literally to come close. By taking more and more digits in the decimal we get a number that is closer and closer to $\frac{1}{3}$. In fact, we can get as close to $\frac{1}{3}$ as we please by simply taking a decimal that is long enough. We describe the situation by saying that the sequence of decimals approaches the number $\frac{1}{3}$ as a limit.

We now have the means of explaining how an infinite decimal may represent a number. An infinite decimal represents a number if the sequence of finite decimals obtained by taking more and more of its digits approaches that number as a limit. With this definition of the meaning of an infinite decimal, it may make as much sense as a finite decimal, and we can now answer the question, "Can every rational number be represented by a decimal?" The answer is, "Yes, by a finite or infinite decimal."

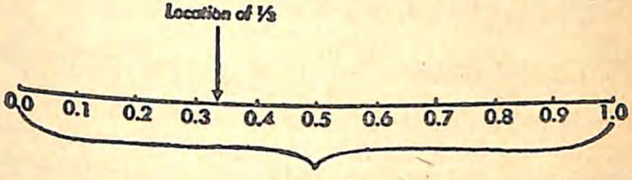
A Nest of Intervals

The meaning of an infinite decimal can also be expressed pictorially in terms of our representation of the rational numbers as points on a line. It can be interpreted as a description of the position of a number, or directions that may be followed in order to find it.

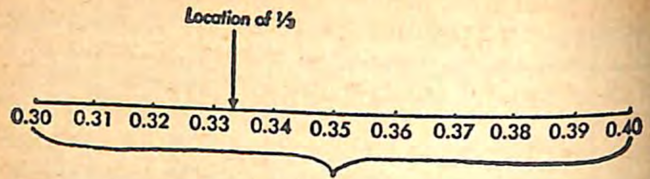
Suppose, for example, we want to describe where the number $\frac{1}{3}$ is located. Since it is a rational number, we can say first that it is somewhere on the line on which the rational numbers are represented. Then we try to specify where it is on the line. The integers on the line divide the line into intervals of unit length. We can assign a name to each interval by using the number that is attached to the end that is nearest to 0. We call the interval between 0 and 1 the "0 interval"; the interval between 1 and 2 the "1 interval," and so on. On the negative side of the line, we call the interval between 0 and -1 the "-0 interval," the interval

between -1 and -2, the "-1 interval." Notice that 0 and -0 refer to different intervals.

Now we can be more specific about where the number $\frac{1}{3}$ is by saying that it is in the 0 interval. So we write down 0 as the first part of a chain of directions. The next step in the description is to say where it is in the 0 interval. For this purpose, we divide the 0 interval into ten equal parts, each with length .1. We label these intervals in order, starting with the one nearest to the 0, as 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9. Now we learn from long division that $\frac{1}{3}$ is more than 0.3, but less than 0.4, so it lies in the interval whose name is 0.3. The name of this interval incorporates both parts of the description we have given so far. It tells us that the number $\frac{1}{3}$ is in the 0 interval, and that within



that interval it is in the sub-interval that has 0.3 as its left end. Now we narrow down its location once more. We divide the sub-interval into ten equal parts, each of width .01, and label them, in order, 0.30, 0.31, 0.32, 0.33, 0.34, 0.35, 0.36, 0.37, 0.38, and 0.39. Again we learn from long division that $\frac{1}{3}$ is more than 0.33, but less than 0.34. So it lies in the interval whose name is 0.33. The name of this interval is a three part description. It says that the number $\frac{1}{3}$ is in the 0 interval, and inside this interval it is in the 0.3 interval, and inside the latter interval it is in the 0.33 interval. Now we subdivide this interval, and continue the process indefinitely.



Each time we pick out the sub-interval that contains the point. According to this scheme, the infinite decimal 0.33333 . . . . represents an infinite sequence of intervals, with the following characteristics: each successive interval lies inside the interval that precedes it in the sequence, and as we move along the sequence, the width of the intervals shrinks toward 0. We call such a sequence of intervals a *nest of intervals.*

In this instance, we chose each interval in the nest as an interval that contains the number $\frac{1}{3}$. So we are sure that the number $\frac{1}{3}$ lies inside every one of the infinite number of intervals in the nest. We can also be sure that no other number can be in there with it, because two different numbers cannot lie in the same nest. This is so, because two different numbers are separated by a distance that is greater than 0. Since the inner intervals in a nest shrink in width toward a width of 0, ultimately the innermost intervals are too narrow to span the distance between two separate points, and cannot enclose them both. For this reason, the nest of intervals represented by the infinite decimal 0.333 . . . . describes the number $\frac{1}{3}$ and no other number. *A nest of intervals represents a number if that number is attached to the only point that lies inside all the intervals of the nest.*

A Nest for Every Rational Number

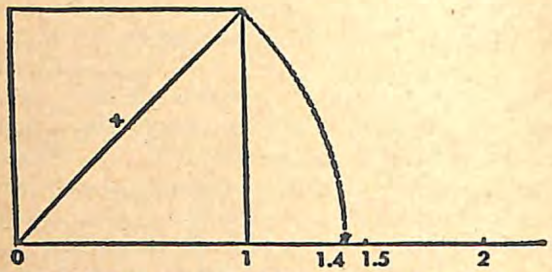
By following the same procedure with any rational number, we can find a nest of intervals that contains the

number, and the nest of intervals can be represented by an infinite decimal. In this way we can get an infinite decimal for every rational number. This is true even for numbers that are represented by finite decimals. In fact, in these cases, the number can be represented by two infinite decimals. Suppose, for example, we look for a nest of intervals and the associated infinite decimal to represent the number 2.3. First we notice that the number lies in the 2 interval which extends from 2 to 3. Now, when we subdivide this interval into ten equal parts, one of the points of division is actually 2.3. As a point of division, it belongs to *two* intervals. It is the right end of the interval whose name is 2.2. It is also the left end of the interval whose name is 2.3. So we may choose either one as the interval to which it belongs. After we have made that choice, we shall not be free to choose again. If we think of the number as belonging to 2.2, then forever after, as we subdivide the intervals, the number will always lie in the last subdivision. The infinite decimal in this case comes out 2.2999999 . . . . with an endless series of nines. If we think of the number as belonging to 2.3, then forever after, as we subdivide the intervals, the number will always lie in the first subdivision. Then the infinite decimal comes out 2.3000000 . . . . with an endless series of zeros. We get two names like this for every number that can be represented as a finite decimal, because a finite decimal turns up, sooner or later, as an endpoint of an interval in the nest. The two different ways of representing the number signify that such a number can be approached from two directions, from the left or from the right.

A Nest That Has No Number

We have answered in the affirmative the question, "Can every rational number be represented by a decimal?" We have found that it can be represented by an infinite decimal, and in some cases by two infinite decimals. In each case the infinite decimal represents a nest of intervals, and the rational number is attached to the only point that is inside the nest.

Now let us reverse the question and ask, "Does every infinite decimal represent a rational number?" Or, to ask the question in another form, "Does every nest of intervals defined by an infinite decimal have a rational number attached to the point inside the nest?" We can see immediately that the answer to this question is, "No." We found before that there are some points on the line that have no rational number attached. The point whose distance from 0 is equal to the length of the diagonal of a unit square is such a point. Its location is pointed out by the arrow in the diagram. However, we can follow the same procedure with this point as we did with the point that represents the fraction $\frac{1}{3}$. We can locate it precisely by means of a nest of



intervals, and the associated infinite decimal. We observe first from the diagram that the point lies between 1 and 2 so the decimal begins with the number 1. Then we subdivide the interval between 1 and 2, and observe that the point lies between 1.4 and 1.5. In fact, we can verify this fact by arithmetic, by observing that $1.4 \times 1.4 = 1.96$, a number that is less than 2, while $1.5 \times 1.5 = 2.25$, a number that is more than 2. So, 1.4 is too small to be the length of the diagonal of the unit square, and 1.5 is too large. By continuing the process of subdivision, we get a nest of intervals and an infinite decimal that represents it. There is one and only one point inside this nest, the point whose distance from 0 is the length of the diagonal of the unit square. But there is no rational number attached to this point. So the infinite decimal that belongs to this nest does not represent a rational number.

The Real Number System

We can think of an infinite decimal as a question. It asks, "If you make the nest of intervals that my digits describe, what number will you find inside the nest?" In the rational number system this question does not always have an answer. So, once again, we ask, "Is there a larger number system in which such a question does always have an answer?" To construct such a number system, we use the same device we have already used twice before. We make the question its own answer. We build a new number system in which each element *is* an infinite decimal. We call this system the system of *real* numbers. In general, each separate infinite decimal is a separate number. But, we know from our experience with the rational number 2.3 that we shall have to provide for some exceptions. We must specify that whenever a decimal ends with an infinite chain of nines, as 2.99999 . . . does, the decimal obtained by replacing the chain of nines by a chain of zeros and adding 1 to the last digit before the chain will represent the same number. Thus 2.300000 . . . . and 2.299999 . . . . will represent the same number. Under this agreement, every real number is either a single infinite decimal, or a pair of infinite decimals either one of which may be used to represent the pair.

When we represented the rational numbers as points on a line, we found that there were gaps on the line, where no numbers were attached. Later we found that every point on the line can be located by an infinite decimal, and now we are converting every infinite decimal into a number. So, at one stroke, we are filling all the gaps on the line. The real number system gives us a number for every point on the line.

So far we have merely defined a collection of elements, each of which is an infinite decimal or a pair of decimals. To convert this collection into a number system we have to give it the structure that every number system must have. We have to define operations of addition and multiplication on these elements, and we must show that the operations obey the five laws. To define the operations, we lean on the smaller number system we already have as a crutch. Just as we used the natural numbers when we built the structure of the system of integers, and we used the integers when we built the structure of the system of rational numbers, we now use the rational numbers when we build the structure of the system of real numbers. We define addition and multiplication of real numbers as follows:

To add two infinite decimals, break up each decimal into a sequence of finite decimals, by using in succession, more and more of the digits in the decimal. These finite decimals represent rational numbers, and may be added by the rules for rational numbers. So, first add the first numbers in each sequence. The sum is a first approximation to your answer. Then add the second numbers in each sequence. This gives you a better approximation to your answer. As you proceed with the sequence of additions, you get a longer and longer finite decimal. The digits at the beginning of the decimal may fluctuate at first, but then settle down so that ultimately a fixed digit is defined for each decimal place. In this way a specific infinite decimal is obtained as the sum. To multiply infinite decimals, multiply the successive pairs of finite decimals in the same way.

For example, to add 0.2222222 . . . . and 0.8888888 . . . ., we proceed by these steps:

$$\begin{array}{r} 0.2 \\ +0.8 \\ \hline 1.0 \end{array} \qquad \begin{array}{r} 0.22 \\ +0.88 \\ \hline 1.10 \end{array} \qquad \begin{array}{r} 0.222 \\ +0.888 \\ \hline 1.110 \end{array} \qquad \begin{array}{r} 0.2222 \\ +0.8888 \\ \hline 1.1110 \end{array} \qquad \begin{array}{r} 0.22222 \\ +0.88888 \\ \hline 1.11110 \end{array} \qquad \begin{array}{r} 0.222222 \\ +0.888888 \\ \hline 1.111110 \end{array}$$

The sum is clearly the infinite decimal 1.111111 . . . . in which all the digits are ones. To multiply these same decimals, we proceed by these steps:

$$\begin{array}{r} 0.2 \\ \times 0.8 \\ \hline 0.16 \end{array} \qquad \begin{array}{r} 0.22 \\ \times 0.88 \\ \hline 176 \\ 176 \\ \hline 0.1936 \end{array} \qquad \begin{array}{r} 0.222 \\ \times 0.888 \\ \hline 1776 \\ 1776 \\ 1776 \\ \hline 0.197136 \end{array} \qquad \begin{array}{r} 0.2222 \\ \times 0.8888 \\ \hline 17776 \\ 17776 \\ 17776 \\ 17776 \\ \hline 0.19749136 \end{array}$$

We see that the product begins with 0.197, and more of the digits are fixed when more of the steps are taken.

At each step in the addition or multiplication just defined, we are using rational numbers. The rational numbers obey the five laws. As a result, the five laws are extended to the real numbers as well. For example, $0.2 + 0.8$ and $0.8 + 0.2$ lead to the same sum; $0.22 + 0.88$ and $0.88 + 0.22$ lead to the same sum; etc. So $0.222 \ldots + 0.888 \ldots$ and $0.888 \ldots + 0.222 \ldots$ lead to the same infinite decimal as sum. In other words, real numbers obey the commutative law of addition. The other four laws can be established by a similar argument.

We Still Have the Rational Numbers

We constructed the real number system in such a way that we have a number for every point on the number line. Among the points on the line are those to which we had previously assigned rational numbers. Let us refer to these as the rational points on the line. The real numbers assigned to the rational points form a subset of the real number system that is isomorphic to the rational number system. For all practical purposes, they are the "same" as the rational numbers. The symbol for a rational number is a fraction or a finite decimal. Such a symbol is easier to write and work with than an infinite decimal, so we use the rational number rather than the infinite decimal to represent the real number attached to a rational point. But first we must learn to recognize which of the real numbers belong to the rational points, and are therefore entitled to this simpler representation.

The infinite decimals that represent rational points are those which, after a finite number of digits in the decimal, simply repeat a fixed block of one or more digits over and over again. For example, the following decimals, in which the repeating block has been italicized, all represent rational numbers: $0.\mathit{3}333 \ldots$, $0.\mathit{12}1212 \ldots$, $2.37\mathit{45}4545 \ldots$. To prove this fact, we have to show, first, that every rational number, represented by a fraction, can be written as a resmaller number system we already have as a crutch. Just as we used the natural numbers when we built the structure of the system of integers, and we used the integers when we built the structure of the system of rational numbers, we now use the rational numbers when we build the structure of the system of real numbers. We define addition and multiplication of real numbers as follows:

To add two infinite decimals, break up each decimal into a sequence of finite decimals, by using in succession, more and more of the digits in the decimal. These finite decimals represent rational numbers, and may be added by the rules for rational numbers. So, first add the first numbers in each sequence. The sum is a first approximation to your answer. Then add the second numbers in each sequence. This gives you a better approximation to your answer. As you proceed with the sequence of additions, you get a longer and longer finite decimal. The digits at the beginning of the decimal may fluctuate at first, but then settle down so that ultimately a fixed digit is defined for each decimal place. In this way a specific infinite decimal is obtained as the sum. To multiply infinite decimals, multiply the successive pairs of finite decimals in the same way.

For example, to add 0.2222222 . . . . and 0.8888888 . . . , we proceed by these steps:

$$\begin{array}{r} 0.2 \\ +\underline{0.8} \\ 1.0 \end{array} \quad \begin{array}{r} 0.22 \\ +\underline{0.88} \\ 1.10 \end{array} \quad \begin{array}{r} 0.222 \\ +\underline{0.888} \\ 1.110 \end{array} \quad \begin{array}{r} 0.2222 \\ +\underline{0.8888} \\ 1.1110 \end{array} \quad \begin{array}{r} 0.22222 \\ +\underline{0.88888} \\ 1.11110 \end{array} \quad \begin{array}{r} 0.222222 \\ +\underline{0.888888} \\ 1.111110 \end{array}$$

The sum is clearly the infinite decimal 1.111111 . . . . in which all the digits are ones. To multiply these same decimals, we proceed by these steps:

$$\begin{array}{r} 0.2 \\ \times\underline{0.8} \\ 0.16 \end{array} \quad \begin{array}{r} 0.22 \\ \times\underline{0.88} \\ 176 \\ \underline{176} \\ 0.1936 \end{array} \quad \begin{array}{r} 0.222 \\ \times\underline{0.888} \\ 1776 \\ 1776 \\ \underline{1776} \\ 0.197136 \end{array} \quad \begin{array}{r} 0.2222 \\ \times\underline{0.8888} \\ 17776 \\ 17776 \\ 17776 \\ \underline{17776} \\ 0.19749136 \end{array}$$

We see that the product begins with 0.197, and more of the digits are fixed when more of the steps are taken.

At each step in the addition or multiplication just defined, we are using rational numbers. The rational numbers obey the five laws. As a result, the five laws are extended to the real numbers as well. For example, 0.2 + 0.8 and 0.8 + 0.2 lead to the same sum; 0.22 + 0.88 and 0.88 + 0.22 lead to the same sum; etc. So 0.222 . . . + 0.888 . . . and 0.888 . . . + 0.222 . . . lead to the same infinite decimal as sum. In other words, real numbers obey the commutative law of addition. The other four laws can be established by a similar argument.

We Still Have the Rational Numbers

We constructed the real number system in such a way that we have a number for every point on the number line. Among the points on the line are those to which we had previously assigned rational numbers. Let us refer to these as the rational points on the line. The real numbers assigned to the rational points form a subset of the real number system that is isomorphic to the rational number system. For all practical purposes, they are the "same" as the rational numbers. The symbol for a rational number is a fraction or a finite decimal. Such a symbol is easier to write and work with than an infinite decimal, so we use the rational number rather than the infinite decimal to represent the real number attached to a rational point. But first we must learn to recognize which of the real numbers belong to the rational points, and are therefore entitled to this simpler representation.

The infinite decimals that represent rational points are those which, after a finite number of digits in the decimal, simply repeat a fixed block of one or more digits over and over again. For example, the following decimals, in which the repeating block has been italicized, all represent rational numbers: 0.*3*333 . . ., 0.*12*1212 . . ., 2.37*45*4545 . . . . To prove this fact, we have to show, first, that every rational number, represented by a fraction, can be written as a repeating decimal. Then, vice versa, we must show that every repeating decimal can be written as a fraction.

To show that a fraction can be written as a repeating decimal, recall that we can convert a fraction into a decimal by long division, dividing the denominator into the numerator. In the long division process there is a step involving subtraction, after which we carry down a digit from the dividend. The dividend is an integer with a finite number of digits. So these are soon exhausted. Then we begin carrying down the zeros that appear after the decimal point. Consider what happens after we reach this stage of carrying down only zeros. In the subtraction step there is a remainder that is less than the divisor, and this remainder determines what the next number in the quotient will be. Since the remainders must be less than the divisor, the list of possible remainders is a restricted finite list. But, as we proceed with the division, we get an endless succession of remainders. So we cannot keep getting a different remainder each time. Sooner or later, a remainder that turned up before is repeated, and the division process begins to repeat itself.

To show that every repeating decimal can be written as a fraction, we shall work out one specific example showing how it is done. It will be clear that the method used can be employed with any repeating decimal whatever.

Suppose we find the fraction that represents the repeating decimal 2.7*151*515 . . . . We first split this decimal into two parts, 2.7, and .0*15*1515 . . . ., separating the non-repeating part from the repeating part. The first part is the fraction $\frac{27}{10}$. Now we find a fraction for the second part. First multiply it by ten, so that the repeating block will begin right after the decimal point. Let us call the result x, and remember that it is ten times as big as the number we are looking for, so after we find x we must divide by ten. x = .151515 . . . . Now multiply both sides of this equation by 100. This has the effect of moving the decimal point two places to the right.

We get $100x = 15.1515 \ldots$, which may also be written as $100x = 15 + .1515 \ldots$. But the decimal in this equation is none other than x all over again. So we may write $100x = 15 + x$. Taking x away from both sides, we find that $99x = 15$. Dividing by 99 on both sides, we find that $x = \frac{15}{99}$ or $\frac{5}{33}$. Now we divide by ten, to find that the second part of our original number is $\frac{5}{330}$. So the infinite decimal 2.7151515 . . . is the sum of $\frac{27}{10}$ and $\frac{5}{330}$. Therefore 2.7151515 . . . . represents the rational number $\frac{896}{330}$ or $\frac{448}{165}$. To check the result, divide 165 into 448.

We Still Have a Field

The rational number system has the structure of a field. Enlarging the number system has not destroyed this structure, because the real number system, too, is a field. We can verify that it has the characteristics of a field, one by one. In the first place it is a commutative group with the operation of addition. The zero element in the group is the infinite decimal 0.0000 . . . ., which we may write briefly as 0, and every infinite decimal has a negative, namely, the infinite decimal written with the same digits in the same order, but having the opposite sign attached. For example, the negative of .333 . . . is −.333 . . . It is also a ring, because the multiplication is distributive with respect to addition. The unity element may be written in two ways: 1.00000 . . . ., or 0.9999 . . . . . . Moreover, every element except 0 has a reciprocal, so the system is a field. To find the reciprocal of an infinite decimal, we can use the successive finite decimals that approximate it, and divide each into the number 1. The quotients we get serve as successive approximations of the reciprocal, and, one by one, we can identify the digits in the decimal that represents it. In special cases we have simplified ways of doing it. For example, to find the reciprocal of $\sqrt{2}$, we write it first in fraction form as $\frac{1}{\sqrt{2}}$. The value is unchanged if we multiply by $\frac{\sqrt{2}}{\sqrt{2}}$, because this multiplier is equal to 1, or the unity element. But then we have $\frac{\sqrt{2}}{2}$, whose decimal equivalent is easily found by dividing 2 into the decimal for $\sqrt{2}$. The decimal for $\sqrt{2}$ begins as 1.414. . . ., so the decimal for its reciprocal begins as .707 . . . .

A Number in Every Nest

The system of real numbers has some special properties that the rational number system does not have. The most convenient way of expressing these properties is in terms of the picture we have set up of numbers as points on a line. When we represented the rational numbers as points on a line, we found that there were gaps that it left unfilled. That is, there were points on the line that had no numbers attached to them. The real number system was deliberately designed to eliminate this defect. In this system, not only do we have a point for every number. We also have a number for every point. There is a one-to-one correspondence between the real number system and the points on the number line. Because of this correspondence, we may think of the real numbers as the points on the line, and can describe the properties of the real number system in terms of relationships of the points on the line.

For example, by making every infinite decimal an element of the real number system, we assured the fact that there would be a real number for every such decimal. This property can also be described in terms of points on the line, as we have seen: an infinite decimal represents a nest of intervals on the line, and for every such nest, there is one and only one point that lies inside every interval of the nest. In this statement, the nest referred to is a nest associated with an infinite decimal. The successive intervals in such a nest

have a special length, viz., 1, .1, .01, .001, etc., and their endpoints are always finite decimals. However, it is possible to form nests of intervals of a more general character by removing these restrictions on the lengths of the intervals and the locations of their endpoints. The only requirements for calling a sequence of intervals a nest are that the successive intervals lie one inside the other and that the lengths of the inner intervals shrink toward 0. It can be proved that in the real number system, all *nests* have the same property we have found for nests associated with infinite decimals: There is one and only one point that lies inside every interval of the nest. In this sense, every nest of intervals defines a single real number. Some of the other properties of the real number system that we shall now examine are closely related to this fact.

Infinite Series

Addition is an operation defined on a pair of numbers, so, initially, we can add only two numbers at a time. However, by performing one addition after another, we can extend the operation to include any finite number of numbers. For example, there is no difficulty about finding the sum of these numbers: $1 + 1 + 1 + 1 + 1 + 1$. The sum is, of course, 6. However, if we permit the series of ones to go on indefinitely, in the infinite series $1 + 1 + 1 + 1 + 1 + 1$ . . . ., then we run into trouble. The step by step addition that we can carry out with a finite number of terms doesn't work here, because it never comes to an end. We are left then with the question: Does it make any sense at all to try to add an infinite series of terms? The answer turns out to be that sometimes it does, and sometimes it doesn't. We can get clues to when an infinite series has meaning as a sum by re-examining an infinite series with which we are already familiar, namely, an infinite decimal.

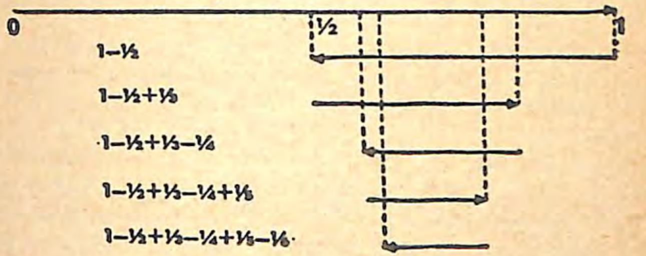
The infinite decimal .3333333 . . . . . . is really an infinite series in disguise. In fact, we may think of it as merely an abbreviated way of writing the infinite series $.3 + .03 + .003 + .0003 + .00003$ . . . . . . On page 97 we also interpreted the infinite decimal as a sequence of finite decimals, .3, .33, .333, .3333, ... These finite decimals are the sums we get when we add a finite number of terms in the infinite series, using the first term alone, then the first two terms, then the first three, and so on. We call these sums the partial sums of the series. We found that these partial sums come closer and closer in value to the fraction $\frac{1}{3}$, approaching this value as a limit, so we assigned the value $\frac{1}{3}$ to the infinite decimal. In a similar way, we can assign a meaning to some other examples of infinite series. *If the partial sums of a series come closer and closer, in the long run, to some definite number, approaching this number as a limit, then we can assign this limiting number as the sum of the infinite series.*

For example, let us consider the series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ in which each term is half the size of the term that it follows. The partial sums are, $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, 1\frac{31}{32}, \ldots$. In this case, the partial sums come closer and closer to the number 2. The first sum differs from 2 by 1. The second sum differs from 2 by $\frac{1}{2}$. The third sum differs from 2 by $\frac{1}{4}$. Successive partial sums differ from 2 by smaller and smaller amounts, and the difference can be made as small as we please if we add up enough terms in the series. So the partial sums approach the number 2 as a limit. We therefore can assign the number 2 as the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$.

Let us try to use the same procedure with the series $1 + 1 + 1 + 1 + \ldots\ldots$ The partial sums for this series form the sequence, 1, 2, 3, 4, 5, 6, ... In this case, there is no number that the partial sums approach as a limit, because they keep increasing without limit. That is, we can get a partial sum to be larger than any number we wish by simply adding enough terms of the series. Because the partial sums do not approach a limit, we cannot assign any number as the sum of this series. So we see that not every infinite series has meaning as a sum. An infinite series has meaning as a sum only if the partial sums of the series approach a limit.

In that case we call the series a convergent series, and the sum of the series is the limit approached by the partial sums.

Now that we know that only some infinite series have meaning as a sum, the next logical question to ask is, "Which ones?" How do we recognize an infinite series that converges? We find that we can recognize some very easily by the fact that they define a nest of intervals. For example, consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$, in which the successive terms decrease toward 0 and the signs alternate between + and −. As we form the partial sums, let us locate them as points on the line of real numbers. The first partial sum is the number 1. Because the next term, $-\frac{1}{2}$, is negative, the second partial sum is to the left of the first. Adding the



third term, $\frac{1}{3}$, brings us back to the right, but not all the way back, because $\frac{1}{3}$ is smaller than $\frac{1}{2}$. As we add in more and more terms, the point representing the partial sum oscillates back and forth, left and right, but never again as far to the left as before, and never again as far to the right. Now consider the intervals that have successive partial sums as their endpoints. The first interval is bounded on the right by 1 and on the left by $1 - \frac{1}{2}$. The second interval is bounded on the left by $1 - \frac{1}{2}$, and on the right by $1 - \frac{1}{2} + \frac{1}{3}$. The third interval is bounded on the right by $1 - \frac{1}{2} + \frac{1}{3}$, and on the left by $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$. Each new interval is inside the one that it follows. Moreover, the width of the intervals is shrinking toward 0. The intervals form a nest, and, as we know, there is one and only one point inside the nest. The partial sums are crowding in toward this point as a limit, so the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is a convergent series.

A similar argument applies to any series in which the terms decrease toward zero, and the signs are alternately + and −. Every such series defines a nest of intervals, and converges to the single point that is inside the nest. This result, which we have established for the real number system, does not hold for the rational number system, because there it is not true that every nest of intervals has a point inside.

Another type of series that is easy to analyze is one in which all terms are positive, and the partial sums are bounded, that is, they are always less than some fixed number. The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is an example of this type, because all partial sums are less than 2. Let us represent the general series of this type by $a_1 + a_2 + a_3 + \ldots$. where the subscripts 1, 2, 3, etc. are labels to identify the position of each term in the series. The partial sums are a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, . . . Since we get each partial sum from the one that precedes it by adding a positive number, the partial sums form an increasing sequence. Let us represent the partial sums by the symbols $S_1, S_2, S_3, S_4, \ldots$ That is, $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and so on.

If we represent the partial sums as points on the line of real numbers, since the numbers are increasing we get a succession of points *moving gradually to the right*. S_2 is to the right of S_1, S_3 is to the right of S_2, and so on. However, since the sums are all less than some fixed number, they cannot move too far to the right. If the fixed number is represented by K on the number line, its position is an upper boundary beyond which the sequence of partial sums cannot go. Now we show how we can use K and the partial sums to define a nest of intervals. The points S_1 and K are the ends of an interval. This is the first interval of the nest. All the S points after S_1 lie to the right of S_1 and to the left of K, so they are inside this interval. Let us divide this interval in half. The points in the sequence of S's are moving to the right. Either they finally enter the second half of the interval, or they do not. If they do enter the second half, once they enter they stay there, because they keep moving to the right, and never

get past K. In that case we use this second half of the interval as the second interval of the nest. If the S points never enter the second half, that means they always remain in the first half. Then we use the first half as the second interval of the nest. Now we repeat the process. We divide the second interval of the nest in half, and pick one of the halves as the third interval of the nest. We choose the second half if the S's ultimately enter it. We choose the first half if they never enter the second half.

In this way we get a sequence of intervals with the characteristics of a nest: the intervals are one inside the other, and the width of the intervals is shrinking toward zero. Then there is a single real number that lies inside all the intervals of the nest. Since we know from the way in which we chose the intervals of this nest that the partial sums enter and remain in each of them, then the partial sums converge toward this single number as a limit. Therefore an infinite series of positive terms whose partial sums are bounded converges to a limit. This result can also be restated in terms of the sequence of partial sums alone, without reference to the series from which the sums were derived: Every increasing sequence of numbers that is bounded on the right converges to a limit.

The two types of convergent series just examined are only special cases. It is not difficult, however, to find a criterion by which all convergent series can be recognized. Suppose the series is represented by $a_1 + a_2 + a_3 \ldots$., where the terms may be either positive or negative. Whenever we form a partial sum, we are breaking the series up into two parts. The first part consists of a finite number of terms at the beginning of the series, taken in order, and added to get the partial sum. Let us call this part the head end of the series. The second part consists of all the remaining terms, that are not used to form this partial sum. Let us call this part the tail end of the series. We can designate the successive partial sums, formed from the head end as more and more terms are included in it, by S_1, S_2, S_3, etc. Let us call the corresponding tail ends T_1, T_2, T_3, etc. Then

$S_1 = a_1$, with tail end T_1: $a_2 + a_3 + a_4 \ldots$

$S_2 = a_1 + a_2$, with tail end T_2: $a_3 + a_4 + a_5 + \ldots$

$S_3 = a_1 + a_2 + a_3$, with tail end T_3: $a_4 + a_5 + a_6 + \ldots$

As we move along the sequence of partial sums, S_1, S_2, etc., we take one term at a time from the tail end and transfer it to the head end to be included in the partial sum. As a result, as n increases, the value of the partial sum, S_n, keeps changing. If the last term that is transferred is positive the change is an increase. If the term is negative, the change is a decrease. If we disregard the sign of the term we get a positive number that tells us the size of the change without regard to whether it is a decrease or increase. This positive number is called the absolute value of the change. Thus, an increase by $\frac{1}{2}$ or a decrease by $\frac{1}{2}$ both have an absolute value of $\frac{1}{2}$.

The series converges if the partial sums approach a limit. If the partial sums approach a limit, they become more and more nearly equal to that limit. This means that, as more and more terms are included in the partial sum, the sum changes less and less. And if n is taken large enough, the partial sum S_n is so close to the limit, that it changes very little no matter how many more terms are transferred from the tail end of the series to the head end. In fact, if n is large enough we can keep the absolute value of this change in the partial sum as small as we please. The partial sum S_n is equal to $a_1 + a_2 + \ldots + a_n$. The tail end T_n consists of the series $a_{n+1} + a_{n+2} + a_{n+3} + \ldots$. If we transfer the first p terms from the tail end to the head end, we add to S_n the sum $a_{n+1} + a_{n+2} + \ldots + a_{n+p}$. So, if the series converges, the absolute value of this sum must shrink toward zero as n increases, no matter how many terms from the tail end it includes. The converse is also true. If the absolute value of the sum of the first p terms of the tail end of the series shrinks to zero as n increases, no matter how large p is, then the series converges. This fact can be proved by showing that, under these conditions, there is a nest of intervals into which the partial sums crowd, so that they converge on the

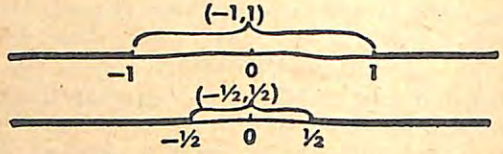
single point that is inside the nest. This criterion for a convergent series is known as the Cauchy criterion, and may be summed up somewhat carelessly in these words: in the real number system, an infinite series converges to a limit if and only if it has a shrinking tail end.

Limit Points and Neighborhoods

So far we have passed through four stages in the enlargement of our number system. We started with the system of natural numbers. Then by successive extensions, we obtained the integers, the rational numbers, and the real numbers. This sequence of extensions, besides showing us how our notion of number has been evolving, also served to introduce us to a variety of mathematical structures. In the natural number system, we first encountered the structure which we have labeled a "*number system*" and which is distinguished by its obedience to the five laws. In the system of integers we found an example of both a *group* and a *ring*. The system of rational numbers was our first example of a *field*. Now, in the system of real numbers, we shall get acquainted for the first time with another type of structure, a *topological space*. A real number is an infinite decimal, and an infinite decimal is an infinite sequence of finite decimals. This fact compelled us to look into the question of when an infinite sequence converges to a limit. Now, as we examine more closely the notion of a limit, it will lead us to the concept of a topological space.

Suppose we look at the sequence of numbers, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, in which the nth number is $\frac{1}{n}$. As n increases, these numbers become smaller and smaller and approach 0 as a limit. If we represent the numbers in the sequence as points on a line, the fact that they approach 0 as a limit shows up in the fact that the points crowd in toward the zero point. This crowding can be described without any reference to the order in which the numbers are arranged in the sequence. In fact, let us discard the sequence and merely think of the numbers as the set of points on the line. Let us give this particular set the name A. To define in what sense the points in the set A crowd in toward the zero point, we first introduce the notion of a neighborhood of a point.

A neighborhood of a point consists of all the points that surround it and are close to it. To make this concept more precise, we have to indicate what we mean by *close.* So we specify degrees of closeness, and each such specification defines a particular neighborhood. For example, we define a neighborhood of zero when we say it consists of all the points whose distance from zero is less than 1. This neighborhood includes all points that are larger than (to the right of) -1 but less than (to the left of) $+1$. In the diagram, we see this neighborhood as the interval between -1 and $+1$, not including its endpoints. We represent this neighborhood



Neighborhoods of zero

by the symbol $(-1, 1)$, in which its boundary points are shown. We define another neighborhood of zero when we specify that it consists of all points whose distance from zero is less than $\frac{1}{2}$. This is a smaller neighborhood than the first one, and is included within it. All of its points lie between $-\frac{1}{2}$ and $+\frac{1}{2}$. We can represent it by the symbol $(-\frac{1}{2}, \frac{1}{2})$.

By picking any positive distance d, we can define a neighborhood of zero as the set of points lying between $-d$ and $+d$, and we represent it by the symbol $(-d, d)$. So we see that the zero point is surrounded by a multitude of neighborhoods of many sizes.

Now we can say exactly what is meant by the fact that the points in the set A $\left(A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\right\}\right)$ crowd in toward zero: If we pick any neighborhood of zero, no matter how small, all but a finite number of these points lie

within that neighborhood. If we choose smaller and smaller neighborhoods, we can exclude some of the points of the set A from the neighborhood. But, no matter how many we exclude, all but a finite number of the points are still inside. This is the meaning of the fact that 0 is the limit point of the set A.

Now let us examine a set in which we find crowding of a different kind. Let B stand for the set

$$\left\{1+\tfrac{1}{2}, 1+\tfrac{1}{3}, 2-\tfrac{1}{3}, 1+\tfrac{1}{4}, 2-\tfrac{1}{4}, \ldots, 1+\frac{1}{n}, 2-\frac{1}{n}, \ldots\right\}.$$

In this set, we find the points crowding around two points of the line, 1, and 2. Here we cannot say that every neighborhood of the point 1 includes all but a finite number of points in the set. In fact, while the neighborhood $(\frac{1}{2}, 1\frac{1}{2})$ includes an infinite number of points of the set B, it also excludes an infinite number, because there are multitudes of them clustering around the point 2. To distinguish this case from the preceding case, we call the points 1 and 2 *cluster points* of the set B. A point is a cluster point of a set of points if every neighborhood of the point includes an infinite number of points of the set. As we see, a set may have more than one cluster point. If a set is confined within a finite interval, and has only one cluster point, then that single cluster point is the limit point of the set.

As a contrast to the behavior of a limit point or a cluster point, let us examine the point -1 in relation to the set A. Surround the point -1 by the neighborhood that extends from $-1\frac{1}{2}$ to $-\frac{1}{2}$. Within this neighborhood, represented by the symbol $(-1\frac{1}{2}, -\frac{1}{2})$, there are no points of the set A. That is, the neighborhood $(-1\frac{1}{2}, -\frac{1}{2})$ excludes all points of the set A. On the other hand, there is no neighborhood of the point 0 that excludes all points of the set A. Similarly, there is no neighborhood of the point 1 that excludes all points of the set B, and there is no neighborhood of the point 2 that excludes all points of the set B. Because they have this property, a limit point and a cluster point are both examples of what we call a point of adherence of a set.

A point is called a *point of adherence* of a set of points if no neighborhood of the point excludes all points of the set. A point need not be a limit point or a cluster point of a set in order to be a point of adherence of the set. For example, the point $1\frac{1}{2}$ is not a cluster point of the set B, but it is a point of adherence. No neighborhood of $1\frac{1}{2}$ can exclude all points of the set B because the point $1\frac{1}{2}$ is itself a member of the set. The points of adherence of a set include all members of the set as well as cluster points that are not in the set. They are the points that cling to a set by either being in it or being crowded by it.

Closed Sets and Open Sets

With the help of the concept of point of adherence, we now define two special kinds of sets of points on the line. A set that contains all of its points of adherence is called a *closed set*. The set A, discussed in the paragraphs above, is not a closed set, because the point 0 is a point of adherence of the set but does not belong to it. However, if we enlarge the set by including 0 as a member, then the enlarged set is closed. Similarly, the set B is not closed. But the enlarged set formed by uniting B with the set $\{1, 2\}$ is closed. Another example of a closed set is the set of points between 0 and 1, *including the points 0 and 1*. Such a set is called a *closed interval*. The set which contains *all* the points on the line of real numbers is also a closed set. We can count the empty set as a closed set, too. It certainly includes all its points of adherence, because there aren't any.

If we delete from the real number line all the members of some closed set, what is left is called an *open set*. Using the terminology defined in Chapter II, we can say that an open set is the complement of a closed set. To see what an open set is like, we have to think of it in relation to the closed set which is its complement. Suppose S is an open set, and C is the closed set which is its complement. Any point of the open set S is not in the closed set C. Therefore any point in the open set S is not a point of adherence of C. (If it were, it would have to be in C, by the definition of a closed set.)

But if a point is not a point of adherence of C, then some neighborhood of the point contains no points of C. This neighborhood, which contains no points of C, must then be part of the open set S, which consists of all the points that are not in C. So we have discovered that every point of an open set is surrounded by an entire neighborhood that is also in the open set. This is a distinguishing feature of open sets on a line.

A neighborhood is itself an example of an open set. For example, the set of points between 0 and 1, *not including 0 and 1*, is an open set. It is called an *open interval* to distinguish it from the closed interval which does include both end points. Another example of an open set is the set formed by uniting into one set all points in any collection of open intervals. The whole line of real numbers is also an open set, because it is the complement of the empty set, which is a closed set. The empty set, too, is an open set, because it is the complement of the whole line, which is a closed set. We see then that the empty set and the whole line are both open and closed.

There are some sets that are neither open nor closed. For example, the set of points between 0 and 1, including the endpoint 0, but not including the endpoint 1, is neither open nor closed. It is not closed, because 1 is a point of adherence of the set, but doesn't belong to it. It is not open, because 0 is a point of adherence of the complement of the set, but does not belong to the complement, so the complement is not closed.

The collection of all open sets on the line has the following properties, some of which we have already noted:

1) The whole line, as well as the empty set, are open sets.
2) The union of any number of open sets is also an open set.
3) The set of points common to two open sets (the intersection of the two sets) is an open set.

Neighbors Make a Neighborhood

The significance of the concept of an open set is that it permits a generalization of the concept of neighborhood.

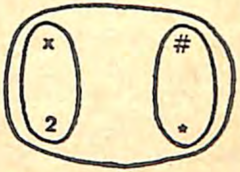
We have already seen that a neighborhood is an open set. Let us agree to extend the word neighborhood to include every open set. The effect of this generalization is to separate the notion of neighborhood from the idea of distance. Then a neighborhood becomes simply a collection of neighbors, with the characteristics of an open set. From this point of view, the whole line may be viewed as a system of interlocking neighborhoods or open sets. The interlocking neighborhoods on the line determine what is called its topological structure, and make it a topological space. They fix a pattern of relationships within the space just as the interlocking stitches in a sweater fix the pattern of the sweater.

In general, any set of objects is called a topological space if a collection of its subsets are singled out so that the collection has the three properties we found in the open sets on the line: 1) The whole space and the empty set belong to the collection; 2) The union of any number of sets in the collection is also in the collection; 3) The intersection of any two sets in the collection is also in the collection. When these three conditions are satisfied, the sets in the collection are called the "open sets" of the "space."

Under this definition, any collection of objects can be converted into a topological space, usually in more than one way. For example, let us consider the set $\{x, 2, \#, *\}$. Its elements were chosen arbitrarily, so they have no relationship to each other beyond the fact that they happen to have been thrown together in the same set. The set is a loose aggregation, and has no structure. However, the set acquires a structure, and the elements become related to each other as neighbors, as soon as we single out certain subsets that will be called neighborhoods or open sets.

For example, we might specify that these four sets should constitute the collection of open sets: $\{x, 2, \#, *\}$, $\{x, 2\}$, $\{\#, *\}$, and $\{\ \}$. This collection satisfies the three requirements listed above. 1) The whole space, $\{x, 2, \#, *\}$, and the empty set, $\{\ \}$, are members of the collection. 2) The union of any number of sets in the collection is also in the collection. For example, the union of $\{x, 2\}$ and $\{\#, *\}$ is $\{x, 2, \#, *\}$,

which is in the collection. 3) The intersection of any two sets in the collection is also in the collection. For example, the intersection of $\{x, 2\}$ and $\{\#, *\}$ is the empty set $\{\ \}$, which is in the collection. Since the requirements are met, this collection of "open sets" defines a topological structure for the set $\{x, 2, \#, *\}$ and converts it into a topological space. The way the topological structure relates the elements of the space to each other as members of neighborhoods is indicated in the diagram below, where each open set is represented by a loop enclosing its members.

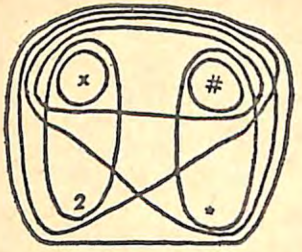


We can give the same set another, different topological structure by picking out other subsets to use as "open sets." We might, for example, decide that we want the collection of open sets to include all those listed above, and in addition the set $\{x, \#\}$. The inclusion of this one additional set among the open sets compels us to include more sets, in order to meet the three requirements. The union of $\{x, \#\}$ and $\{x, 2\}$ is $\{x, 2, \#\}$. To meet requirement 2), we must include it among the open sets. The intersection of $\{x, \#\}$ and $\{x, 2\}$ is the set $\{x\}$. To meet requirement 3), we must classify it as an open set. For similar reasons, we have to include $\{x, \#, *\}$ and $\{\#\}$ among the open sets. We find that all three requirements are met now by the enlarged collection consisting of these nine sets:

$$\{x, 2, \#, *\},\ \{x, 2\},\ \{\#, *\},\ \{\ \},$$
$$\{x, \#\},\ \{x, 2, \#\},\ \{x, \#, *\},\ \{x\},\ \{\#\}.$$

The enlarged collection of open sets gives the space a more complicated structure of interlocking neighborhoods, as shown in the diagram on page 122.

A third topological structure can be defined for the same



set by specifying that *every* subset shall belong to the collection of open sets. This definition meets the three requirements, because the whole set and the empty set are both subsets; the union of any number of subsets is a subset; and the intersection of any two subsets is a subset. Under this definition, the set of four elements becomes a topological space containing sixteen open sets, including the empty set. The three topological structures we have defined convert the same set of four elements into three different topological spaces. They are different as spaces because, in each of the topological structures, the elements hang together differently as members of interlocking neighborhoods.

Here is another example of a topological space. Consider all the straight lines that can be drawn in a plane parallel to some fixed direction. We can convert this set of lines into a topological space, in which each of the lines is a "point" of the space, by defining for the space a structure of interlocking open sets. Let an open set consist of a set of lines between any two lines, or a union of any number of such sets. In this space, a set of lines between any two lines is analogous to what we called an open interval in the real number system, namely, a set of points between any two points.

A more familiar looking example of a topological space is the circumference of a circle, in which arcs play the same role that intervals do on a straight line. On a circle, there are two arcs joining any two of its points. The interior (excluding the endpoints) of the arc may serve as an "open" arc just as the interior of a segment serves as an open interval on a straight line. An open set on the circle can then

be defined as the union of any number of such open arcs. With this definition, the circle is furnished with a topological structure that meets the three requirements listed on page 120. This topological structure on a circle can also be derived from the topological structure on a straight line in which each open set is a union of open intervals. Take a segment of a straight line, and loop it to make a circle by joining the two ends together. Then open intervals on the straight line become open arcs on the circle.

By isolating the topological structure of the real number system for separate study, mathematicians have obtained a deeper insight into the characteristics of that system. They could separate from each other those properties that the system has by virtue of the fact that it is a topological space, those that it has by virtue of the fact that it is a field, and those that it has because it happens to be both. The separate study of topological structures also develops a body of knowledge that is applicable to other topological spaces, no matter what the elements of these spaces may be.

Rubber-Sheet Geometry

Now that we have introduced the notion of a topological space, it is relevant to ask, "When are two topological spaces essentially the same?" The problem is analogous to the problem we encountered when we were talking about groups, rings, or fields, and has a similar answer. We said that two fields are essentially the same, or are isomorphic, if there is a one-to-one correspondence between them that preserves the field structure embodied in the operations of addition and multiplication. Similarly, we say that two topological spaces are essentially the same, or are *homeomorphic*, if there is a one-to-one correspondence between them that preserves the topological structure embodied in the system of interlocking open sets. That is, two topological spaces are homeomorphic if, under a reversible mapping which establishes a one-to-one correspondence between them, an open set in either space has as its image an open set in the other. For example, imagine the circumference of a circle being stretched like a rubber band, into a distorted shape. Each point of the circle may take up a new position. Distances between points change, but the interlocking system of open sets remains. As a result the distorted curve obtained is homeomorphic to the original circle. Because the topological structure of a space is not changed when it is stretched without tearing, the study of topological structure has sometimes been referred to as "rubber-sheet geometry."

DO IT YOURSELF

1. By an argument similar to the one used on page 94, prove that there is no rational number equal to the square root of 3.
2. Find the repeating decimal that represents the rational number $\frac{2}{13}$.
3. What rational number is represented by the repeating decimal .181818. . . . ?
4. Define a topological structure for the set of five elements $\{a, e, i, o, u\}$ by designating some but not all of the subsets as "open" sets. (See example on page 120.) Be sure your open sets meet the requirements for a topological structure listed on page 120. Which sets are closed sets (complements of open sets) in this structure?
5. Every even integer has the form $2k$, and every odd integer has the form $2k + 1$, where k is an integer. Use this notation to prove that the square of every even integer is an even integer, and the square of every odd integer is an odd integer.
6. Find the repeating decimal that represents
   a) $\frac{1}{7}$ b) $\frac{3}{7}$ c) $\frac{5}{11}$ d) $\frac{2}{17}$ e) $\frac{5}{12}$
7. Find the rational number represented by the repeating decimal:

a) .555. . .	b) .777. . .	c) .060606. . .
d) .050505. . .	e) .121212. . .	f) .161616. . .
g) .123123123. . .	h) .025025025. . .	i) 6.2575757. . .
j) 3.2777. . .		

CHAPTER VI

Spilling Into the Plane

IN THE system of rational numbers we were able to solve the equation $x^2 - 1 = 0$, but we were unable to solve the equation $x^2 - 2 = 0$. We remedied this defect by constructing a larger number system, the system of real numbers, in which the equation $x^2 - 2 = 0$ does have a solution. In fact, it has two solutions in the real number system, viz., $+\sqrt{2}$, and $-\sqrt{2}$. But the real number system has some defects of its own. For example, although the equation $x^2 + 1 = 0$ does not look any more complicated than the other two equations mentioned above, it has no solutions in the real number system. We can see why this is so, by restating the question that the equation asks us. First we add -1 to both sides of the equation, and we get the equivalent equation $x^2 = -1$. This equation says, "Find a number which, when multiplied by itself, gives -1 as the product." The real number 0 does not meet the requirement, because 0 multiplied by itself gives 0 as the product. A positive real number can not meet the requirement, because a positive number multiplied by itself gives a positive number as the product. A negative real number cannot answer the question either, because a negative number multiplied by itself also gives a positive product. For example $(-1) \cdot (-1) = +1$. So there is no real number that can satisfy the equation $x^2 + 1 = 0$, or $x^2 = -1$. Our next goal is to construct an expanded number system in which this equation does have a solution.

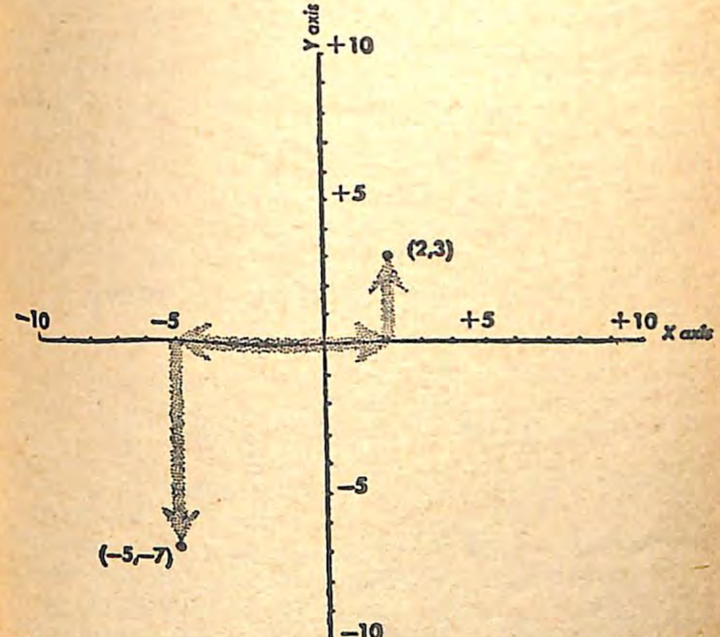
At each stage in the expansion of the number system so far we have represented numbers pictorially as points on a line. With the construction of the real number system, we finally achieved our purpose of having a number for every point on the line. Another possible goal for the next step in the expansion of the number system would be to spill over into the plane, by constructing a number system that gives us a number for every point in the plane. It turns out that these two goals coincide. We shall reach the goal of finding a solution to the equation $x^2 + 1 = 0$ by building a number system that supplies a number for every point in the plane.

Number Pairs and Arrows

The construction of a number system that has a number for every point in the plane will be carried out in two installments. We shall do only half of the job in this chapter, by creating an appropriate system of elements, and defining an addition operation on these elements. We shall finish the job in Chapter VIII when we define a multiplication operation for these elements. The elements we shall use are ordered pairs of real numbers, like $(0, 1)$, $(-2, 3\frac{1}{2})$, or $(1, \sqrt{2})$. We assign such a pair to each point in a plane by the familiar method used in elementary algebra to identify the co-ordinates of a point on a graph. First we draw in the plane a straight line called the x axis. We furnish it with a scale on which measurements can be made by assigning a real number to each point on the line, in the manner described in the preceding chapters. Then we draw another line passing through the 0 of the x axis, and crossing the axis at right angles. This new line is called the y axis. The point where the axes cross is called the *origin*. We put a scale on the y axis, too, using the origin as its zero point, and putting the positive numbers on the upper half of the axis.

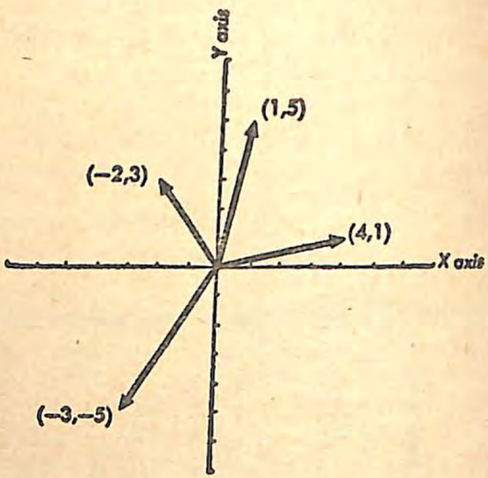
Now we assign a pair of numbers to each point in the plane in the following way. Drop a perpendicular from a given point to the x axis. We can reach the given point from the origin by moving toward it in two steps. First move from the origin to the foot of the perpendicular. The distance moved supplies the first number of the ordered

pair. It is a positive number if the motion is to the right of the origin. It is a negative number if the motion is to the left of the origin. Now, from the foot of the perpendicular, move along the perpendicular to the given point. The distance moved in this second step supplies the second number of the ordered pair. It is a positive number if the motion is up from the x axis. It is a negative number if



the motion is down from the x axis. The pair of numbers is written inside parentheses. The first number in the pair is called its x component. The second number is called its y component. The ordered pairs that belong to some points in the plane are shown in the diagram below. Notice that the ordered pair $(0, 0)$ belongs to the origin. For all points on the x axis, the y component is zero. For all points on the y axis, the x component is zero. We are going to convert the system of ordered pairs of real numbers into a number system in its own right by giving it the necessary structure.

The point to which an ordered pair is assigned may be thought of as a picture of the pair. We can also associate another kind of picture with each pair in the following way. Draw an arrow from the origin to the point, with the arrowhead pointing away from the origin. We may think of the arrow as the second picture of the ordered pair. Each

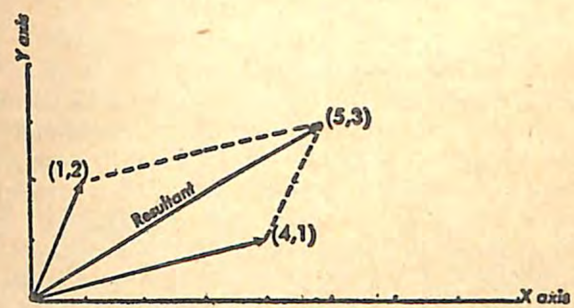


ordered pair, except (0, 0) has an arrow of its own. The arrow has a definite length, and a definite direction. We can even assign an arrow to the pair (0, 0) by giving it an arrow whose length is zero. However, we cannot assign any direction to this zero arrow.

Addition of Arrows

The representation of ordered pairs as arrows is very helpful, because it guides us to an appropriate definition of addition for our system of ordered pairs. There are many practical situations where we encounter just such arrows,

and where there is a natural kind of addition that takes place. For example, in physics, a force can be represented as an arrow. The length of the arrow indicates the strength of the force, and the direction of the arrow indicates the direction of the force. If two forces act on a body at the same point, the effect is the same as though the body were acted on by a single force called the *sum* or *resultant* of the two forces. One way to find this sum or resultant is to draw the two forces as arrows at the origin, and then complete the parallelogram that has the two arrows as sides. The diagonal of the parallelogram that can be drawn from the origin is the resultant of the two forces.



However, there is another, simpler way of getting the resultant, too. Each force, considered as an arrow, has an x component and a y component. To add the two forces, all we have to do is add their x components separately, and add their y components separately. For example, the forces being added in the diagram above can be represented as arrows, or as the ordered pairs, (4, 1), and (1, 2). The x components are 4 and 1, and the y components are 1 and 2. Adding them separately, we find that the resultant or sum of the forces belongs to the ordered pair (5, 3). This example suggests how we should define addition of ordered pairs, if we want them to be useful for the solution of practical problems. We therefore give the following definition of addition in the new system: To add two ordered pairs, add their components separately. In symbols, this definition says, $(a, b) + (c, d) = (a + c, b + d)$. Because the sum of two ordered pairs is also an ordered pair, this addition is a binary operation.

There are several facts that we can observe about this binary operation immediately. First, it is associative. That is, $[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$. This follows from the fact that we carry out the addition by adding the components, which are real numbers, and addition of real numbers is associative. For example,

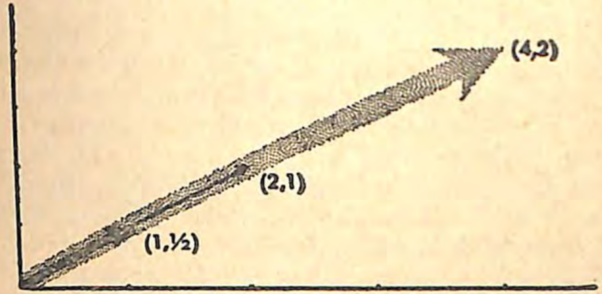
$$[(2, 3) + (3, 1)] + (4, 8) = (5, 4) + (4, 8) = (9, 12).$$

But $(2, 3) + [(3, 1) + (4, 8)] = (2, 3) + (7, 9) = (9, 12)$, too. Secondly, the addition of ordered pairs is commutative. That is $(a, b) + (c, d) = (c, d) + (a, b)$. This follows from the fact that addition of real numbers is commutative. For example, $(2, 3) + (3, 1)$ and $(3, 1) + (2, 3)$ both yield the same sum, $(5, 4)$. Third, the system has a zero element, because $(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b)$. Finally, for each element in the system, there is a negative in the system, too, with the usual property that a negative is supposed to have. That is, the sum of any ordered pair and its negative is equal to the zero element. The negative of any ordered pair (a, b) is the ordered pair $(-a, -b)$. For example, the negative of $(2, 3)$ is $(-2, -3)$ because $(2, 3) + (-2, -3) = (0, 0)$. With these four characteristics, the system of number pairs we have constructed meets all the requirements for being an abelian group, with the group operation denoted by the plus sign.

With this observation, we complete the first half of the job of converting these elements into a number system. We shall go on with the second half of the job in Chapter VIII. Meanwhile we take a detour along a road pointed out by our system of arrows. As we follow this road we shall encounter some more mathematical structures that play an important part in modern mathematics.

Stretching the Arrows

In the diagram below we see an arrow representing the ordered pair (2, 1). If the arrow is stretched so that its length is doubled while its direction remains unchanged, we get a new arrow (the shaded arrow in the diagram). If the arrow is contracted to half its original length we also get an arrow pointing in the same direction. In the first case, the length of the arrow was multiplied by a factor of 2. In the second case it was multiplied by a factor of $\frac{1}{2}$. Because of this fact, it is natural to think of the operation of stretching or shrinking an arrow as a kind of multiplication of the arrow by a real number. To see how we may



define the operation in terms of ordered pairs, notice that when the arrow (2, 1) was doubled, each of its components was doubled, giving the arrow (4, 2) as the result. So we define a special kind of multiplication for our system of ordered pairs as follows. To multiply an ordered pair of real numbers by a real number, multiply each of its components separately by that real number. In symbols, the definition says $a \cdot (b, c) = (a \cdot b, a \cdot c)$. It is important to notice that this kind of multiplication differs in one important respect from all the other multiplications defined so far in this book. In each of the earlier cases, the multiplication was an operation on two elements drawn from the same system. In this case, the two elements being multiplied are drawn from *two different systems.* One of the multipliers is drawn from the field of real numbers. The other multiplier is drawn from the abelian group consisting of *ordered pairs* of real numbers. In order to sharpen the distinction between these two systems, we introduce some special names for them. We call the elements of the real number field *scalars*. The ordered pairs, or their pictorial representations as points in the plane or as arrows, will be called *vectors*. The kind of multiplication in which a vector is multiplied by a scalar to form a product which is also a vector is called *scalar multiplication*. We can verify from our definitions that scalar multiplication is *distributive with respect to vector addition*. That is, if s is a scalar, and (a, b) and (c, d) are vectors, then $s \cdot [(a, b) + (c, d)] = s \cdot (a, b) + s \cdot (c, d)$. To prove this rule, observe first that the expression on the left instructs us to do the vector addition first, and the scalar multiplication afterwards. So $s \cdot [(a, b) + (c, d)] = s \cdot (a + c, b + d) = (s \cdot [a + c], s \cdot [b + d]) = (s \cdot a + s \cdot c, s \cdot b + s \cdot d)$. The expression on the right instructs us to do two scalar multiplications first, and then add the results by vector addition. Following these instructions, we find that $s \cdot (a, b) + s \cdot (c, d) = (s \cdot a, s \cdot b) + (s \cdot c, s \cdot d) = (s \cdot a + s \cdot c, s \cdot b + s \cdot d)$. The proof is completed by observing that the two results are the same.

Scalar multiplication also satisfies another distributive law. It is distributive with respect to the addition of scalars. That is, if s and t are scalars, and (a, b) is a vector, then $(s + t) \cdot (a, b) = s \cdot (a, b) + t \cdot (a, b)$. To prove this rule, we observe that carrying out the indicated operations on both sides of the equals sign leads to the same result: $(s + t) \cdot (a, b) = ([s + t] \cdot a, [s + t] \cdot b) = (s \cdot a + t \cdot a, s \cdot b + t \cdot b)$. $s \cdot (a, b) + t \cdot (a, b) = (s \cdot a, s \cdot b) + (t \cdot a, t \cdot b) = (s \cdot a + t \cdot a, s \cdot b + t \cdot b)$.

Scalar multiplication also obeys a *mixed associative law*, in which two types of multiplication appear: 1) scalar multiplication, in which a scalar multiplies a vector, and 2) multiplication of scalars, in which a scalar multiplies a scalar. This law says that, if s and t are scalars, and (a, b)

is a vector, then $(s \cdot t) \cdot (a, b) = s \cdot [t \cdot (a, b)]$. To prove this law, observe that $(s \cdot t) \cdot (a, b) = ([s \cdot t] \cdot a, [s \cdot t] \cdot b) = (s \cdot t \cdot a, s \cdot t \cdot b)$. But $s \cdot [t \cdot (a, b)] = s \cdot (t \cdot a, t \cdot b) = (s \cdot [t \cdot a], s \cdot [t \cdot b]) = (s \cdot t \cdot a, s \cdot t \cdot b)$.

Scalar multiplication also has the property that the number 1, which is the unity element for multiplication of a scalar times a scalar, is also the unity element for multiplication of a scalar times a vector. This is seen from the fact that, if (a, b) is a vector, $1 \cdot (a, b) = (1 \cdot a, 1 \cdot b) = (a, b)$.

We shall now summarize these properties of scalar multiplication in a new abbreviated notation. So far we have always written a vector as an ordered pair, in which its two components are put on display. In the abbreviated notation, we represent a vector by a single symbol, with a little arrow over it to remind us that it stands for a vector. Thus, $\vec{x}$, $\vec{y}$, and $\vec{z}$ represent vectors. Symbols like r, s, and t, written without arrows, will represent scalars. In this notation, the properties of scalar multiplication are expressed in this form:

I. Distributive laws: $r \cdot (\vec{x} + \vec{y}) = r \cdot \vec{x} + r \cdot \vec{y}$

$$(r + s) \cdot \vec{x} = r \cdot \vec{x} + s \cdot \vec{x}$$

II. Mixed Associative law: $r \cdot (s \cdot \vec{x}) = (r \cdot s) \cdot \vec{x}$

III. Unity element: $1 \cdot \vec{x} = \vec{x}$

Because of the characteristics we have observed in the system of ordered pairs (pictured as arrows), this system serves as an example of the kind of mathematical structure that is called *a vector space*. The name vector space is used to describe any system of elements that has these properties: 1. It is an abelian group. 2. There is associated with this group another system of elements that is a field, and is called the scalar field. A scalar multiplication exists, in which a scalar from the field multiplies a vector from the

vector space, and the product is a vector in the vector space.
3. The scalar multiplication has properties I, II, and III listed above.

Some Other Vector Spaces

Vector spaces won recognition as a special kind of structure worthy of separate study when mathematicians realized that there are many systems that have this kind of structure. In the example that we have been observing, each element in the vector space was an ordered pair of real numbers. We can construct another vector space, by using as elements ordered triples instead, like (1, 3, −2). In this system, each element has three components. We could define vector addition and scalar multiplication for this system in the same way that we did for the system of ordered pairs: To add two vectors, add their separate components separately; to multiply by a scalar, multiply each component separately by that scalar. For example, $(2, 3, -5) + (1, 4, 2) = (3, 7, -3)$. And $2 \cdot (2, 3, -5) = (4, 6, -10)$. With these definitions of vector addition and scalar multiplication, the system of triples meets all the requirements for being a vector space. The system of ordered pairs is an example of a vector space of two dimensions, and is represented pictorially by points in a plane. The system of ordered triples is an example of a vector space of three dimensions, and can be represented pictorially by the points in three dimensional space. By using ordered quadruples as elements, we can construct a vector space of four dimensions. In general, we can construct a vector space of n dimensions by using as elements ordered sets of real numbers with n numbers in each set. If we take $n = 1$, the vectors are single real numbers, and the vector space is one dimensional. In other words, the real number system may be thought of as a vector space of one dimension whose associated scalar field is also the real number system.

Another familiar system that has the structure of a vector space is the system of all polynomials whose terms

are powers of x multiplied by some real number. Typical elements in this system look like this:

$$5x^4 + 3x^2 - 2x + 7$$

$$\sqrt{2}\, x^2 - 3x + 9$$

The two polynomials shown here are of the fourth and second degree respectively. There are also polynomials that do not contain x at all. They are just ordinary real numbers like 5, 7, −4, $\sqrt{3}$, and so on. We call them polynomials of zero degree. In this system, each polynomial is a vector. The real number system is the associated field of scalars. Vector addition is carried out by adding polynomials according to the rules taught in elementary algebra. The zero element for this addition is 0, a polynomial of no degree. Scalar multiplication is carried out by multiplying a polynomial by a real number according to the rules taught in elementary algebra. With these two operations, the system of polynomials meets all the requirements for being a vector space. In this vector space, each distinct power of x is a separate component. Any one polynomial contains only a finite number of components. But, since any positive power of x, like x^9, or x^{187}, may be used in a polynomial, there is an infinite number of components to choose from in constructing a polynomial. Therefore this system is an example of a vector space of *infinite dimension.*

The *i, j, k* Notation

In the study of physics it is found that such things as forces, velocities, and rotations can be represented as vectors in a three dimensional vector space. To carry out computations in this vector space the physicist uses a special notation in which all vectors are expressed in terms of three unit vectors called i, j, and k. They are defined as follows: $i = (1, 0, 0)$; $j = (0, 1, 0)$; $k = (0, 0, 1)$. The vector (2, 3, 5) can be expressed in terms of these units in this way: $(2, 3, 5) = (2, 0, 0) + (0, 3, 0) + (0, 0, 5) = 2 \cdot (1, 0, 0) + 3 \cdot (0, 1, 0) + 5 \cdot (0, 0, 1) = 2i + 3j + 5k$.

In general, the vector $(a, b, c) = ai + bj + ck$. When this notation is used, all steps in vector addition and scalar multiplication become simple exercises in elementary high school algebra. For example:
Vector addition:

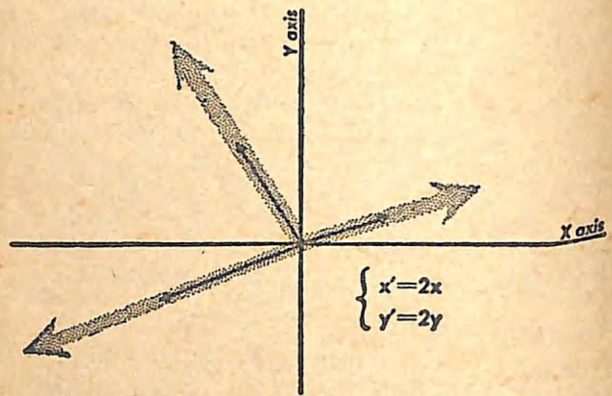
$$(2i + 3j + 5k) + (3i - 4j + 2k) = 5i - j + 7k$$

Scalar multiplication:

$$2(2i + 3j + 5k) = 4i + 6j + 10k$$

Mapping the Plane Into Itself

On page 131, we saw that multiplying the vector $(2, 1)$ by the scalar 2 has the effect of doubling the length of the associated arrow without changing its direction. If we multiply each of the vectors in the plane by 2, every one of the

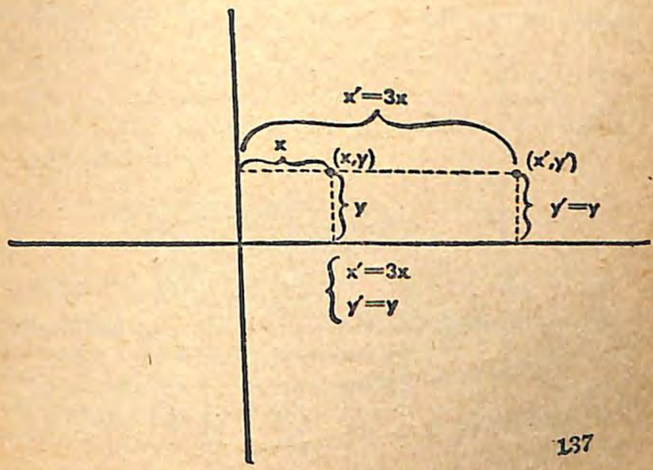


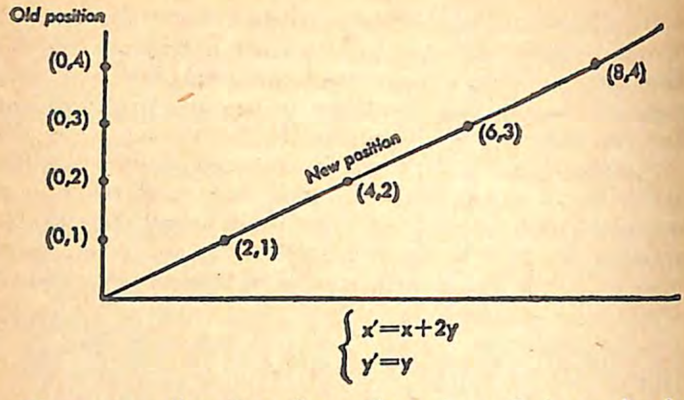
Solid arrow represents (x,y)
Shaded arrow represents (x',y')

associated arrows has its length doubled. The effect of the multiplication is to stretch the entire plane uniformly in all directions, so that each point of the plane is pulled to a new position, twice as far from the origin as it was before. We can represent a typical vector by the symbol (x, y), where x is its x component, and y is its y component. As a

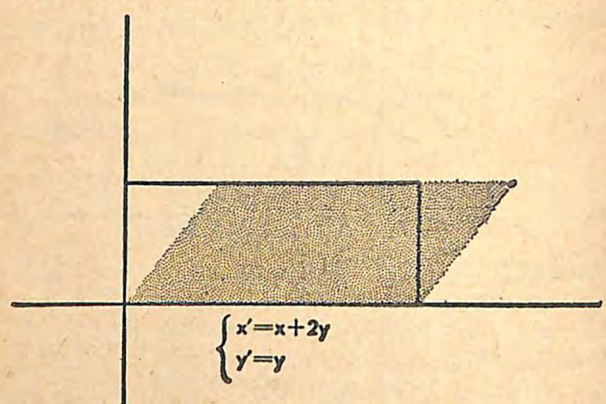
result of the stretching, the vector (x, y) is changed into the vector $(2x, 2y)$. If we call the components of the new vector x' and y', the relationship between the old components and the new components is expressed in the equations: $x' = 2x$, and $y' = 2y$.

The stretching of the plane is an example of a mapping of the vector space into itself, whereby each element in the vector space is mapped into a particular image. Such a mapping is called a *transformation*. The equations above serve to define the transformation precisely, by giving directions for calculating, from the components of any vector, the components of the image into which it is mapped. If the stretching of the plane is one in which each arrow has its length tripled, the equations are $x' = 3x$ and $y' = 3y$. If we contract the plane, to shrink each arrow to half its size, the equations are $x' = \frac{1}{2}x$, and $y' = \frac{1}{2}y$. If we reverse the direction of each arrow in the vector space, the equations are $x' = -x$, and $y' = -y$. Such a reversal of direction is called a reflection. In general, equations of the form $x' = kx$ and $y' = ky$, where k is some fixed real number, define a stretching or contraction of all vectors, with or without a reversal of direction.





There are other transformations, too, that can be described pictorially in terms of motions or deformations of the plane. Each is defined precisely by an associated pair of equations that shows how the new components (of the image vectors under the mapping) are computed from the old components. The equations $x' = 3x$, $y' = y$ define a transformation which stretches the plane horizontally, while leaving each point as close to the x axis as it was before. The equations $x' = x + 2y$, $y' = y$ define a transformation which also leaves points at the same height above



or below the x axis, while moving them horizontally through varying distances that depend on their height. As a result this mapping moves the points of the y axis into new positions on a sloping line, as shown in the first diagram. The effect is a shearing of the plane, which deforms rectangles into parallelograms, as shown in the second diagram. The equations $x' = x \cos\theta - y \sin\theta$, $y' = x \sin\theta + y \cos\theta$ define a rotation of the plane around the origin as center. All these mappings are special examples of a particular family of transformations whose equations have this simple form:

$$x' = a_1x + a_2y$$
$$y' = b_1x + b_2y$$

Mappings that can be written in this form, where a_1, a_2, b_1, b_2 may be any fixed real numbers, are called *linear mappings.*

Another kind of mapping is obtained when the entire plane is moved in the direction of some particular vector through a distance equal to the length of that vector. Such a mapping is called a *translation*. For example, the equations $x' = x + 2$, $y' = y + 3$ define a translation that moves every point in the plane in the direction and through the distance specified by the vector $(2, 3)$. In general, equations of the form $x' = x + h$, $y' = y + k$, where h and k are fixed real numbers, represent translations of the plane. A more complicated type of transformation is obtained when a linear mapping is followed by a translation. Then the equations look like this:

$$x' = a_1x + a_2y + h$$
$$y' = b_1x + b_2y + k$$

Reversible Mappings

The mapping defined by the equations $x' = x + 2$, $y' = y + 3$ moves every point in the plane two units to the right and three units up. This mapping is easily reversed. If, after the mapping, each point is moved two units to the left and three units down, the points all return to their original positions. The reversibility of the mapping can be expressed in terms of the equations, too. The equations show how to calculate the new components x' and y' from the old components x and y. If we solve the equations for x and y, we get equations in this form: $x = x' - 2$, $y = y' - 3$. These equations give directions for calculating the old components from the new, so they are the equations of the reverse mapping. In this reverse mapping, considered as a mapping in its own right, x' and y' are the components of a vector in its initial position, and x and y are the components in the final position. To conform to the original notation, in which the primed symbols are used to represent the new position, we represent the reverse mapping in this form: $x' = x - 2$, $y' = y - 3$. In general, for every translation, represented by the equations $x' = x + h$, $y' = y + k$ there is a reverse translation whose equations are $x' = x - h$, $y' = y - k$.

The Translations Form a Group

In the equations $x' = x + h$, $y' = y + k$, every time we pick definite values for h and k, we have equations representing some definite translation. For example, the pair of equations $x' = x + 2$, $y' = y + 3$ represents one translation. Let us call this translation P. The pair of equations $x' = x + 5$, $y' = y + 7$ represents another translation. Let us call it Q. If we take all possible values for the numbers h and k, we get all possible translations of the plane into itself. These translations form a system of elements in which an operation of multiplication can be defined.

Just as we did in the case of the symmetries of the triangle, back in Chapter III, we define the product of two translations as the result of performing one translation after another, with the second one taking over where the first one leaves off. The combined effect of the two translations is equivalent to a single transformation, which, in fact, is also a translation.

For example, suppose we designate by $Q * P$ the transformation we get when P is performed first, and Q is per-

formed next on the results of P. P transforms components x and y by adding 2 and 3 to them respectively, so the new components are $x + 2$, and $y + 3$. Q transforms these components by adding 5 and 7 to them, respectively, so the final components are $x + 7$ and $y + 10$. So the equations for the product $Q * P$ are: $x' = x + 7$, $y' = y + 10$. These are of the form $x' = x + h$, $y' = y + k$, with the special values 7 and 10 for h and k. So $Q * P$ also belongs to the system of translations. Because the product of two translations is itself a translation, the operation $*$ is a binary operation in the system of translations. The operation is associative, because, if P, Q, and R are three translations, $(R * Q) * P$ and $R * (Q * P)$ both mean P followed by Q followed by R. The translation defined by the equations $x' = x + 0$, $y' = y + 0$ doesn't move the plane at all, and is the identity element for the operation $*$. If we designate it by I, then $P * I = I * P = P$. We have already observed that every translation is reversible. Let us designate by P^{-1} the translation that is the reverse of P. If it is applied after P, it brings every point in the plane back to its original position. So the product $P^{-1} * P$ is equal to the identity element I. Similarly, $P * P^{-1} = I$. So P^{-1} is the inverse of P with respect to the operation $*$. Because of these properties, the system of translations constitutes a group with respect to the operation $*$. In fact, it is an abelian group, because the operation turns out to be commutative.

The Linear Group

In the equations

$$x' = a_1x + a_2y$$
$$y' = b_1x + b_2y,$$

every time we pick definite values for a_1, a_2, b_1 and b_2, we get equations representing a particular linear mapping of the plane into itself. If we take all possible real values of a_1, a_2, b_1, and b_2, we get all possible linear mappings. They form a system of elements in which the operation $*$, defined in the same way as for translations, is a binary operation, because the product of two linear mappings is itself a linear mapping. For example, if P stands for the mapping

$$x' = 2x + 3y$$
$$y' = 5x - y$$

and Q stands for the mapping

$$x'' = 3x' + 4y',$$
$$y'' = 2x' + 7y'$$

we get the mapping $Q * P$ by applying the mapping Q to the results of P as follows:

$$x'' = 3(2x + 3y) + 4(5x - y) = 6x + 9y + 20x - 4y$$
$$y'' = 2(2x + 3y) + 7(5x - y) = 4x + 6y + 35x - 7y.$$

So the transformation $Q * P$ has the equations

$$x'' = 26x + 5y$$
$$y'' = 39x - y$$

These are the equations of a linear mapping in which a_1, a_2, b_1, and b_2 have the special values 26, 5, 39 and -1 respectively.

However, the system of all linear mappings does not form a group. It fails to qualify as a group because not all the linear mappings are reversible. The reason for this difficulty can be seen by comparing one of the troublesome linear mappings with a translation. A translation maps each point into an image in such a way that no two points have the same image. We can reverse the translation by carrying each point back to the single point of which it is the image. However, the linear mapping defined by the equations

$$x' = 0x + 0y = 0$$
$$y' = 0x + 0y = 0,$$

behaves differently. It carries all points into the origin. As a result, the origin is the image of not one point, but

many points. A many-to-one mapping, as we saw on page 15, is not reversible.

However, there are some linear mappings that are reversible. If we put aside those linear mappings that are not reversible, and keep only those that are reversible, then we get a subset of the system of linear mappings that does constitute a group. In this subset, every linear mapping has an inverse, and all the other requirements for qualifying as a group are satisfied. We can find out which linear mappings are reversible by actually trying to reverse one, and seeing what conditions must be fulfilled to attain success. Suppose we take the equations of a linear mapping in general form:

$$x' = a_1x + a_2y$$
$$y' = b_1x + b_2y.$$

To reverse the mapping means to solve for x and y in terms of x' and y'. Let us first write the equations with the x' and y' on the right hand side, and then solve for x and y by the usual method for solving simultaneous equations by eliminating one of the unknowns. To eliminate y, we multiply the first equation by b_2, multiply the second equation by $-a_2$, and then add the resulting equations:

$$\begin{array}{r}
a_1b_2x + a_2b_2y = b_2x' \\
-a_2b_1x - a_2b_2y = -a_2y' \\
\hline
a_1b_2x - a_2b_1x = b_2x' - a_2y'
\end{array}$$

The distributive law permits us to rewrite the left hand side of the equation in factored form: $(a_1b_2 - a_2b_1)x = b_2x' - a_2y'$. The next step in solving for x would be to divide both sides of the equation by $(a_1b_2 - a_2b_1)$ to get as a result

$$x = \frac{b_2x' - a_2y'}{a_1b_2 - a_2b_1}$$

However, this step is not always possible. We know from our discussion on page 74 that division makes sense only when the divisor is not zero. So we can succeed in reversing the linear mapping if and only if $a_1b_2 - a_2b_1$ is not equal to zero. The number $a_1b_2 - a_2b_1$ is called the determinant of the linear mapping. So we can say that a linear mapping is reversible if and only if its determinant is different from zero. For example, suppose linear mappings A and B are defined as follows:

$$A: \begin{cases} x' = 2x + 3y \\ y' = 4x + 6y \end{cases} \qquad B: \begin{cases} x' = 2x - 3y \\ y' = 3x + y \end{cases}$$

The determinant of mapping A is $2 \cdot 6 - 3 \cdot 4 = 0$, so A is not a reversible mapping. The determinant of mapping B is $2 \cdot 1 - (-3) \cdot 3 = 11$, which is different from zero, so B is a reversible mapping. In fact, if we solve for x and y in terms of x' and y', we get

$$x = \frac{1}{11}x' + \frac{3}{11}y'$$

$$y = \frac{-3}{11}x' + \frac{2}{11}y'.$$

These equations have the right form to qualify as a linear mapping, which we may designate as B^{-1}, or the inverse of B. Moreover, B^{-1} is reversible, because solving for x' and y' leads back again to the original equations of B.

The system of reversible linear mappings is known as the *linear group*. It includes all rotations, stretches, and shears of the plane.

If a reversible linear mapping is followed by a translation, the product is called an *affine* transformation. The system of all such products also turns out to be a group, and is known as the affine group. If a uniform stretching of the plane is followed by a translation, the product is called a *similitude*. The system of all such products constitutes another group, known as the group of similitudes. If we put together in one set of transformations all rotations, translations, and reflections, this set, too, is a group, and is known as the *Euclidean* group.

The fact that transformations of a plane into itself can be associated with each other in families of transformations, some of which have a group structure, has led to a new insight into the meaning of geometry. In the geometry we study in high school, considerable time is devoted to the study of congruent figures. We try to find out what figures are congruent to each other. We also investigate properties of a figure that it has in common with any other figure to which it is congruent. Such properties include lengths of corresponding lines, sizes of corresponding angles, area, etc. Two figures were defined as being congruent if they could be made to coincide. This definition implied the use of a motion to carry one figure onto the other. To assure ourselves that the figure would not be deformed while it was being moved, we calmed our fears with the "axiom" that a geometric figure can be moved from place to place without changing its form or size. The effect of this axiom was to banish from the realm of legitimate motions all stretches and shears. At the same time, it singled out as the only legitimate motions those that we have called rotations, translations and reflections. But these are the motions that make up the Euclidean group of transformations. This fact makes it possible to define more precisely what is meant by congruence. Two figures are congruent if one can be mapped onto the other by a transformation that belongs to the Euclidean group. This definition also gives a new meaning to such concepts as length, area, and so on. They turn out to be among the characteristics of a figure that remain unchanged when it is transformed by a mapping that belongs to the Euclidean group.

Another subject treated in high school geometry is that of similar figures. The concept of similarity can also be defined in terms of a group of transformations. Two figures are similar if one can be mapped onto the other by means of a transformation that belongs to the group of similitudes. A similitude does not leave the length of a line unchanged, but it does leave unchanged such things as angles, and ratios of lengths. The fact that the two traditional concerns of plane geometry can be described best in terms of groups of transformations has led to the modern notion of what a geometry is. A geometry is now defined as the study of figures which can be mapped into each other by a group of transformations, and of the properties of figures that remain unchanged when the transformations in the group are applied. In the sense of this definition, what we studied in high school was not geometry, but *some geometries*. When we studied congruence, we were studying Euclidean geometry, associated with the Euclidean group. When we studied similarity, we were studying a different geometry, associated with the group of similitudes. Moreover, there are other geometries which we did not study in high school at all. For example, there is an *affine geometry* associated with the affine group. Because there are many groups that may operate on the same vector space, there is a multiplicity of geometries belonging to one and the same space.

DO IT YOURSELF

1. Use the definition of vector addition given on page 130 to find the following vector sums:

$$(3, 2) + (-1, 2) \qquad (8, -5) + (-5, 8)$$

$$(4, 7) + (-4, -7) \qquad (2, 0) + (0, 3)$$

2. Use the definition of vector addition to prove that $(a, b) + (c, d) = (c, d) + (a, b)$. (Commutative Law of Addition)
3. Locate the points $(2, 3)$, $(-5, 4)$, and $(-6, -2)$ on a graph with coordinates measured from an x axis and a y axis that are perpendicular to each other. Carry out the following scalar multiplications, and locate the products on the graph: $3 \cdot (2, 3)$; $2 \cdot (-5, 4)$; $-\frac{1}{2} \cdot (-6, -2)$. Verify from the graph that the first two scalar multiplications change the length of the arrow belonging to the vector, without changing its direction. Verify that the

third one changes the length and reverses the direction.
4. Represent these three-dimensional vectors in the i, j, k notation:

$$(4, 6, 1) \qquad (-2, 1, 0) \qquad (0, 5, 7)$$

5. Two linear mappings P and Q (mapping a plane into itself) are defined as follows. Find the equations that define the product $Q * P$.

$$P: \begin{cases} x' = x + y \\ y' = x - y \end{cases} \qquad Q: \begin{cases} x' = 2x + y \\ y' = x - 3y \end{cases}$$

Find the equations that define the product $P * Q$. Is $Q * P = P * Q$? Is the operation $*$ for such linear mappings commutative?
6. Find the inverse of the mapping P defined in question 5.
7. Express as an ordered pair:
   a) $(2, 4) + (x, 6)$. b) $(3, -1) + (7, 2)$.
   c) $(6, -4) + (-2, 3) + (1, 1)$. d) $4(2, -1) + 3(4, -2)$.
   e) $2(-3, 1) + 4(5, -2)$. f) $t(2, 1) + (5, 7)$.
8. Solve the following equations for the vector (x, y):
   a) $(x, y) + (2, -3) = (0, 0)$.
   b) $(x, y) + (4, 1) = (7, 2)$.
   c) $2(x, y) + (3, -2) = (4, 1)$.
9. Solve for the scalars x and y:
   a) $x(2, 3) + y(4, -1) = (16, 3)$.
   b) $x(2, 4) + y(9, 6) = (4, 4)$.
   c) $x(1, 0) + y(0, 1) = (0, 0)$.
10. Find the determinant of each of these linear transformations:

$$A: \begin{cases} x' = 2x + 3y \\ y' = 4x + 6y \end{cases} \qquad B: \begin{cases} x' = 3x - 2y \\ y' = 4x - 3y \end{cases}$$

$$C: \begin{cases} x' = 2x + 3y \\ y' = 3x + 5y \end{cases} \qquad D: \begin{cases} x' = x + y \\ y' = 2x - y \end{cases}$$

11. a) Which of the transformations in Ex. 10 have an inverse?
    b) Find these inverses. (See p. 143.)

CHAPTER VII

The Rank and File: Matrices

IN THIS chapter we re-examine from another point of view the linear mappings of a plane into itself. In the course of this re-examination we shall get acquainted with another type of mathematical structure that is important. At the same time we shall acquire some equipment that will be useful to us in the next chapter where we finish the job of constructing a number system that supplies a number for every point in the plane.

Some typical linear mappings of the plane are listed below, with a capital letter assigned to each one as its name:

$$P: \begin{cases} x' = 1x + 1y \\ y' = 1x - 1y \end{cases} \qquad Q: \begin{cases} x' = 2x + 1y \\ y' = 1x - 3y \end{cases}$$

$$O: \begin{cases} x' = 0x + 0y \\ y' = 0x + 0y \end{cases} \qquad I: \begin{cases} x' = 1x + 0y \\ y' = 0x + 1y \end{cases}$$

The mapping called O could have been written more briefly in the form $x' = 0$, $y' = 0$. Similarly, the mapping called I could have been written as $x' = x$, $y' = y$. However, showing the zero coefficients explicitly, as we have in the longer way of writing them, has the advantage of stressing the fact that all linear mappings have the same form. In each mapping, the new x and the new y are obtained by adding some multiple of the old x to some multiple of the old y. Two different linear mappings differ from each other only by

virtue of the fact that they use different multiples. What distinguishes one linear mapping from another, then, is the set of four coefficients that appear in the equations of the mapping. This fact suggests that we can represent a mapping in an abbreviated notation in which we delete the letters x, y, x' and y', and merely list these four coefficients, arranging them in a square array, just as they are arranged in the equations written above. Such a square array of numbers is called a *matrix*. In this case, because the matrix has two rows and two columns, we call it a 2 by 2 matrix. Every linear mapping of the plane is associated with such a 2 by 2 matrix, and, vice versa, every 2 by 2 matrix of real numbers belongs to some linear mapping of the plane. To show this correspondence, we shall use as the name of a matrix the name of the mapping that it belongs to. Here are the matrices of the mappings P, Q, O, and I:

$$P = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \quad Q = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} \quad O = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Matrices Form a Vector Space

Now that we have a collection of matrices, we can disregard their origin as sets of coefficients belonging to linear mappings, and may think of them as constituting an independent system of elements. We proceed to give this system a structure in a simple and natural way. We convert it into a vector space by methods like those we used in the last chapter. Just as the vector (2, 3) is made up of two components, each distinguished by the place it occupies in the ordered pair, each matrix is made up of four components, each distinguished by the place it occupies in the square array. So we can define addition of matrices the way we defined addition of ordered pairs. *To add two matrices, add the corresponding components separately.* We can also define a scalar multiplication for matrices, using the field of real numbers as the field of scalars, in the same way that we defined scalar multiplication for ordered pairs. *To multiply a matrix by a scalar, multiply each of its components by* *that scalar*. Examples of addition and scalar multiplication for matrices are shown below:

$$P + Q = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = \begin{vmatrix} 1+2 & 1+1 \\ 1+1 & (-1)+(-3) \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix}$$

$$2 \cdot Q = 2 \cdot \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = \begin{vmatrix} 2(2) & 2(1) \\ 2(1) & 2(-3) \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & -6 \end{vmatrix}$$

With addition and scalar multiplication defined in this way, the system of 2 by 2 matrices becomes a vector space. To verify this fact, we have to show that it has all the characteristics of a vector space, listed on page 133. First, we observe that it is an abelian group with respect to addition. This is proved below by showing that addition is associative and commutative, that it has a zero element, and that every matrix has a negative:

Addition is associative:

$$\text{Let } P = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad Q = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \quad R = \begin{vmatrix} e_1 & e_2 \\ f_1 & f_2 \end{vmatrix}$$

$$(P + Q) + R = \begin{vmatrix} a_1 + c_1 & a_2 + c_2 \\ b_1 + d_1 & b_2 + d_2 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 \\ f_1 & f_2 \end{vmatrix}$$

$$\begin{vmatrix} (a_1 + c_1) + e_1 & (a_2 + c_2) + e_2 \\ (b_1 + d_1) + f_1 & (b_2 + d_2) + f_2 \end{vmatrix}$$

$$P + (Q + R) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 + e_1 & c_2 + e_2 \\ d_1 + f_1 & d_2 + f_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 + (c_1 + e_1) & a_2 + (c_2 + e_2) \\ b_1 + (d_1 + f_1) & b_2 + (d_2 + f_2) \end{vmatrix}$$

But $(a_1 + c_1) + e_1 = a_1 + (c_1 + e_1)$, etc., because addition of real numbers is associative. Therefore $(P + Q) + R = P + (Q + R)$.

Addition is commutative:

$$P + Q = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} = \begin{vmatrix} a_1 + c_1 & a_2 + c_2 \\ b_1 + d_1 & b_2 + d_2 \end{vmatrix}$$

$$Q + P = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} c_1 + a_1 & c_2 + a_2 \\ d_1 + b_1 & d_2 + b_2 \end{vmatrix}$$

But $a_1 + c_1 = c_1 + a_1$, etc., because addition of real numbers is commutative. Therefore $P + Q = Q + P$.

There is a zero element:

$$0 + P = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} 0 + a_1 & 0 + a_2 \\ 0 + b_1 & 0 + b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = P$$

Therefore the matrix $0 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$ is the zero element for matrix addition, and is called the zero matrix.

Every matrix has a negative:

In fact, the negative of $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is $\begin{vmatrix} -a_1 & -a_2 \\ -b_1 & -b_2 \end{vmatrix}$

$$\text{because } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} -a_1 & -a_2 \\ -b_1 & -b_2 \end{vmatrix} = \begin{vmatrix} a_1 + (-a_1) & a_2 + (-a_2) \\ b_1 + (-b_1) & b_2 + (-b_2) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \text{the zero matrix}$$

We observe next that scalar multiplication for matrices obeys the two distributive laws, the mixed associative law, and the law that the number 1 serves as a unity element for scalar multiplication:

Distributive laws:

$$r(P + Q) = r\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}\right) = r\begin{vmatrix} a_1 + c_1 & a_2 + c_2 \\ b_1 + d_1 & b_2 + d_2 \end{vmatrix}$$

$$= \begin{vmatrix} r(a_1 + c_1) & r(a_2 + c_2) \\ r(b_1 + d_1) & r(b_2 + d_2) \end{vmatrix} = \begin{vmatrix} ra_1 + rc_1 & ra_2 + rc_2 \\ rb_1 + rd_1 & rb_2 + rd_2 \end{vmatrix}$$

$$rP + rQ = r \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + r \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}$$

$$= \begin{vmatrix} ra_1 & ra_2 \\ rb_1 & rb_2 \end{vmatrix} + \begin{vmatrix} rc_1 & rc_2 \\ rd_1 & rd_2 \end{vmatrix} = \begin{vmatrix} ra_1 + rc_1 & ra_2 + rc_2 \\ rb_1 + rd_1 & rb_2 + rd_2 \end{vmatrix}$$

Therefore $r(P + Q) = rP + rQ$.

$$(r + s)P = (r + s) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} (r + s)a_1 & (r + s)a_2 \\ (r + s)b_1 & (r + s)b_2 \end{vmatrix}$$

$$= \begin{vmatrix} ra_1 + sa_1 & ra_2 + sa_2 \\ rb_1 + sb_1 & rb_2 + sb_2 \end{vmatrix}.$$

$$rP + sP = r \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + s \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} ra_1 & ra_2 \\ rb_1 & rb_2 \end{vmatrix} + \begin{vmatrix} sa_1 & sa_2 \\ sb_1 & sb_2 \end{vmatrix}$$

$$= \begin{vmatrix} ra_1 + sa_1 & ra_2 + sa_2 \\ rb_1 + sb_1 & rb_2 + sb_2 \end{vmatrix}$$

Therefore $(r + s)P = rP + sP$.

Mixed associative law:

$$r(sP) = r\left(s \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) = r \begin{vmatrix} sa_1 & sa_2 \\ sb_1 & sb_2 \end{vmatrix}$$

$$\begin{vmatrix} r(sa_1) & r(sa_2) \\ r(sb_1) & r(sb_2) \end{vmatrix} = \begin{vmatrix} rsa_1 & rsa_2 \\ rsb_1 & rsb_2 \end{vmatrix}$$

$$(rs)P = (rs) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} (rs)a_1 & (rs)a_2 \\ (rs)b_1 & (rs)b_2 \end{vmatrix}$$

$$= \begin{vmatrix} rsa_1 & rsa_2 \\ rsb_1 & rsb_2 \end{vmatrix}$$

Therefore $r(sP) = (rs)P$.

Unity element for scalar multiplication:

$$1 \cdot P = 1 \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} 1a_1 & 1a_2 \\ 1b_1 & 1b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = P$$

Multiplication of Matrices

In the last chapter we defined an operation of multiplication for linear mappings. Since each mapping has an associated matrix, we can transfer this operation to the matrices, and in this way define a multiplication operation for matrices. First, let us write down two matrices P and Q, and the mappings that are associated with them:

$$P = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \text{associated mapping} \quad \begin{cases} x'' = a_1x' + a_2y' \\ y'' = b_1x' + b_2y' \end{cases}$$

$$Q = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \qquad \begin{cases} x' = c_1x + c_2y \\ y' = d_1x + d_2y \end{cases}$$

To get the product mapping $P * Q$, we first perform the mapping Q, and then perform the mapping P on the new components we get as a result of having used Q.

$$x'' = a_1(c_1x+c_2y)+a_2(d_1x+d_2y) = a_1c_1x+a_1c_2y+a_2d_1x+a_2d_2y$$
$$y'' = b_1(c_1x+c_2y)+b_2(d_1x+d_2y) = b_1c_1x+b_1c_2y+b_2d_1x+b_2d_2y$$

By rearranging the terms to bring the x terms together, and the y terms together, and then factoring out the x and y, we find that the product mapping $P * Q$ takes this form:

$$x'' = (a_1c_1 + a_2d_1)x + (a_1c_2 + a_2d_2)y$$
$$y'' = (b_1c_1 + b_2d_1)x + (b_1c_2 + b_2d_2)y$$

The matrix that belongs to this mapping is

$$\begin{vmatrix} a_1c_1 + a_2d_1 & a_1c_2 + a_2d_2 \\ b_1c_1 + b_2d_1 & b_1c_2 + b_2d_2 \end{vmatrix}.$$

We call it the product matrix, and designate it by PQ. (We omit the $*$ just as in elementary algebra it is customary to omit the multiplication sign.)

Now, by comparing the matrix PQ with the matrices P and Q, we can observe a simple rule by which the multiplication can be carried out without reference to the equations of the mappings. Notice that in the product matrix PQ, the component in the *first row* and *first* column is $a_1c_1 + a_2d_1$. This expression can be obtained from the pairs (a_1, a_2) and (c_1, d_1) by first multiplying corresponding terms in the pairs, and then adding the products. The pair (a_1, a_2) is the *first row* of matrix P. The pair (c_1, d_1) is the *first column* of the matrix Q. So we get the component in the first row and first column of the product PQ by multiplying corresponding components of the first row of P and the first column of Q, and adding the results. In order to be able to refer to it in fewer words, let us designate this kind of operation as multiplication of the first row of P by the first column of Q. We can now use this language to describe the procedure for getting all the components of the product matrix. To get the component in the first row and second column, multiply the first row of P by the second column of Q. To get the component in the second row and first column, multiply the second row of P by the first column of Q. To get the component in the second row and second column, multiply the second row of P by the second column of Q. In general, to get the component of the product matrix PQ that is in any particular row and column, multiply the corresponding row of P by the corresponding column of Q.

If we apply this rule for multiplying matrices to get the product QP, we find that

$$QP = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} c_1a_1 + c_2b_1 & c_1a_2 + c_2b_2 \\ d_1a_1 + d_2b_1 & d_1a_2 + d_2b_2 \end{vmatrix}.$$

Notice that this product is not the same as PQ, so matrix multiplication is *not commutative*.

The Matrices Form a Ring

Although matrix multiplication is not commutative, it is associative. We do not have to give a special proof of this fact, because we have already observed that the operation $*$, performed on linear mappings, is associative, and matrix multiplication is merely another way of expressing the same operation. It is also possible to show that matrix multiplication is distributive with respect to matrix addition. Consequently, the system of 2 by 2 matrices meets all the requirements for being a ring, just as the natural numbers, the integers, the rational numbers, and the real numbers did. However, unlike these other systems, it is not a commutative ring, since matrix multiplication is not commutative.

There is a unity element for matrix multiplication. In fact, the unity element is the matrix which we have called I. Since multiplication is not, in general, commutative, we have to verify separately that I behaves as a unity element when it is used as a multiplier either from the left or from the right.

$$IP = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} 1a_1 + 0b_1 & 1a_2 + 0b_2 \\ 0a_1 + 1b_1 & 0a_2 + 1b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = P$$

$$PI = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} a_1 \cdot 1 + a_2 \cdot 0 & a_1 \cdot 0 + a_2 \cdot 1 \\ b_1 \cdot 1 + b_2 \cdot 0 & b_1 \cdot 0 + b_2 \cdot 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = P$$

Therefore $IP = PI = P$.

It is natural to ask whether the ring of 2 by 2 matrices is also a field. We find that it is not a field, because it contains zero divisors, and a field may not have zero divisors, as we saw on page 87. Recall that zero divisors are non-zero elements that have a zero product. The matrices $\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$ and $\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$ are different from the zero matrix, but their product is equal to the zero matrix:

$$\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \cdot 0 + 2 \cdot 0 & 0 \cdot 3 + 2 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 3 + 0 \cdot 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

In fact, each of these matrices has the peculiar property that when it is multiplied by itself, the product is zero. If we use the letter T to represent the matrix $\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$, then

$$T^2 = T \cdot T = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \cdot 0 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 2 + 0 \cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

An element like T that has the property that there is a power of it that is equal to zero is called a *nilpotent* element. In the rings we met before, the zero element itself was the only nilpotent element. However, there are many rings which, like the ring of 2 by 2 matrices, have nilpotent elements that are different from zero.

The Matrices Form an Algebra

The ring of 2 by 2 matrices has a double structure. With matrix addition and scalar multiplication, it has the structure of a vector space. With matrix addition and matrix multiplication, it has the structure of a ring. A system that has such a double structure is known as an *algebra*. Roughly, an algebra may be described as a ring that also

has a scalar multiplication defined for it, or a vector space that also has a binary operation of multiplication defined for it. The algebra we have just examined is only one of many possible algebras. In fact, by using definitions of addition, scalar multiplication, and matrix multiplication analogous to those given for 2 by 2 matrices, we can construct an algebra of 3 by 3 matrices, an algebra of 4 by 4 matrices, etc.

Rectangular Matrices

So far we have considered as matrices only square arrays of numbers. However, it is possible to extend the concept of a matrix to include rectangular arrays as well. If a rectangular array of numbers has two rows and five columns, for example, it is known as a 2 by 5 matrix. A vector space can be constructed out of all 2 by 5 matrices by defining matrix addition and scalar multiplication in the same way that we did for square matrices. In general, we can give a vector space structure in this way to the system of all rectangular matrices that have any fixed number of rows and any fixed number of columns. From this point of view, the vectors we first described in the last chapter turn out to be special cases of matrices. An ordered pair like (2, 3) is nothing but a 1 by 2 matrix. An ordered triple is a 1 by 3 matrix, and so on. It is also possible to have a 2 by 1 matrix that has two rows and one column, or a 3 by 1 matrix that has three rows and one column.

The definition of matrix multiplication can also be extended to rectangular matrices, with a special restriction that arises from the way in which matrix multiplication is carried out. To multiply two matrices, we have to multiply each of the rows of the first matrix by each of the columns in the second matrix. This is possible only if a row of the first matrix has as many components as a column does in the second matrix. This means that multiplication of matrices is possible only when the number of columns in the first matrix is equal to the number of rows in the second matrix. For example, we can multiply a 2 by 3 matrix times a 3 by 4 matrix, but we cannot multiply a 3 by 4 matrix times a 2 by 3 matrix.

Rectangular matrices give us a very powerful condensed language in which systems of several equations may sometimes be expressed as a single simple matrix equation. For example, consider these equations for a linear transformation that maps the ordered triples (x_1, x_2, x_3) into the ordered triples (y_1, y_2, y_3):

$$y_1 = a_1x_1 + a_2x_2 + a_3x_3$$

$$y_2 = b_1x_1 + b_2x_2 + b_3x_3$$

$$y_3 = c_1x_1 + c_2x_2 + c_3x_3$$

These three equations are equivalent to the single matrix equation:

$$\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

as you can verify by carrying out the matrix multiplication. If we introduce names for the matrices, as follows,

$$Y = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} \quad P = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad X = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix},$$

then the equation takes the particularly simple form, $Y = PX$. Working with this single equation according to the rules of matrix algebra then takes the place of working with the three original equations.

Matrix algebra, one of the youngest branches of mathematics, is now one of the most widely used. Besides being an indispensable tool in higher mathematics, it is also employed in such diverse fields as psychology, chemistry, physics, economics, and electrical engineering.

DO IT YOURSELF

1. Find the sums $P + Q$ and $Q + P$, and the products PQ and QP of the following 2 by 2 matrices:

$$P = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} \quad Q = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Compare your answers to verify that $P + Q = Q + P$, but PQ is not equal to QP.

2. a) Find the scalar product $2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

b) Find the matrix product $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Compare the results of both multiplications. Notice that the matrix $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$, used with matrix multiplication, behaves like the scalar 2, used with scalar multiplication.

3. Let $P = \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \quad Q = \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} \quad R = \begin{vmatrix} 2 & -1 \\ 6 & 5 \end{vmatrix}$

a) Find $P(Q + R)$. That is, add Q and R, and then multiply by P from the left.
b) Find $PQ + PR$. That is, find the products PQ and PR, and then add them.
c) Compare the results of a) and b) to verify that the distributive law is obeyed.

4. Let $T = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

a) Find $T^2 = TT$.
b) Find $T^3 = T^2T$. What kind of element is T? (See page 156)

CHAPTER VIII

Arrows That Are Numbers

NOW WE return to unfinished business. In Chapter VI we had set out to accomplish a double purpose. We wanted to construct an extension of the real number system that would supply a number for every point in a plane, and would at the same time contain a number that satisfies the equation $x^2 = -1$. As the first step toward accomplishing this purpose we constructed a system in which each element is an ordered pair of real numbers, like (1, 4), or $(-\sqrt{2}, \frac{1}{2}\pi)$. We defined an operation of addition for these elements, and an operation we called scalar multiplication. With these two operations the system became an example of the type of structure we called a vector space.

In this vector space, we have an element for every point in the plane. However, this does not mean that we have already accomplished our purpose. Our goal was to find a *number system* that supplies an element for every point in the plane. The vector space we have constructed does not yet qualify as a number system. A structure is entitled to be called a number system only if it has two binary operations called addition and multiplication, such that each of these operations is associative and commutative, and multiplication is distributive with respect to addition. In the vector space whose elements are ordered pairs, we have an addition operation that is associative and commutative. We also have the operation called scalar multiplication, but it does not help the system qualify as a number system, because scalar multiplication is not a binary operation. In scalar multiplication, we multiply a vector by an element that is outside the system of vectors. In a multiplication

that is a binary operation, we would have to multiply a vector by a vector, and get a product that is also a vector. So, to complete our construction, we now define a multiplication of this kind.

Ordered Pairs Become Numbers

We define the multiplication of ordered pairs by the following equation: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. (In this equation, ac means $a \cdot c$, in accordance with the custom of sometimes omitting the multiplication sign in indicated products.) With this definition, for example, the product $(2, 3) \cdot (4, 1)$ would be equal to $(2 \cdot 4 - 3 \cdot 1, 2 \cdot 1 + 3 \cdot 4) = (5, 14)$. Now we show that this operation is commutative and associative, and is distributive with respect to addition.

To show that multiplication of ordered pairs is commutative, we compare $(a, b) \cdot (c, d)$ with $(c, d) \cdot (a, b)$.

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$
$$(c, d) \cdot (a, b) = (ca - db, cb + da).$$

But, in the real number system, $ac - bd = ca - db$, and $ad + bc = cb + da$. So the two products are the same, and the commutative law is obeyed.

To show that multiplication of ordered pairs is associative, we compare $[(a, b) \cdot (c, d)] \cdot (e, f)$ with $(a, b) \cdot [(c, d) \cdot (e, f)]$.

$$\begin{aligned}
[(a, b) \cdot (c, d)] \cdot (e, f) &= (ac - bd, ad + bc) \cdot (e, f) \\
&= ([ac - bd]e - [ad + bc]f, [ac - bd]f + [ad + bc]e) \\
&= (ace - bde - adf - bcf, acf - bdf + ade + bce). \\
(a, b) \cdot [(c, d) \cdot (e, f)] &= (a, b) \cdot (ce - df, cf + de) \\
&= (a[ce - df] - b[cf + de], a[cf + de] + b[ce - df]) \\
&= (ace - adf - bcf - bde, acf + ade + bce - bdf).
\end{aligned}$$

The first component in each of these products is the sum of four terms. Notice that they are the same four terms, merely written in a different order. Therefore these components are equal. Similarly, the second components are equal, and therefore the two products are equal. So multiplication of ordered pairs obeys the associative law.

To show that multiplication of ordered pairs is distributive with respect to addition, we compare $(a, b) \cdot [(c, d) + (e, f)]$ with $(a, b) \cdot (c, d) + (a, b) \cdot (e, f)$.

$$\begin{aligned}(a, b) \cdot [(c, d) + (e, f)] &= (a, b) \cdot (c + e, d + f)\\ &= (a[c + e] - b[d + f], a[d + f] + b[c + e])\\ &= (ac + ae - bd - bf, ad + af + bc + be).\end{aligned}$$

$$\begin{aligned}&(a, b) \cdot (c, d) + (a, b) \cdot (e, f)\\ &\quad= (ac - bd, ad + bc) + (ae - bf, af + be)\\ &\quad= ([ac - bd] + [ae - bf], [ad + bc] + [af + be])\\ &\quad= (ac - bd + ae - bf, ad + bc + af + be).\end{aligned}$$

The comparison shows that both expressions lead to the same result, and the distributive law is obeyed.

With the addition defined in Chapter VI, and with this new kind of multiplication, the system of ordered pairs of real numbers obeys the five laws that are characteristic of a number system. So now, at last, we have a number system that supplies a number for every point in a plane. We saw in Chapter VI that each ordered pair can be represented pictorially by an arrow, and the operations defined for ordered pairs may be interpreted as operations with the arrows. So we have, in effect, converted the system of arrows into a number system. We call it the system of *complex numbers.*

We have already observed, in Chapter VI, that the system of ordered pairs is an abelian group with respect to addition. Now that we have for this system a binary operation of multiplication that is associative and obeys the distributive law, the system meets the requirements for being a ring. But we already know that it is a vector space, equipped with a scalar multiplication. So the system of

complex numbers has a double structure and therefore qualifies as an *algebra* in the sense of the definition given on page 156.

The Complex Number System Is a Field

To qualify as a field as well as a ring, the complex number system must meet the requirements of having a unity element, and having a reciprocal for every element except the zero element. We observe first that the complex number $(1, 0)$ is a unity element for the system, because when it multiplies any number in the system, it leaves that number unchanged:

$$(1, 0) \cdot (a, b) = (1 \cdot a - 0 \cdot b, 1 \cdot b + 0 \cdot a) = (a, b).$$

Now we shall produce a reciprocal for every complex number that is different from the zero element. The zero element in the complex number system is $(0, 0)$. So a complex number (a, b) is the zero element only if $a = 0$ and $b = 0$. If a complex number is different from the zero element, then a and b are not both zero. In that case, $a^2 + b^2$ is not zero, so that division by $a^2 + b^2$ is possible. So, if (a, b) is not the zero element, the symbols $\dfrac{a}{a^2 + b^2}$ and $-\dfrac{b}{a^2 + b^2}$ represent actual real numbers, and the ordered pair $\left(\dfrac{a}{a^2+b^2}, -\dfrac{b}{a^2+b^2}\right)$ is a complex number. We now show that it is the reciprocal of (a, b) by showing that their product is equal to the unity element:

$$\begin{aligned}&(a, b) \cdot \left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right)\\ &= \left(a\left[\frac{a}{a^2+b^2}\right] - b\left[-\frac{b}{a^2+b^2}\right], a\left[-\frac{b}{a^2+b^2}\right] + b\left[\frac{a}{a^2+b^2}\right]\right)\\ &= \left(\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}, -\frac{ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}\right)\\ &= \left(\frac{a^2 + b^2}{a^2 + b^2}, 0\right) = (1, 0).\end{aligned}$$

For example, the reciprocal of (3, 4) is $(\frac{3}{25}, -\frac{4}{25})$.

This fact can be verified separately by multiplication:

$$(3, 4) \cdot (\tfrac{3}{25}, -\tfrac{4}{25}) = (3 \cdot [\tfrac{3}{25}] - 4[-\tfrac{4}{25}], 3[-\tfrac{4}{25}] + 4[\tfrac{3}{25}])$$

$$= (\tfrac{9}{25} + \tfrac{16}{25}, -\tfrac{12}{25} + \tfrac{12}{25}) = (1, 0)$$

We Still Have the Real Numbers

When we constructed the system of integers, we showed that there is a subset of the integers, the positive integers, that is isomorphic to the natural number system, and therefore can take its place for all practical purposes. In this sense, the system of integers includes the system of natural numbers, and is therefore an extension of the natural number system. In the same way we showed that the rational number system includes a subset isomorphic to the integers, and the real number system includes a subset isomorphic to the rational numbers. Now we show that the complex number system includes a subset isomorphic to the real numbers, and is therefore an extension of the real number system.

Before we display this subset, let us first introduce a new notation analogous to the *i, j, k* notation used for three dimensional vectors in Chapter VI. The complex number system is a two dimensional vector space, and all of its elements can be expressed in terms of the two unit vectors (1, 0) and (0, 1). We have already seen that (1, 0) is the unity element of the system, so let us call it u, to remind us of that fact. We assign the name i to the vector (0, 1). Then, the complex number $(a, b) = (a, 0) + (0, b) = a \cdot (1, 0) + b \cdot (0, 1) = au + bi$. These equalities follow from the rules for vector addition and scalar multiplication. Now we shall show that the real number system is isomorphic to the subset of the complex number system consisting of those ordered pairs in which the second component is 0. All the numbers in this subset have the special form $(a, 0) = a \cdot (1, 0) = au$. To prove the isomorphism we have to produce a one-to-one correspondence in which the image of a

sum is the sum of the images, and the image of a product is the product of the images.

The one-to-one correspondence we use is the mapping $a \leftrightarrow au$. That is, we associate with each real number a the complex number that is equal to a times the unity element u. Under this mapping, the image of 1 is $1u$, the image of 5 is $5u$, the image of $\frac{1}{2}$ is $\frac{1}{2}u$, the image of $\sqrt{3}$ is $\sqrt{3}u$, the image of -2 is $-2u$. The image of 0 is $0u = 0(1, 0) = (0, 0) =$ the zero element of the complex number system. The image of $a + b$ is $(a + b)u$, and the image of ab is $(ab)u$.

Now let us compare the image of a sum with the sum of the images. The image of a is au. The image of b is bu. The sum of the images is $au + bu = a(1, 0) + b(1, 0) = (a, 0) + (b, 0) = (a + b, 0) = (a + b)\ (1, 0) = (a + b)u$. But this is precisely the image of the sum $a + b$ under the mapping. So the image of a sum is the sum of the images, and the mapping preserves addition.

Now let us compare the image of a product with the product of the images. The product of the images is $(au) \cdot (bu) = [a(1, 0)] \cdot [b(1, 0)] = (a, 0) \cdot (b, 0) = (ab - 0 \cdot 0, a \cdot 0 + b \cdot 0) = (ab, 0) = (ab)\ (1, 0) = (ab)u$. But this is precisely the image of the product ab. So the image of a product is the product of the images, and the mapping preserves multiplication. Therefore, in so far as the binary operations of addition and multiplication are concerned, the complex number au behaves just like the real number a, and, in particular, the unity element u behaves just like the real number 1. However, we have one more comparison to make, before we can agree that real numbers and complex numbers of the form au are interchangeable. The real number system does a special job in relation to the complex number system when it serves as the field of scalars for scalar multiplication. We have to check whether complex numbers of the form au can serve as scalars, too. To answer this question, we multiply any complex number (c, d) by the scalar a, using scalar multiplication, of course. Then we multiply the same complex number (c, d) by the complex image of a, namely au. Since both (c, d) and au

are complex numbers, this multiplication will be the binary operation defined for the complex number system. Then we compare the results.

Scalar multiplication: $a(c, d) = (ac, ad)$.

Complex number multiplication:

$$(au) \cdot (c, d) = [a(1, 0)] \cdot (c, d) = (a, 0) \cdot (c, d)$$
$$= (ac - 0 \cdot d, ad + 0 \cdot c) = (ac, ad).$$

The products are the same! Multiplication by the complex number au produces the same result as scalar multiplication by the real number a. So the complex numbers of the form au and the real numbers are completely equivalent, and the complex number system is an extension of the real number system. We can now dispense with the special symbol u, and replace it by 1, the symbol for the real number that it is equivalent to. Similarly, we write a for the complex number au. So now, the complex number (a, b), which we have written as $au + bi$, may be written as $a + bi$. The a is referred to as its real part and the bi is referred to as its *imaginary part*, and the complex number i is called the imaginary unit. In this notation, a real number is a complex number $a + bi$ whose imaginary part has $b = 0$.

The fact that scalar multiplication by the real number a is equivalent to multiplication by the complex number au (now also designated by a) makes it unnecessary to retain scalar multiplication as a separate operation for the algebra of complex numbers. Any time we need scalar multiplication, we merely think of it as the special case of the multiplication of complex numbers that arises when one of the multipliers happens to be a real number.

The Question Is Answered

One of our purposes in constructing the system of complex numbers was to have a system in which the equation

$x^2 = -1$ has a solution. We can now show that this purpose has been achieved. First let us be sure that we can recognize -1 in this system when we see it. The number -1, as an element in the complex number system, is equivalent to what we called $-1u$ before, which is $-1(1, 0)$, or $(-1, 0)$. It is in this form that we shall encounter it. The equation that we are trying to solve asks the question, "What number multiplied by itself gives -1 as the product?" We now show that the imaginary unit i provides the answer to the question.

$$i \cdot i = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)$$
$$= (-1, 0) = -1.$$

Now that we have established the fact that $i^2 = i \cdot i = -1$, we can discard the cumbersome apparatus of ordered pairs, and work with complex numbers in a particularly easy way. We write them in the form $a + bi$, as we have already done, and we add them and multiply them according to the rules of elementary high school algebra, making use of the special rule that $i^2 = -1$. How addition and multiplication of complex numbers are carried out in this notation is shown in the following examples:

To add $(2 + 3i) + (-5 + 6i)$:

$$\begin{array}{r} 2 + 3i \\ \underline{-5 + 6i} \\ -3 + 9i \end{array}$$

To multiply $(2 + 3i)(-5 + 6i)$:

$$\begin{array}{rl} & 2 + 3i \\ & \underline{-5 + 6i} \\ & -10 - 15i \\ & \underline{\quad\quad + 12i + 18i^2} \\ & -10 - 3i + 18(-1) \\ = & -28 - 3i \end{array}$$

Bad Names for Good Numbers

Complex numbers were first introduced by the Italian mathematicians of the sixteenth century, who found, as we have, that they could not solve certain equations without them. However, mathematicians in those days were accustomed to think in terms of only one number system of positive and negative numbers. They considered the numbers in this system genuine, and therefore called them "real." The complex numbers clearly did not belong to the "real" number system, so, although they used them as a convenience, they considered them to be spurious, or unreal, and called them "imaginary." In fact, the great seventeenth century mathematician, René Descartes, was so doubtful of the reality of these numbers that he rejected them altogether. Today we realize that there is not only one number system. There are many number systems. All are equally genuine, although they differ from each other. Nevertheless we still use the old names, "real" numbers and "imaginary" numbers. So we must be careful not to be influenced by the old prejudices that are expressed in these names. When we use the term "imaginary numbers" now, we must bear in mind that it is a technical term, and should not be interpreted as a derogatory epithet casting doubt on the genuineness of the numbers.

Imaginary numbers are genuine numbers. We have demonstrated their existence by constructing them in the form of ordered pairs of real numbers. However, in order to dispel any lingering doubts that may remain about their genuineness, we now proceed to construct them by two other methods. After constructing complex numbers in three different ways, we ought to feel certain that they are genuine, though not "real."

Complex Numbers as Matrices

The second construction of the complex number system requires no new act of creation on our part. We find that the complex numbers are actually hidden in one of the structures we built up before. They form a subset of the algebra of 2 by 2 matrices defined in Chapter VII. In fact, the complex numbers are nothing else but the 2 by 2 matrices of the form $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$, in which the components in the upper left hand corner and the lower right hand corner are the same real number, and the other two components have the property that each is the negative of the other. Examples of this type of matrix are:

$$\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} \quad \begin{vmatrix} -6 & -5 \\ 5 & -6 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}.$$

It may seem strange for this to be true, in view of the fact that multiplication of matrices is not commutative, while multiplication of complex numbers is commutative. However, while multiplication may not be commutative for a system as a whole, it may be commutative for some special subset in the system. In any case, we can easily prove that multiplication is commutative for the 2 by 2 matrices that have this special form. We compare

$$\begin{vmatrix} a & b \\ -b & a \end{vmatrix} \cdot \begin{vmatrix} c & d \\ -d & c \end{vmatrix} \text{ with } \begin{vmatrix} c & d \\ -d & c \end{vmatrix} \cdot \begin{vmatrix} a & b \\ -b & a \end{vmatrix},$$

and show that the products are the same.

$$\begin{vmatrix} a & b \\ -b & a \end{vmatrix} \cdot \begin{vmatrix} c & d \\ -d & c \end{vmatrix} = \begin{vmatrix} a(c) + b(-d) & a(d) + b(c) \\ -b(c) + a(-d) & -b(d) + a(c) \end{vmatrix}$$
$$= \begin{vmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{vmatrix}.$$

$$\begin{vmatrix} c & d \\ -d & c \end{vmatrix} \cdot \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = \begin{vmatrix} c(a) + d(-b) & c(b) + d(a) \\ -d(a) + c(-b) & -d(b) + c(a) \end{vmatrix}$$
$$= \begin{vmatrix} ac - bd & bc + ad \\ -ad - bc & -bd + ac \end{vmatrix}.$$

An examination of the two products, component by component, shows that they are the same.

Actually, it is not necessary to give this special proof that multiplication of these matrices is commutative. All we really have to do is show that the system of these special matrices is isomorphic with the complex number system we already have. Once we have established this fact, then it follows that the system of special matrices has all the properties that the complex number system has, including the commutativity of multiplication.

To prove that the two systems are isomorphic, we make use of the mapping $a + bi = (a, b) \longleftrightarrow \begin{vmatrix} a & b \\ -b & a \end{vmatrix}$.

Under this mapping, $(0, 0)$ is associated with $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$, $(1, 0)$ is associated with $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, and $(0, 1)$ is associated with $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$. This mapping is clearly a one-to-one correspondence. Now we show that it preserves addition and multiplication. The ordered pairs (a, b) and (c, d) have the sum $(a + c, b + d)$. Their images in the system of matrices are $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ and $\begin{vmatrix} c & d \\ -d & c \end{vmatrix}$. These images have the matrix sum $\begin{vmatrix} a + c & b + d \\ -b - d & a + c \end{vmatrix}$, which is the image of the ordered pair $(a + c, b + d)$ under the mapping. So the sum of the images is the image of the sum.

The ordered pairs (a, b) and (c, d) have the product $(ac - bd, ad + bc)$. Their images, $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ and $\begin{vmatrix} c & d \\ -d & c \end{vmatrix}$, have the matrix product $\begin{vmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{vmatrix}$, as shown on page 169. But this product is the image of the

ordered pair $(ac - bd, ad + bc)$ under the mapping. So the product of the images is the image of the product. Therefore this system of special 2 by 2 matrices is isomorphic to the complex number system. It is nothing but the complex number system masquerading in a different style of dress. In this representation, the number 1 appears as $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. The real numbers have the form $\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$. The numbers bi, whose real part is zero, have the form $\begin{vmatrix} 0 & b \\ -b & 0 \end{vmatrix}$. Consequently, the number i is $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$, and the number -1 is $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$.

We can verify directly by matrix multiplication that the number i in this form satisfies the equation $x^2 = -1$:

$$i^2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 \cdot 0 + 1(-1) & 0 \cdot 1 + 1 \cdot 0 \\ -1 \cdot 0 + 0(-1) & -1 \cdot 1 + 0 \cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = -1.$$

Complex Numbers as Residue Classes

The third method of constructing the complex number system follows a procedure we have already used before. In Chapter III we divided the system of integers into residue classes modulo 3, by putting together into the same class all integers that have the same remainder when we divide them by 3. We defined operations of addition and multiplication for these residue classes. Then we found that, with these operations, the residue classes formed a number system. In Chapter IV we found out that the system of residue classes modulo 3 is also a field. We are now going to go through these steps in the same order. First, we shall construct a ring that will play the same part here that the ring of integers did in Chapter III. Then we shall divide it into residue classes, by putting into the same class all members of the ring that have the same remainder when we divide by a particular member of the ring specially chosen for our purpose. Then, with addition and multiplication of residue classes defined as it was in Chapter III we shall find that the system of residue classes is a number system, and a field. Finally, we shall show that this number system is nothing but the complex number system in disguise.

The elements that we use for the ring that we start with are the polynomials whose terms are powers of x with coefficients that are real numbers. In this system, each separate real number, like $\frac{1}{2}$, or 2, or $\sqrt{3}$, is a polynomial of zero degree. Then there are polynomials of the first degree, like $2x + 5$, and $\sqrt{3}x - 7$. There are polynomials of the second degree, like $x^2 - 1$, $x^2 + 1$, and polynomials of higher degree like $2x^3 - 5x^2 + 6x + 7$, $x^5 - 4x + 1$, etc. On page 134, we saw that this system of polynomials is a vector space. The addition in this vector space is the ordinary addition of polynomials we learned in high school algebra. The polynomials form an abelian group with respect to this operation of addition. The zero element for this operation is the real number 0, considered as a polynomial of no degree. Now we introduce an operation of multiplication of polynomials by using the rules of multiplication taught in elementary high school algebra. For example, we obtain the product $(2x + 1)(3x - 5)$ in this way:

$$\begin{array}{r} 3x - 5 \\ \underline{2x + 1} \\ 6x^2 - 10x \phantom{{}-5} \\ \underline{+\ 3x - 5} \\ 6x^2 - 7x - 5 \end{array}$$

This operation of multiplication of polynomials is associative and commutative, and it is distributive with respect to

addition. So the system of polynomials has the structure of a ring. The unity element for multiplication of polynomials is the real number 1 considered as a polynomial of zero degree.

The next step we take is influenced by the purpose we are trying to accomplish. Our goal, you will recall, is to construct a number system in which the equation $x^2 + 1 = 0$ has a solution. The polynomial that appears in this equation plays a special part in the next step of our construction. We use it as a divisor, and divide it into every other polynomial in the system. Where the division comes out even, we get a quotient, and a remainder equal to zero. Where the division does not come out even, the remainder is different from zero. For example, when we divide $x^4 + 3x^2 + 2$ by $x^2 + 1$, the quotient is $x^2 + 2$, and the remainder is 0. When we divide $x^2 + 1$ by $x^2 + 1$, the quotient is 1, and the remainder is 0. When we divide 0 by $x^2 + 1$, the quotient is 0, and the remainder is 0. When we divide $x^2 + 2$ by $x^2 + 1$, the quotient is 1 and the remainder is 1. When we divide x by $x^2 + 1$, the quotient is 0, and the remainder is x. When we divide $x^2 + 3x + 5$ by $x^2 + 1$, the quotient is 1, and the remainder is $3x + 4$.

In arithmetic, when we divide one integer by another, the remainder is always smaller than the divisor. There is an analogous rule for the division of polynomials. When we divide one polynomial by another, the remainder is always a polynomial of lower degree than the divisor. This is so because, as long as the remainder has the same degree as the divisor, or a higher degree, the division can be carried out for at least one more step. This is seen in the succession of steps for dividing $2x^3 - 3x^2 + 4x - 1$ by $x^2 + 1$:

$$\begin{array}{r|l} & 2x - 3 \\ \cline{2-2} x^2 + 1 & 2x^3 - 3x^2 + 4x - 1 \\ & \underline{2x^3 \phantom{{}- 3x^2} + 2x \phantom{{}-1}} \\ & - 3x^2 + 2x - 1 \\ & \underline{- 3x^2 \phantom{{}+ 2x} - 3} \\ & \phantom{2x^3 - 3x^2 +{}} 2x + 2 \end{array}$$

The remainder in this case is $2x + 2$. Since the divisor $x^2 + 1$ is of the second degree, all remainders we get will be of the first degree or the zero degree. They will therefore be in the form $a + bx$, where a and b are real numbers. The remainder is of zero degree in the cases where b happens to be 0.

Now we sort all the polynomials in the ring into residue classes, by putting together in the same class all polynomials that have the same remainder when they are divided by $x^2 + 1$. We shall designate each class by a symbol of the form C_P, where P is a polynomial that belongs to the class. Thus each class has many names, and we recognize that two names stand for the same class when we see that the polynomials written as subscripts have the same remainder when we divide by $x^2 + 1$. For example, C_0, C_{x^2+1}, and $C_{x^4+3x^2+2}$ all stand for the same class, because the polynomials 0, $x^2 + 1$, and $x^4 + 3x^2 + 2$ all have the remainder 0 when they are divided by $x^2 + 1$. The polynomial x belongs to the class C_x. The polynomial a, where a is a real number, belongs to the class C_a. Since different real numbers have different remainders when they are divided by $x^2 + 1$ (each is its own remainder), no two real numbers belong to the same class.

We define operations of addition and multiplication of residue classes in the same way that we did for residue classes modulo 3 in Chapter III. To add two residue classes, pick a polynomial from each class, and add them. Then identify the class to which the sum of the polynomials belongs. That class will be the sum of the two classes. In the notation we are using, a member of each class is always put on display as a subscript in the name of the class. So we can add two classes by merely adding their subscripts. For example, $C_2 + C_3 = C_5$. $C_{2x} + C_7 = C_{2x+7}$.

To multiply two residue classes, we multiply a polynomial in one class by a polynomial in the other, and find the class that the product belongs to. Since a member of each class is put on display as a subscript in the name of the

class, we can multiply two classes by merely multiplying their subscripts.

With these definitions of addition and multiplication, the system of residue classes has the structure of a ring and a number system. This follows from the fact that addition and multiplication in the system are commutative and associative, and multiplication is distributive with respect to addition. We can verify the commutativity of addition very easily. $C_P + C_Q = C_{P+Q}$, and $C_Q + C_P = C_{Q+P}$. But addition of polynomials is commutative, so $P + Q = Q + P$, and therefore $C_{P+Q} = C_{Q+P}$. The other properties are established by a similar argument. The zero element in the ring is C_0, because if C_P is the class of any polynomial P, $C_0 + C_P = C_{0+P} = C_P$.

Within this number system, there is a subset that is isomorphic to the real number system. This subset consists of all classes of the form C_a, where a is a real number (that is, a polynomial of zero degree). To prove that it is isomorphic to the real number system, we have to produce a one-to-one correspondence that preserves the operations of addition and multiplication. The mapping we use for this purpose is $a \leftrightarrow C_a$. The mapping is a one-to-one correspondence, because no two different real numbers belong to the same class. Under this mapping, if a and b are two real numbers, their images are C_a and C_b. The sum of the images is $C_a + C_b = C_{a+b}$, which is the image of the sum. The product of the images is $C_a \cdot C_b = C_{ab}$, which is the image of the product. Therefore the set of classes of the form C_a, where a is a real number, is isomorphic to the real number system. Since isomorphic systems are the same for all practical purposes, we can use the symbols of one system to represent the other. So now, instead of writing C_a for the class to which a real number a belongs, we shall simply write a. Thus 0 will stand for C_0, 1 will stand for C_1, and -1 will stand for C_{-1}.

Now that we have found the real numbers within our new system, we know that it is an extension of the real number system. Next we observe that it contains an element that is a solution to the equation $x^2+1=0$. To show that this is so, we must produce a class C_P which has the property that $(C_P)\cdot(C_P)+1=0$. The class that has this property is the class C_x. In fact, $C_x \cdot C_x + 1 = C_x \cdot C_x + C_1 = C_{x^2} + C_1 = C_{x^2+1} = C_0 = 0$. Notice that we made use of the fact that when x^2+1 is divided by x^2+1, the remainder is 0, so that $C_{x^2+1} = C_0$.

The class C_x therefore has the same property as the complex number we called i before. In fact, since x^2 has the remainder -1 when we divide it by x^2+1, $C_x \cdot C_x = C_{x^2} = C_{-1} = -1$. Let us therefore use the symbol i to stand for C_x, since the rule $i^2=-1$ holds for C_x as well.

Now only one more step remains to show that this system of residue classes is essentially the same as the complex number system we built up before. We show that it is isomorphic to the complex number system. First let us remind ourselves that every class in the system can be represented by the remainder that the members of the class have in common. Secondly, we recall that all remainders, when we divide by x^2+1, are first degree polynomials of the form $a+bx$ where a and b are real numbers. So every member of our new class of numbers can be represented in the form C_{a+bx}. Now, using the rules for addition and multiplication of residue classes, we see that C_{a+bx} can be written as $C_a + C_b \cdot C_x$. We have already agreed to write a instead of C_a, and b instead of C_b, because classes of this special form behave like real numbers. We have also agreed to write i for C_x, because the rule that $i^2=-1$ correctly describes the behavior of this class. So we can write $C_a + C_b \cdot C_x$ in the form $a+bi$, with the understanding that a and b behave like real numbers and i has the property $i^2=-1$. When we write it this way we see that the residue class C_{a+bx} is nothing but the complex number $a+bi$ in disguise. The system of residue classes modulo (x^2+1) is therefore the field of complex numbers. This completes the third construction of the complex number system.

We have now seen the system of complex numbers in

three forms of dress. We have seen it as a system of ordered pairs, as a system of two by two matrices, and as a system of residue classes in a ring of polynomials. In each of these systems, an individual complex number has a different appearance. But the structure of the three systems, in so far as their addition and multiplication tables are concerned, is the same. So we recognize them as merely different representations of one and the same number system, whose elements are usually represented in the convenient form $a+bi$.

No More Extensions Needed

We have expanded our number system four times. Each expansion to a more extensive system was made necessary by the fact that a certain type of equation could not be solved in the less extensive system. In the natural number system, we could not solve an equation like $x+5=3$. So we constructed the system of integers, where all equations of the form $x+b=a$ have a solution. In the system of integers, we could not solve an equation like $2x=3$. So we constructed the system of rational numbers, in which all equations of the form $ax=b$ have a solution, as long as a is not equal to 0. At this stage, we found that we could always solve equations of the first degree, whose general form is $ax+b=0$. But we could not say the same for equations of the second degree. To be able to solve the equation $x^2-2=0$, we had to construct the real number system. To be able to solve the equation $x^2+1=0$, we had to construct the complex number system.

Now it is worth asking what equations we can solve in the complex number system. So far we have examined only equations of the first and second degree. If we try other algebraic equations of higher degree, will we find any that cannot be solved in the complex number system? Will it be necessary to expand the number system again and again, as we try more and more complicated algebraic equations? If you have had visions of an endless chain of extensions of the number system, you can dismiss them at once. As far as algebraic equations are concerned, we have reached the end of the road. The complex number system not only gives us a solution to the equation $x^2 + 1 = 0$. It gives us a solution for *every other algebraic equation* as well. To be more specific, it is known that every algebraic equation of the form

$$a_n x^n + a_{n-1} x^{n-1} \cdots a_1 x + a_0 = 0,$$

whose coefficients are complex numbers, has a solution in the complex number system. In fact it has as many solutions as the degree of the equation indicates. A first degree equation has one solution, a second degree equation has two solutions, a third degree equation has three solutions, and so on. The fact that algebraic equations can always be solved within the complex number system is known as the Fundamental Theorem of Algebra. It was first proved by the great German mathematician, Carl Friedrich Gauss (1777-1855).

The Old in the New

We have accomplished the purpose we aimed for at the beginning of the book. We have seen how, through successive extensions of the number system, mathematicians have eliminated its defects while losing none of its virtues. We have been introduced to a variety of mathematical structures, like groups, rings, fields, vector spaces, and topological spaces, that are being explored vigorously in mathematics today. We have found that, although their names are new, and at first sound unfamiliar, they are closely related to such familiar things as addition and multiplication of numbers, and collections of points on a line. Although the world of modern mathematics is a new world in many ways, it has never lost contact with the old world of number and space from which it has grown.

DO IT YOURSELF

1. Use the definition of multiplication of ordered pairs (see page 161) to find the following products:
   $(2, 3) \cdot (4, 1)$ $(2, 3) \cdot (2, -3)$ $(2, 0) \cdot (0, 3)$

2. Write each of these ordered pairs in the form $a + bi$, making use of the convention that 1 stands for $(1, 0)$, and i stands for $(0, 1)$:
   $(2, 3)$ $(2, -3)$ $(3, 0)$ $(0, 2)$
3. Verify by multiplication that $\frac{1}{2} - \frac{1}{2}i$ is the reciprocal of $1 + i$. (Remember that $i^2 = -1$)
4. Let 1 stand for the two by two matrix $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, and let i stand for the two by two matrix $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$.
   Write the two by two matrices that represent $2 + 3i$ and $2 - 3i$. Verify by matrix multiplication that $(2 + 3i) \cdot (2 - 3i) = 13 = \begin{vmatrix} 13 & 0 \\ 0 & 13 \end{vmatrix}$.
5. Use the rules for addition and multiplication of residue classes modulo $(x^2 + 1)$ to verify the following sums or products:
   $C_{2x^2+1} + C_1 = C_0$. $C_x \cdot C_{2x} = C_{-2}$.
   $C_{x+1} \cdot C_{x+2} = C_{3x+1}$. $C_{x^2} + C_3 = C_2$.
6. Multiply out, and express the answer in the form $a + bi$:
   a) i^3 b) $(2 + i)^3$
   c) i^4 d) $(3 + 4i)(5 - 2i)$
   e) $(1 + i)^2$ f) $i(2 + 3i)$
   g) $(2 - 3i)(2 + 3i)$ h) $(c + di)(e + fi)$
7. Prove that $(a + bi)(a - bi)$ is a real number.
8. Find the reciprocal of
   a) i b) $6 + 8i$
   c) i^3 d) $1 + i$
   e) $5 + 12i$ f) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
9. If $1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ and $i = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$, write each of these complex numbers as a two by two matrix:
   a) $1 + i$ b) $2 + 2i$ c) $2 + i$
   d) $2 - i$ e) $3 + i$ f) $2 - 2i$
   g) $-1 + i$ h) $3 - i$ i) $-1 - i$

10. Use the results of Ex. 9 to find these products by matrix multiplication; then check by the method of multiplication shown on p. 167:

a) $(1 + i)(3 + i)$ b) $(2 - 2i)(3 + i)$
c) $(2 - i)(2 + i)$ d) $(2 - 2i)(1 + i)$
e) $(-1 + i)(-1 - i)$ f) $(2 - i)(1 + i)$
g) $(3 + i)(3 - i)$ h) $(2 - 2i)(-1 - i)$

11. In any field, we can define division as follows: $A \div B = A \cdot \left(\frac{1}{B}\right)$. With the help of this definition, express in the form of $a + bi$:

a) $\dfrac{1+i}{3+4i}$ b) $\dfrac{2-i}{5+12i}$ c) $\dfrac{3+i}{1+i}$ d) $\dfrac{i}{1-i}$

12. If $\bar{x} = a + bi$, $a - bi$ is called the conjugate of x, and is denoted by x. If x and y are complex numbers, prove:

a) $\overline{(\bar{x})} = x$.
b) $\bar{x} + \bar{y} = \overline{x + y}$.
c) $(\bar{x}) \cdot (\bar{y}) = \overline{(x \cdot y)}$.
d) $\bar{x} - \bar{y} = \overline{(x - y)}$.
e) $\dfrac{\bar{x}}{\bar{y}} = \overline{\left(\dfrac{x}{y}\right)}$.
f) $x + \bar{x}$ is a real number.
g) $\dfrac{x - \bar{x}}{i}$ is a real number.

Bibliography

For readers who wish to learn more about the subjects discussed in this book:

Elements of Algebra, by Howard Levi
Chelsea Publishing Co., New York
Higher Algebra for the Undergraduate, by Marie J. Weiss
John Wiley & Sons, New York
A Survey of Modern Algebra, by Garrett Birkhoff & Saunders MacLane, The Macmillian Company, New York

Summary of Basic Definitions

Zero element: A is a zero element for addition if $A + X = X + A = X$, for every X.

Unity element: U is a unity element for multiplication if $U \cdot X = X \cdot U = X$, for every X.

Identity element: E is an identity element for the operation $*$ if $E * X = X * E = X$, for every X.

Negative: A is the negative of B if $A + B = B + A = 0$.

Reciprocal: A is the reciprocal of B if $A \cdot B = B \cdot A = 1$.

Inverse: A is the inverse of B with respect to the operation $*$ whose identity element is E, if $A * B = B * A = E$.

Commutative: Addition is commutative if $x + y = y + x$. Multiplication is commutative if $x \cdot y = y \cdot x$.

Associative: Addition is associative if $(x + y) + z = x + (y + z)$. Multiplication is associative if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Distributive: Multiplication is distributive with respect to addition if $x \cdot (y + z) = x \cdot y + x \cdot z$.

Number system: A set of elements is a number system if it has two binary operations called addition and multiplication, each of which is commutative and associative, and if multiplication is distributive with respect to addition.

Group: A system is called a group if it has a binary operation that is associative, has an identity element for the operation, and has an inverse for every element. If the operation is commutative, the group is called abelian.

Ring: A system is called a ring if it has two associative binary operations called addition and multiplication, is an abelian group with respect to addition, and if the multiplication is distributive with respect to addition.

Field: A ring is called a field if it has a unity element for multiplication and contains a reciprocal for every element except 0.

Topological Space: A system of elements is called a topological space if a collection of its subsets is singled out to be called "open sets," and the collection has these properties: 1) The whole space and the empty set belong to the collection. 2) The union of any number of sets in the collection is also in the collection. 3) The intersection of any two sets in the collection is also in the collection.

Vector Space: A system is a vector space if it is an abelian group with respect to addition, is subject to a scalar multiplication by elements from an associated field of scalars, and if the scalar multiplication obeys these laws:

$$r \cdot (\vec{x} + \vec{y}) = r \cdot \vec{x} + r \cdot \vec{y}.$$

$$(r + s) \cdot \vec{x} = r \cdot \vec{x} + s \cdot \vec{x}$$

$$r \cdot (s \cdot \vec{x}) = (r \cdot s) \cdot \vec{x}$$

$$1 \cdot \vec{x} = \vec{x}.$$

Algebra: A system is an algebra if it is provided with binary operations of addition and multiplication, and a scalar multiplication, that make it both a vector space and a ring.

Geometry: A geometry is a study which identifies those figures in a space which are equivalent to each other under a group of transformations, and determines what properties equivalent figures have in common.

Do-It-Yourself Supplement

CHAPTER I

Ex. 1-4 are on pp. 33-4 of the text.

5. A multiplication operation is defined for the system of Ex. 2 by this table:

$\cdot$	a	b
a	a	a
b	a	b

a) Does this system have a unity element?
b) Show that multiplication is commutative in this system.
c) Show that multiplication is associative in this system. (Write the eight possible statements of the form $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, and verify them from the table.)
d) Show that this multiplication is distributive with respect to the addition defined in Ex. 2. (Write the eight possible statements of the form $x \cdot (y + z) = x \cdot y + x \cdot z$, and verify them from the tables.)
e) Using the same tables, show that addition is not distributive with respect to multiplication. (Write the eight possible statements of the form $x + (y \cdot z) = (x + y) \cdot (x + z)$, and show that they are not all true.)
f) What conclusion follows from the results of 2b, 4a, b, and 5b, c, d?

6. Define addition and multiplication for the system of two elements 0 and 1 as follows:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

a) Show that 0 is a zero element and that 1 is a unity element for the system.
b) Show that, with the operations defined by these tables, the system is a number system.
c) Show that this system is isomorphic to the system defined in Ex. 2 and 5.

7. Define addition and multiplication for the system of three elements a, b and c as follows:

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

a) Show that a is a zero element and b is a unity element.
b) Show that the tables define a number system.

8. If a and b are both zero elements in a number system, prove that $a = b$.
9. If a and b are both unity elements in a number system, prove that $a = b$.
10. Show that $(20\ av\ 24)\ av\ 36 \neq 20\ av\ (24\ av\ 36)$. (See p. 21.)
11. Define a binary operation D for the system of natural numbers (including 0) as follows: If $a = b$, $aDb = 0$. If $a \neq b$, $aDb =$ the larger of the numbers minus the smaller one. (For example, $5D5 = 0$; $6D2 = 4$; $3D4 = 1$.)
   a) Prove that the operation D is commutative.
   b) Prove that the operation D is not associative.
12. Define a binary operation S for the system of natural numbers as follows: $aSb = 0$, if a is less than b; $aSb = 1$, if a is greater than b; and $aSa = a$.
   a) Construct the table for the operation S.
   b) Prove that the operation S is neither commutative nor associative.
13. Determine whether the binary operation $*$ defined for the system of natural numbers is commutative or associative if
   a) $x * y = x + 3y$.
   b) $x * y = x + y + 5$.
14. a) Use ordinary addition and multiplication as operations on the set $\{0, 3, 6, 9, \cdots\}$ containing all the natural numbers that are multiples of 3. Show that, with these operations, the system of multiples of 3 is a number system.
   b) $k \longleftrightarrow 3k$ defines a one-to-one correspondence between the system of natural numbers and the system of multiples of 3. Show that this correspondence is *not* an isomorphism.

CHAPTER II

Ex. 1-3 are on p. 43 of the text.

4. Assign names to the subsets found in Ex. 3.
   a) Construct a table for the union operation for this system of subsets.

b) Construct a table for the intersection operation.

5. Assign names to the subsets of $\{x\}$ as follows: $I = \{x\}$, $0 = \{\ \}$. Construct a table for the union operation and intersection operation for this system of subsets.

6. Let $I = \{a, b, c, d, e, f, g, h\}$. If X is a subset of I, let X' represent the complement of X in I.
Let $R = \{a, b, e, f\}$ $S = \{a, b, g, h\}$ $T = \{a, b, c, d\}$
$U = \{g, h\}$ $V = \{a, b, c\}$ $W = \{a, e, h\}$
$Z = \{b, d, g\}$
List the elements in each of these sets:
a) $R', S', T', U', V', W', Z'$.
b) $R \cup S, R \cup T, R \cup U, R \cup V, R \cup W$.
c) $R \cap S, R \cap T, R \cap U, R \cap V, R \cap W$.
d) $R \cup Z, S \cup T, S \cup U, S \cup V, S \cup W$.
e) $R \cap Z, S \cap T, S \cap U, S \cap V, S \cap W$.
f) $S \cup Z, T \cup U, T \cup V, T \cup W, T \cup Z$.
g) $S \cap Z, T \cap U, T \cap V, T \cap W, T \cap Z$.
h) $U \cup V, U \cup W, U \cup Z, V \cup W, V \cup Z, W \cup Z$.
i) $U \cap V, U \cap W, U \cap Z, V \cap W, V \cap Z, W \cap Z$.
j) $R \cup S', T \cup W', U \cup V', U \cup Z', W \cup Z'$.
k) $R' \cup S, T' \cup W, U' \cup V, U' \cup Z, W' \cup Z$.
l) $R \cap S', T \cap W', U \cap V', U \cap Z', W \cap Z'$.
m) $R' \cap S, T' \cap W, U' \cap V, U' \cap Z, W'' \cap Z$.

CHAPTER III

Ex. 1-6 are on pp. 71-2 of the text.

7. Add $(6 \sim 2) + (2 \sim 5)$; $(10 \sim 1) + (1 \sim 10)$; $(7 \sim 4) + (12 \sim 3)$.
8. Multiply $(6 \sim 2) \cdot (2 \sim 5)$; $(10 \sim 1) \cdot (1 \sim 10)$; $(7 \sim 4) \cdot (12 \sim 3)$.
9. Use the test for equality of integers to prove that $(5 \sim 2) = (7 \sim 4)$. (See p. 48.)
10. a) Add $(5 \sim 2) + (x \sim y)$.
b) Add $(7 \sim 4) + (x \sim y)$.
c) Use the text for equality of integers to show that the answers to a) and b) are equal.
11. a) Multiply $(5 \sim 2) \cdot (x \sim y)$.
b) Multiply $(7 \sim 4) \cdot (x \sim y)$.
c) Use the test for equality of integers to show that the answers to a) and b) are equal.
12. Prove $(a \sim b) + [(c \sim d) + (e \sim f)] = [(a \sim b) + (c \sim d)] + (e \sim f)$.
13. Prove $(a \sim b) \cdot [(c \sim d) \cdot (e \sim f)] = [(a \sim b) \cdot (c \sim d)] \cdot (e \sim f)$.

14. Prove $(a \sim b) \cdot [(c \sim d) + (e \sim f)] = (a \sim b) \cdot (c \sim d) + (a \sim b) \cdot (e \sim f)$.
15. Prove $(a + d \sim b + d) = (a \sim b)$.
16. If A is an integer, prove $(-1) \cdot A = -A$. (Hint: Let $A = (x \sim y)$. Then $-A = (y \sim x)$. $-1 = (0 \sim 1)$.)
17. Prove that in any ring with a unity element $(-1) \cdot A = -A$. (Hint: $(-1) + 1 = 0$. Multiply both sides by A on the right.)
18. Prove that if A and B are elements of a ring,
a) $(-A) \cdot (B) = -(A \cdot B)$.
b) $(A) \cdot (-B) = -(A \cdot B)$.
c) $(-A) \cdot (-B) = A \cdot B$.
d) $-(A + B) = (-A) + (-B)$.
(Hint: Use the fact that $A + (-A) = 0$, and $B + (-B) = 0$.)
19. Define eight permutations on the letters A, B, C, D as follows:

I	P	Q	R	S	T	U	V
$A \to A$	$A \to B$	$A \to C$	$A \to D$	$A \to A$	$A \to C$	$A \to B$	$A \to D$
$B \to B$	$B \to C$	$B \to D$	$B \to A$	$B \to D$	$B \to B$	$B \to A$	$B \to C$
$C \to C$	$C \to D$	$C \to A$	$C \to B$	$C \to C$	$C \to A$	$C \to D$	$C \to B$
$D \to D$	$D \to A$	$D \to B$	$D \to C$	$D \to B$	$D \to D$	$D \to C$	$D \to A$

a) If consecutive vertices of a square are called A, B, C, D respectively, show that I, P, Q, R, S, T, U, V are the symmetries of the square.
b) Define * as in Ex. 4, and construct the multiplication table for this set of permutations.
c) Prove that they form a group with the operation *.
d) Prove that I, P, Q, R form a subgroup.
e) Prove that I, S form a subgroup.
f) Find four more subgroups.
20. Use the associated remainders, 0, 1, 2, 3, as the names of the residue classes modulo 4. Construct addition and multiplication tables for these residue classes. What element is a zero divisor in this system?
21. Show that the set of all multiples of 4 is a subgroup of the group of integers with respect to the operation +. Show that it is a subring of the ring of integers. Show that it is an ideal of the ring of integers.
22. Every integer can be written in the form $3n + r$, where $r = 0, 1$, or 2, and n is an integer. $3n + r \to r$ defines a mapping of the ring of integers onto the residue class ring modulo 3. Show that this mapping is a ring homomorphism.
23. Show that the subgroup I, P, Q, R in Ex. 19 is isomorphic to the residue class group modulo 4 with + as group operation. (Ex. 20.)

24. Let K = the subgroup $\{I, P, Q, R\}$ in Ex. 19.
Denote the set $\{S * I, S * P, S * Q, S * R\}$ by $S * K$.
Denote the set $\{I * S, P * S, Q * S, R * S\}$ by $K * S$.
$S * K$ is called a left coset of K, and $K * S$ is called a right coset of K.
   a) Show that in this case, $S * K = K * S$.
   b) Show that $S * K = T * K = U * K = V * K$.
   c) Denoting the coset obtained in b) by L, define the operation $*$ for the set $\{K, L\}$ as on p. 67, and construct the multiplication table for the set $\{K, L\}$ with this operation. Show that the set $\{K, L\}$ with the operation $*$ is a group (a quotient group). Show that it is isomorphic to the subgroup $\{I, S\}$.

CHAPTER IV

Ex. 1-4 are on p. 89 of the text.

5. Prove that multiplication of rational numbers is associative.
6. Prove that multiplication of rational numbers is distributive with respect to addition.
7. a) Add $\left(\frac{-3}{2}\right) + \left(\frac{x}{y}\right)$
   b) Add $\left(\frac{-9}{6}\right) + \left(\frac{x}{y}\right)$
   c) Use the test for equality of rational numbers to show that the answers to a) and b) are equal.
8. a) Add $\left(\frac{a}{b}\right) + \left(\frac{x}{y}\right)$
   b) Add $\left(\frac{a \cdot c}{b \cdot c}\right) + \left(\frac{x}{y}\right)$
   c) Use the test for equality of rational numbers to show that the answers to a) and b) are equal.
9. a) Multiply $\left(\frac{-3}{2}\right) \cdot \left(\frac{x}{y}\right)$
   b) Multiply $\left(\frac{-9}{6}\right) \cdot \left(\frac{x}{y}\right)$
   c) Use the test for equality of rational numbers to show that the answers to a) and b) are equal.
10. a) Multiply $\left(\frac{a}{b}\right) \cdot \left(\frac{x}{y}\right)$
   b) Multiply $\left(\frac{a \cdot c}{b \cdot c}\right) \cdot \left(\frac{x}{y}\right)$
   c) Use the test for equality of rational numbers to show that the answers to a) and b) are equal.

11. Prove that $\left(\frac{a}{c}\right) + \left(\frac{b}{c}\right) = \left(\frac{a+b}{c}\right)$
12. If $c \neq 0$, prove that $\left(\frac{a}{b}\right) = \left(\frac{a \cdot c}{b \cdot c}\right)$
13. Define "greater than" (denoted by $>$) for the rational number system as follows: $a > b$ if a-b is positive. (See p. 54.) If a and b are rational numbers, and $a > b$, let $m = \frac{1}{2}(a + b)$. Prove that $a > m$, and $m > b$.

CHAPTER V

Ex. 1-7 are on p. 124 of the text.

8. The method used on page 107 for expressing a repeating decimal as a fraction can be used for finding the sum of some convergent series. For example, if $x = 1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots\ldots$, then $x = 1 + \frac{1}{2}[1 + \frac{1}{2} + (\frac{1}{2})^2 + \ldots]$ and $x = 1 + \frac{1}{2}x$. Solving this equation for x, we find that $x = 2$. Use this method to find the sum of each of these series:
   a) $1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\ldots$
   b) $1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\ldots$
   c) $1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\ldots$
   d) $2 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots$
9. Which of these sets of real numbers are open? closed? neither open nor closed?
   a) $\{1, 2, 3, 4, 5\}$
   b) The set of all integers.
   c) The set of all numbers x such that $2 < x < 3$.
   d) The set of all numbers x such that $2 \leqq x < 3$.
   e) The set of all numbers x such that $2 \leqq x \leqq 3$.
   f) The set of numbers of the form $5 - \frac{1}{n}$, where n is a positive integer.

CHAPTER VI

Ex. 1-11 are on pp. 146-7 of the text.

12. Using A, B, C, D, as defined in Ex. 10,
   a) find the products $A * B$ and $B * A$.
   b) find the products $A * C$ and $C * A$.
   c) find the products $A * D$ and $D * A$.

d) find the products $B * C$ and $C * B$.
e) find the products $B * D$ and $D * B$.
f) find the products $C * D$ and $D * C$.

CHAPTER VII

Ex. 1-4 are on p. 159 of the text.

5. Prove that
$$r\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} r & 0 \\ 0 & r \end{vmatrix} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6. Prove that
$$r\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

7. The transformation B is defined by: $x' = 2x + y$
$y' = 3x + 2y$.
   a) Find B^{-1}.
   b) Write the matrices of B and B^{-1}.
   c) If we denote these matrices by B and B^{-1} respectively, show that $B\,B^{-1} = B^{-1}\,B = I$.

8. If $A = \begin{vmatrix} r & 0 \\ 0 & s \end{vmatrix}$, and $B = \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix}$, prove that $AB = BA$.

9. If $A = \begin{vmatrix} r & 0 \\ 0 & r \\ 0 & r \end{vmatrix}$, and $B = \begin{vmatrix} x & y \\ z & w \end{vmatrix}$, prove that $AB = BA$.

10. If $A = \begin{vmatrix} r & 0 \\ 0 & s \end{vmatrix}$, and $B = \begin{vmatrix} x & y \\ z & w \end{vmatrix}$, find AB and BA.

11. If $A = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$, and $B = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$, prove that $AB = BA$.

12. If $A = \begin{vmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{vmatrix}$, and $B = \begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}$, prove that $AB = BA$.

13. If $A = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$, and $B = \begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}$, find AB and BA.

Index

What are "real" numbers?

What are "imaginary" numbers?

Can there be a number system *without* numbers?

We learn to count almost as soon as we learn to speak. What we rarely learn are the *hows* and *whys* of our workday numbers. Irving Adler's book is the first to explain the familiar 1, 2, 3, and to take out of hiding the basic and profound concepts that are found there. He traces advanced theories from their beginning in grammar-school arithmetic, and helps the layman to discover the meaning of the revolutionary developments in the world of modern mathematics.

IRVING ADLER is an Instructor in Mathematics in the School of General Studies at Columbia University and the author of more than a dozen books, including **How Life Began, Magic House of Numbers,** and **The Stars.** The hardcover editions of his books are published by The John Day Company.

PUBLISHED BY THE NEW AMERICAN LIBRARY