking. They often do.

Supply the missing words and signs in each of these equations:

If no reduction is made for buying a large quantity —

1. Cost per quart = (cost per bushel) ÷ . . .
2. Cost per quart = (*total cost*) . . . (number of quarts).
3. Cost per quart = $\dfrac{\text{cost per}\ldots}{8}$.
4. Cost for 20 articles = 20 . . . (the cost per article).
5. Cost for 8 articles = 8 . . . (the cost per article).

If we let n stand for the number of articles bought,

6. Cost for n articles = n . . . (the . . .   . . . article).

In a RECTANGLE, area = $l \times w$.

7. Area in sq. in. = (length in inches) × (. . . .).
8. Perimeter in inches = (2 × length in inches) . . . (2 × width in inches) or

Perimeter = sum of the lengths of the four sides.

In a PARALLELOGRAM, area = $b \times a$.

9. Area in sq. in. = (base in inches) . . . (altitude in inches).
10. Perimeter = . . . of the lengths of the . . .

In a TRIANGLE,

11. Area in sq. in. = (base in inches) . . . ½ (. . . in inches).
12. Perimeter = sum of . . . of . . .   . . .
13. Distance traveled in miles = (time in hours) . . . (miles per hour).
14. Time required in hours = (distance traveled in miles) . . . (miles per hour).

If we remember to use hours, minutes, and seconds, and miles, rods, yards, and feet correctly, we may think:

15. Distance = time . . . rate of motion.
16. Time = distance . . . rate of motion.
17. Rate of motion = distance . . . time.

The use of such statements is defended by some teachers on the ground that they show whether a pupil really understands what he has been doing—that he is not working by mere habit. A test with similar problems changed in their superficial appearance will show this better, however; for a pupil may understand the solution of a problem and still become confused in the linguistic task of explaining it. With at least nineteen out of twenty problems the pupil's task should be simply to get the right answer.

INDIVIDUAL DIFFERENCES

The work of computation in arithmetic grade by grade can be learned and done by all or almost all who have been promoted to that grade, though some will require very much longer to learn to do it and very much longer to do it than others. With problem solving, this is not the case. If a hundred problems graded in difficulty in ten steps over a fairly wide range are assigned to, say, a thousand children in Grade 6, there will be some pupils who can (except for occasional slips) solve the entire hundred, and some who cannot solve more than fifty—not if they struggle for hundreds of hours. Problems of a certain degree of complexity and abstractness they simply cannot solve, just as they cannot jump over a fence five feet high or lift a weight of five hundred pounds. Similar but smaller differences hold from the ablest to the least able of the group. If we give problems that nearly all can solve, the abler pupils are left idle; problems that make the abler pupils work are enigmas to the less able.

These facts are recognized informally by textbooks and by good teachers. The textbooks usually give in any one lesson made up of problems a variation in difficulty; and the good teacher does not expect that the dull pupils will have many right! It may be well to recognize the fact more openly and honestly, as by lessons of the sort shown on the next page.

Earning, Spending, and Saving

Each pupil writes on the blackboard two problems about earning, spending, and saving. One problem is hard, the other is easy. The pupils solve either one or both as they choose. If you think you can solve the hard one, try to do so. If you think it is too hard for you, solve the easy one. If you have time, solve them both.

EXERCISES

Examine the problems in these ten pages: I, 166, 168, 172, 174, 183, 195, 197, 204, 220, 221.

1. List ten in which the data of the problem are present to the pupil's senses nearly or quite as fully as they would be in corresponding problems in real life.

2. List ten in which the pupil might fairly be expected to identify himself with the person concerned and realize what the problem was nearly or quite as well as if he really had had the experience or made the plan.

3. Could you have found more than ten of each sort?

4. In finding the area of a real triangular field is the chief difficulty remembering that it is obtained by $\frac{1}{2}$ (Altitude × Base) or is it knowing what altitude means and how to determine it, and using the right altitude with the right side?

5. What is the defect in this problem? A triangular field has a base of 40 rd. and an altitude of 21 rd. What is its area in sq. rd.?

6. Which fits better for life, A or B? Why?

A

On January 1, 1920, John gave the bank a non-interest-bearing note for $100 due March 1, 1920. Should the bank give him about $99 or about $100 or about $101 for it? Would the bank give him more for it or less if he kept the same note until February 1 and gave it to the bank then?

How much must John pay the bank on March 1, 1920, if he gives the bank the note on January 1, 1920?

Must he pay more, or less, or just the same on April 1, if he keeps the note till February 1 before giving it to the bank?

B

What are the proceeds of a note made January 1, 1920, due March 1, 1920, if it is discounted on January 1, 1920, the interest rate being 6 percent?

What are the proceeds if it is discounted February 1, 1920, at the same rate?

What amount must the maker of the note pay at its maturity?

7. Examine and solve the problems printed below.

Which of them require experience of facts which few elementary-school children will have had? Mark these *Ex*.

Which of them encourage erroneous ideas? Mark these *W*.

Which of them require an understanding of language which is beyond the ability of many elementary-school children of the grade for which the problem is suitable? Mark these *L*.

Which of them are problems which not over one pupil in a hundred will ever encounter after school? Mark these *Unr*.

(The same problem may of course be assigned more than one of these demerits.)

a. A pile of wood in the form of a cube contains 31 cords. What are the dimensions to the nearest inch?

b. In this picture you see one kitten on the ground and one kitten on the stump. If you should ask me how many kittens are on the stump, what would be my answer?

c. What is the cost of $\frac{6}{7}$ of 42 eggs at 25 cents a dozen?

d. At $\frac{5}{8}$ of a cent apiece how many eggs can I buy for $60?

e. Three bodies move uniformly in similar orbits around the same center in 87, 224, and 365 days, respectively. Supposing all three in conjunction at a given time, find after how many days they will be in conjunction again.

f. You see a flageolet and a violin. They are musical instruments. One musical instrument and one other musical instrument are how many?

g. John has $\frac{7}{8}$ as many hens as Mary, who has 24. How many has John?

h. If asked your age, would you answer in years or in weeks? If asked how long before you would go home today, how would you answer?

i. Eight times the number of stripes in our flag is the number of years from 1800 until Roosevelt was elected President. In what year was he elected President?

j. A tree fell and broke into four pieces $9\frac{3}{8}$ feet, $13\frac{2}{9}$ feet, $16\frac{11}{18}$ feet, and $14\frac{1}{2}$ feet long. How tall was the tree?

k. Sound travels 1100 ft. per second. How long after a cannon is fired at New York will it be heard in Philadelphia, which is 90 miles from New York?

l. A fisherman caught 968 fish. One-eighth were haddock and the rest were cod. How many were cod?

m. If a horse trot 9 miles in one hour, how far will he travel in 10 hours?

n. What is the duty, at 20 percent, ad valorem, on 40 bales of merino wool that cost 25¢ a pound, the bales averaging 400 lbs. each, and the tare being 5 percent?

o. George Washington was born February 22, 1732. How old would he be if he were living February 22, 1898?

p. Mr. A owns 8 horses, which are $\frac{4}{31}$ of the number of cows he had and $\frac{7}{21}$ of the number of sheep. How many animals has he?

q. If the pressure of the atmosphere is 15 lb. per sq. in. what is the pressure on the teacher's desk in your room?

r. How many lines must you make in drawing 8 triangles and 6 squares?

s. What sum must a broker pay for $200 in gold at a premium of $\frac{5}{8}$ percent?

t. One train had 7 cars and another train had 15 cars. How many more cars did the latter have than the former?

CHAPTER VIII

TEACHING AS GUIDANCE

In previous chapters we have examined the general procedure of the newer methods in adapting the teaching of arithmetic to the nature of the learner and the needs of life. We have learned what the general principles are that guide the teacher in choosing topics, in arousing and utilizing interest, in securing understanding of the science of arithmetic, ability to compute, and ability to apply arithmetic to the problems of the real world, and in organizing arithmetic into a series of instructive experiences and activities.

The newer methods do more than provide such general principles. They seek to apply them, and also all the helpful conclusions that classroom experience and scientific studies of the learning process have reached, to every detail of the teaching of arithmetic. The rest of this book will be concerned chiefly with such details. This chapter will present some facts under the general head of generalship or guidance.

We have learned to think of teaching as providing the most instructive experiences and the most instructive activities, so organized and arranged as to produce maximum knowledge of arithmetic as a science and skill in arithmetic as an art. The teacher may be thought of as a general who protects his army against such and such dangers, extricates them from this or that trap, and provides them with the best weapons and ammunition. Or she may be likened to a guide who prevents his party from taking wrong paths, helps them out of pitfalls and crevasses, and provides them with proper ropes, staffs, and axes. So the teacher's work includes measures to avoid misunderstandings and false steps, the diagnosis and cure of difficulties, and the selection or invention of just the best means for learning each topic.

BLOCKING WRONG PATHS

Confusing cardinal with ordinal numbers. Almost all children, except the very dullest, know the meanings of "one," "two," and "three" at entrance to school and have a true but vague sense of the meanings of some other numbers. A few, however, confuse "two" with "the second," "three" with "the third," and so on; and more of the dull pupils will fall into this error in using larger numbers, because of the numbering of pages in their reader, and the use of counting by cardinal numbers.

The preventive is to make sure that they have sufficient experience with cardinal numbers in their primary use, and to teach them that the 22 on page 22 means there are 22 pages in all so far, that the 8 on the ruler means that from the end to that point equals 8 inches, that, when they count play money, 5 means that the five pennies so far counted are 5. The use of "first" and "second" should also be taught. The use of third, fourth, fifth, etc., as ordinal numbers need not be stressed, for, by the unfortunate constitution of our language, these words are used both for position in a series and for fractional parts.*

In all objective presentations, the primary use of cardinal numbers should be emphasized. Figures 11 and 13 are right. Figures 12 and 14 are wrong or, at least, inadvisable.

FIGURE 11

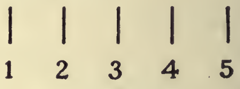

FIGURE 12

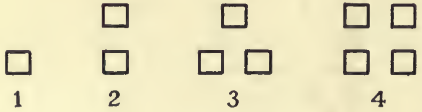

FIGURE 13

FIGURE 14

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*For this reason it may be well to use "the third one," "the fourth one," "the fifth one," "the sixth one," in place of "the third," "the fourth," etc., as ordinal numbers in the lower grades.

Adding by counting forward by ones. Counting is wisely used so as to derive the early addition combinations and to check the results as the pupil is learning to trust his memory of them. To prevent it from becoming a fixed habit, the teacher should use "hidden" addition,* and should force speed, as by drills in which she gives pairs of numbers with 5 seconds between, then with 4 seconds, then with 3, then with 2½, then with 2, the pupils answering in turn. The same procedure is also used, but with the pupils writing answers in a column on ruled paper, leaving a blank when they are not sure. The gifted pupils may teach individually pupils who cannot reach 2 seconds speed, giving two numbers and announcing the answer if it is not given at once by the learner and having him at once repeat (e. g., 9 and 6, 15"), continuing until he can answer without any time for counting, and never giving him time to count.

A pupil who avoids this bad habit with the fundamental combinations may still fall into it with the additions to higher decades (12+9, 13+4, 25+8, and the like). The forcing of speed is again the preventive.

Subtracting by counting backward by ones. The pupil should never under any circumstances be allowed to do this, or even know that it is a way to get answers in subtraction. At the very beginning he should derive the subtraction facts, not from counting backward by ones, but by selecting the addition

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*"Hidden addition" means addition where real objects are presented so that the problem is sure to be understood and so that the result can be verified by counting, but where they are hidden during the act of adding, so that the pupil must *think the numbers* and *add* them. He cannot *see* the objects and *count* them. For example:

Take 10 cents of play money, put 4 cents in a pile, and put your hand over the pile. Put 2 cents more in the pile under your hand. How many cents are in the pile under your hand? Look at them to see that your answer was right.

Put 6 cents in a pile under your hand. Put 2 cents more under your hand. How many cents are under your hand? Is your answer right?

Put 3 cents in a pile under your hand. Put 2 cents more under your hand. How many cents are under your hand?

fact that fits. (5+ . . . =8; he thinks of the "5+" facts until he has the one that fits; when he has it he learns it.)

Subtracting by counting forward by ones to make up the difference. The preventives are as in the case of addition.

Serial memorizing of the multiplication and division tables. When taught by customary methods, pupils confronted by "8 times 3 = ?" often have to start with "one times three, two times three," and go on through the table until they come to the desired fact. Speed drills of the pattern described for addition with the facts in a random order are one preventive, and perhaps an adequate one. It seems probable, however, that the early learning of the multiplication facts in a tabular form is undesirable. If they never are learned in series, the pupil will not be tempted to resort to memory of the series when he needs one fact from it. Of course the pupil should be aware of the system and, if he does not remember a required fact, should be able to derive it. He may, however, more usefully derive it from the original source, successive additions,

$$\text{in the usual form}\quad \begin{array}{r}3\\3\\\hline 6\end{array}\quad \begin{array}{r}3\\3\\3\\\hline 9\end{array}\quad \begin{array}{r}3\\3\\3\\3\\\hline 12\end{array}.\quad \text{as well as in the verbal form}$$

3, 6 (two 3's), 9 (three 3's), 12 (four 3's). At the first derivation, the facts should be thoroughly learned as separate associations. Very early, after the times 2's, times 3's, times 4's, and times 5's are learned, the multiplication facts should be put to use, first, in work like

$$\begin{array}{r}32\\3\\\hline\end{array}\quad \begin{array}{r}43\\2\\\hline\end{array}\quad \begin{array}{r}321\\4\\\hline\end{array}\quad \begin{array}{r}523\\3\\\hline\end{array}$$

and then in work like

$$\begin{array}{r}524\\7\\\hline\end{array}\quad \begin{array}{r}345\\6\\\hline\end{array}\quad \begin{array}{r}232\\5\\\hline\end{array}\quad \begin{array}{r}415\\9\\\hline\end{array}$$

The division facts should not be presented at first in tables, but should be derived each from the corresponding multiplication fact. They should at once be thoroughly learned as separate associations. They should soon be put to use in division with a remainder, and, a little later, in divisions of three-place and four-place numbers by a one-place number. In these uses it is extremely unprofitable to have to stop to track through a table or derive a product or quotient. Hence, even the very dull pupils can be led to realize the need for mastery of the multiplication and division facts.

Adding denominators. When a child says or writes that $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ or $\frac{3}{8}+\frac{5}{8}=\frac{8}{16}$, he is led astray by the habit, acquired with the addition of integers, of adding everything in sight. There is nothing intellectually perverse or demoniacal in his doing so. On the contrary, all pupils would tend to answer $\frac{3}{4}+\frac{3}{4}$ by $\frac{6}{8}$ or $\frac{7}{7}$ or 14 if some contrary force did not prevent.

The preventives are thorough knowledge of the meanings of the fractions in question, the learning of $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$ or 1, $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$, $\frac{1}{4}+\frac{1}{4}=\frac{2}{4}$ or $\frac{1}{2}$, $\frac{1}{4}+\frac{3}{4}=\frac{4}{4}$ or 1, and a few other common addition combinations of fractions, just as we learn that $5+3=8$ or $7+5=12$, and the verification of additions of fractions by objective measurement.

DIAGNOSING DIFFICULTIES

In order to block wrong paths most effectively we need to know why the pupil inclines to take them. In order to help him when he is blundering or at a loss what to do, we need to know why he is misled, perplexed, and confused. That is, we need to diagnose his difficulty.

As a first case of this, we may take division by a fraction, the topic which, considering the amount of time spent upon it, pupils used to learn least well. The older methods approached this topic with painstaking efforts to show the pupil the following: (1) If fractions a and b have the same denominator, dividing the numerator of b by the numerator of a, and using the result as the answer, is the right thing to do. (2) If they have not the same denominator, it is right to reduce them to fractions having the same denominator and then divide the numerator of b by the numerator of a. (3) Since no use is made of the common denominator, it is merely necessary to multiply the numerator of b by the denominator of a and to multiply the numerator of a by the denominator of b and then divide the resulting numerator for b by the resulting numerator for a. (4) This is done conveniently by inverting the terms of a and multiplying b by the result.

All such explanations are based on the belief that the pupil's difficulty lay in an unwillingness to "invert" a fraction and multiply when told to divide by it, even if this did always give an answer which he felt was right, *unless he could see some deeper reasons why it must give the right answer.* Was this his difficulty? I think not, for two reasons: first, because these explanations were of little aid in curing it, and, second, because pupils seemed entirely willing to invert the wrong fraction, or even both fractions, and multiply! Most pupils, it may be safely asserted, would be entirely willing to invert one or both fractions or turn them sidewise or swap numerators or do anything else that always brought an approved answer. What was the difficulty?

Put yourself in the place of a child who has divided thousands of times and on every occasion found the answer to be much smaller than the number divided—who has had "make much smaller" as the one uniform associate of "*divide.*" He now is told so to operate that $16 \div \frac{1}{8}$ gives a result far greater than 16, that $1\frac{2}{3} \div \frac{3}{8}$ gives a result much greater than $1\frac{2}{3}$. There is a natural and, in a sense, a commendable reluctance to attach confidence to a procedure that produces a result so contrary to what he has always found previously to be the essence of division. This lack of confidence is very unfavorable to learning. Moreover, the older explanations, being directed so exclusively toward justification of inverting and multiplying, neglected to teach clearly that you should invert only one, and which one that was.

The cure for the difficulty will consist, then, in revising the old attitude toward division to make it fit this new case, making the right answers seem right to the pupil, and, incidentally, teaching him always to invert the right fraction. How to bring about this cure we shall see in a later chapter.

Another common difficulty is with the quick solution of problems like: "If it takes 5 days for a team to haul 36 tons of coal a certain distance, how long will it take the same team to haul 48 tons the same distance?" Or, "If 3½ lb. roast beef cost $1.12, what will 4¾ lb. cost?" The quick solutions, of course, require the comparison of 48 with 36 in the form of "times as long," and of 4¾ with 3½ in the form "times as expensive."

It is common for teachers to assume that the difficulty in these cases is due to the pupil's failure to try to think. They often insist, "You could have done that, only you would not think." This is sometimes the difficulty, but only rarely. Usually it is very different.

One part of the difficulty is the requirement of intelligence to see that the "times" comparison must be made and to attach the right adjective (as "long," as "quick," as "expensive," as "cheap," as "heavy," as "light," or whatever it may be) to it. The other part is the requirement of familiarity with the "times" comparison and with division as the means of making it. This second part of the difficulty is preventable or curable by enough practice. This practice has not been given, partly because teachers and textbook makers have exaggerated the ability of pupils to infer that "How many times as many as 5 is 40?" or "40 is how many times as much as 5?" means "divide 40 by 5," and partly because they have rightly disliked to use these awkward forms of question.

The newer methods avoid this awkwardness by the use of the "omitted number" form—"40 is . . . times as many as 5," and "40 = . . . ×5," and take pains to give much practice with division as the means of answering such. Samples of such practice are shown below:

How Many Times as Tall, Long, Large, Heavy, etc.

1. Tell the missing numbers:

A.	B.	C.
84 = . . . ×21	½ in. = . . . ×¼ in.	6 = . . . of 24
44 = . . . ×4	¾ lb. = . . . ×¼ lb.	6 = . . . of 18
1.6 = . . . ×2	1 hr. = . . . ×¼ hr.	6 = . . . of 48
1.6 = . . . ×20	$1.25 = . . . ×25¢	6 = . . . of 9

Oral Review

Practice with these until you can give all the quotients in 3 min.

A.	B.	C.
a. 10 = . . . ×2	*i.* 15 = ¼× . . .	*q.* 20 = ⅘ of . . .
b. 10 = . . . ×3⅓	*j.* 15 = ½ of . . .	*r.* 20 = ⅚ of . . .
c. 12 = . . . ×2	*k.* 15 = 3× . . .	*s.* 20 = ⅔ of . . .
d. 12 = . . . ×3	*l.* 16 = . . . ×2	*t.* 24 = 2× . . .
e. 12 = 1½× . .	*m.* 18 = . . . ×2	*u.* 24 = 1½× . . .
f. 15 = . . . ×5	*n.* 18 = . . . ×3	*v.* 24 = 3× . . .
g. 15 = 2× . . .	*o.* 20 = 2× . . .	*w.* 24 = 50% of . . .
h. 15 = ⅓× . . .	*p.* 20 = 5× . . .	*x.* 24 = 10% of . . .

D.	E.	F.
$32=\frac{1}{2}$ of...	$16=$...of 24	$6=$...of 9
$32=\frac{1}{4}$ of...	$16=$...of 48	$6=$...of 10
$32=\frac{2}{3}$ of...	$16=$...of 32	$6=$...of 11
$32=50\%$ of...	$16=$...of 160	$6=$...of 12
$32=10\%$ of...	$16=$...% of 200	$6=$...of 15

The equation form with an empty space to signify the number to be determined, we may note, is used by the newer methods again and again. As a substitute or alternative for "What part of 8 is 6?" we have "6= . . . of 8." As a substitute or alternative for "Of what number is 6 three-fourths?" we have "6 = $\frac{3}{4}$ of . . .?" "What must you multiply $2\frac{1}{2}$ by to obtain $3\frac{3}{4}$?" becomes ". . . $\times 2\frac{1}{2}=3\frac{3}{4}$"; and similarly with many other verbal forms. The equation form is the simplest and clearest way to state a quantitative problem. It is one of the best ways to retain arithmetical facts in memory. Its use stimulates and trains the habit of inspecting obtained results to see that they really do meet the stated requirements. It prepares the pupil to understand formulae and equations of all sorts. It is a model for brief, clear, decisive thinking.

Pupils often make errors when the work involves using or obtaining percents over 100, though they are competent in similar work with percents under 100. There are two very simple causes for this. First, they have not any sure and ready understanding of what these large percents mean. Second, the practice given is for a long time exclusively with percents under one hundred and mostly under 50. Consequently the pupils are at a loss when asked to do something with 140 percent or 205 percent or the like, or when the computation requires that they express 1.40 or 2.05 or the like as percents, and feel that something is wrong when their computation gives them such a percent as an answer. And they have no sure and ready knowledge of meanings to suggest the right action.

An inspection of the ordinary textbook confirms this view of the matter. For example, the first book that came to hand showed only two percents over 100 in the entire 31 pages of the first treatment of the subject.

The preventive or cure, then, consists in providing work like that shown below fairly early, and thereafter including a few cases of percents over 100 in the general practice.

1. Supply the missing numbers, as in the first two lines:

A.	B.
15% of 200 = .15×200, or 30	25% of 40 = .25×40, or 10
115% of 200 = 1.15×200, or 230	125% of 40 = 1.25×40, or 50
125% of 200 = 1.25×200, or . . .	110% of 40 = 1.10×40, or . . .
150% of 200 = 1.50×200, or . . .	120% of 40 = 1.20×40, or . . .
200% of 200 = 2×200, or . . .	210% of 40 = 2.10×40, or . . .
210% of 200 = 2.10×200, or . . .	310% of 40 = 3.10×40, or . . .

Estimating Percents

1. Name something which weighs about 1 percent of a man's weight.

2. Something which weighs about 10% of a man's weight.

3. Something which weighs about 50% of a man's weight.

4. Something which weighs about 200% of a man's weight.

5. Something which weighs about 500% of a man's weight.

6. Alice's little sister weighed 8 lb. when she was born, and weighs 22 lb. now on her first birthday. What percent is her weight now of her weight when she was born?

7. If she gains 150 percent of 22 lb. in the next five years, how many pounds will she gain, and how much will she weigh on her 6th birthday?

8. Helen's sister weighed 24 lb. when she was a year old, and gained 125 percent in the next five years. How many pounds did she gain, and how much did she weigh when she was six years old?
9. Estimate quickly what 205 percent of 650 is, approximately. Then multiply to find exactly what it is.
10. How near was your estimate to the exact percent?
11. Estimate quickly what 125 percent of $15.00 is, approximately. Then find what it is exactly.
12. How near was your estimate to the exact percent?
13. Which of these increases in weight about 3 percent? Which increases about 100 percent? Which increases about 200 percent? Which increases about 1000%?

    A baby that grows from 7 lb. to 21¼ lb.
    A young turkey that grows from 2 lb. to 3.96 lb.
    A girl who grows from 80 lb. to 82.4 lb.
    A calf that grows from 75 lb. to nearly 850 lb.

14. Tell some things that increase about 1000 percent in a year or even more than 1000%.
15. What percent of 120 is 30? 300? 600?
16. What percent of 40 is 16? 160? 80? 180?

These three illustrations of better diagnosis of pupils' difficulties by closer attention to just what the situation is and just what their experience and attitude are in respect to it are sufficient for our purpose in this connection. They point clearly to the general principle which the newer methods everywhere accept, viz.: study the learner as well as the lesson. Consider the situation as his mind meets it and the tendencies which he has toward it as well as the responses which you wish him to make.

PROVIDING THE BEST MEANS FOR LEARNING

It is clear that certain illustrations are better than others, that certain computations show principles more clearly than others, that certain facts serve better as centers for problems than others. The progress of teaching has hit upon many such, and their use has been made a part of accepted best practice. Thus in the early stages of multiplication by a two-place number, multipliers like 22, 33, 44, etc., are used oftener than they would be by chance, because they throw into relief the difference in place value of the partial products. Thus inches are specially helpful in teaching about eighths; and inches and pounds are specially helpful in teaching about sixteenths; fifths are best made interesting by means of seconds. Thus there is coming to be general agreement that the first multiplications to be systematically taught should be the $\times 5$'s, not only because they are very easy, but chiefly because they make a clear and large distinction from addition (2×2 is the same as $2+2$, and 6×2 is not much more than $6+2$, but 5×5 and 6×5 are far removed from $5+5$ and $6+5$).

The newer methods search deliberately for the best tool for each feature of arithmetical learning. They examine carefully the games of childhood, the familiar objects of the home, and the other studies of the school with a view to finding better means of providing reality, increasing interest, illustrating a meaning, or applying a process. They are not content with anything unless it is the best means that they can find, or one of several means which are equally good. They inspect every detail used in teaching, in the hope that there may be a better means of attaining the particular result desired and that they may find it. Some results of their search will show the possibilities of improving the teaching of arithmetic in this way.

It is not, of course, claimed that these samples represent the best means that will be found for the learning in question. On the contrary, the newer methods look for continued advance.

Case I. The best means to introduce exact division by a one-place number with a fraction in the quotient, which hitherto has been expressed with "and . . . . remainder." (Late in Grade 4.) This seems to be the computation of average school marks. In the world in general nine-tenths (probably more) of exact quotients are cases of averages, so the ability is being formed in the way in which it will really be used. School marks are familiar vital facts, and the meaning of average is better understood through them than by heights, weights, costs, or lengths. There is a genuine interest in the exact quotient, for if Mary is 90+ and Jennie is also 90+, it is a matter of concern to them and their friends and enemies to know which ranked higher. The same lesson unit serves to insure understanding of "averages," a term of very great usefulness thereafter, as well as to teach the new process.

Case II. An effective introductory problem and genuine uses for division of a compound number by an integer. (Grade 5.) Can we do better than ask the average length of a set of pieces of cloth or wire or string, which average nobody would probably even ask for and nobody certainly would care about? Consider the following:

Division

1. The heights of the eleven players of the Clinton High School football team added together make 62 ft. 9 in. What is the average height for a boy on this team?

$$\begin{array}{r} 5\text{ ft. }8\tfrac{5}{11}\text{ in.} \\ 11\overline{)62\text{ ft. }9\text{ in.}} \end{array}$$

$62 \div 11 = 5$ ft. and 7 ft. remainder.
$7 \times 12 = 84$.
$84 + 9 = 93$. 93 in. $\div\, 11 = 8\tfrac{5}{11}$ in.

2. In five trials at the mile run, Dick made records of 6 min. 12 sec., 5 min. 58 sec., 5 min. 34 sec., 6 min. 10 sec., and 6 min. 18 sec. What was his average time?

3. Joe made records of 6 min. 15 sec., 6 min. 10 sec., 5 min. 53 sec., 6 min. 28 sec., and 5 min. 50 sec. What was his average time?

4. A steamboat went from New York to Liverpool in 7 da. 6 hr. on one trip, 7 da. 4 hr. on the second trip, 6 da. 18 hr. on the third trip, and 6 da. 11 hr. on the fourth trip. What was the average time?

5. The heights of the players of the Clinton High School girls' basket-ball team are: 5 ft. 6 in., 5 ft. 7 in., 5 ft. 3 in., 5 ft. 4 in., and 5 ft. 8 in. What is the average height for a girl on this team?

Case III. What are the best connections to make at first between "What percent of *a* is *b*?" and real things? (Grade 6.) The best seem to be with percent of games won (or lost, or tied), and percent of words spelled or problems answered correctly in school tests. The data are familiar so that the issue is clear. The uses are real. The results are subject to a very rapid and convenient check, which also is easily understood because of familiarity with the facts. Interest may easily be aroused in the boys and some girls by the use of the records of well-known athletic teams, and in the girls and some boys by the use of tests actually taken by the pupils.

Case IV. What is an effective introduction to the procedure of expressing quantities as decimals so as to make them comparable? (Grade 6 or later.) Consider this:

The boys were trying to decide which of these was the longest jump:

R. Locke 14.75 ft.
D. Wade 14 ft. 8 in.
V. Lavisse (a boy in France) 4.41 meters (1 meter = 39.37 in.)
S. Beach $3\frac{3}{4}$ yd.

1. Which was the longest?
2. Which was the next longest?

Some girls were trying to decide which of these is the largest blanket:

$2\frac{1}{4}$ yd. $\times 2\frac{1}{12}$ yd.
2 yd. 4 in. $\times$2 yd. 6 in.
2 meters long $\times$2 meters wide
80 in. $\times$78 in.

3. Which is the largest?
4. Which is the next largest?

This introduction is weak in one respect, that it uses rather unreal measures of the facts in question (the decimal of a foot and the yard for a jump). It does this, of course, to make the need for reduction to comparable units of measure more striking. The variety of measures of the same kind of fact does this. The problems (except as just noted) are genuine. They are fairly interesting. The numbers are so chosen that the pupil cannot obtain the right answer except by reducing. The pupil must think concerning what he is to reduce to, and this reinforces the principle that he must so reduce as to make the measures comparable.

Case V. How can we give interesting and vital and varied practice in arranging personal accounts and still have all the class working with the same items so that they can be supervised by the teacher and can themselves compare and criticize their arrangements? Most children in Grade 6 or 7 have no accounts to keep; and, if they had, work by each on his own accounts would be unsociable and practically impossible to supervise or correct. Printed stories of receipts and expenditures lack interest. If pupils report actual receipts and expenditures, these are likely to lack unity. The best solution seems to be to have pupils in turn report *imagined* accounts, the conditions being so specified as to secure interest, variety, and a need for good arrangement. For an example of such treatment, see page 162.

Keeping Accounts of Receipts and Expenditures

One child tells her receipts and expenditures like this:

"Play that I am Helen, a rich man's daughter. It is Monday. I have $6.52 brought forward from last week. I receive an allowance of $1.00 for the week. My Uncle Roger gives me $2.00 Wednesday. On Tuesday I spend 50¢ for a book, and 30¢ for a violin string. On Thursday I buy 4 sundaes for 10¢ each. On Saturday I spend $1.50 to go to a concert. On Sunday I give 10¢ at Sunday school."

The other children write out Helen's account for the week, as fast as she tells what she received and spent, and find how much money she has left at the end of the week.

Then some child plays that he is an energetic boy who earns much money in all sorts of ways, and tells what his receipts and expenditures for a week might be.

Then some child plays that she is an excellent singer who receives money for singing at concerts and spends money for music and music lessons.

The other children write out the accounts as fast as they hear what the person received and spent.

Practice with these sums so that you can play the game well.

1.	2.	3.	4.	5.	6.	7.	8.
7.16	5.08	9.12	8.31	4.16	.72	3.70	4.96
7.69	1.08	1.98	9.33	9.54	3.95	4.94	8.64
.75	.22	.49	.36	1.25	.68	.70	.18
.48	.33	.95	1.00	.42	.27	.65	4.49
2.65	6.18	.36	.56	.88	7.48	6.21	.42
.54	.45	9.10	.88	.36	1.30	.34	1.32
.56	2.25	.21	8.75	.92	2.18	1.75	.95
.33	.42	1.20	7.56	8.56	.97	1.40	5.68
5.24	.95	.92	1.10	.95	9.36	.45	1.88

These five cases show the teacher searching through the environment in general for the best means to help in some special feature of learning. In just the same way a teacher in some particular city or village may search through that particular environment. She will thus look, when board measure is to be taught, for a house that is being built. She will know that such and such a field or park is about 2 acres. She will know what the drug store around the corner has to offer that will help in the learning of arithmetic. She will know the games commonly played by the children, and how these may be used. She will know in detail just what she has to teach and will be learning year by year better and better means of teaching it.

EXERCISES

1. If pupils are given much practice with additions like

$$\begin{array}{cc} 8 & 9 \\ 4 & 5 \\ 9 & 6 \\ \underline{3} & \underline{7} \end{array}$$

with written answers before "carrying" is learned, what wrong paths are they likely to follow when you begin to teach them to carry?

2. Would it be better to have only oral answers in such additions?

3. Pupils sometimes tend to add 1 to the next column in addition regardless of whether the sum of the column just added was a number in the teens, the twenties, or the thirties. What is the probable cause of this? How would you prevent it?

4. There are many cases where absolute mastery of and confidence in certain bits of knowledge helps to the understanding of certain matters of theory and procedure. How would such mastery of $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$, $\frac{1}{4}=.25$, and $\frac{1}{8}=.125$ help the pupil in learning the placing of the decimal point in multiplication with decimals?

5. Give another case where knowledge of facts helps the learning of procedures.
6. After studying numbers to one hundred, one pupil wrote sixteen, seventeen, eighteen, and nineteen as 61, 71, 81, and 91. Another wrote them as 60, 70, 80, and 90. Another wrote them as 610, 710, 810, 910. Another wrote 6, 7, 8, and 9 and said he knew there should be something more, but did not know what it should be. What one habit, good in itself, predisposed the pupils to all these errors?
7. Give another case of a habit, good in itself, which predisposes pupils to error?
8. Is it wise to have nineteen out of twenty of the first exercises in long division give quotients without remainder? Justify your answer.
9. In a "ladder" test with 10 steps graded in difficulty, 5 examples at each step, two pupils scored as follows:

		Number Correct	
		Pupil A	Pupil B
Step	1	5	3
	2	5	5
	3	5	2
	4	4	3
	5	5	4
	6	4	3
	7	1	5
	8	0	4
	9	0	0
	10	0	0
Total		29	29

Which pupil knew most about the processes? Which was the more careless?

10. What concrete material would you use in teaching $\frac{\quad}{16}$s? In teaching $\frac{\quad}{5}$s?

11. Observe the material used for $\frac{}{12}$s and $\frac{}{24}$s (II, 17), $\frac{}{7}$s and $\frac{}{9}$s (III, 133); $\frac{}{32}$s and $\frac{}{64}$s (III, 262). Note any other real uses of these fractions that would be instructive in the elementary school, if you think of such.

12. Which is the better set of multiplication facts to teach first, the 5's or the 1's?

13. What are some specially good applications of "*a* is what percent of *b?*" Compare with your answer II, 191, 192, and 196.

14. Examine the means taken to secure a good attitude toward learning the computation of areas (II, 221). Think of other means to accomplish this same purpose.

15. Observe the choice of means in the teaching of graphs (III, 30, 81, 164, 166, 177, 182, 194–196, 230, 231) with consideration of the value of the arithmetical work associated with them, their interest, and their practical value, as well as of their service in teaching elementary principles of the graphic presentation of quantitative facts.

CHAPTER IX

SOME HARD THINGS

Many of the difficulties that pupils experience are unnecessary. Good teaching can, as we have seen, avoid them by teaching the right subject matter at the right time in the right way. Some things in arithmetic, however, are essentially hard and always will be. All that we can hope to do is to reduce the difficulty to what is necessary. The newer methods seek to do this by ascertaining just what the essential difficulty is, and what is the best way for children's minds to meet and conquer it. We shall describe what they have to offer in four typical cases—long division, the so-called zero difficulties, division by a fraction, and square root.

LONG DIVISION

Long division is hard (1) because it requires "judgment" in selecting the figure to try as a quotient figure, (2) because it is complex, requiring shift from division to multiplication to subtraction to bringing down the right figure, and (3) because it has few uses which appeal to children as important and which can be used to infuse the work with interest.

The selection of the number to multiply by is made easy at the start by the use of divisors like 21, 31, 41, or 19, 29, 39, but sooner or later the pupil must "judge" for himself. This judgment requires (for two-place divisors) ready knowledge of the products of 2 to 9 and 20 to 90, by 2 to 9, expertness in addition to the higher decades or in mental addition of two-place numbers, and a power of coördination or thinking things together, so that, for example, on seeing $76\,|\overline{6125}$ the pupil will quickly realize that 9× is impossible, 8× is very close, and 7× is fairly likely. That is, he thinks $9\times70=630$, $8\times70=560$, and $7\times 70=490$ all in one pulse of thought, and holds the essentials of the 560 and 490 in mind while thinking "either 7 or 8 times 6 is a good-sized number." In deciding whether to try 8 or 7 he may actually multiply mentally far enough to think or know that the 8 is safe. While doing so he needs to have the 612 in view. Such thinking of facts together is hard.

Consider the process in this way. If all steps were carried out in a rational order but without abbreviation, the pupil would think: "$76|\overline{6125}$ $9\times 70=630$, no, 8 likely, $8\times 6=48$, $8\times 7=56$, $56+4=60$, 608, 8 all right." Or he would think "$76|\overline{6125}$ $7\times 70=490$, may be, $8\times 70=560$, may be, $7\times 6=42$, $8\times 6=48$," and one or more steps further to decide. Or he would review some equally elaborate series of facts. At any point he may risk a decision. Thus in $74|\overline{4276}$ the $6\times 70=420$ with the 4 of 74 and the 7 of 427 in view would probably lead the reader to decide at once to try 5. If so, it is by a quick coördination of facts or probabilities.

To select the right figure, you must either (A) write down many facts and look them over, or (B) keep in your head many facts and think them over, or (C) make a decision on the basis of a part of these facts as soon as you dare risk it. If children are taught to do A, the work is very tedious and fairly hard. If they are taught to do B, it is less tedious but harder. If they are taught to do C, it is much less tedious, but very much harder to do correctly every time. There is, however, no reason to require that you should choose the right number correctly every time. The best practice, all things considered, is to risk decision, multiply, and try another number if your choice was too large or too small. Expert computers do so. Pupils should be encouraged, even urged, to do so. The selection of the quotient figure to try should not be by any prescribed routine, but by a general inspection of the situation with enough mental calculation of whatever sort seems most useful to lead you to a probable estimate. If, in cases like the above, the first trial multiplications (of 70×8 and 7 in that case) lead you to correct decisions nine times out of ten, or even four times out of five, you will save much more time than you lose in retrials. In children's language, the rule should be, "Guess as soon as you think you can guess right," though the procedure is, of course, in no true sense a random idea or guess. It is rather that action of a person's whole repertory of ability in respect to the situation which we call judgment or tact or insight or sagacity.

Children will do this and enjoy it, if they are taught to, but at the outset it is repugnant to their arithmetical habits. Most of what they have done has been by strict routine. The new methods have given them some preparation in selective thinking, by teaching them to derive $7+\ldots=11$ and the like by trying likely facts from their knowledge of addition until they find the $7+4$ that fits; and further by teaching them to derive their division facts by selection from the multiplication facts. Even so, they are reluctant at first to go ahead, take responsibility, and make the best decision they can in long division. If some suggestion is made about using 70 as a "guide" if 71 or 72 or 73 or 74 is the divisor, and 80 as a "guide" if 79 or 78 or 77 or 76 is the divisor, they tend to accept that as a rule to follow implicitly. So the newer methods say little or nothing about "guides," but stress the element of initiative, the decision as soon as you dare, the method of "trial and success," the "guess." After the general procedure is mastered with divisors like 21, 31, 41, 29, 39, 49, work is given like that shown below:

Find the quotients and remainders. Sometimes you may think of a wrong figure for the quotient. Then you must see whether it is too large or too small and change it. But try to think of the right number the first time.

11. $28\overline{)817}$ Are there 3 28's in 81 or only 2?

12. $47\overline{)992}$ Are there 2 47's in 99 or only 1?

13. $27\overline{)538}$ Are there 2 27's in 53 or only 1?

14. $17\overline{)476}$ Are there 3 17's in 47 or only 2?

15. $35\overline{)81062}$ Try 2 as the quotient figure. How do you know 2 is right and not 3?

16. $13\overline{)9276}$ Try 1. Why is 2 wrong?

17. $312\overline{)1249}$ Try 4. Why is 3 wrong?

18. $151\overline{)375}$ Shall you try 3 or 2?

19. $123\overline{)375}$ Shall you try 3 or 2?

20. $225\overline{)650}$ Shall you try 3 or 2?

21. $25\overline{)425}$ Shall you try 2 or 1?

22. $15\overline{)470}$ Shall you try 4 or 3?

23. $15\overline{)615}$ Shall you try 4 or 3?

24. $211\overline{)495}$ Shall you try 7 or 6 or 5?

The complexity of the process, shifting from the selection of the quotient figure to multiplication, to correct placing of the product, to subtraction, to bringing down the right figure, cannot be avoided or much mitigated. Fairly wide spacing between the digits of the dividend will help somewhat. Previous mastery of the separate processes prevents an exaggeration of the difficulty. But the procedure in long division is complex and we must expect a pupil to need time and care to master it.

The same is true of the lack of clear practical usefulness. Long division is useful chiefly in planning, cost accounting, and scientific work. Problems in planning like those shown on pages 170–172 are the best the teacher has to offer under "The Uses of Long Division." Nor is there any strong appeal to the intellectual interests in mental activity and achievement.

All that can be done is: First, accept this situation and try to turn it to some account by telling the pupils that long division is useful as a sort of examination or test of abilities acquired, and that if they do it well it will be proof that they can use their multiplications, additions, and subtractions. Second, to postpone the dreary work with very large quotients until after decimals are learned, giving in Grade 4 mainly divisions with only one or two figures in the quotient; with a few longer examples to teach the process—to show that you keep on the same round of *multiply*, *subtract*, *bring down*. Great speed in long division is not at all necessary for a pupil in Grade 4 or 5 or even later. He will very rarely have long-division problems in life. If he is sure of what to do and how to do it, we have done our duty. It is wise to give training in one-figure quotient work which will help him to choose quotient figures, and to judge approximately how many times as large as another one number is. Some such work is shown below and on pages 171, 172.

A Christmas Party

It is Christmas time and the children are getting ready to give a Christmas party.

1. They plan to cut out 100 gold stars. The teacher says: "I will make one for a sample. You make the rest." There are 33 children. How many stars should each child make?

2. They wish to make 12 paper chains, each chain to have 50 links. The teacher makes 6 links for samples. They make the rest. How many links should each child make?

[**3.** Is a problem in multiplication. **4.** Is a problem in subtraction.]

5. 16 of the children divide equally the work of making eight dozen cornucopias. How many should each of the 16 children make?

Dividing by Large Numbers

1. Besides the land used for paths, the school garden has 6100 sq. ft. for the children to plant. There are 254 children. How many sq. ft. will each child have for his garden if the 6100 sq. ft. are divided? How many sq. ft. will be left over?

$254\overline{)6100}$ *Think how many 254's there are in 610.*
Three is wrong, for 3 × 254 = 762,
which is more than 610.

Uses of Long Division

(*Use pencil and paper when you need to.*)

1. 14 boys plan to buy a football together. It costs 98¢. How much must each boy pay?
2. They plan to buy a second-hand catcher's mask for 70¢. Must each boy pay 7¢ or 6¢ or 5¢?
3. How many yards of ribbon that costs 15¢ a yard do you get for 60¢? For 45¢? For 75¢? For 90¢? For $1.05?
4. Tickets to the concert cost 75¢ each. How much do three tickets cost? How many tickets will $2.00 buy, and how much will there be left over? How many tickets will $3.00 buy? How many tickets will $4.00 buy, and how much will be left over? Will $5.00 buy 7 tickets? Will $6.00 buy 8 tickets?

Earning and Saving

1. John wishes to earn \$17.25 to buy a bicycle. He can get \$.75 a week for working at the store. In how many weeks can he earn enough to buy the bicycle?

$75\overline{)1725}$ *The quotient means weeks.*

2. Mary, who is in high school, earns \$14.00 every month by working evenings. In how many months will she earn enough to buy a typewriter for \$70.00?

$14\overline{)70}$ *The quotient means months.*

To find out how many times a certain amount of money is contained in some other amount of money, write both amounts as cents or write both amounts as dollars. Then divide.

On Booster Day the stores will sell any 25-cent article for 19¢.

3. How many 25-cent articles can be bought on Booster Day for 75¢? How many cents will be left over?

4. How many can you buy for \$1.25, and how many cents will you have left over?

5. How many for \$1.00? **6.** For \$4.50? **7.** For \$4.75? **8.** For \$2.50?

The stores sell any 50-cent article for 39¢ on Booster Day.

9. How many 50-cent articles can be bought for \$1.00, and how many cents will be left over?

10. For \$2.50? **11.** For \$8.75? **12.** For \$5.00? **13.** For \$1.25?

14. In how many weeks can you save \$21.00, if you save 12¢ per week? **15.** If you save 25¢ per week? **16.** 28¢ per week? **17.** 75¢ per week?

Find the quotients and remainders:

$23\overline{)100}$	$96\overline{)750}$	$87\overline{)520}$	$62\overline{)500}$	$36\overline{)300}$
$48\overline{)125}$	$24\overline{)200}$	$35\overline{)225}$	$93\overline{)682}$	$85\overline{)600}$
$42\overline{)350}$	$78\overline{)500}$	$52\overline{)400}$	$35\overline{)125}$	$38\overline{)250}$
$85\overline{)715}$	$44\overline{)100}$	$66\overline{)250}$	$53\overline{)500}$	$84\overline{)625}$

THE ZERO DIFFICULTIES

Adults like to see zeros in their arithmetical work because zeros make computation easy. They make the understanding and mastery of procedures hard, however; and it is often wise to take special precautions when 0 appears on the scene. Thus $\begin{array}{r}5000\\-3475\end{array}$ will cause difficulty, though $\begin{array}{r}5213\\-3475\end{array}$ is handled perfectly.

Thus $\begin{array}{r}208\\\times\ \ 9\\\hline\end{array}$ is not surely mastered, when $\begin{array}{r}218\\\times\ \ 9\\\hline\end{array}$ has been.

Thus $\begin{array}{r}564\\\times20\\\hline\end{array}$ $\begin{array}{r}619\\\times30\\\hline\end{array}$ $\begin{array}{r}225\\\times40\\\hline\end{array}$ require additional teaching, though'

$\begin{array}{r}514\\\times2\\\hline\end{array}$ $\begin{array}{r}691\\\times3\\\hline\end{array}$ $\begin{array}{r}225\\\times4\\\hline\end{array}$ and $\begin{array}{r}514\\\times23\\\hline\end{array}$ $\begin{array}{r}691\\\times35\\\hline\end{array}$ $\begin{array}{r}225\\\times46\\\hline\end{array}$ are mastered.

In $\begin{array}{r}564\\\times207\end{array}$ $\begin{array}{r}619\\\times305\end{array}$ $\begin{array}{r}225\\\times408\end{array}$ we have a new difficulty. In $\begin{array}{r}309\\6\overline{)1854}\end{array}$ we have trouble again, and in long divisions with 0 in the quotient, like $\begin{array}{r}205\\25\overline{)5125}\\\underline{50}\\125\\\underline{125}\end{array}$, we have still more. In multiplying by numbers like .054 or .0028, and in quotients which require prefixing or annexing zeros, we have another group of serious difficulties.

There are two main reasons for the zero difficulties. First, 0 is peculiar. 0 is not, from the child's experience and point of view, a number like 2 or 3 or 4; it does not amount to anything! "0×5 cents is 0 cents" is unreal compared with "2×5 cents is a dime." 0 is peculiar arithmetically in that it has a separate set of habits of its own, such as: 0 in column addition, neglect it; any number minus 0 is unchanged; 0 times any number = 0; any number times 0 = 0; 0 divided by any number = 0.

Second, the operations with 0 are not uniform. In $6|\overline{1818}$ we do not write 03 in the first division of 18, but we must in the second. In $3|\overline{21}$ we do not write 07, but in $3|\overline{.21}$ we must. In subtracting 625 from 625 we just write 0 or even no figure at all, but in $\begin{array}{r}3002\\ 4\\ \hline\end{array}$ we must write 00. Each habit of use has to be related to the particular conditions where it is appropriate. Its proper use requires comprehension of the general system of decimal notation and place value more than is the case with any other number.

It would be possible to reduce these difficulties by giving much experience with 0 as equivalent for "no" or "not any" and by using long forms such as:

$$\begin{array}{r}0305\\ 7|\overline{2134}\end{array}\qquad\qquad\begin{array}{r}715\\ 208\\ \hline 5720\\ 000\\ 1430\\ \hline\end{array}\qquad\qquad\begin{array}{r}2004\\ 28|\overline{56125}\\ 56\\ \hline 1\\ 00\\ \hline 12\\ 00\\ \hline 125\\ 112\\ \hline 13\end{array}$$

A certain amount of everyday use of the word *zero* in place of the words "not any" is probably in fact desirable; and, in the case of a process of rather rare use in life, such as long division, the use of a long form using the familiar habits might on the whole be profitable. The general opinion, however, is against the latter as a permanent method. So, on the whole, the treatment of zero is and will be difficult. We must expect to give time and attention to it.

DIVISION BY A FRACTION

We learned in an earlier chapter that, in learning to divide by a fraction intelligently, a pupil has to counteract the now harmful habit of expecting quotient to be smaller than dividend, and has to have a basis for, and practice in, choosing which number to invert. If proper treatment is applied, the difficulty is reduced to perhaps one-quarter or even one-tenth of what it was by the older methods; but even so it demands consideration.

The proper treatment of the now harmful habit is to replace it by the more adequate habit of thinking and acting in accord with these rules:

When you divide a number by something more than 1, the result is smaller than the number.

When you divide a number by 1, the result is the same as the number.

When you divide a number by something less than 1, the result is larger than the number.

If divisor is more than 1, quotient is smaller than dividend.

If divisor is 1, quotient = dividend.

If divisor is less than 1, quotient is larger than dividend.

The replacement is made by using exercises like those shown on page 176.

Dividing by Numbers Smaller Than 1

1. Read, supplying the right numbers where the dots are:

A. For 5¢ you can get . . . balls at 5¢ each.
For 5¢ you can get . . . apples at $2\frac{1}{2}$¢ each.
For 5¢ you can get . . . sticks of candy at 1¢ each.
For 5¢ you can get . . . glass marbles at $\frac{1}{2}$¢ each.
For 5¢ you can get . . . clay marbles at $\frac{1}{8}$¢ each.

B. $5 \div 5 =$ . . .
$5 \div 2\frac{1}{2} =$ . . .
$5 \div 1 =$ . . .
$5 \div \frac{1}{2} =$ . . .
$5 \div \frac{1}{8} =$ . . .

C. In 4 in. there are . . . 2-in. lengths.
In 4 in. there are . . . 1-in. lengths.
In 4 in. there are . . . $\frac{1}{2}$-in. lengths.
In 4 in. there are . . . $\frac{1}{4}$-in. lengths.
In 4 in. there are . . . $\frac{1}{8}$-in. lengths.

D. 3 pies = . . . half-pies.
3 pies = . . . quarters
3 pies = . . . sixths.

E. 7 dimes = . . . nickels
7 dimes = . . . cents.

2. Read, supplying the right numbers where the dots are:

A. In 8 there are . . . 4's.
In 8 there are . . . 2's.
In 8 there are . . . 1's.
In 8 there are . . . $\frac{1}{2}$'s.
In 8 there are . . . $\frac{1}{4}$'s.
In 8 there are . . . $\frac{1}{8}$'s.

B. 6 = . . . 3's.
6 = . . . 2's.
6 = . . . 1's.
6 = . . . $\frac{1}{2}$'s.
6 = . . . $\frac{1}{3}$'s.
6 = . . . $\frac{1}{4}$'s.
6 = . . . $\frac{1}{5}$'s.

C. $12 \div 6 =$
$12 \div 4 =$
$12 \div 1 =$
$12 \div \frac{1}{2} =$
$12 \div \frac{1}{3} =$
$12 \div \frac{1}{4} =$

D. 2 lb. = . . . $\frac{1}{2}$-lb. weights.
2 lb. = . . . $\frac{1}{4}$-lb. weights.
2 lb. = . . . $\frac{1}{8}$-lb. weights.
2 lb. = . . . $\frac{1}{16}$-lb. weights.

E. $2 \div \frac{1}{2} =$
$2 \div \frac{1}{4} =$
$2 \div \frac{1}{8} =$
$2 \div \frac{1}{16} =$

3. Do the work of this page again.

4. Tell the missing quotients:

A.
1 = . . . $\frac{1}{2}$'s
1 = . . . $\frac{1}{3}$'s
1 = . . . $\frac{1}{4}$'s
1 = . . . $\frac{1}{5}$'s
1 = . . . $\frac{1}{6}$'s
1 = . . . $\frac{1}{7}$'s
1 = . . . $\frac{1}{8}$'s

B.
$2 \div \frac{1}{2} =$
$2 \div \frac{1}{3} =$
$2 \div \frac{1}{4} =$
$3 \div \frac{1}{2} =$
$3 \div \frac{1}{3} =$
$3 \div \frac{1}{4} =$
$3 \div \frac{1}{8} =$

C.
$3 \div \frac{1}{6} =$
$4 \div \frac{1}{5} =$
$6 \div \frac{1}{3} =$
$2 \div \frac{1}{8} =$
$5 \div \frac{1}{8} =$
$7 \div \frac{1}{2} =$
$10 \div \frac{1}{4} =$

D.
$12 \div \frac{1}{2} =$
$5 \div \frac{1}{3} =$
$9 \div \frac{1}{4} =$
$3 \div \frac{1}{8} =$
$8 \div \frac{1}{2} =$
$6 \div \frac{1}{4} =$
$5 \div \frac{1}{2} =$

E.
2 = . . . $\frac{1}{4}$'s
4 = . . . $\frac{1}{12}$'s
3 = . . . $\frac{1}{12}$'s
20 = . . . $\frac{1}{2}$'s
5 = . . . $\frac{1}{6}$'s
10 = . . . $\frac{1}{3}$'s
8 = . . . $\frac{1}{4}$'s

The treatment of the matter of choice of which fraction to invert is part of a larger issue. The newer methods reduce this difficulty (and several others), and also teach one of the universal laws of numbers, by giving one procedure for division with a fraction as divisor, division with a fraction as dividend, and division with fractions both as dividend and as divisor.

Instead of the mere trick of "inverting," it teaches, as shown below, the general law that *to divide by any number, you may multiply by its reciprocal*, teaching also, of course, what *reciprocal* means. After decimals are learned, this same reciprocal rule is shown to be the basis for time-saving operations with aliquot parts of 100 or $1.00.

Learn this:

2 is the reciprocal of $\frac{1}{2}$
3 is the reciprocal of $\frac{1}{3}$
4 is the reciprocal of $\frac{1}{4}$

To divide by a fraction, multiply by its reciprocal.

$\div \frac{1}{8} = \times 8$ $\div \frac{1}{4} = \times 4$ $\div \frac{1}{6} = \times 6$
$\div \frac{1}{12} = \times 12$ $\div \frac{1}{3} = \times 3$ $\div \frac{1}{2} = \times 2$

1. Compare the result of $12 \div 3$ with the result of $12 \times \frac{1}{3}$.
2. Compare the result of $16 \div 8$ with the result of $16 \times \frac{1}{8}$.
3. Compare the result of $10 \div 2$ with the result of $10 \times \frac{1}{2}$.

$\frac{1}{2}$ *is the* ***reciprocal*** *of 2.* $\frac{1}{3}$ *is the* ***reciprocal*** *of 3.*

$\frac{1}{12}$ *is the* ***reciprocal*** *of 12.* $\frac{1}{16}$ *is the* ***reciprocal*** *of 16.*

4. What is the reciprocal of 8? Of 6? Of 4? Of 10?
5. Learn these lines:

$\div 2$ means $\times \frac{1}{2}$. **$\div 3$ means $\times \frac{1}{3}$.** **$\div 4$ means $\times \frac{1}{4}$.**
$\div 5$ means $\times \frac{1}{5}$. **$\div 6$ means $\times \frac{1}{6}$.** **$\div 7$ means $\times \frac{1}{7}$.**

To divide by a number is the same as to multiply by its reciprocal.

Find the quotients. Express the $\div 4$ or $\div 5$ or $\div 6$, etc., as $\times\frac{1}{4}$ or $\times\frac{1}{5}$ or $\times\frac{1}{6}$, etc. Cancel when you can.

6. $\frac{9}{16}\div 3$ *Write* $\frac{\overset{3}{\cancel{9}}}{16}\times\frac{1}{\cancel{3}}$. $\frac{3}{16}$ *is the correct result.*

7. $\frac{8}{9}\div 4$

8. $\frac{3}{8}\div 2$

9. $\frac{1}{2}\div 3$

10. $\frac{9}{2}\div 3$

11. $\frac{15}{24}\div 5$

12. $\frac{1}{3}\div 4$

13. $\frac{2}{3}\div 5$

14. $\frac{3}{4}\div 3$

15. $\frac{5}{6}\div 7$

16. $\frac{16}{5}\div 8$

17. $\frac{10}{3}\div 5$

18. $\frac{3}{4}\div 9$

19. $1\frac{4}{5}\div 9$ Use $\frac{9}{5}$ for $1\frac{4}{5}$.

20. $6\frac{2}{3}\div 10$ Use $\frac{20}{3}$ for $6\frac{2}{3}$.

21. $4\frac{1}{8}\div 3$ Use $\frac{33}{8}$ for $4\frac{1}{8}$.

22. $1\frac{2}{3}\div 5$ Use $\frac{5}{3}$ for $1\frac{2}{3}$.

1. Read these lines:

The reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$ or 2.
The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.
The reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.
The reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$.
The reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$ or 4.
The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.
The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.
The reciprocal of $\frac{5}{12}$ is $\frac{12}{5}$.

2. Name the reciprocals of:

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{5}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{1}{6}$ $\frac{5}{6}$ $\frac{1}{8}$ $\frac{3}{8}$

$\frac{5}{8}$ $\frac{7}{8}$ 2 3 4 $\frac{7}{5}$ 5 6 $\frac{3}{2}$ 24 $\frac{5}{12}$ $\frac{15}{8}$ $\frac{9}{12}$

Find each quotient. Multiply by the reciprocal. Cancel when you can.

1. $32\div 24$ Use $32\times\frac{1}{24}$.

2. $2\frac{1}{2}\div 2$ Use $\frac{5}{2}\times\frac{1}{2}$.

3. $6\div\frac{1}{3}$ Use 6×3.

4. $3\frac{3}{4}\div\frac{3}{4}$ Use $\frac{15}{4}\times\frac{4}{3}$.

5. $3\div\frac{5}{8}$ Use $3\times\frac{8}{5}$. Check your result by multiplying it by $\frac{5}{8}$.

6. $2\frac{5}{8}\div 1\frac{1}{2}$ Use $\frac{21}{8}\times\frac{2}{3}$. Check your result by multiplying it by $1\frac{1}{2}$.

7. $50 \div 16$ Use $50 \times \frac{1}{16}$.

8. $\frac{9}{16} \div \frac{3}{8}$ Use $\frac{9}{16} \times \frac{8}{3}$.

9. $\frac{2}{3} \div \frac{2}{9}$ Use $\frac{2}{3} \times \frac{9}{2}$.

10. $\frac{5}{6} \div \frac{5}{12}$ Use $\frac{5}{6} \times \frac{12}{5}$.

Find each quotient. Multiply by the reciprocal. Cancel when you can. Express mixed numbers as fractions.

1. $10 \div 6$

2. $4 \div \frac{2}{3}$

3. $4\frac{1}{8} \div 3$

4. $\frac{8}{5} \div \frac{2}{5}$

5. $2\frac{1}{2} \div \frac{5}{8}$

6. $9 \div 12$

7. $5 \div 25$

8. $\frac{3}{4} \div 3$

9. $100 \div \frac{4}{5}$

10. $\frac{3}{2} \div \frac{5}{6}$

11. $\frac{15}{16} \div \frac{1}{8}$

12. $10\frac{1}{2} \div 1\frac{3}{4}$

Remember this:

To divide by a number is the same as to multiply by the reciprocal of the number.

The pupil thus has abundant practice in using the divisor when he divides, and the fact that the divisor is the number to be modified is much more clearly impressed upon his mind than by the older method.

SQUARE ROOT

Computing square roots is hard, but it has been made unnecessarily so by confusing the learning of the procedure with the learning of the proof that the procedure is correct, and treating the procedure as a matter of absolute necessity. It is better to separate these two matters, and to teach the procedure frankly as a matter of estimating divisor and quotient for a given dividend with the requirement that they be alike. The treatment by this method is shown on pages 180 and 181.

First the estimating is done by the pupil's own judgment; then the systematic procedure is taught. He will probably forget the systematic procedure long before he ever has to compute a square root by it in life, but he will remember the procedure in estimating by his own judgment nearly or quite as long as he remembers what square root means. If later

Estimating the Square Root of a Number

To find what the square root of a number is look in a table of square roots, if you have one. Such tables are printed in the handy books which mechanics, surveyors, engineers, and other workers use in their work. If you do not happen to have a table of square roots to use, you can find the square root for yourself, by estimating and correcting your estimate.

It will often be useful to estimate one figure at a time like this:

To find the square root of 186:

$$\begin{array}{r}13\\ 13\overline{)186}\end{array}$$

First estimate what number times itself will give about 180 as a result. For example, think "10 × 10 = 100. It is more than 10. 20 × 20 = 400. It is less than 20. So I write 1 in the tens place as the first figure. 12 × 12 = 144. So 12 is too small. I know that 13 × 13 = 169, 14 × 14 = 196. It is more than 13 and less than 14. 14 is the nearest whole number." If you wish a closer result, try 13.5 × 13.5. The result is 182.25. Or try 13.6 × 13.6. The result is 184.96. Or try 13.7 × 13.7. The result is 187.69.

1. It will usually save time in finding square roots if you make out first a little table of the squares of the numbers from 13 to 29.

Copy and complete this table to use in Ex. 3, 4, 5, 6, 7, and 8.

13 × 13 =
14 × 14 =
15 × 15 =
16 × 16 =
17 × 17 =
18 × 18 =
19 × 19 =
20 × 20 =
21 × 21 =
22 × 22 =
23 × 23 =
24 × 24 =
25 × 25 =
26 × 26 =
27 × 27 =
28 × 28 =
29 × 29 =

2. Read the results of your table, supposing that you are finding the squares of 1.3 × 1.3, 1.4 × 1.4, 1.5 × 1.5, etc.

3. Using your table, make an estimate of the square root of 350. Try your estimate by multiplying it by itself. Then estimate again until your estimate multiplied by itself gives a result between 349 and 351.

4. Estimate from your table what the square root of 450 is. Correct your estimate until the square is between 449 and 451.

5. Close the book and find the square root of 186 by yourself.
6. Estimate the square root of 151 to the nearest whole number.
7. Estimate (to the nearest whole number) the square root of—
   a. 255 *b.* 318 *c.* 47 *d.* 85 *e.* 500 *f.* 632 *g.* 975
8. Estimate the square root of 32 to the nearest tenth.

Estimating Square Roots to the Second Decimal Place

Finding a square root is like dividing, except that you have only the dividend to start with, and have to find both the divisor and the quotient and have them be the same number.

You may save time in estimating the correct square root by estimating in this way:

To find the square root of 75 to the second decimal place:

$$\begin{array}{rl}
 & 8.66 \\ \hline
 & 75.0000 \\
 & \underline{64} \\
166 & 1100 \\
 & \underline{996} \\
1726 & 10400
\end{array}$$

Think of 8 × 8 and 9 × 9. Write 8 as the first figure.

Subtract 64 from 75.

Bring down **two** *figures (00), making 1100-*

Think how many times 16 (2 × 8) is contained in 110.

$$16\overline{)110}^{\;6}$$

Write 6 as the second figure.

Think "6 × 166 = 996."

Subtract 996 from 1100.

Bring down **two** *figures (00), making 10400.*

Think how many times 172 (2 × 86) is contained in 1040.

$$\begin{array}{r|l}
 & 6 \\ \hline
172 & 1040 \\
 & 1032
\end{array}$$

Write 6 as the third figure.

Place the decimal point where it belongs. Check your result by multiplying 8.66 × 8.66. The result should be very close to 75.

1. Find the square root of 62.4 to the second decimal place.
2. Find the square root of 9.94 to the second decimal place.
3. Find the square root of 38 to the second decimal place.

it is desired to teach any geometric or algebraic proofs, they may be taught as a matter of general mathematical training.

The stock explanation, shown below, does not, with the great majority of pupils in Grade 8, explain or justify the procedure for numbers which are not perfect squares, which is what really requires explanation and justification. Let the reader rewrite this stock explanation for the case of finding the square root of 600 or 650, and try to teach it to a class!

Ex. 1. I wish to arrange 625 tiles, each of which is 1 foot square, into a square pavement; what will be the length of one of the sides?

Ans. 25 feet.

OPERATION

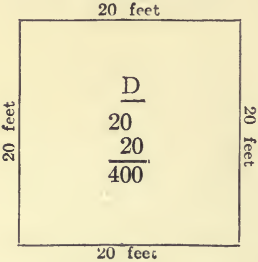
625|25, Ans.
4
25|225
225

We must extract the square root of 625 to obtain one side of the pavement, in feet. (Art. 280.)

Beginning at the right hand, we point off the number into periods, by placing a point over the right-hand figure of each period; and then find the greatest square number in the left-hand period, 6 (hundreds), to be 4 (hundreds), and then its root is 2, which we write in the quotient. As this 2 is in the place of tens, its value is 20, and represents the sides of a square, the area or superficial contents of which are 400 square feet, as seen in Fig. 1.

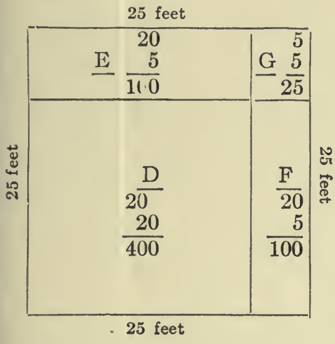
We now subtract 400 feet from 625 feet, and have 225 feet remaining, which must be added on two sides of Fig. 1, in order that it may remain a square. We therefore double the root 2 (tens) or 20, one side of the square, to obtain the length of the two sides to be enlarged, making 40 feet; and then inquire how many times 40, as a divisor, is contained in the dividend 225, and find it to be 5 times. This we write in the quotient or root, and also on the right of the divisor, and it represents the width of the additions *E* and *F* to the square, as seen in Fig. 2.

Fig. 1



The width of the additions multiplied by 40, in the length of the two additions, makes the contents of the two additions *E* and *F* to be 200 square feet, or 100 feet for each. The space *G* now remains to be filled, to complete the square, each side of which is 5 feet, or equal to the width of *E* and *F*. We square 5, and have the contents of the last addition, *G*, equal to 25 square feet. It is on account of the last addition that the last figure of the root is placed in the divisor; for we thus obtain 45 feet for the length of all the additions made, which being multiplied by the width (5 ft.), the last figure in the root, the product, 225 square feet, will be the contents of the three additions, *E*, *F*, and *G*, and the equal to the feet remaining after we had found the first square.

FIG. 2



Hence, we obtain 25 feet for the length of one side of the pavement, since 25×25=625, the number of tiles to be arranged, and equal to the sum of the several parts of Fig. 2; thus 400+100+100+25=625.

This illustration and explanation is founded upon the principle, *That the square of the sum of two numbers is equal to the squares of the numbers plus twice their product.* Thus, 25 being equal to 20+5, its square is equal to the squares of 20 and 5, plus twice the product of 20 and 5, or to 400+2×20×5+25=625.

EXERCISES

1. Name some things in arithmetic which are hard in the sense that they require much drill for mastery.
2. Name some that are hard in the sense of "hard to understand so as to know what to do."
3. Name some that are not really hard but are long and tedious.
4. The practices actually used in business (except some parts of cost accounting) generally sacrifice great exactitude for ease, as in not computing with fractions of a cent.

How does this help in the case of compound interest? (See III, 59.)

5. Name some cases where the difficulty lies in connecting the right meaning with a word.

6. If a pupil can add single pairs of numbers, one being a one-figure number, so as to be right 99 times out of a hundred, and "carry" correctly 99 times out of a hundred, about what percent of correct answers will he obtain in exercises with ten five-place numbers as in A (with no checking)?

A

65284
36956
97128
23807
72650
58175
82579
43196
38100
97374

7. About what percent of correct answers will he obtain in exercises with five five-place numbers as in B (with no checking)?

B

31765
65294
97170
48683
30921

8. About what percent of correct answers will he obtain in an exercise with five two-place numbers as in C (with no checking)?

C
76
28
50
94
35
—

9. About how long would an addition task have to be so that he would never get a right answer except occasionally by chance?
10. Criticize the practice of giving much work in arithmetic that is hard simply in the sense that the pupil must do some simple operation correctly 999 times out of a thousand to be sure of eight or nine right answers out of ten.
11. Bank discount and ratio are two topics famous for difficulty that might have been treated along with long division, the zero difficulties, division by a fraction, and square root. Examine the treatment of bank discount (III, 146–149) and the early treatment of ratio (II, 137, and III, 12 and 77–79), observing especially the definitions used and the means taken to connect knowledge about bank discount and ratio with the realities where that knowledge is needed.

CHAPTER X

SOME COMMON MISTAKES

ABSTRACT AND CONCRETE NUMBERS

The older methods made much of the distinction between what they called abstract numbers (such as 4, 7, 25, $\frac{1}{3}$, $\frac{5}{8}$) and what they called concrete numbers (such as 9 inches, 21 ft., 32 cents, $4.75). They spent much time in teaching the pupils principles based on these distinctions, such as: "In adding and subtracting with concrete numbers, the numbers must be of like denominations, that is, must mean the same sort of thing. In multiplication the multiplier must be an abstract number. The multiplicand may be either abstract or concrete. The product must be of the same denomination as the multiplicand."

The newer methods find this distinction between abstract numbers and concrete numbers undesirable in teaching, and find much better ways of attaining the desired ends. The valuable distinction is between a *number* (or abstract number as the older methods would call it) and a *quantity*, or number of units of a certain sort. 4, 7, 25, $\frac{1}{3}$, and $\frac{5}{8}$ are just numbers; 4 inches, 7 ft., 25 men, $\frac{1}{3}$ of a cent, and $\frac{5}{8}$ of a collection of 200 stamps are quantities. What we add and subtract and multiply and divide with are numbers. If we wish to know what seven boxes, each containing 144 pieces of chalk, contain in all, we do not multiply pieces of chalk by boxes or pieces of chalk by 7. We multiply 144 by 7. We may keep mental or written notes of what quantities the numbers meant so as to be able to know what quantity the number obtained as product means. Thus we may write $\begin{array}{r}144\\7\\\hline\end{array}$ pieces of chalk. We do not write "7 boxes," because it makes no difference to the meaning of the answer whether the 7 stands for boxes, bags, pans, pails, rooms, or days.

If we wish to know the average weight in pounds of six thirteen-year-old, seventh-grade, Chicago girls in a basket-ball team, the total weight being 838 lb., we divide 838 by 6, doing whatever is desirable to help us to remember that the quotient will, for the answer to our problem, mean a number of pounds. If we needed to be sure to remember the team, sex, age, grade, and locality of which the $139\frac{2}{3}$ lb. is the average weight, it would be equally allowable to write

$$\text{6 girls, 13 yr., 7 Gr., Chi.}\,\left|\overline{838}\right.\text{ lb.}$$

We operate with numbers, taking such measures as are useful to enable us to know what quantity the resulting number means, in case it does, by the conditions of the problem, mean some quantity.

The rules of the older methods were valuable only as means to help the pupil to know what quantity his answer meant. They were not very valuable in this respect, because consistency required that by them the pupil should solve "How much do 268 2-cent stamps cost?" by

$$\begin{array}{r} 2\\ \underline{268}\\ 16\\ 12\\ \underline{4}\\ 536 \end{array}$$

and the like. They also, if consistent, confused him by elaborate casuistry in showing that though you must not multiply feet by feet to get square feet, and the like, yet you must do something which to him seemed indistinguishable therefrom.

As general principles they were simply false. You *can* add a number of feet to a number of inches, or a number of cents to a number of dollars. In fact, you can add slices of bread to pieces of butter, a large part of addition in algebra being concerned with just such tasks. You *can* have multipliers that mean quantities just as well as multiplicands. In "8 yards = how many inches?" the 8 means yards just as surely as the

36 means inches. It is of course true that you do not add feet to inches or multiply inches by yards, according to rigorous linguistic usages of the words. As stated above, it is a number that is added, subtracted, multiplied, or divided. As ordinary folks use words, however, we *do* add cents to dollars, subtract acres from square miles, and multiply inches by yards.

There were two extensions of this unwise doctrine about abstract and concrete numbers, which were of doubtful value. One was the doctrine that only "like numbers," numbers meaning just the same sort of thing, can be added or subtracted, taught as the basis for putting units under units, tens under tens, etc., and for reducing fractions to a common denominator. As a general doctrine, this is false. You can add 4 hundreds and 7 ones; you can add $\frac{1}{2}$ to $\frac{1}{4}$ without thinking of the $\frac{1}{2}$ as $\frac{2}{4}$. The matter is not one of mathematical necessity, but of practical convenience. Also, it seems better to teach all these matters positively as useful habits, rather than negatively as "You must not do this in any other way."

The other concerned the treatment of rules and formulae for computing areas and volumes. The older methods spent much time in teaching the pupil that he must not say or think miles times miles = square miles, or feet times feet = square feet, or other forms of $l \times w = \text{area}$, in the case of a rectangle. Time that should have been spent in teaching him what to do, and how to choose units of measure and express his answer in harmony with his choice of units, was used in confusing him by efforts to teach him why he should not think of these formulae in certain ways. These ways, though perhaps improper linguistically, are the ways which the best intellects of the world in science and engineering habitually use, and should be left undisturbed. It is sheer pedantry to object to them. In fact, common sense got the better of pedantry, even in the older methods, by the time the circle was reached, and the pupil was freely taught that $2 \times \pi r^2$ equaled the area of a circle in square feet, *r* being in linear feet. Feet × feet equaled square feet for a circle, though not for a rectangle!

The newer methods save all this waste time for use in exercises which teach the necessity for correct choice and use of units and interpretations of results, as shown below and on pages 190–191.

Units of Measure

Whatever quantity is called 1 is the unit of measure.

1. Read. Supply the missing words as is shown in the first two lines.

a. A half mile is ½ if we are using *a mile* as the unit of measure.

b. A half mile is 160 if we are using *a rod* as the unit of measure.

c. A half mile is 880 if we are using....as the unit of measure.

d. A half mile is 2640 if we are using...as the unit of measure.

e. This square is 2×2 if we use an....as the unit of length.

f. This square is $\frac{1}{6}\times\frac{1}{6}$ if we use a....as the unit of length.

g. An hour is 1 if we use an ...as the unit of measure.

h. An hour is $\frac{1}{24}$ if we use... as the unit of measure.

i. An hour is 60 if we use... as the unit of measure.

Any quantity is a multiple of some unit.

Thus 9 mi. is 9×1 mi., 10½ mi. is 10½×1 mi., 3¾ lb. is 3¾×1 lb.

In using the dimensions of any surface to find its area, express both dimensions as multiples of the same unit. Choose a convenient unit.

2. Supply the missing words:
   - *a.* Length of a rectangle in. . . . ×width in. . . . =area in sq. in.
   - *b.* Length of a rectangle in. . . . . × . . . . . =area in sq. ft.
   - *c.* Length of a rectangle in yards× . . . . . =area in. . . . .
   - *d.* Base of a parallelogram in inches×altitude in. . . . . = area in. . . . . .
   - *e.* Base of a parallelogram in miles×altitude in. . . . . = area in. . . . .
   - *f.* Base of a triangle in feet × ½ of. . . . . . = ...sq. ft.
   - *g.* Average of two parallel sides of a trapezoid × altitude= area. If dimensions are in inches, area is in. . . . If dimensions are in feet, area is in. . . . . . . If dimensions are in miles, area is in. . . . .

(*With pencil.*)

3. How many square feet are there in a road 2.4 miles long and 18 ft. wide, counting the road as perfectly straight?
4. How many square yards of material are there in a big flag 5 yards long and 10 ft. wide?
5. What fraction of a square mile is the area of this park?

In using the dimensions of any box or bin or solid to find its capacity or volume, express all dimensions in the same unit.

6. How many cubic feet will a rectangular trough contain that is 10 ft. long, 2 ft. 6 in. wide, and 18 in. deep?
7. A rectangular pile of wood 4 × 4 × 8 ft. equals 1 cord of 4-ft. wood. How many cords of 4-ft. wood are there in a pile 4 ft. wide, 4 ft. high, and 24 yards long?

8. How many cubic yards are excavated in digging a hole 40 ft. by 24 ft. by 8 ft.?

In solving any problem, think what the units of measure mean.

9. The Merchants' Express goes 220 miles in 4 hr. 24 min. The Continental goes at the rate of a mile in 80 seconds. Which goes faster? Prove that your answer is right.
10. Helen can add 100 two-place numbers in 248 seconds. Alice can add the numbers at the rate of 30 a minute. Which girl adds more rapidly? Prove that your answer is right.

Units of Measure in Division

1. Read, supplying the missing words:

To find how many times a certain amount of money is contained in another amount of money, express both as cents or express both as......; then divide.

To find how many times a certain area is contained in another area, express both as multiples of the same unit (both as sq. in., or both as......, or both as......, or both as......, or both as......); then divide.

In every case, to find how many times as large as another one quantity is, express both quantities as multiples of the same unit. Then divide.

(*With pencil.*)

2. How many 15-cent toys can be bought for \$3.75?
3. How many gal. of water will a square can 16 × 16 × 16 in. hold? (1 gal. = 231 cu. in.)
4. How many sheets, each 1 ft. by $1\frac{1}{2}$ ft., can be cut from a roll of paper 2 ft. wide and 10 yd. long?
5. How many feet equal 28 percent of a mile and a half?
6. How many square feet are there in a floor 4 yd. long and $3\frac{1}{2}$ yd. wide?

NEGLECT OF THE EQUATION

Mention has been made elsewhere of the very great value of the equation-form with an empty space to represent the number or quantity to be found, as in the exercises shown below:

I

State the missing numbers:

A.	B.	C.
....7's=70	14 days=....weeks	35=....7's
....7's=63	70 days=....weeks	21=....7's
....7's=14	42 days=....weeks	14=....7's
....7's=28	21 days=....weeks	28=....7's
....7's=35	49 days=....weeks	56=....7's
....7's=21	56 days=....weeks	42=....7's
....7's=49	63 days=....weeks	63=....7's
....7's=56	28 days=....weeks	49=....7's

II

1. 25 pt. =..qt. and..pt.
2. 19 qt. =..gal. and..qt.
3. 29 ft. =..yd. and..ft.
4. 20 in. =..ft. and..in.
5. 9 gills=..pt. and..gills.
6. 35 qt. =..pk. and..qt.
7. 20 da. =..wk. and..da.
8. 25 qt. =..gal. and..qt.
9. 100 sec. =..min. and..sec.
10. 15 pt. =..qt. and..pt.
11. 15 pt. =..gal. and..pt.
12. 38 pt. =..gal. and..pt.
13. 50 qt. =..pk. and..qt.
14. 50 pk. =..bu. and..pk.
15. 200 min.=..hr. and..min.
16. 30 da. =..wk. and..da.

III

1. Read and supply the missing numbers:

A.	B.	C.	D.
$16=\frac{1}{\ }$ of 32	$8=\frac{1}{\ }$ of 24	$2=\frac{1}{2}$ of...	$2=\frac{1}{\ }$ of 48
$8=\frac{1}{\ }$ of 32	$4=\frac{1}{\ }$ of 24	$2=\frac{1}{3}$ of...	$4=\frac{1}{\ }$ of 48
$4=\frac{1}{\ }$ of 32	$2=\frac{1}{\ }$ of 24	$2=\frac{1}{4}$ of...	$6=\frac{1}{\ }$ of 48
$2=\frac{1}{\ }$ of 32	$4=\frac{1}{\ }$ of 48	$2=\frac{1}{16}$ of...	$8=\frac{1}{\ }$ of 48

Supply the missing numbers, as is done in the first two. Express all fractions in lowest terms.

A.

$9=\frac{3}{4}$ of 12
$16=\frac{2}{3}$ of 24
$2=\frac{1}{2}$ of . .
$2=\frac{1}{3}$ of . .
$2=\frac{2}{3}$ of . .
$3=\frac{3}{4}$ of . .
$3=\frac{1}{2}$ of . .
$4=\frac{1}{2}$ of . .
$9=\frac{1}{3}$ of . .
$10=\frac{2}{3}$ of . .
$12=\frac{1}{2}$ of . .
$12=\frac{2}{3}$ of . .
$12=\frac{3}{4}$ of . .
$12=\frac{4}{5}$ of . .

B.

$7=\frac{1}{3}$ of 21
$8=\frac{4}{5}$ of 10
4= . . of 16
4= . . of 8
4= . . of 6
4= . . of 10
4= . . of 5
2= . . of 4
2= . . of 6
5= . . of 15
6= . . of 8
5= . . of 10
10= . . of 15
15= . . of 20

C.

$23=\frac{23}{24}$ of 24
$10=\frac{5}{8}$ of 16
21= . . of 24
11= . . of 12
10= . . of 12
9= . . of 12
8= . . of 12
11= . . of 16
7= . . of 8
6= . . of 18
20= . . of 24
30= . . of 40
22= . . of 24
12= . . of 24

D.

$25=\frac{5}{6}$ of 30
$15=\frac{1}{10}$ of 150
$8=\frac{4}{5}$ of . . .
$8=\frac{2}{3}$ of . . .
$15=\frac{3}{4}$ of . . .
$15=\frac{15}{16}$ of . . .
$15=\frac{5}{6}$ of . . .
$10=\frac{2}{3}$ of . . .
$18=\frac{3}{4}$ of . . .
$18=\frac{9}{10}$ of . . .
$18=\frac{1}{2}$ of . . .
$24=\frac{4}{5}$ of . . .
$24=\frac{3}{4}$ of . . .
$21=\frac{7}{8}$ of . . .

The equation form with an unknown quantity to be determined, or a missing number to be found, should be connected with its meaning and with the problem attitude long before a pupil begins algebra, and in the minds of pupils who never will study algebra. Children who have just barely learned to add and subtract learn easily to do such work as the following:

Write the missing numbers:

$4+8=\ldots$
$5+\ldots=14$
$\ldots+3=11$
$\ldots=5+2$
$16=7+\ldots$
$12=\ldots+5$

The equation form is the simplest uniform way yet devised to state a quantitative issue. It is capable of indefinite extension if certain easily understood conventions about parentheses and fraction signs are learned. It should be employed widely in accounting and the treatment of commercial problems, and would be except for outworn conventions. It is the chief contribution of algebra to business and industrial life and one that arithmetic can make as well. It saves more time in the case of drills on reducing fractions to higher and lower terms alone than was required to learn its meaning and use.

Mention has also been made of the value of experience in putting problems into equation form as an aid to solving them, and of framing generalized equations for typical problems such as those relating to costs, profits, time-distance-rate relations, and the like.

There is a third field for useful experience with equations, to neglect which would also be a mistake. This is with the formulae or equations of mensuration. Teachers commonly consider these formulae for areas, triangles, trapezoids, circles, etc., for volumes of prisms, cylinders, spheres, etc., and for the areas of the surface of spheres, cylinders, etc., in too limited ways. They think of them either as facts to be learned and remembered to enable the pupil to compute items about flower beds, silos, wells, haycocks, and the like; or as rules whose derivation or proof from geometrical principles is a beautiful exercise of thought; or in both of these ways. They fail to think of them as equations to be interpreted, and as means of training in the elements of general symbolic representation. They fail, except incidentally, to use them so.

Consider these three uses of the subject matter of mensuration. It is certainly useless for a pupil to learn and remember the formulae if he cannot understand what one means after he has recalled it. It is doubtful whether it is very useful for elementary-school pupils in general to remember them. Very few of our most useful lawyers, clergymen, business executives, surgeons, statesmen, farmers, clerks, or managers of households do remember them.

Not one in thirty graduates of the elementary school will probably ever have to compute the area of the surface of a sphere or its volume; if they need to know the lateral surface of a cylinder, they will not, as a rule, measure the diameter of its base, but measure around it with string! So far as the mere knowledge of facts goes, we should probably find it wise to stop with the circumference and area of a circle and the volume of a cylinder, leaving pyramids, spheres, and cones to be studied as matters of special trade knowledge.

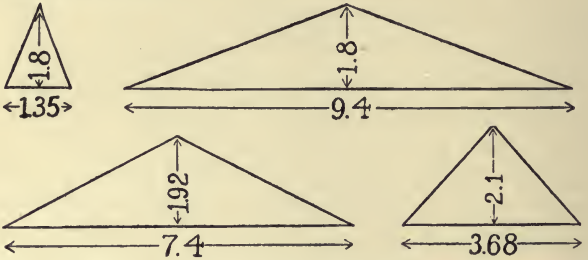
The easier derivations and proofs of some of the formulae (notably parallelogram and trapezoid) are admirable intellectual exercises for the brighter half of the pupils, but those depending on the theory of limits are too hard for all save a very, very few. How many of my readers even can give the derivation and proof of *area of circ.* $= \pi r^2$ and of *vol. of sphere* $= \frac{4}{3}\pi r^3$?

Shall we then never let a pupil see these and similar formulae? If their only value was to be remembered and used, or to be derived and proved, we might reasonably omit them, as some radical courses of study do. They have great value, however, for training in understanding and using equations. The newer methods so use them, the pupil's work being, not to remember them, but simply to read, understand, and use them when they are there before his eyes, treating these rarer formulae for mensuration just as he treats formulae for the strength of beams, or the rim speed of a wheel, and facts such as "The weight of 1 cu. ft. water $= 62\frac{1}{2}$ lb." or "1 meter $= 39.37$ in." That is, the most important formulae, notably for circumference of circle, area of circle, and volume of cylinder, are studied both for meaning and for fairly permanent knowledge and use. The less important formulae are studied only for meaning and use, by exercises of the sort shown on pages 196 and 197.

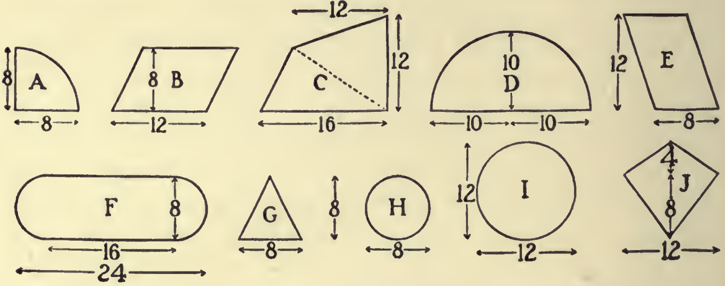
Equations

1. What equation do you use to find the area of a triangle?

2. Find the area of each of these triangles. The dimensions represent miles.



3. Which of the surfaces shown below are parallelograms? Which are sectors of circles? Which are composed of a rectangle and two semicircles?



4. Use the equations printed below to find the area of each of the surfaces shown.

The dimensions represent feet.

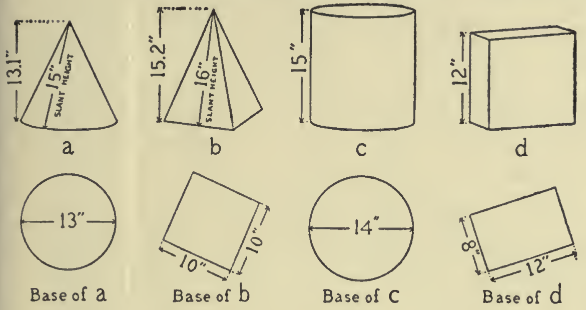
5. Use equations that you know (or use your common-sense) to find the perimeter of A, D, F, H, and I.

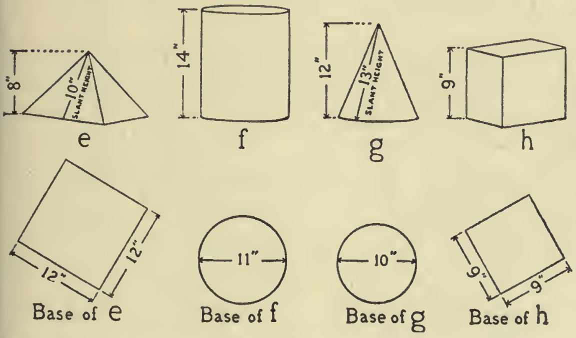
Area of circle $= \pi r^2$. *Area of parallelogram = alt.* $\times$ *base.*

Area of sector $= \frac{1}{2} r \times$ *arc. Area of any surface bounded by straight lines = sum of areas of triangles composing it.*

Optional Exercises in Understanding Equations

1. Which of the solids shown below are spheres? Which are cones? Which are pyramids? Which are cylinders? Which are rectangular prisms (shaped like a box or beam)?





2. Use the equations printed below to find the volume of each solid and the area of its surface.

In a cylinder, volume = alt. × *area of base.*

In a cone, volume = $\frac{1}{3}$ *alt.* × *area of base.*

In a sphere, volume = $\frac{4}{3}\pi r^3$.

In a pyramid, volume = $\frac{1}{3}$ *alt.* × *area of base.*

In a cylinder, surface = $2 \times$ *(area of base) + (alt.* $\times\ 2\pi r$*).*

In a cone, surface = area of base + ($\frac{1}{2}$ *slant height* × *circumf. of base). The pictures show what the slant height is.*

In a sphere, surface = $4\pi r^2$.

In a pyramid, surface = area of base + ($\frac{1}{2}$ *slant height* × *perimeter of base).*

UNDUE USE OF "CRUTCHES"

Many teachers use "crutches," or methods that are easily explained and learned but must sooner or later be supplemented by methods which are better for eventual use. Adding and subtracting by counting on the fingers, writing the number to be carried, and writing the sign + or — or × to guide you in computing, in cases like $\begin{array}{r}568\\+321\\\hline\end{array}$ $\begin{array}{r}568\\-321\\\hline\end{array}$ $\begin{array}{r}568\\\times 321\\\hline\end{array}$, are familiar illustrations. They would agree to the principle "Other things being equal, form no habits that will have to be broken." The question is about the inequality of the other things. They would defend the use of the crutch on the ground that it saved more time in learning than it cost later for replacement by the regular habit.

In some cases methods are taught for permanent use which other teachers regard as valuable, if at all, for temporary use. The question then is whether or not the habit is one that must be broken.

Teachers are tempted to be shortsighted, sacrificing the future for the present. In order to evade some difficulty at the outset, they are tempted to involve the teachers in later grades in worse difficulty. Every school, therefore, will profit by a definite policy with respect to each such real or alleged crutch. The tendency is to use too many of them, and use them too long. Let us, therefore, consider some of the most popular ones.

Counting forward by ones as a crutch in adding integers may be used for a few days in deriving the additions and for a few weeks as a check to verify additions. Then it should disappear. Counting backward by ones should not be used as a crutch in subtraction. The reason is that any child capable of learning arithmetic at all can learn the addition facts and subtraction facts and will save much more than the time required to learn them in a short time thereafter. These counting crutches are popular with very few teachers, but with many pupils.

Adding and subtracting by reference to some familiar combination (as $9+7=16$ by "$10+7$ would be 17, 9 is 1 less than 10, so $9+7=16$"; or $11-5=6$ by "$10-5$ would be 5, 11 is 1 more than 10, so $11-5=6$") may be called intelligent wastes of time. They are wastes of time because children who can do such reasoning could very quickly learn the combinations direct. They may be called intelligent because they replace a more mechanical process by a more thoughtful one. They do little harm, because they tend to "telescope" into the direct process. It is to be doubted, however, whether they really are as easy to teach and to learn as the direct processes.

Using +, —, or × as a sign of what you are to do in computations like $\begin{array}{r}596\\ \underline{214}\end{array}$ on the blackboard and in books, and teaching the child to write +, —, or × in his own computations, are popular practices with teachers, but seem surely inadvisable. It seems much better to write at the head of the page, row, or column, "Find the sums," or "Find the differences," or "Find the products," than to attach a sign to each pair of numbers. It seems much better for the pupil to think what he is to do as he would in ordinary life. For, other things being equal, we should form a habit in the way in which it is to be used, presenting the situation as life will present it, and requiring the response which life will require. It seems that the practice of using the signs +, —, and × was adopted partly as a crutch to save teaching the meanings of add, sum, subtract, difference, multiply, and product, and partly as a crutch to save the pupil the work of remembering what he had decided to do with certain numbers. Experiments are needed for a final decision, but they will almost certainly show that neither saving is large enough to warrant the custom.

Writing the number that is to be "carried" in addition is perhaps the most popular crutch. Perhaps it is not a habit that must be broken, but should be permitted in life as well as in school. Perhaps, though to be abandoned later, it is worth employing in the lower grades. It is one of the better crutches, since it does not introduce new sources of error, or much mar the appearance of clerical work, and makes checking quicker, though a bit less trustworthy. On the other hand, children can learn to add without it even in the third or second grade, and nobody has demonstrated that it is of very great help in early training. It is a sample of a question which we might debate endlessly without decision. Experiments are needed to measure what good it does and what harm it does.

Writing the numbers as changed in subtraction seems almost indefensible except for a few times at the beginning to show the procedure. Whether the form used is $\begin{array}{r}\scriptstyle 49\ \ 1\ \ \\ \$50.25\\ \underline{19.67}\end{array}$ or $\begin{array}{l}\$50.25\\ \scriptstyle 20\ \ 7\\ \underline{19.67}\end{array}$, the crutch seems to do much more harm by confusion to the eye and mind, distraction and loss of time in writing the figure, and interference with the later habit's formation, than it does good by delaying a difficulty until pupils are older and abler. Since the change is always -1 or $+1$, there is no burden of remembering *what* it is, such as there is in addition. To remember *that* the change is to be made requires only that the pupil should keep in mind what he did in the previous step; and this seems reasonable training for a pupil in Grade 3.

The same procedure used with carrying in multiplication seems much worse. Writing the numbers to be carried in this case, though permitted in many excellent schools, seems a clear case of shortsightedness in teaching. If the habit is not broken before "long" multiplication is studied, we have computations appearing as shown below in which the eye is distracted by irrelevant figures and in which there is a possibility of frequent error.

$$\begin{array}{r} 389 \\ \underline{276} \\ 55 \\ 2334 \\ 66 \\ 2723 \\ 11 \\ \underline{778} \end{array} \qquad\qquad \begin{array}{r} .47 \\ 685 \,|\, \overline{\$325.00} \\ 32 \\ \underline{2740} \\ 5100 \\ 53 \\ \underline{4795} \end{array}$$

Whereas in addition you write one "crutch" figure to represent a more or less long series of operations, here a crutch figure is written for every multiplication except those few resulting in products under 10 and those at the end of each partial product. There is thus an appreciable waste of time in writing the numbers and in looking at them so as to add them. The crutch is practically valueless in checking, since there would be no advantage in a checking back of the products themselves.

The "regular" habit of holding the number in mind and adding it mentally not only must be acquired some time; it also offers a form of specific training in keeping things in mind which seems well suited to pupils in Grade 3 and Grade 4. The regular procedure is in no sense harder to understand, or less indicative of the general rationale of the process of multiplication. It is harder only in so far as it requires the pupil to keep more facts and relations under control.

Finally, it is very hard to replace the crutch habit by the regular habit or in any way transform the former into the latter. Some crutches easily grow into the regular habits, either by mere dropping out of steps or by dropping a little at one place and adding on a little at another; but if a pupil has learned not to think of what he is to carry, but to write it, he has to drop off just what he learned to put on, and put on just what he learned to drop off.

It used to be almost universal to teach pupils in adding or subtracting unlike fractions to rewrite them as reduced to a common denominator somewhat after this fashion:

Add

$$\begin{array}{r|l} 5\frac{1}{2} & \frac{4}{8} \\ 9\frac{3}{8} & \frac{3}{8} \\ 8\frac{1}{4} & \frac{2}{8} \\ 6\frac{5}{8} & \frac{5}{8} \\ \hline 29\frac{3}{4} & \end{array}$$

Subtract

$$\begin{array}{r|l} 27\frac{3}{4} & \frac{9}{12} \\ 9\frac{1}{3} & \frac{4}{12} \\ \hline 18\frac{5}{12} & \end{array}$$

For the sort of additions and subtractions that used to be taught, this was necessary. Indeed it was often necessary also to do much computing to find some convenient common denominator for them. When rare collections of fractions are to be added together, such as $\frac{}{2}$s, $\frac{}{3}$s, and $\frac{}{8}$s, or $\frac{}{2}$s, $\frac{}{3}$s, $\frac{}{4}$s, and $\frac{}{5}$s, we should encourage any reasonable written work that will insure accuracy.

Modern good practice, however, puts almost all of its time upon securing mastery with the additions and subtractions that will be used. Its "regular" method in adding $\frac{}{2}$s, and $\frac{}{4}$s, or $\frac{}{2}$s and $\frac{}{3}$s, or $\frac{}{4}$s and $\frac{}{8}$s, or even $\frac{}{2}$s, $\frac{}{4}$s, and $\frac{}{8}$s, is to *think* of them as reduced to $\frac{}{4}$s or $\frac{}{6}$s or $\frac{}{8}$s, as the case may be, but not to rewrite them. The problem then arises whether what used to be the regular method shall be retained as a crutch. It is an interesting problem, which I shall leave the reader to answer, first calling his attention to certain facts.

One answer might be right for addition and the opposite answer for subtraction. When there are only two fractions, both ones in common use, the choice of a common denominator, the mental reduction of the two fractions to it, and the subtraction using these remembered fractions are not a very hard task. Surely nobody would advocate the crutch in these four cases:

$$\begin{array}{r|l} 12\frac{3}{4} & \frac{3}{4} \\ 9\frac{1}{2} & \frac{2}{4} \end{array} \qquad \begin{array}{r|l} 7\frac{1}{2} & \frac{2}{4} \\ 3\frac{1}{4} & \frac{1}{4} \end{array} \qquad \begin{array}{r|l} 11\frac{7}{8} & \frac{7}{8} \\ 6\frac{1}{2} & \frac{4}{8} \end{array} \qquad \begin{array}{r|l} 17\frac{5}{8} & \frac{5}{8} \\ 6\frac{1}{4} & \frac{2}{8} \end{array}$$

When, on the other hands, there are a dozen numbers in $\frac{\ }{2}$s, $\frac{\ }{4}$s, and $\frac{\ }{8}$s to be added, the task is much increased. A compromise crutch of the type shown below might be worth considering:

$$\begin{array}{rc} 6\frac{3}{8} & 3 \\ 8\frac{1}{4} & 2 \\ 7\frac{5}{8} & 5 \\ 9\frac{3}{4} & 6 \\ 5\frac{1}{2} & 4 \\ \underline{2\frac{5}{8}} & \underline{5} \\ & \frac{25}{8} \end{array}$$

The use of the crutch hinders the learning and use of direct fraction combinations such as:

$$\begin{array}{cccccccc} & & \frac{1}{4} & & \frac{1}{4} & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\ \underline{\frac{1}{2}} & \underline{\frac{1}{4}} & \underline{\frac{1}{2}} & \underline{\frac{1}{2}} & \underline{\frac{1}{2}} & \underline{\frac{1}{2}} & \underline{\frac{1}{2}} & \underline{\frac{1}{2}} \end{array}$$

From the discussion of these different crutches, it should be clear: first, that crutches vary very greatly in merit or demerit; second, that the objection to the objectionable ones is not that it is childish to write instead of think, but that on the whole they waste more time than they save or weaken the learner more than they facilitate learning.

EXERCISES

1. Consider this definition and this rule: Numbers applied to the same unit are called like numbers. Thus \$9 and \$43 are like numbers, \$9 and 43 cents are unlike numbers, 9 dollars and 43 boys are unlike numbers. Only like numbers can be added, and the sum is like the addends.

   Try to follow the rule in solving this problem:

   "Mr. Jones has 7 horses, 9 cows, and 23 sheep. Mary gave each one a name. How many names did she give in all?"
2. What useful purpose does the rule serve? What harm does it do?
3. Consider this rule: The multiplier must be thought of as abstract, and the product is like the multiplicand. What useful purpose does the rule serve? What harm does it do?
4. How must you think of this equation: "The number of volts times the number of amperes = the number of watts," in order to make it fit the rule quoted in 3?
5. Give other practices approved in science which seem incorrect according to the rule.
6. What is the purpose of the following exercises?

I

Let r = the number of miles per hour traveled.
Let d = the distance traveled (in miles).
Let t = the time (in hours).

Tell clearly in words the meaning of each of these. tr means t times r.

$d = tr \qquad r = \frac{d}{t} \qquad t = \frac{d}{r} \qquad 3d = 3tr \qquad \frac{1}{3}d = \frac{1}{3}tr$

II

Let I = the number of dollars of interest.
Let P = the number of dollars on which interest is paid.
Let R = the rate per year.
Let T = the time in years.

Which of these are correct equations or formulae showing the value of I?

$$I = P \times R \times T \qquad I = P \times R \times P \qquad I = TRP$$

$$I = \frac{P}{R} \times T \qquad I = R \times P \times T \qquad I = \tfrac{1}{2}(P \times R) \times \frac{T}{360}$$

III

Study each of these equations or formulae until you know how to use it in solving the problems below.

Let S = the rim speed or the number of feet per minute that a particle on the circumference of a wheel travels.
Let R.P.M. = the number of revolutions or complete turns that the wheel makes per minute.
Let r = the radius of the wheel, in feet or fractions of a foot.

Then $S = 2\pi r \times R.P.M.$

To find the number of revolutions per minute required to give a certain rim speed, you therefore use

$$R.P.M. = \frac{S}{2\pi r}$$

Let wt. = the number of pounds that is a safe load for a beam to hold up when the weight is distributed evenly over the beam.
Let b = the width of the beam (in inches).
Let d = the depth of the beam (in inches).
Let l = the distance between supports (in feet).

$$\text{Then, for chestnut, wt.} = (120 \times b \times d^2) \div l,$$
$$\text{for hemlock, wt.} = (110 \times b \times d^2) \div l,$$
$$\text{for Georgia pine, wt.} = (200 \times b \times d^2) \div l.$$

Let S = the breaking strength of manila rope (that is, the number of pounds that a manila rope will just hold without breaking).
Let r = ½ the diameter of the rope in inches.

Then $S = 720 \times$ *the square of* $2\pi r$. $S = 720(2\pi r)^2$.

Let Ls = a safe load (in pounds) for manila rope working at slow speed.
Let S = the breaking strength of the rope.

$$\text{Then } Ls = \frac{S}{7} \quad \text{or} \quad Ls = \frac{720(2\pi r)^2}{7}.$$

1. What is the rim speed in feet per minute of each of the grindstones or emery wheels represented in the picture on p. 217, at 750 revolutions per minute? [Picture is omitted here.

2. (*a*) Answer the same question for 1200 revolutions per minute.

   (*b*) How many times as great is the rim speed of the 16-inch wheel as the speed of the 10-inch wheel?

   [And further problems.]

7. Compare some of the exercises with missing numbers to be supplied with the same exercises as they would be if put as questions to be answered. Use II, 56, II, 119, or II, 250, and others if time permits.

8. Make a list of any arithmetical "crutches" you have seen employed by books or by teachers, other than those described in the chapter.

9. Name two or more crutches that do more good than harm.

10. Name two or more crutches that do more harm than good.

11. Consider the topic "The Undue Use of Short Methods." Write a short chapter or section, describing some of the short methods often taught which seem to you to represent an unwise expenditure of time with elementary-school pupils. Consider especially grouping to make tens in column addition as shown here:

$$\begin{array}{rl} 3 & \\ 2 & \\ 4 & \\ 7 & \quad \text{Think } 10+10+9+4 \\ 9 & \\ 8 & \\ \hline \end{array}$$

and the multiplication by 31, 41, 51, 61, etc., in either of these forms:

$$\begin{array}{r} 2184 \\ 31 \\ \hline 6552 \\ \hline 67704 \end{array} \qquad\qquad \begin{array}{r} 6552 \\ 2184 \\ 31 \\ \hline 67704 \end{array}$$

12. A pupil who solved correctly problems such as, "What is the area of a triangle of base 14½ inches and altitude 5¼ inches?" was entirely unable to find the area of a triangle cut out of paper. What probably had been the mistake in the teaching?

CHAPTER XI

SOME INSTRUCTIVE DISPUTES

There are three questions which probably cause more argument among teachers of arithmetic than any other dozen questions. They are: Should subtraction be taught by the "subtractive" or by the "additive" method? Should placing of the decimal point in the quotient in division by a decimal be taught by moving the decimal point in divisor and dividend so as to make the divisor a whole number or by marking off to the right of the decimal point in the dividend as many places as there are decimal places in the divisor? Should the pupils be given keys of the answers?

There is a close division of opinion on each of these questions, about as there is between Republican and Democratic parties. And the partisans on either side may become passionately devoted to their doctrine, so that the decision as to which method to adopt in a school often causes a regular campaign, with much excitement, and unfortunately often with neglect of more important matters.

Decision as to which alternative is the better in each of these three cases is not very important. Common sense suggests that when nearly half of the persons who ought to know think one way and a little over half the other, there cannot be a very great superiority for either way. We shall see that this is true of these three disputes. Each side has about an equal number of points in its favor. Decision to do one or the other does not compare in importance with decision to eliminate unreal problems, to reduce the eye strain in copying numbers, to encourage the verification of each new process by those already learned, to provide motive for drills, or to organize topics from the point of view of the learner.

In such cases it is probable that each of the contrasted procedures has its advantages, and there is a possibility that instead of quarreling about which is the better we may be able to devise a way that will have some or all of the advantages of both. The case may be like that of two parties wishing to get to the other bank of a river, one arguing that it is better to walk ten miles and cross the bridge than get wet by wading, the other arguing that it is better to get a little wet than to be all tired out by walking and lose so much time—when it may be possible to find a boat near by!

It is then instructive to consider these three typical cases of disputes with difficult decision, partly in the hope of finding a still better way, and partly as an exercise in weighing one advantage against another.

TWO METHODS FOR SUBTRACTION

The two processes are, of course, in essence, as follows:

SUBTRACTIVE

370520
160875
209645

Change 0 to 10. Decrease 2 to 1. 5 from 10 = 5.
Change 1 to 11. Decrease 5 to 4. 7 from 11 = 4.
Change 4 to 14 and change 0 to 10. Decrease 10 to 9 and 7 to 6. 8 from 14 = 6.
0 from 9 = 9
6 from 6 = 0
1 from 3 = 2

ADDITIVE

370520
160875

Increase 0 to 10. 5 from 10 = 5. Increase 7 to 8.
Increase 2 to 12. 8 from 12 = 4. Increase 8 to 9.

Increase 5 to 15. 9 from 15=6. Increase 0 to 1.
Increase 0 to 10. 1 from 10=9. Increase 6 to 7.
7 from 7=0.
1 from 3=2.

The "borrowing" process depends on the axiom that if you add a number to the minuend and also subtract it from the minuend, the value of the minuend is unchanged. The "additive" process depends on the axiom that if you add the same number to both minuend and subtrahend, the difference between them is unchanged.

Observe that in both methods we say 5 from 10= ..., 7 from 11= ..., 8 from 12= ..., etc., and not 5 and ... = 10, 7 and ... =11, 8 and ... =12. We could have just as well used 5 and ... =10, 7 and ... =11, 8 and ... =12. That is, the choice between decreasing in the minuend and increasing in the subtrahend is entirely independent of whether we think of the elementary subtraction facts as, "2−1, 1 taken away, 1 will be left," etc., or as, "2−1, 1 needs 1 more to make 2," etc. Observe further that, although we think in subtraction "10, 5, 5" directly without any thought that 5 added to 5 makes 10, we may have *originally learned* "10, 5, 5" by thinking "5 and ... =10," and using our knowledge of addition to work out the result.

The dispute concerning subtractive and additive methods includes, in reality, three very different questions:

(1) Shall we learn the subtraction combinations by the aid of the addition combinations using the form, $a+\ldots=b$?

(2) If so, shall we retain this form in thought until we have learned to think the facts directly?* Until we have learned when subtracting to just think "10, 5, 5," "14, 6, 8," etc., shall we think "5 and ... =10," "6 and ... =14," or shall we change to the words "5 from 10" or "10 minus 5" or "10 less

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*Every competent teacher will agree that eventually the pupil, when subtracting, should have these elementary facts operate directly without *any* attention to *and, from, less, minus,* or other verbal accompaniments.

5," etc., in our inner speech, or shall we use "5 and . . . = 10" or "5 from 10 = . . ." or "10 minus 5 = . . . ," according to the problem?

(3) Shall we decrease in the minuend or increase in the subtrahend?

Let us consider these in order.

The advantages of using the pupil's knowledge of addition to help derive the subtraction combinations are that time is saved and, what is much more important, that the pupil has more stimulus to active thinking and less to memory or counting. The disadvantages are that he may confuse the processes, adding when he should subtract, and vice versa. "2 and what = 5?" is more easily confused with "2 and 5 = what?" than "2 from 5 = what?" is. It appears that a large majority of experts would favor the original learning of the elementary subtraction facts by derivation from the addition facts (aided by objective subtraction, of course) if they could be assured that the pupil would distinguish subtraction sharply from addition, give it its proper name, understand that it is finding the answer to "Had lost, had left," "Cooked, ate, remained," "earned, spend, had," etc., as well as to "How much larger, older, longer?" "Have, wish, must get," etc., and soon learn to think the subtraction results directly when needed. With proper teaching the use of the "and . . ." should not hinder any of these accomplishments. A bigoted use of the additive form, which would narrow the idea of subtraction to cases of making up the balance, or fail to introduce the minus sign, or leave the pupils helpless when asked, "What is left when 7 is taken from 13?" and the like, would hinder them. On the whole, there seems to be a balance in favor of *learning* the subtraction combinations in the additive form, but at no cost to certain important features of subtraction in general.

So we face the second question, "Until we have learned to think the subtraction result directly from the situation, without any verbal form, shall we retain the 'and ...' or displace it by 'from,' 'less,' or 'minus,' or permit the use of more than one verbal form?" Before discussing this question, we may best admit frankly that we shall not attain a sure answer to it. There are too many conflicting forces.

First of all, subtraction has two main uses. In one set of uses the result is clearly a remainder; in the other set the result is a difference. The former is the more striking and dramatic use; the latter is the wider. All remainders can, though sometimes only by some strain of thought, be considered as differences, but differences cannot be thought of as remainders without much greater strain. The remainder usage more often evokes unreal problems. Thus, "Had, lost, had left" or "Made, ate, were left" would be answered in reality, not by counting the *had's* and *made's* and counting the *lost* and *eaten* and subtracting, but simply by counting the pennies or cakes that remained. In real life these would become difference problems. "I had ... , I now have ... , I must have lost ... ." The *lost* is not naturally thought of as the remainder, but as the difference. Similarly with "I made .... There are now .... We have eaten ... ." In fact, except where the minuend and subtrahend are both known without counting, or the minuend is known and the subtrahend is smaller than the remainder, the determination of real remainders by subtraction is an imbecility.

The difference problems as "How much taller, longer, heavier, older?" "How much more will it cost?" "How much must I save?" "What is my profit or loss?" and the like, will be found to be much more numerous and important in the life of children and adults than remainder problems. From these considerations alone the "and ..." form would seem preferable, if only one form is to be used, for it fits the idea of obtaining a difference much better than does the *from* or *less* form, just as that fits the idea of obtaining a remainder better than "and . . ." does. From these same considerations, however, one can argue strongly for using *both* forms. The two ideas, finding the difference in general, and finding the difference when it is the remainder left by taking away something, are so different, it may be said, that they should have different verbal forms. Is it not well worth the time required to know 15 minus 7=8 as an equivalent in result to 7 and . . . =15, because its meaning of "take 7 away, what is left?" will fit the remainder problems so much better and be suggested by them so much more surely? This argument introduces the general question of learning to do the same thing in more than one way.

Many teachers will object violently to teaching two ways of thinking of the "10, 5, 5" fact. They will rightly demand that weighty reasons be given. We should all admit the principle "Do not form two or more habits when one will do as well," but we may argue that the learning is not of the same thing, that "10, 5 of the 10, what is the remainder?" and "10, 5, what is the difference?" cannot be handled as well by any one habit as by two. Whatever we decide about this argument, it is certain that good teaching of subtraction demands adaptation to both the remainder and the difference problems, especially the latter.

Returning to the question of which form shall be chosen if only one is used and which shall be chief if both are used, we ask which is easier for children to use. Here we encounter a widespread fallacy. An adult almost invariably thinks that the form in which he learned to think of the subtraction facts is much easier and more "natural." If he learned to think *from*, he insists that you naturally think "5 taken from 10" in subtraction just as you do "5 added to 10" in addition. "The contrast makes it easy," he says, and may add, naïvely, "It would be simply absurd to think of it as '5, what more to make 10?'" If he has learned to think "and . . . ," he insists that it is an added and unnatural burden to think of taking 5 from 10, when you know that 5 more make 10. He may add, as naïvely as his opponent, "It is silly to bother with taking 5 from 10 when you know the answer already." Of course the way we have practiced thousands of times is easier and more natural—for us! As to the actual facts, there is a growing testimony from teachers that the "and . . ." form is almost or quite as easy and natural for children who have learned by it as the *less* or *from* form is for children who have learned by it.

On the whole, there seems a balance in favor of using the "and . . ." form as the chief or "regular" verbal form, giving also abundant practice with genuine remainder problems and their objective verification, and with exercises like $10-5=\ldots$, $9-3=\ldots$, $16-7=\ldots$, in which the sign is read as *minus*, and permitting pupils to use *minus* or *from* in their verbal statements whenever they do so appropriately.

We are now back to our starting point, the choice between decreases in the minuend and increases in the subtrahend. As was stated, this is entirely independent of whether the pupil thinks of his store of subtraction facts in the verbal form of *and* . . . , *from*, *less*, or *minus*.*

The sum and substance of the matter is: Decreasing in the minuend has the advantage of being more likely to teach the pupil about the nature of our system of decimal notation and place value. Increasing in the subtrahend has the advantage of a little greater ease in operation.

In learning to "borrow" one ten and change it to ten ones, to borrow one hundred and change it to ten tens, etc., the pupil, if he has sufficient intellect, receives a valuable lesson in place value. If he "borrows" the ten ones in the minuend and "pays it back" by adding one ten in the subtrahend, this

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*It should be said, however, that *from* is a little better than *less* or *minus* in column subtraction, since it is a little better to have the mental process start with the subtrahend number. The reason is that "5, 2, change the 2 to 12," is a little easier than "2, 5, change the 2 to 12."

lesson, though possible, is likely to be hidden. It is not the case, as sometimes claimed, that decreases in the minuend are a logical consequence of our system of numbers, whereas increases in the subtrahend are a mere mechanical procedure which happens to work out correctly always. Neither one is a jot or tittle more logical than the other; but the logic of keeping out of error by subtracting from a number as much as you have added to it is more apparent than the logic of adding to the subtrahend as much as you have added to the minuend. More children probably would perceive it.

The difference in ease of operation of the two procedures is due to the fact that in cases with successive 0's in the minuend, the first method often requires the pupil to perform the elaborate change of one hundred to 9 tens and 10 ones, or of one thousand to 9 hundreds, 9 tens, and 10 ones, before beginning to write any part of the corresponding figures in the answer, and to keep in mind what he has done until he has written all the corresponding figures. He has really to learn and use different procedures for $\begin{array}{r}30\\ \underline{16}\end{array}$, $\begin{array}{r}300\\ \underline{16}\end{array}$, $\begin{array}{r}3000\\ \underline{2116}\end{array}$, whereas in the second method the one habit of adding one in the subtrahend after each addition of 10 in the minuend suffices. Since subtractions from even numbers of dollars, and from $10.00, $20.00, etc., are very common, this is a matter of some consequence. Partly to balance it, we have the cases where the addition of 1 is to 9, making 10 and requiring the pupil occasionally to increase an *unwritten* 0 to 1, which is troublesome. These cases of 9 in the highest place in the subtrahend are, however, much rarer than those of successive zeros in the minuend.

On the whole, it would be very rash to claim that increasing the numbers in the subtrahend was essentially either 5 percent better or 5 percent worse than decreasing them in the minuend. The present writer favors the former because it seems to have a slight essential advantage and also because when subtraction is taught by this method teachers seem to give more attention to accurate operation and less to long explanations about why you must borrow; and this seems desirable. They also seem less prone to let pupils use written crutches; and this also seems desirable.

Some of my readers will wonder why no mention has been made of "making change." Have we not omitted the chief argument for the additive method—that it is the one to be used in the commonest of all uses of subtraction, making change? It is entirely true that change should be made in stores by adding cents, nickels, dimes, etc., to the purchase price until the amount of the coin tendered is reached. It is also true that the pupil who has been taught the form "and . . . " will be a little more likely to invent the correct method of making change than one who has been taught the form *from* or *minus*. It is also true that making change facilitates and is facilitated by learning subtraction somewhat more if that is thought of in the "and . . . " form than if it is in the *from* or *minus* form. This is, of course, restricted to cases of 5, 10, and 15 as minuends. But, in general, making change with coins is not subtraction with numbers, and the fact that in making change with coins, say 17 cents from $1.00, we find it profitable to add 3, 5, 25, and 50, certainly does not imply that we should find the difference between 17 cents and $1.00 by thinking "and 3 is 20, and 5 is 25, and 25 is 50, and 50 is $1.00." This would be much more foolish than to make change by thinking 17 cents from $1.00=83 cents, and then taking 50 cents+25 cents +5 cents +3 cents to make up the 83 cents. If we had only to *tell* the customer what his change was, not give it to him, and if he handed us, not always certain coins and bills, but checks for $2.89, $4.15, and the like, we should find it very unprofitable to count up by 1's, 5's, 10's, etc. We should then usually subtract in regular fashion, except when the difference was small.

Making change with coins is one thing. You then must give coins of values 1, 5, 10, 25, 50, 100, etc., amounting to the difference, but need not know what it is. Subtraction is a very different thing. There you must know what the difference is, expressed in one number, that is, in a sequence of figures meaning ones, tens, hundreds, etc. For making change you need to know only a few of the combinations, and to know them in the regular addition form $a+b=\ldots$.* For subtraction you need to know all the combinations and to know them in the form $a+\ldots=c$, or $c-a=\ldots$. Making change is not subtraction; it is *addition* until certain defined points are reached. Making change to reach points 5, 10, and 15 is beneficial to learning addition and subtraction, somewhat more so to the latter if that is taught in the "and . . ." form. Making change further to points 25, 50, and 100 may, if taught early, do much *harm* to subtraction, by confusing the pupil and interfering with his mastery of the general procedure in subtraction.

On the whole, the argument from making change is just a slight argument in favor of learning the combinations by derivation from the additions and in the "and . . ." form. It has nothing whatever to do with the merits of increasing in the subtrahend rather than decreasing in the minuend.

TWO METHODS FOR PLACING THE DECIMAL POINT

Examine these two rules:

A

Make a mark in the dividend as many places to the right of the decimal point as there are decimal places in the divisor. Place the decimal point in the quotient directly above this mark.

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*Two extra ones have to be known, of course, namely, $25+25=50$ and $50+50=100$.

In using this rule, if the dividend is a whole number, place a decimal point at the right of the units figure and annex 0 or 00 or 000 as is needed.

B

Change the divisor to a whole number by multiplying it by 10 or 100 or 1000 or 10,000. Multiply the dividend by the same number. Then divide.

The former has three slight advantages: (1) In business or scientific work the items are not changed from their original form. (2) The decimal point can be placed correctly even if the pupil has forgotten the rule for placing it when dividing a decimal by an integer. The third advantage will be stated farther on. The latter rule is much simpler to state, and the method is said to have the advantage of greater ease in learning, and was introduced because of the difficulty pupils had in using the other rule. Opinion is almost evenly divided between these two.

This dispute is instructive because in reality neither rule is of very great importance; neither hits the vital spot in mastery of placing the decimal point; expert computers use neither in most of their work.

The important rule is that (C) *Divisor* × *quotient must* = *dividend.* Consequently (D) *Number of decimal places in the divisor* + *number of decimal places in the quotient must equal number of decimal places in the dividend.* In most divisions with decimals we do not even need to count decimal places at all. We divide as with whole numbers and then place the decimal point where it should be to make divisor × quotient = dividend. Thus in $3.5|\overline{8.75}$ we know it cannot be .25 or 25, and must be 2.5; in $3.5|\overline{87.5}$ we know it cannot be so large as 250; or so small as 2.5. It is only in cases like $.035|\overline{.0875}$, $.035|\overline{875}$, or $.35|\overline{.0875}$, where the divisor and dividend are not subject to a rough comparison with surety, that we need to stop to count decimal places at all.

For the fundamental general training in placing the decimal point in dividing by a decimal what is needed is some such procedure as the following:

1. How many minutes will it take a motorcycle to go 12.675 miles at the rate of .75 mi. per minute?

$$\begin{array}{r} 16.9 \\ .75\overline{)12.675} \\ 7\,5 \\ \hline 5\,17 \\ 4\,50 \\ \hline 675 \\ 675 \\ \hline \end{array}$$

2. Check by multiplying 16.9 by .75.

3. How do you know that the quotient cannot be as little as 1.69?

4. How do you know that the quotient cannot be as large as 169?

5. Find the quotient for $3.75 \div 1.5$.

6. Check your result by multiplying the quotient by the divisor.

7. How do you know that the quotient cannot be .25 or 25?

8. Look at this problem: $.25\overline{)7.5}$

How do you know that 3.0 is wrong for the quotient?
How do you know that 300 is wrong for the quotient?

State which quotient is right for each of these:

.021 or .21 or 2.1 or 21 or 210
9. $1.8\overline{)3.78}$

.021 or .21 or 21 or 210
10. $1.8\overline{)37.8}$

.03 or .3 or 3 or 30 or 300
11. $1.25\overline{)37.5}$

.03 or .3 or 3 or 30 or 300
12. $12.5\overline{)37.5}$

.05 or .5 or 5 or 500
13. $1.25\overline{)6.25}$

.05 or .5 or 5 or 500
14. $1.25\overline{)6.25}$

After the pupil has learned to examine divisor, quotient, and dividend, and place the decimal point so that divisor × quotient will not give a preposterous result in these ordinary cases, he may examine Rule D and use it for ten or twelve quotients, still aided by inspection of the numbers. Then he may learn either Rule A or Rule B as a device which will always put the decimal point in its proper place. He should verify it by inspection of the numbers until he sees that it always does do this. He then uses it whenever the proper place for the decimal point is not easily determined by inspection. The pupil so taught is not at a loss if he does forget the rule. Nor will he be so likely to forget it; for he has himself verified its correctness. He also has an understanding of principles which is worth much more than memory of the rule.

If the pupil is thus taught rules as means to making divisor × quotient = dividend, Rule A has its third advantage. It connects somewhat better with that general principle than Rule B does. Rule B, however, might be made to have a special advantage if taught in a modified form as B_2.

B_2

If you cannot be sure where the decimal point belongs by inspection, multiply or divide both divisor and dividend by 10, or by 100, or by 1000 until you can be sure.

The pupil confronted by $.035|\overset{25}{\overline{.0875}}$ could then multiply both by 100, and, having 3. and 8., know that 2.5 was right. With $.035|\overset{25,}{\overline{875}}$ he would multiply both by 100 and know that with 3 and over 80,000, 25,000 was right.

Either Rule A or Rule B may be taught, but the essential principle of the matter should be taught first, and is the chief thing to be taught.

THE USE OF KEYS

Here once more opinion is somewhat equally divided. Some excellent schools provide all pupils with keys; others absolutely forbid the sale of keys to pupils.

The chief arguments in favor are, of course, that the pupil works with more zest when he can know as he works whether or not he has succeeded, and that he is prevented from doing an entire assignment incorrectly and so forming a bad habit. Two minor arguments are that the time of the school session is not used in inspecting answers, and that the less gifted pupils keep more nearly up to the required progress because they work longer at each assignment, repeating tasks which they find by the key are in error.

The chief arguments against the use of keys are that some pupils will use them not simply to check their results, but in unfair ways to obtain the results, and that they weaken the persistence and self-reliance of even the best pupils, it being too much to expect of human nature that children should fight to the bitter end with checks and verifications of their own, when the key is there ready to help.

It is not our intention here to discuss these and other arguments, but only to call attention to the bearing of certain features of the newer methods upon them, and to two facts of great importance which both those who do and those who do not permit the use of keys seldom bear in mind.

The newer methods use much more work with small numbers and less with large numbers, more short-column work in addition and less work with very long columns, more work with fractions in common use and less with rare fractions, more work with the simple computations frequently found in business and less with transactions involving unusual amounts, times, and rates. The newer methods also teach the pupil much more fully and systematically to check his results. That is, they make the work easier to check and give more practice in checking. Consequently pupils will suffer much less from not having a key when taught by the newer methods.

The newer methods much more often work toward a defined ability than did the older methods. The pupil's task is often stated, not as "to do such and such exercises," but "to practice with such and such *until you can do them all* correctly in . . . minutes.*" In such cases the folly of misusing the key becomes apparent. The pupil is working to reach a certain ability, not to obtain a certain quantity of answers. If he works with sufficient care and checking to obtain correct answers independently, he knows that he can pass the test. If he trusts too much to the key, and repeats only those exercises he finds wrong by the key, he will obtain wrong answers to them or others in the test, and fail. Consequently the pupils will be less likely to misuse a key if they do have it.

The first fact of importance referred to above is that, other things being equal, *much more work* should be required of classes whose pupils are permitted to check their results by a key than of classes whose pupils rely on themselves alone. For example, take the case of multiplications of a three-place by a three-place number, such as $\begin{array}{r}469\\325\\\hline\end{array}$. Compare pupils who are required to hand in 20 such examples correct as to answer and as to the three partial products in each case, with and without the aid of the key. Assume that all the pupils who use the key are honest, not taking advantage of it to avoid adding any partial products, or even to locate any error, but using it only to see if the result they obtain is correct, and doing the computation all over again if it is not. Assume that the pupils who use no key check by reversing the multiplicand and multiplier, and leave a result as satisfactory when the first checking tallies with the original, or when, if it does

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* A reasonable allowance will of course be made for such occasional "lapses" or "slips of the mind" as even the best computers make.

not, a second checking tallies with either one of the first two results.*

Then pupils not using the key will, according to their ability, do from 150 to 200 percent as much as those using the key. For example, a pupil getting 14 out of 20 right in the first trial and using the key will on the average do 20, + 6 repeated, + 2 of these repeated, or 28. If he uses his key he will do 20, + 20 first checking, + 6 rechecked, + 2 checked once more, or 48, or 171 percent as much. The greater mastery pupils have, the higher this percentage will be. The exact percentage will also depend on the ratio of the labor of checking to the labor of the first computation.

The teacher should then consider, in making assignments, whether a key is used. Other things being equal, an assignment to be checked by the pupil should be from half to three-fourths as long as one to be checked by a key.

The second fact of importance, often neglected by teachers, is that classes which use keys require more tests and more careful tests than classes not permitted to use keys, other things being equal. If the teacher puts the responsibility of the use of answers upon the pupil, she must accept the responsibility of more careful measurement of his arithmetical achievement when it is required to operate without these aids.

EXERCISES

Consider each of the following disputes, trying to estimate the advantages of each side, and, if possible, to plan to secure a large measure of the advantages of both:

1. All work possible should be done without pencil and paper.
Pupils should write at least the answers to almost all the exercises and problems they work.

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*This would insure right answers in practically every case for pupils able to get 14 or more out of 20 original answers correct.

2. The great majority of problems should be made up by the teacher from the life of the community.

The great majority of problems should be taken from textbooks prepared by experts with a view to the maximum of interest and instructiveness to pupils of the grade in question.

3. The multiplication tables should not be learned beyond 10 × 10.

The multiplication tables should be learned to 12 × 12.

4. Decimal fractions should be taught as a special case of common fractions with denominators of 10, 100, 1000, etc.

Decimal fractions should be taught as an extension of the system of thousands, hundreds, tens, ones, to tenths, hundredths, thousandths, etc.

CHAPTER XII

TERMS, DEFINITIONS, AND RULES

As with explanations, drills, and problems, so with terms, definitions, and rules, the aim is to help children to learn arithmetic and use it for life's needs. If learning about *orders* and *periods* does not help pupils to read, write, and understand numbers, these terms should not be taught, for life itself will never say "Separate the number into periods" or "Of what order is 7 in 27,468?" The guiding principle for the teacher is helpfulness—to choose such terms, definitions, and rules as will help and so to teach them that their help will be a maximum.

TERMS

The newer methods use fewer technical terms than used to be customary. The following, which were in rather frequent use in 1900, they would not use at all: *abstract number*, *antecedent*, *base* (used for the number of which a percent is computed), *common multiple*, *composite factor*, *composite number*, *compound fraction*, *compound number*, *concrete number*, *consequent*, *couplet*, *denominate number*, *denomination*, *exact interest*, *extremes*, *greatest common divisor* (*or highest common factor*), *least common denominator*, *least common multiple*, *like numbers*, *means*, *notation*, *numeration*, *order*, *period*, *reduction ascending*, *reduction descending*, *separatrix*, *simple fraction*, *terms* (of a ratio).

The following they would teach, but for somewhat special reasons: *addend*, *altitude*, *hypotenuse*, *minuend*, *percentage*, *perimeter*, *subtrahend*.

These seven terms are not of much help in learning, but they are likely to be used by teachers and in books of reference, so that lack of knowledge of them may indirectly hamper the pupil. Learning them takes not much time and, if proper guidance is given, will not confuse the pupil or encourage him to use words instead of real knowledge. Minuend and subtrahend are perhaps the worst of the seven. Almost no persons save teachers and almost no books save arithmetics ever use them.

The only technical terms added by the newer methods are *reciprocal* and *decimal mixed number*.* The newer methods do, however, take pains to teach (not by definitions, usually, but by adroit use in a context that gives the meaning) certain words and phrases, such as *together*, *in all*, *both*, *as much as*, *times as*, *total*, *equal*, *equals*, *divided equally*, which a pupil needs to know and may not have learned in his general experience at home and in school.

A term should be taught at the time when learning it will be most helpful, all things considered. Other things being equal, this will be just as soon as the fact itself is understood. If the name is given before then, memory of the words may be used by the pupil in place of knowledge of the fact; if it is delayed much after then, the fact lacks a convenient hook on which to hang. The other things are not always equal. Sometimes the term should be given first, the experience of the fact coming as an answer to the problem "What does . . . mean?" Sometimes it comes much later. This is often desirable, especially when the term is simply a technical substitute for a clear but longer or less customary name. Thus *product* in place of "result when you multiply" may be delayed until after four or five of the multiplication tables are used. Similarly *multiplicand* and *multiplier* may be delayed until late in Grade 3 and taught together with *dividend* and *divisor*. The child's thinking of "number multiplied" and "number I multiply by" or even "upper number" and "lower number" does no harm until that time—or even later.

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*These are not strictly additions, since they have been used hitherto, though not commonly.

Other things being equal, the newer methods avoid arithmetical "baby-talk"—the use of inaccurate, or clumsy, or unusual substitutes for technical terms because they are easier to understand at the time. Such are "9, take away 2," or "9, lose 2." It is not hard to learn the right name if the fact itself is known. Having the right name, the pupil can profit more from the talk of older pupils and parents about arithmetical facts and from such references as appear in books.

DEFINITIONS

A definition in words cannot replace real experience. Definitions are to coöperate with the pupil's experiences. They may occasionally precede the experience in question, serving as a stimulus or preparation or guide. Even then the definition should usually rest on some previous experience that gives it at least partial meaning. Usually it comes along with or after the real experience as a convenient summing up or shorthand representation thereof. Thus the definition of *quotient* is given after the fundamental division facts are learned and the process of division of two- and three-place numbers has been taught in its first steps. Thus the definitions of *numerator* and *denominator* are taught after pupils have used fractions somewhat in addition and subtraction. Thus the definition of *to reduce to higher terms* is given as a summary of the experience of multiplying both numerator and denominator by the same number.

Definitions are considered, not only as final perfected furniture of a mind whose education is completed, but also as working tools, or as representatives of ideas that are growing. As was made clear in an earlier chapter, all that the pupil learns about any topic should be true, but he need not learn all that is true. Thus, after certain problems in computing the number of square feet in certain flower beds, we have, "We call the number of square feet in a flower bed its *area*."

After further experience the definition of area is widened and refined.

Observe that care is taken not to say that "the area of a flower bed is always the number of square feet in it and nothing but that is its area." Definitions are subsidiary to meanings, and all that was said in chapter vi about the gradual extension and improvement of knowledge of meanings applies to definitions.

Since definitions are to be learned only when they will be helpful, not everything known needs to be defined in words. It would not be very helpful in learning to play chess to learn definitions of all the pieces, moves, and combinations of moves. It is not very helpful in learning to shoot with a gun to know verbal definitions of each part of the gun and each part of the landscape at which one shoots. The ability to respond correctly to an arithmetical fact is what we wish, and the ability to talk about it in correct language is only one small feature of the ability to respond. Correct responses in computations and problems outweigh correct responses by definitions at least a hundred to one. Many facts are and should be defined in the pupil's mind, not by words, but by the tendencies to think and act which are associated with them.

Consequently the newer methods spend much less time upon verbal definitions than the older methods did. Material such as follows under A would be omitted because the pupil acquires the ability to respond correctly to the fact in question irrespective of learning its verbal definition. Material such as follows under B would be omitted because it does not seem helpful in any way whatsoever.

A

Notation is the art of writing numbers.
Numeration is the art of reading numbers.
Any standard used in counting or in measuring is called a unit.

A figure is a character used to express number.

A fraction is one or more of the equal parts of a unit.

Addition is the process of finding a number equal to two or more numbers.

Compound division is the process of finding one of the equal parts of a compound number.

Evolution is the converse of involution.

B

A unit is one.

An abstract number is not applied to any particular thing.

Denomination is the name of a unit of measure.

Like numbers are numbers having the same kind of unit or that express the same kind of quantity.

The area of any surface is the number of units of area the given surface contains.

Time is a measured portion of duration.

RULES

The treatment of rules is much the same as for definitions. A rule should often grow out of a pupil's experience and sum up what he has already learned or is learning to do in the way most helpful for memory and future guidance. The older methods had a pupil learn a rule and then learn to operate in accordance with it. The newer methods often have him learn to operate in a certain way, with or without a set rule, as is most helpful, and then state what he has learned to do in a rule that will help him to remember how to do it and to learn to do other things.

Thus pupils who have learned the products of 10×1 . . . . to 9, and to obtain the products of 20, 30, 40, 50, etc., ... 90, when multiplied by 1, 2, 3, ... 9; and of 200, 300, 400, etc., ... 900 when multiplied by 1, 2, 3, ... 9, are given the experience shown on page 230.

Supply the missing numbers:

A. $10 \times 4 = \ldots$
   $10 \times 8 = \ldots$
   $10 \times 6 = \ldots$
   $10 \times 3 = \ldots$
B. 100×2 or $2 \times 100 = \ldots$
   100×5 or $5 \times 100 = \ldots$
   100×7 or $7 \times 100 = \ldots$
   100×4 or $4 \times 100 = \ldots$
C. $10 \times 2 = \ldots$
   $100 \times 2 = \ldots$
   $10 \times 7 = \ldots$
   $100 \times 7 = \ldots$

Then comes the rule:

To multiply by 10, annex 0.
To multiply by 100, annex 00.

Then come exercises and problems giving further practice in its use.

Thus the rule for placing the decimal point in the product when multiplying with decimal numbers should be taught after abundant experience with cases where common sense locates the decimal point correctly, and with exercises such as:

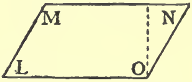
The distance to the school and back is 1.13 mi. How far does Alice go in making 4 trips to school and back?
Check your answer by adding four 1.13's.
How can you know just by looking at $\begin{array}{r} 1.13 \\ 4 \\ \hline \end{array}$ that the product is not so little as .452 mi.?
How can you know that the product is not so much as 45.2 mi.?

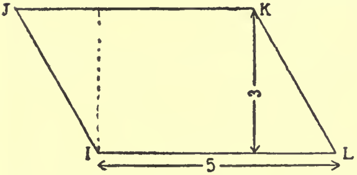
Thus the rule for computing the area of a parallelogram is preceded by experiences as shown below and on page 231:

1. Using paper ruled in inch squares, cut out a rectangle like A B C D, with base 6 in. and height 3 inches.

Cut out a parallelogram like L M N O with base 6 in. and height 3 in., the same as in the rectangle. What is the area of the rectangle?

2. See how the parallelogram compares in size with the rectangle by cutting off one end of the parallelogram, as shown by the dotted line, and placing the two pieces to cover the rectangle.





3. Make a rectangle with a base of 5 in. and height or altitude 3 in. Make a parallelogram like I J K L, of the same base and altitude as the rectangle, slanting like the diagram. Cut and compare as you did before.

4. Compare other sizes of rectangles with other parallelograms of equal base and altitude until you are sure that —

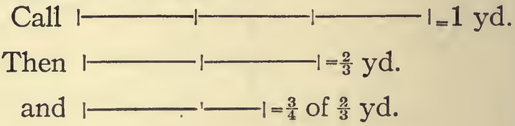
The area of any parallelogram = the area of a rectangle of the same base and altitude.

The area of any parallelogram = the product of its base and altitude. Base and altitude must be expressed in the same unit of measure before multiplying.

Occasionally a rule is given rather as a guide to and aid in interpretation of experience than as a statement of the result of the experience. Added experience then often appears in connection with the verification of the rule. Such is the case with the rule for multiplying a fraction by a fraction as taught on the following page.

Remember that "$\frac{1}{2}\times$" means "$\frac{1}{2}$ of."

1. What part of a dollar is $\frac{1}{2}\times\frac{1}{2}$ dollar?
2. Check your result by finding $\frac{1}{2}$ of 50 cents, and finding what part of 100 cents 25 cents are.
3. What part of a pound is $\frac{3}{4}\times\frac{1}{2}$ lb.?
4. Check your result by finding $\frac{3}{4}\times 8$ oz. and finding what part of 16 ounces 6 ounces are.
5. What part of a foot is $\frac{2}{3}\times\frac{3}{4}$ ft.?
6. Check your result by drawing a line $\frac{3}{4}$ ft. long, and finding $\frac{2}{3}$ of it.
7. What part of a dozen is $\frac{1}{2}\times\frac{1}{2}$ dozen?
8. What part of a yard is $\frac{3}{4}\times\frac{2}{3}$ yard?
9. Check your result by a drawing like this:



In multiplying with fractions:

Write the product of the numerators as numerator. Write the product of the denominators as denominator. Cancel when you can.

Find the products. You may check your results by drawing and measuring with a foot rule.

A.

$\frac{1}{2}\times\frac{1}{4}$ in. =
$\frac{1}{2}\times\frac{1}{2}$ in. =
$\frac{1}{2}\times\frac{3}{4}$ in. =
$\frac{1}{2}\times 1$ in. =
$\frac{1}{2}\times 1\frac{1}{4}$ (or $\frac{5}{4}$) in. =
$\frac{1}{2}\times 1\frac{1}{2}$ (or $\frac{3}{2}$) in. =
$\frac{1}{2}\times 1\frac{3}{4}$ (or $\frac{7}{4}$) in. =

B.

$\frac{2}{3}\times\frac{1}{4}$ ft. =
$\frac{2}{3}\times\frac{1}{2}$ ft. =
$\frac{2}{3}\times\frac{3}{4}$ ft. =
$\frac{2}{3}\times 1$ ft. =
$\frac{2}{3}\times 1\frac{1}{4}$ (or $\frac{5}{4}$) ft. =
$\frac{2}{3}\times 1\frac{1}{2}$ (or $\frac{3}{2}$) ft. =
$\frac{2}{3}\times 1\frac{3}{4}$ (or $\frac{7}{4}$) ft. =

C.

$\frac{3}{4}\times\frac{1}{3}$ yd. =
$\frac{3}{4}\times\frac{2}{3}$ yd. =
$\frac{3}{4}\times 1$ yd. =
$\frac{3}{4}\times 1\frac{1}{3}$ yd. =
$\frac{3}{4}\times 1\frac{2}{3}$ yd. =
$\frac{4}{5}\times\frac{1}{2}$ in. =
$\frac{4}{5}\times 1$ in. =

[Then follows a page of verification by drawing and comparing areas.]

A rule should be a part of a growing, active ability, widening its scope as the ability widens. Each rule taught should be true, but the entire truth about an operation need not be taught the first time that the operation is used. The older methods made learning fit a fixed set of rules. The newer fit rules to learning.

Thus the pupil learns in order, "When adding $\frac{\quad}{2}$s and $\frac{\quad}{4}$s, think of $\frac{1}{2}$ as $\frac{2}{4}$." "When adding or subtracting $\frac{\quad}{2}$s and $\frac{\quad}{8}$s, think of $\frac{1}{2}$ as $\frac{4}{8}$." "When adding or subtracting $\frac{\quad}{2}$s, $\frac{\quad}{4}$s, and $\frac{\quad}{8}$s, think of $\frac{1}{2}$ as $\frac{4}{8}$; think of $\frac{1}{4}$ as $\frac{2}{8}$; think of $\frac{2}{4}$ as $\frac{4}{8}$." "When you add or subtract with $\frac{\quad}{2}$s and $\frac{\quad}{3}$s and $\frac{\quad}{6}$s, think of $\frac{1}{2}$ as $\frac{3}{6}$, $\frac{1}{3}$ as $\frac{2}{6}$, $\frac{2}{3}$ as $\frac{4}{6}$." "When you add or subtract with fractions, express them as fractions having the same denominator."

Thus the pupil learns to compute areas of rectangles and volumes of rectangular solids first when all the dimensions are expressed in the same unit of measure. Only when the more general case arises is the more general rule given. "In multiplying to find area or volume or capacity, express all dimensions in the same unit of measure."

Thus the pupil learns first (in Grade 4), "To find out how many times a certain amount of money is contained in some other amount of money, write both amounts as cents or write both amounts as dollars. Then divide."

A half year later this is replaced by:

"When both the dividend and the divisor mean amounts of money, express both as dollars or both as cents. Then divide.

"When both the dividend and the divisor mean lengths, express both in the same unit. Write both as inches, or both as feet, or both as yards, or both as rods, or both as miles. Then divide.

"When both the dividend and the divisor mean areas, express both in the same unit. Write both as sq. in., or both as sq. ft., or both as sq. yd., or both as sq. rd., or both as acres, or both as sq. mi. Then divide."

A year or more later this is generalized as:

"To find how many times a quantity contains another quantity, express both in the same unit of measure before dividing."

If a pupil has mastery of an operation as summed up in a rule, and if thought about the rule will not help him to learn anything else, there is no need of any rule. The newer methods give no rules for rules' sake.

Thus it is superfluous to give a rule for writing the tens-place figure to the left of the units-place figure, or for writing 1 above the line and 2 below it in writing one-half in figures, after a pupil has mastered these processes. It is useless to give the rules before then, since imitation and practice, not rules, are efficacious. Thus, if a pupil really understands percentages and the meaning of discount, it is useless to give a rule for computing a single discount. With successive discounts the case may be very different.

Like the facts or procedures which they embody in verbal form, rules vary enormously in importance. Some represent the very flesh and blood of the science and art of arithmetic. Some only tell the story of useful customs. This matter has been already discussed in chapter vi; the difference will be clear from the samples shown below.

SAMPLES OF ESSENTIAL RULES

In subtracting, smaller number plus difference should = larger number.
Divisor × quotient should = dividend.
To multiply a number by 10 annex 0.
To multiply a number by 100 annex 00.
To multiply a number by 1000 annex 000.

If both terms of a fraction are multiplied by the same number, the value of the fraction is not changed.

If both terms of a fraction are divided by the same number, the value of the fraction is not changed.

To multiply by a fraction, multiply by its numerator and divide by its denominator.

To divide by any number, you may multiply by its reciprocal.

The amount represented by a figure depends upon the place it occupies.

To find how many times a quantity contains another quantity, express both in the same unit of measure before dividing.

In solving any problem, think what the units of measure mean.

SAMPLES OF USEFUL CUSTOMS

In addition, always begin with the right-hand column.

Add each column upward. Check by adding downward.

In multiplying a number which means dollars and cents, multiply it as if it were without the decimal point and meant cents. Divide the answer obtained by 100 and prefix the dollar sign.*

To multiply by 5, multiply by 10 and divide by 2.

When a decimal is multiplied by an integer, point off in the product as many decimal places as there are in the multiplicand.

EXERCISES

1. Score each of the definitions printed below for truthfulness, letting 0 mean that it gives a definitely wrong impression, letting 4 mean that it is entirely true, and letting 1, 2, and 3 be intermediate grades.

Score each of them for comprehensibility by children in the grade indicated. Let 0 mean that the children of that grade could hardly understand the definition at all. Let 4 mean that you cannot yourself frame a more comprehensible definition. Let 1, 2, and 3 be intermediate grades.

Score each of them for helpfulness in learning arithmetic. (Of course, if they are 0 in comprehensibility or truthfulness, they will be 0 in helpfulness, but they could be 4

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*A still more useful rule than this could be framed.

in comprehensibility or truthfulness or both and still be low in helpfulness.) Let 0 mean that the definition will be of almost no aid in learning or remembering anything of value in arithmetic. Let 4 mean that you cannot yourself frame a more helpful definition of the thing in question. Let 1, 2, and 3 be intermediate grades.

Grade 2 or 3

a. 0 means *no* or *not any*. 0 boys means no boys at all. 40 means 4 tens and not any ones.
b. 0 is called naught and is used to fill vacant orders.
c. The figure 0 means none to count.
d. 0 when standing alone has no value.
e. The other nine figures express each one or more units and are called significant.

Grade 3

f. Multiplication is the process of taking one number as many times as there are units in another.
g. Multiplication is the process of taking a number a number of times.
h. Multiplication is a short operation of finding the sum of as many expressions of one number as there are units in another number.
i. You multiply when you find the answers to questions like:

How many are 9×3?
How many are 3×32?
How many are 8×5?
How many are 4×42?

If you add 3 to 32, you have 35. 35 is the sum.

If you subtract 3 from 32, the result is 29. 29 is the difference or remainder.

If you multiply 3 by 32 or 32 by 3, you have 96. 96 is the product.

Grade 4 or 5

j. Fractions are numbers that count fractional units.

k. The equal parts into which a number or thing is divided are called fractions.

l. A fractional number is a number of the equal parts of some quantity considered as a unit.

m. Equal parts of a unit are fractions.

n. The area of any surface is the number of units of area the given surface contains.

o. The area of any surface is the quantity of surface.

Grade 5

p. A decimal fraction is one or more of the decimal divisions of a unit.

q. Numbers like .24, .07, .692, .8, .475, .2782 are called decimal fractions.

r. A decimal fraction is a fraction whose unit is divided into tenths, hundredths, thousandths, etc.

s. A decimal fraction is one or more tenths, hundredths, thousandths, etc., written like the orders of integers.

Grade 5 or 6

t. Addition is the process of combining two or more numbers into one number.

u. A period is a group of three orders of units counting from left to right.

v. An integer is a number that is not, in whole or in part, a fractional number.

w. Like numbers are numbers of the same kind or order.

Grade 6

x. A line is that which has length only.

y. A level or plane surface bounded by four straight lines is called a quadrilateral.

z. A plane figure bounded by four straight lines is called a quadrilateral.

aa. A rectangle is a flat surface with four straight sides and four square corners.

bb. A rectangle is a quadrilateral having four right angles.

Grade 6 or 7

cc. Interest is compensation for the use of money.

dd. Interest is money paid for the use of money.

ee. Interest is percentage allowed for the use of money or for value received.

Grade 7 or 8

ff. Ratio is the relation between two numbers of the same denomination, expressed by the quotient of the first divided by the second.

gg. Ratio is the expression of the relative magnitude of two similar quantities.

hh. Ratio is relation by quotient.

2. Score each of the rules printed below for truthfulness, comprehensibility, and helpfulness as you did the definitions of Exercise 1. Score them also for importance, letting 0 mean that the rule is of almost no importance, letting 4 mean that it is one of the twenty or thirty more important rules in arithmetic, and letting, 1, 2, and 3 mean intermediate grades.

Grade 3

a. In reading integral numbers, the primary unit should be most prominent in consciousness.

b. One million equals a thousand thousands.

c. In reading a number expressed by three figures, the hundreds, tens, and units are read together as so many units.

Grade 3 or 4

d. Rule for notation. Begin at the left and write the figures of each period in their proper orders, filling all vacant orders and periods with ciphers.

e. The unit of any period is equal to 1000 units of the next lower period.

f. When a figure is moved one place to the left, it represents units of ten times the value it did before.

g. Any order of units can be subtracted only from a like order of units.

h. The smaller number plus the difference should equal the larger number.

i. 0 times any number is 0.

j. The multiplier is always an abstract number.

k. To multiply by 10, annex 0.

l. To multiply by 100, annex 00.

(Note that rules *k* and *l* are for Grades 3 and 4 before decimals are in question).

Grade 5

m. To subtract a mixed number from a whole number or from a mixed number, we add such a fraction to the subtrahend as will make it a whole number, and add the same fraction to the minuend; then we subtract.

n. To subtract mixed numbers, first subtract the fractions, and then the integers.

o. To reduce fractions to similar fractions with a given common denominator, we divide the given common denominator by the denominator of the first fraction, and multiply the quotient by its numerator, and this will be the required numerator of the first fraction. In the same way we find the numerator of each of the other fractions.

p. To reduce or change a fraction to higher terms, multiply both the numerator and the denominator by 2 or 3 or 4 or 5 or some larger number.

q. To reduce a fraction to lower terms, divide both the numerator and the denominator by 2 or 3 or 4 or some larger number.

r. To add fractions, we change the fractions to similar fractions (if they are not similar), and write the sum of the numerators of the similar fractions over the common denominator. We reduce the resulting fraction to its lowest terms; and if it is an improper fraction, we reduce it to a whole or mixed number.

s. When you add or subtract with fractions, express them as fractions having the same denominator.

t. To change a mixed number to an improper fraction, we multiply the whole number by the denominator of the fraction, and to the product add the numerator; under this sum we write the denominator.

u. Multiplying both terms of a fraction by the same number does not change the value of the fraction.

v. To divide a fraction by a fraction, reduce the fractions to equivalent fractions, and divide the numerator of the dividend by the numerator of the divisor.

w. To divide a fraction by a fraction, invert and multiply.

x. To divide by a fraction, invert the fraction and multiply.

y. To divide by a fraction, invert the divisor and multiply.

z. To divide by a fraction, multiply by its reciprocal.

aa. A change in the dividend by multiplication or division produces a like change in the quotient; but such a change in the divisor produces an opposite change in the quotient.

bb. Divisor times quotient should equal dividend.

cc. The number of decimal places in the divisor plus the number of decimal places in the quotient should equal the number of decimal places in the dividend.

dd. Before beginning to divide, place a separatrix (**v**) in the dividend immediately after that figure in the dividend that is of the same denomination as the right-hand figure of the divisor. When in the process of division this separatrix is reached, the decimal point must be written in the quotient.

ee. In dividing with integers and decimal numbers, your result will be correct if —

I. You make sure that each estimate is right before going on to the next.

II. You multiply the divisor correctly by each quotient digit and write each product in the right place.

III. You subtract correctly and "bring down" the right digit or digits from the dividend.

IV. You put the decimal point in the right place.

Unless you are a very accurate worker, it is best to check every result in long division by multiplying the quotient by the divisor.

ff. "Percent" means "hundredths."

gg. "What percent of?" means "how many hundredths of?"

"What percent of 60 is 32?" means "how many hundredths of 60 is 32?"

hh. Percent means by or on a hundred.

ii. In computing with percents be careful in changing from hundredths expressed with a decimal point to percents expressed with %.

CHAPTER XIII

TESTS AND EXAMINATIONS

PURPOSE

Tests and examinations may be used for at least seven different purposes. These are:

(1) To inform the teacher of the relative ability of each pupil, so that she knows who did least well, who did best, and so on, in the abilities tested.

(2) To inform the pupil of his relative ability.

(3) To inform the teacher of the absolute ability of each pupil so that she knows which things he can do, or how hard things he can do, or how accurately, or rapidly, or both, he can do certain things.

(4) To inform the pupil of his absolute ability.

Relative ability and absolute ability are used here to mean position in comparison with others, and position measured as so much advance from zero ability.

(5) To spur the teacher to help her class do better work.

(6) To spur the pupils to do better work.

(7) To train, as well as test, the pupils.

It will be admitted by any competent thinker that the third and fourth aims are more important than the first and second—that a pupil's progress from zero ability and from his own past positions is more important to measure and report than his progress ahead of somebody who previously was ahead of him or his falling back behind somebody whom he previously led. The order of merit within a class is of some human interest, especially to those competing for the top, but it can always be derived from the absolute abilities if these are known. They cannot be derived from it. They measure actual achievements and progress. It measures only ranks.

It will be admitted further that specific information which will help the teacher to know the detailed abilities and weaknesses of her pupils is more important than a general score which simply informs her of pupils' fitness for promotion. For the same reasons the pupil profits more from specific knowledge of his own abilities than from a monthly or annual report of general competence.

In view of these purposes and facts the newer methods are in search of something more than collections of fair tests, grade by grade, in which a pupil receives a total score.

GRADED OR "LADDER" TESTS

First of all they use tests which, beginning with a very easy task, progress to harder and harder tasks of the same sort, and which are given with a sufficient time allowance to enable the pupil to do whatever he can do. One of these "ladder" tests is shown below and on page 244. Others have been shown in other connections on pages 23 and 24.

A Division Ladder

Begin at the bottom of the division ladder (on page 244) and climb to the top without making a mistake. Express any common fractions or mixed numbers in your results in lowest terms.

Step 11. Find quotients to the third decimal place:

a. $39.37|\overline{80}$ *b.* $11.25|\overline{6\quad}$ *c.* $360|\overline{.50}$ *d.* $293\frac{7}{8} \div 61.5$

Step 10. Find exact quotients:

a. $2.5|\overline{1200}$ *b.* $.25|\overline{3.55}$ *c.* $.045|\overline{20.25}$ *d.* $.05|\overline{42.3}$

Step 9. *a.* $6\frac{3}{4} \div \frac{3}{4}$ *b.* $\frac{1}{4} \div \frac{3}{8}$ *c.* $10\frac{1}{2} \div \frac{7}{8}$ *d.* $1\frac{3}{4} \div \frac{1}{2}$ *e.* $\frac{3}{4} \div \frac{1}{8}$

Step 8. Find the missing numbers:

a. \$10 = . . × 66⅔¢ *b.* \$25 = . . × 16⅔¢ *c.* \$5 = . . × 62½¢
d. \$50 = . . × 75¢ *e.* \$10 = . . × 37½¢

Step 7. Find quotients to the third decimal place:

a.	*b.*	*c.*	*d.*	*e.*
$19\overline{)390.6}$	$13\overline{)400}$	$14\overline{)859.15}$	$35\overline{)2941}$	$45\overline{)180.135}$

Step 6. Find exact quotients as integers or mixed numbers. Do not extend quotients to any decimal places.

a.	*b.*	*c.*	*d.*	*e.*
$36\overline{)1000}$	$18\overline{)725}$	$24\overline{)2000}$	$16\overline{)2500}$	$17\overline{)6075}$

Step 5. Find exact quotients:

a.	*b.*	*c.*	*d.*
$3\overline{)\text{5 hr. 9 min.}}$	$4\overline{)\text{10 ft. 8 in.}}$	$5\overline{)\text{8 lb. 2 oz.}}$	$5\overline{)\text{2 lb. 3 oz.}}$

Step 4. Find quotients and remainders. Do not extend quotients to any decimal places.

a.	*b.*	*c.*	*d.*	*e.*
$7\overline{)1499}$	$9\overline{)6310}$	$8\overline{)6458}$	$6\overline{)28236}$	$5\overline{)2705}$

Step 3. Find quotients to the nearest cent:

a.	*b.*	*c.*	*d.*	*e.*
$5\overline{)\$10.40}$	$7\overline{)\$25.75}$	$9\overline{)\$15.00}$	$8\overline{)\$36.00}$	$6\overline{)\$10.00}$

Step 2. Find quotients as integers or mixed numbers. Do not extend quotients to any decimal places.

a.	*b.*	*c.*	*d.*	*e.*
$20\overline{)740}$	$80\overline{)1375}$	$40\overline{)7500}$	$90\overline{)72000}$	$30\overline{)965}$

Step 1. Find quotients and remainders. Do not extend quotients to any decimal places.

a.	*b.*	*c.*	*d.*	*e.*
$3\overline{)196}$	$4\overline{)215}$	$5\overline{)92}$	$7\overline{)252}$	$6\overline{)127}$

The arrangement in a graded series by difficulty has several advantages. The pupil is given confidence at the beginning and works with a better attitude. He has five chances at each degree of difficulty so that the examination is admittedly fair, and so that the teacher can quickly distinguish lack of knowledge of the process from carelessness. (One or two wrong out of five will practically always mean knowledge of the process; four or five wrong will practically always mean ignorance of it.) The form lends itself well to training in thoroughness and care for 100 percent efficiency in those tasks which the pupil claims to be able to do at all. Pupils can be led by such tests to see that, since they know enough to get three right at a certain step, they could have got all five right by care and checking. This lesson can be reinforced to any extent desired by the method of scoring. For instance, as a thoroughness score, a credit of 0 could be given for a step which has only 0, 1, or 2 right, a credit of 2 for 3 right, a credit of 5 for 4 right, and a credit of 10 for all 5 right. Perfect work on the five easiest steps would thus be valued as high as 80 percent correctness on ten steps.

There is an efficient adaptation to individual differences. The less able pupils are not disheartened by a job nothing of which they can do well. The gifted pupils are not bored by work which is mere routine. For they may be allowed to do the highest third or half of such a test, with credit for the easier steps if they succeed with these harder ones. The teacher, by inspecting the class records step by step, can decide where the class as a whole, and where individuals, need further teaching and practice.

INVENTORY TESTS

In the cases where the work does not fall into a series from easy to hard, these step-up tests may be replaced by inventory tests in which each step represents one set of facts or kind of work. Such inventory tests are but a systematization of what every good teacher does from time to time. A simple specimen is shown below:

Review

1. Add 296 to each of these numbers:

231 509 625 474 382 528 189 398

2. Subtract 468 from each of these numbers:
682 721 500 735 898 668 934 929

3. Multiply each of these numbers by 9:
78 106 54 29 27 45 111 110

4. Find the quotients and remainders when you divide each of these numbers by 6:
472 976 800 608 849 675 550 345

SPEED TESTS

Second, the newer methods use speed tests, but with special conditions and for a special purpose. The conditions are, as a rule, (1) that the test measure a specified set of bonds, and (2) that the system of credits and penalties be such that the work is substantially errorless. The purpose is to gain information about whether the specified set of bonds (e. g., the fundamental multiplication facts) are perfected sufficiently for their later uses. The speed in and of itself is of little importance, but as a symptom of mastery it is important. So we often assign a drill lesson as: "Say the missing numbers. Practice with them until you can say them all correctly in three minutes."

TRAINING IN ALERTNESS AND ADAPTABILITY

Third, the newer methods seek to make examinations useful in giving training in alertness and adaptability. The examination should not always take up a topic in the ways in which it was learned or has been reviewed.

The following are plans which will interest the pupils and give training in alertness and adaptability:

Selection tests. Exercises and problems, each with five or more answers attached, from which the pupil is to select the right answer, he being assured that one of the five is right.

PART OF "SELECTION" TEST, GRADES 6–8

$\frac{3}{4}+\frac{1}{4}$	$\frac{3}{16}$	3	$\frac{3}{4}$	$\frac{4}{3}$	1
$.21\,\overline{)33.6}$	.016	.16	1.6	16	160
1 Acre	5280 sq. ft.	27225 sq. ft.	43560 sq. ft.	528000 sq. ft	4356 sq. ft.
$\frac{9\times8\times7\times6}{4\times4\times3\times3\times3}$	7	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$3\frac{1}{2}$

"Matching" tests. Tests wherein two series of eight or more items each are given, the pupil to state which item in Series A goes with each item in Series B. For example, Series A may be ten definitions and Series B ten cases illustrating them. Or Series A may be ten surfaces and Series B their areas. Or Series A may be ten statements and Series B ten equations representing them.

MATCHING TEST, GRADE 8

Write each formula after the words it fits. Remember that $\pi=\frac{22}{7}$. You have to decide for yourself what a, b, r, s_1, and s_2 mean.

Diameter........................$a\times b$
Circumference...................$\frac{1}{2}a\times b$
Area of a circle..................$\frac{22}{7}\times a\times r^2$
Length of hypotenuse............$\frac{22}{21}ar^2$
Area of a triangle................$a\times\frac{s_1+s_2}{2}$
Area of a parallelogram..........$2r$
Area of a trapezoid...............$\frac{44}{7}r$
Area of a square..................$\frac{22}{7}r^2$
Volume of a cylinder..............$\frac{88}{21}r^3$
Volume of a sphere...............$s_1{}^2$
Volume of a cone ....$\sqrt{s_1{}^2+s_2{}^2}$

Completion tests. Formulae, equations, statements, etc., with omitted words, numbers, or signs to be supplied.

PART OF A COMPLETION TEST, GRADE 5 OR 6

Write the missing words, numbers, or signs to make each statement true.

To multiply by a fraction, ... by its numerator and .... by its ....

When both the dividend and divisor mean amounts of money, express both as .... or express both as .... Then ....

To .... by a ...., multiply by its reciprocal.

"Same-different" tests. Series of pairs of numbers, quantities, and numerical expressions, each pair to be marked *S* or *D* according as the two numbers do or do not have the same value.

PART OF A "SAME-DIFFERENT TEST," GRADE 6 (LATE) OR 7 OR 8

Look at the two numbers, or quantities, or words on each line.

Write *S* if they have the same value or meaning.
Write *D* if they do not.

$16\frac{2}{3}$¢	one-sixth of a dollar
$4\frac{1}{2}$ per cent of	.045 ×
$.62\frac{1}{2}$	$\frac{5}{8}$ of 100
.4 ×	40 per cent of
\$$\frac{3}{4}$	75¢
$\frac{1}{8}$ of $\frac{2}{3}$ of	$\frac{1}{24}$ of

"True false" tests. Statements, equations, and the like to be marked true or false.

PART OF A "TRUE FALSE" TEST

Examine each equation.
Write *Y* if it is true.

Write *N* if it is false.

$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{12}$
$\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{6}{9}$
$.4 + .04 + .004 = .444$
$\frac{3}{8} + \frac{3}{8} + \frac{3}{8} = \frac{9}{8}$

Any of these can be given as a "power" test to find how hard things a pupil can do, or as an "inventory" test to find which things he can do, or as a speed test to find how readily he can use his abilities under the novel circumstances of the test.

STANDARDIZED TESTS

The newer methods also favor the use of efficient "standardized" tests as fast as such are available. We call a test or examination standardized if

(1) the exact difficulty of it is known, or
(2) the exact difficulty of each of its steps is known, or
(3) the scores made in it by pupils of different grades are known, or
(4) the scores made in it by pupils of different ages are known, or
(5) the scores which should be made in it to fit pupils for certain specified work in school or in life are known, or
(6) if two or more of these facts are known.

The Courtis tests and the Woody tests are the best known. The teacher who is not familiar with them should become so. The Courtis tests* include tests in the four operations with integers.† The Woody tests‡ are "power" tests in the four operations with integers, fractions, decimals, and denominate numbers, beginning with the very easiest tasks and extending up to tasks as hard as life will probably offer.

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* Obtainable from S. A. Courtis, 82 Eliot Street, Detroit, Michigan.

† With certain subsidiary tests for use in interpretation.

‡ Obtainable from the Bureau of Publications, Teachers College, Columbia University, New York.

By dividing the score made by a pupil in a standardized test (× 100) by the average score for a pupil of his age, we obtain what may be called his "educational quotient" or E. Q. If his score is 16 and the standard for his age is 20, his E. Q. in that particular ability is 80. If his score were 26, his E. Q. would be 130. These E. Q.'s for various abilities are of great importance in connection with his general intelligence quotient or I. Q., obtained in a suitable test for general intelligence. For example, consider these pupils, A, B, and C, all obtaining E. Q.'s of 100 in arithmetical computation. Their I. Q.'s are: A, 100; B, 110; C, 135. A is doing as well in his arithmetical computation as his general intelligence would make it reasonable to demand, but B is not, and C is far below what his general intelligence suggests that he should attain. The educational achievement which it is fair to expect of a pupil or class is evidently conditioned somewhat by their natural abilities. $\text{E. Q.} - \text{I. Q.}$ or $\frac{\text{E. Q.}}{\text{I. Q.}}$ is evidently to some extent a measure of the quality of the pupil's effort and of the teaching he has received. When his E. Q.'s are not up to his I. Q., there is usually evidence of need of improvement in one or the other of these.*

With respect to the age and grade standards in the Courtis, Woody, and other tests, it should be remembered that these standards are of achievement under present conditions of time spent on arithmetic, methods used, and the like. By better methods we may hope to raise these standards, or reduce the time required in learning or both. In particular, the accuracy in these standards is below what we can and should try to obtain by better methods of teaching.

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*The matter is not so simple as it is put here, for any particular E. Q., since the pupil may have some degree of *special* ability or disability in that line. Also the average of his E. Q.'s may be lowered by special circumstances, e. g., deafness, foreign residence, race, special devotion to music, and the like.

THE TEST OF LIFE

The newer methods are everywhere watchful to protect the learner against artificial examinations made up merely to examine and rate pupils, using tasks which life will rarely or never require. The tests which the newer methods have constantly in view are the tests which the pupil meets and will meet in the home, the factory, by the counter, and elsewhere in connection with his career as a worker, citizen, and thinker. In examinations, as in lessons, they favor the situations which life itself will offer and the responses which life itself will require.

It is, of course, true that work may be useful as a test which would not be useful as training—that work may be used to measure achievements that would not be used to improve them. But there is a risk in this. Teaching tends to prepare for testing. Whatever sort of examinations are given—for these teachers will consciously or unconsciously train their pupils. If we desire training for life rather than for the traditional examinations with unreal problems, then we should give no such unreal problems in our examinations. The examination that really counts is fifty years long; its situations are real things and events; it demands mastery of a few things rather than 60 per cent efficiency with many. We cannot, of course, duplicate it exactly under schoolroom conditions, but we can make our examinations much more like it than they have been. Examinations, like explanations, drills, definitions, and rules, should be for the learner and for life.

EXERCISES

1. Examine the ladder reviews in III, 5, 11, 131, and 132. Note in each case how well the test might serve as a rough inventory test.
2. Note any changes you would make in these reviews to make them serve better as examinations.
3. Make an "Interest Ladder."

4. Examine the "Angle, Area, Volume" test in III, 133, 134. Is this primarily a ladder test or an inventory test?

5. What does it test besides certain principles of mensuration?

6. Make an inventory test for all the important abilities in mensuration, but with all computations very simple and easy.

7. Make a test of ability with fractions on the "Select the right answer" plan.

8. Make a test of ability with fractions on the "Same-Different" plan.

9. (*a*) What do you think is the purpose of the test in addition below?

[This test is printed. The papers are distributed bottom side up. The pupils write their names, then, at the signal "Go," turn the sheets over and start. After 40 seconds the teacher says, "Begin B now." After 80 seconds she says, "Begin C now." After 120 seconds she says, "Stop." The pupils have had tests similar in plan so that they understand the necessity of beginning promptly, working steadily, and writing the figures rapidly.]

A. Add 9 to each of these numbers. Write the answers as fast as you can.

4
7
3
11
19

16
28
32
5
56

43
36
8
24
37

29
35
17
12
18

B. Add 8 to each of these numbers. Write the answers as fast as you can.

7
17
32
3
11

5
24
12
18
37

56
19
4
16
43

29
35
36
38
7

C. Add 7 to each of these numbers. Write the answers as fast as you can.

7	4	5	24
38	12	8	56
36	18	17	19
43	37	32	35
16	11	29	3

(*b*) What would you recommend as teaching for each of these pupils (Albert, Bertha, and Charles)?

(*c*) Is it probable that Charles adds by counting on his fingers?

Albert does 12 in A, 13 in B, 12 in C; and has 9, 9, and 8 right.

Bertha does 12 in A, 12 in B, 14 in C; and has 12, 12, and 14 right.

Charles does 4 in A, 5 in B, 5 in C; and has 4, 5, and 5 right.

10. Criticize each of these examination questions from the point of view of similarity to life's tasks.

a. Simplify: $\frac{3\times\frac{2}{7}\times4.2}{\frac{5}{18}\times\frac{20}{27}}$.

b. Divide $\frac{3}{4}$ of $\frac{5}{6}$ of $7\frac{1}{5}$ by $3\frac{2}{9}$.

c. Find the least common multiple of 153, 204, and 510.

d. Define numerator, denominator, divisor, factor, proportion.

e. A man bought a watch and chain for $140. One-half of the cost of the watch equals $\frac{2}{3}$ of the cost of the chain. What was the cost of each?

f. A boy had a pole 20 feet long. A piece equal to 15 per cent of its length broke off.

What percent of the whole pole was the part remaining?

What percent of the part remaining was the part broken off?

What percent of the part broken off was the part remaining?

g. A and B are 48 miles apart and walk toward each other; A walks $2\frac{1}{2}$ miles an hour and B $3\frac{1}{2}$. How far will B have walked when they meet?

h. A circular field contains 20 acres. What is the circumference?

i. Find the contents in bushels of a wagon box 10 ft. long, 42 in. wide, and 4 ft. high. Allow 2150.42.

i. If $\frac{5}{8}$ of $\frac{2}{3}$ of a piece of land cost $420, what is the value of the whole?

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