WHY SCHOOL MATHEMATICS SHOULD BE TAUGHT IN A CONTEMPORARY SETTING

by

Howard F. Fehr

Is there a "new" mathematics? Reading the current educational literature and the titles of textbooks for the elementary and secondary schools one would come to believe so. The words "new", "modern", "contemporary" and the like have now been bandied around in a very nebulous and ambiguous fashion for a number of years. Yet most school programs contain practically the same mathematics that appeared in school instruction fifty years ago, with some modification in presentation and with a few new symbols. So, at the start, it must be emphasized that practicing mathematicians make no distinction between "new" and "old" mathematics. To them, mathematics today is what it is as a result of gradual development and critical appraisal over a long period of time. With increased knowledge there comes deeper insight into the nature of their subject. The contemporary setting has come about from a very serious study, made over the last one hundred fifty years, of the fundamental basic concepts with which they were dealing.

To show the development of mathematics over these 150 years in all detail is a Herculean task and all that can be done here is to give a sketch of the few highlights that reflect the present nature of our subject. For the purpose of contrast I shall briefly review the historical development of what I shall call the classical setting.

The Classical Setting

Traditionally, mathematics grew out of a need for understanding the physical environment and thus created systems for counting and measuring (arithmetic), and an idealization of sensory physical space (Euclidean geometry). Further generalization of these topics led to the study of algebra. The early paradoxes of zero, the summing of areas and volumes, and the study of tangents to curves led to the development of infinite processes which by 1750 A.D. had become a separate field called Analysis. More than two hundred years ago, the gradual development of these branches led to the traditional organization of mathematical content into four main divisions: Arithmetic, Albegra, Geometry, and Analysis--each considered a closed and separate field of investigation. Under each of these branches a profliferation of separate subjects came into existence. Indeed, until quite recently, if one asked a mathematician what he considered his field of investigation he would respond: "I am an algebraist" or "I am an analyst" or the like. This does not imply however that he did not know the other fields. This is the classical setting of mathematics(Figure 1).

It is easy to understand how this traditional organization became the pattern for organizing the school curriculum in mathematics. The first item an adult needed to know was number, using it in counting, measuring, and computation applied to business and social affairs of everyday life. During the 14th and 15th centuries arithmetic was the principal study of the

CLASSICAL MATHEMATICS

ARITHMETIC	ALGEBRA
Theory of numbers	Computational algebra
Combinatorial analysis	Matrices and determinants
Calculus of probabilities	Theory of equations
Nomography	Polynomial algebra
Interpolation	Higher algebra
Etc.	Etc.

GEOMETRY	ANALYSIS
Euclidean synthetic	The calculus
Analytic	Numerical analysis
Descriptive	Theory of functions
Projective	Differential equations
Non-Euclidean	Calculus of variations
Vector analysis	Harmonic analysis
Differential	Tensor analysis
Etc.	Etc.

Fig. 1

Classical Structure of Mathematics

European university. As new knowledge, such as algebra, entered
the university study, arithmetic descended to the secondary
school and finally down to the elementary school. Very early
in the history of man, measurement and explanation of the
universe about him became a necessary activity of his life. This
knowledge was organized into a practical geometry and later
idealized into abstract space we now call Euclidean geometry.
Until the middle of the 19th century this geometry was university
study, and then it gradually descended into the secondary school
curriculum. Today, the geometry of the physical universe is a
part of elementary school study.

Algebra, throughout the middle ages, was a study for the
most learned of scholars. But as commerce and navigation
increased, there developed on the part of educated workers a
need to understand general algebraic formulas and trigonometric
solutions of triangles. The subject then became a university
study and by the end of the last century had descended into
the secondary school curriculum. In the early part of this
century at the encouragement of Felix Klein, calculus, a
graduate and university study, became a subject of instruction
in the German gymnasium. Within the last fifty years it has
become a secondary school subject in the scientific line for
all developed countries. However, in this downward descent
all these subjects took on aspects of a prefabricated and
eternally lasting knowledge to be passed on one generation to
the next. The traditional arithmetic of computation, algebra

of manipulation, geometry of Euclidean space, and formula operative calculus became the program, in all schools of all countries, and has remained fairly constant since the beginning of the 20th century.

While maintaining a sort of classical program, the other developed countries of the world--Europe, Russia, Japan--have at least integrated the subjects to the extent that algebra, geometry, calculation and analysis are all taught in each of the years of secondary school study. The United States of America is the only developed country that teaches one year of algebra and no other mathematics during this year and then drops the subject; then teaches a year of geometry (mostly synthetic Euclidean geometry) and nothing else and then drops the study of pure geometry forever. Then, in the U.S.A. we follow this with another year of algebra, one-third of which is spent reteaching the first year of algebra which has been forgotten in the interim. A fourth year is spent on a variety of topics preparing students for collegiate study. For an exceptionally small group of the mathematically elite, this program is begun a year earlier (eighth grade) allowing the senior year for advanced-placement calculus. This program is inefficient, and it is out of touch with contemporary thought.

The Genesis of Contemporary Mathematics

Up to 1800, mathematicians were so engrossed in doing mathematics that they paid little attention to the nature of the elements with which they were working. However, during the past 150 years mathematicians have been studying their

subject, what its ambitions are, and what limits are imposed
on these ambitions. The strides that have been taken have
literally shaken apart the foundations of classical mathematics.
The revelation of the nature of number by Peacock, Hamilton,
Dedekind, and Weierstrass, the development of groups by Abel
and Galois, the discovery of matrices by Sylvester and their
algebraic development by Cayley, the new concepts of space
initiated by Lobachewsky, Bolyai, and Riemann, the arithmetization
of space by Grassmann--all contributed to new insights and
relations among the classical branches. This activity reached
a climax with Cantor's discovery of set theory, Hilbert's
development of formalism, and an entirely new structural point
of view developed by the Bourbaki.

The last, a group of mathematicians, founded their associa-
tion in France in the 1930's. The main objective was (and still
is) to reconstruct the whole of mathematics--classical and modern--
on a broad general basis, so as to encompass the whole of it
as one unified study. They broke down the barriers of the
classical organization by founding mathematics on the theory of
sets, relations, and functions. On this base they erected two
great structures--the algebraic and the topological, in each
case partitioning the structures into sub-structures. The
algebraic structures included groups, rings, modules, fields, and
the topological structures included groups, compact spaces, convex
spaces, metric spaces and others. Both these structures, are
strongly united in vector space structure. The Bourbaki structure
is complex, but it gives an excellent overview of all mathematics
(Figure 2).

STRUCTURES OF MODERN MATHEMATICS

Theory of Numbers

Lie Groups

Functional Equations

Distributions

Geometric Algebra

Differential Geometry

Functions of Complex Variable

Polynomials

Integration

Normed Spaces

Algebras

Algebraic Equations

Geometry

Functions of Real Variable

Vector Spaces

Topological Vector Spaces

Differential Algebra

Measure

Complex Numbers

Fields

Real Numbers

Metric Spaces

Modules

Ideals

Rings

Topological Algebra

Algebra of Boole

Ordered Structures

Compact Spaces

Groups

Topological Groups

Algebraic Topology

Locally Compact Groups

Semigroups

Algebraic Structures

Finite Sets

Combinatorics

Graphs
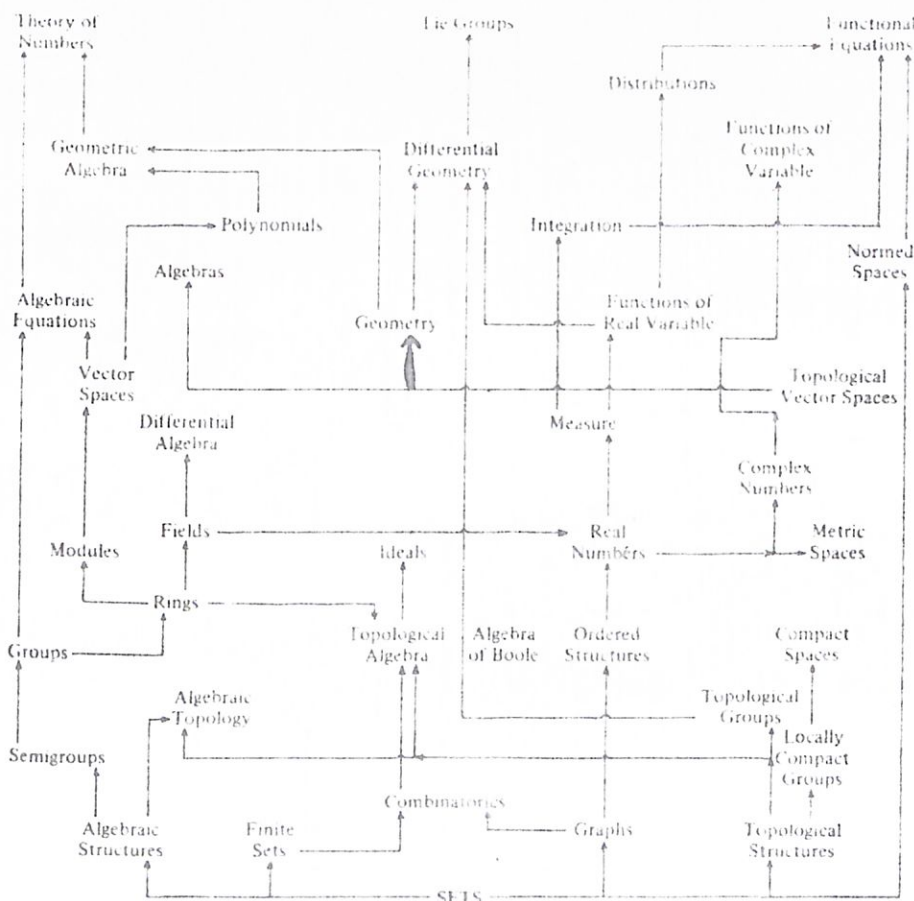
Topological Structures

SETS

Figure 2

It is this structure, or any similar one, that reveals to us the contemporary setting of mathematics. While recently developed topics are shown in the structure, all the important traditional mathematics is also embedded in it, clothed with new language and symbolism. In order to obtain a sharper picture of the Bourbaki unification, we now recall some significant steps leading to the contemporary view of mathematics.

The development of classical algebra, from that of the Babylonians and Egyptians, through the work of the Greeks,

notably Heron and Diophantus, extended by the Hindus and Arabs
in the middle ages and brought to a head by the Italian School
(Fibonacci, Ferrari, Cardan, Bombelli) and the French (Vieta,
Fermat, Descartes, DeMoivre) was epitomized in Euler's "Intro-
duction to Algebra" (1770) in which algebra is defined as:

<u>The Theory of Calculation with Quantities</u>

The subject matter in this textbook is an assortment of topics
such as can be found in most present day textbooks on the subject.
To this content a hundred years later Serret added all the classi-
cal theory relating roots of an equation to the solution of
equations. In his <u>Algebra</u> (1860) he defines the subject as

<u>The Analysis of Equations.</u>

Most of the content of this book is found in the great spate
of textbooks published from 1880 to 1940 called 'Theory of
Equation". There is in Serret's book, however, one of the
first milestones in the development of modern algebra; namely,
Galois theory.

The Development of Modern Algebra

The development of modern algebra was the result of the
gradual merging of three streams of mathematical endeavor in
one great confluence. This merger of algebraic, geometric, and
arithmetic creative research exposed fundamental concepts that
are the base of the foundations of all mathematics. The geometric
stream started with the geometric explanation of the complex
numbers via the Wessel-Argand diagram. Gauss also developed the
complex numbers as a system of ordered pairs of real numbers,
thus setting the stage for the development of structure. By

1840, vectors as oriented line segments or arrows were used in physics, and this use was developed by mathematicians into vector geometry, and vector analysis of two and three dimensions. There thus developed an algebraic geometry, which completed the liberation of geometry from the shackles of purely synthetic treatment. Finally, the use of transformations led to the use, through Klein in 1872, of groups to distinguish one geometry from another.

Arithmetic, as could be expected, became the greatest contributor to abstract structure. The logical development by Gauss and Hamilton of complex numbers as ordered pairs marked the turning point in mathematical thought, for it opened the way to the postulational method in algebra and also suggested a procedure for the explanation of ordered triples, ordered quadruples, and finally, ordered n-tuples. Gauss also introduced modular arithmetic and the equivalence relation for classification of the integers. Cauchy did the same thing for polynomial functions, that is $F \equiv R$ (mod M), and for the modulo (x^2+1) found that his residual classes R, of the form a+bx, had all the formal properties of complex numbers with x replacing i. He constructed a whole real number algebra identical with, or having the same structure as, the algebra of complex numbers, thus paving the way for the important concept of isomorphism.

The next breakthrough was made by Hamilton and his creation of quaternions. His great contribution was the rejection of commutativity of an operation as a necessary requirement for a number system. Cayley at the same time was developing the

arithmetic of matrices and showed a complete model of the com-
plex numbers in the arithmetic of 2x2 matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = ai + bj .$$

He also developed the theory of quaterions using 2x2 matrices

of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Later quaternions were developed as

ordered pairs of complex numbers, all of these ideas giving
further reinforcement to the concept of isomorphism.

Grassmann in 1844 took the giant step which placed him
beyond the confines of Euclid's three dimensions with his theory
of extension in which he developed real space of n-dimensions,
or manifold of n-dimensions, as the set of ordered n-tuples
$(a_1, a_2, a_3, \ldots a_n)$. The stage was now set for the creation
and extension of linear algebra that took place from 1880 to
1940, in which vector space structure was developed. The only
other significant creation was made by the characterization of
a number system by its properties. This was accomplished by
Dedekind in 1879 when he developed the _field_ properties of the
real numbers, although he did not use the word "field"!

The algebraic stream began with Lagrange's permutation of
the three roots of a cubic equation into two cyclic subsets.
Galois extended this study by examining the set of permutation
of the n roots of a polynomial. He thus developed the theory
of permutation groups where the operation was that of composition,
and for the first time in its history mathematics had a binary
operation performed on objects other than variables or numbers.

From then on, mathematicians busied themselves to the nth degree with the development of abstract groups. With the development of group structure for both the operations + and $\cdot$, as well as other abstract operations, and the recognition of the field structure of real and complex numbers, it still required about twenty-five years more to obtain the precise formulation that was given by Whitehead and Russell in terms of isomorphism of two sets.

The convergence of these three streams--arithmetic, algebra, and geometry gave birth to a new algebra, a unique study different from that of classical algebra. The date of birth may be given as 1910 with the publication of Steinitz's The Algebraic Theory of Fields (Theorie Algebriques des Corps). It came of age with the work of Emmy Noether, Emil Artin, and others at the University of Hamburg, and for the first time this modern definition of structure was revealed in Modern Algebra (Moderne Algebra), by Van der Waerden in 1931. The first such book on algebraic structures in English was A Survey of Modern Algebra by Birkhoff and MacLane (1941). These dates confirm the use of the word "modern".

Today algebra is a study of structures. Problems that previously were insolvable by the techniques of classical algebra are now examined from a different point of view and, in many cases, solved. In geometry and in analysis these structures become a unifying thread. The university instructor who gives a course in modern analysis finds that unless his students have a good foundation in contemporary algebra, he cannot discuss

important concepts, e.g., that of an operator. In geometry the concept of group enabled mathematicians to define different spaces in terms of possible transformations. The new concept of algebra became, through a study of its morphism, a tool for further expansion of mathematics--e.g., categories and functors.

Another achievement of abstract algebra is in its description of physical phenomena. In 1890 a Russian crystallographer, E. S. Fedorov, applied group theory to the classification of systems of points in space to define the atomic structure of crystals. This was the first time that group theory had been applied to solve a previously unresolved problem in science. Later, J. W. Gibbs, an American scientist, used an algebra of ordered triples to develop the theory of refining oil by cracking crude oil. Recent applications of matrices and linear algebra have solved technological problems in all the sciences--physical, behavioral, biological--all of which have confirmed the basic requirement of acceptance of any 'new' mathematics, that is, its successful use outside mathematics per se.

The current view of algebra can be described as the study of structures for which there is

1. A set of undefined elements (the set).

2. A set of statements relating the elements (the structure).

3. A logic for drawing inferences (the language).

4. A series of propositions that can be proved (the theory).

5. Realizations and applications within itself or in other disciplines via models (the use).

From this necessarily brief description of modern algebra, it is clear that algebra today represents a conception completely different from that now taught in the schools. Today, numbers, variables, expressions, functions, conditional sentences, and the host of activities and applications to which they are put to use are all subservient to the structures from which they are derived. From 1910 to 1960 abstract structures and linear algebra were conceived of as graduate or undergraduate advanced study, far removed from secondary school instruction. However, following the traditional pattern of descent to lower level study, we recognize that this modern algebra must be at the very core of what we teach in high school--of course, presented in a form adaptable to secondary school maturity. Moreover, the unity of mathematics demands that this algebra be taught with every possible intervention into geometry and analysis.

The Development of Geometry

From around 325 B.C. to 1827 A.D. only one geometry existed as a means to study space, namely Euclid's synthetic axiomatic geometry. During all this time the only controversial question was the possibility of proof of the parallel postulate, and this occupied the energies of great mathematicians--Omar Khayyam, Wallis, Saccheri, Lambert, Lengendre, Gauss, Bolyai, and Lobachewsky. The works of these men paid high tribute to the genius of Euclid in accepting this postulate, for through their efforts to prove it, non-Euclidean geometries were invented, and for the first time there arose the obvious implication:

There is more than one geometry.

The mathematical world did not at first accept the conclusions of these men. They could not understand any space other than that described by Euclid. But in 1854, Bernhard Riemann generalized the concept of space by creating new geometries. The immediate result of Riemann's paper: <u>On the Hypotheses Which Lie at the Base of Geometry</u>, published posthumously in 1868, was a burst of activity in the development of different types of geometry. A new light on space interpretation was presented by Fleix Klein in 1872 when he showed that one geometry may be distinguished from another by its group of transformations. A geometry may determine a group and a group determines a geometry. For example, the group of similitudes and the group of isometries lead respectively to affine and Euclidean spaces. (However, it must be noted that there are geometries that do not possess a group structure.)

Riemann, in his first paper, also pointed out some of the flaws in Euclid's list of postulates, thereby initiating a spate of activity among mathematicians to clear Euclid of all blemishes. This task was completed first by Pasch in 1882 and subsequently by Peano, Pieri, Hilbert (1899),and Veblen, 1901 With the problem of perfecting Euclid's synthetic geometry solved, outside of the possible discovery of some more exceptional points, lines, or circles, the study reached a dead-end.* However, the solution resulted in a lengthy set of axioms deemed far too complicated and abstract to be used in secondary school instruction. In Europe and America there followed a sixty year

---

*It can be argued, however that general topology developed out of it.

period of sporadic efforts to do something about the subject
to retain it as a secondary school study. Euclid had come
down from the collegiate study to the secondary school,
and the feeling persisted that "Euclid must be saved!" To
save Euclid, a commonly used modification of Hilbert's axioms
was given by Birkhoff in 1929 when he assumed all the properties
of real numbers and created the "real ruler" and "real protractor"
axioms.

Today the development of geometry and its counter-part,
topology, is going on in all directions. Its pervasiveness
in mathematics and science may be sensed by a partial list of
geometries such as: affine, projective, Euclidean, hyperbolic,
elliptic, combinatorial, absolute, analytic, differential,
algebraic, Minkowskian, integral, transformation, vector, linear,
topological, conformal, optical, relativistic, and so on,
involving infinite dimensional, convex, metric, finite dimensional,
and compact spaces. It is thus obvious that geometry today
represents a conception quite different from that exhibited
in the contemporary high school program.

Today, geometry is the study of spaces where each space is
a (set, structure). This concept of geometry has evolved slowly
over the last 140 years as a result of two phenomena. The first
was the discovery and development by mathematicians of the
geometries mentioned before, and "spaces" such as topological,
vector, Banach, and Hilbert. The second and most influential
phenomenon occured as a result of advances in science and
technology. Prior to 1900 the only geometry used outside of

pure mathematics was Euclid's. The advent of relativity changed this state of affairs. After Einstein developed the concept of matter in a space-time relationship described by a fourth dimensional model of Riemann space, various other spaces and non-Euclidean geometries found application in physics, astronomy, biology, and economics. As a result, today Euclidean geometry is only a small part of either pure of applied mathematics. To teach it as the only geometry is to give our students a distorted picture of what is going on in the world of mathematics.

Linear Algebra and Vector Spaces

The concept of a vector space is now the fundamental building block for many of the "new" areas of mathematics and science. It is a way of unifying algebra, geometry, and analysis. Jean Dieudonne has claimed that there cannot exist an elementary geometry which is separated from linear algebra and from vector space. All developed countries of the world are in the process of including its study in secondary school mathematics. As an example of the power of the concept, consider the following stages. Starting with a <u>vector space</u> (a set with the vector space structure) we can introduce an inner product to get an <u>inner product space</u>. We can extend the structure further by considering the norm which arises from the inner product $\left[\|x\| = \sqrt{(x,x)}\right]$ and now we have <u>normed vector space</u>. Depending on the function which defines the inner product and the vector, we get various other spaces from this procedure.

pure mathematics was Euclid's. The advent of relativity changed
this state of affairs. After Einstein developed the concept
of matter in a space-time relationship described by a fourth
dimensional model of Riemann space, various other spaces and
non-Euclidean geometries found application in physics, astronomy,
biology, and economics. As a result, today Euclidean geometry is
only a small part of either pure of applied mathematics. To
teach it as the only geometry is to give our students a distorted
picture of what is going on in the world of mathematics.

Linear Algebra and Vector Spaces

The concept of a vector space is now the fundamental building
block for many of the "new" areas of mathematics and science.
It is a way of unifying algebra, geometry, and analysis. Jean
Dieudonne has claimed that there cannot exist an elementary
geometry which is separated from linear algebra and from vector
space. All developed countries of the world are in the process
of including its study in secondary school mathematics. As an
example of the power of the concept, consider the following
stages. Starting with a vector space (a set with the vector
space structure) we can introduce an inner product to get an
inner product space. We can extend the structure further by
considering the norm which arises from the inner product
$$\left[\|x\| = \sqrt{(x,x)} \right]$$ and now we have normed vector space. Depending
on the function which defines the inner product and the vector,
we get various other spaces from this procedure.

For example, if we start with $R^3 = \{(x_1, x_2, x_3): x_1, x_2, x_3 \in R\}$ and define

1. $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$

2. $\alpha(x_1, x_2, x_3) = (\alpha x_1,\ \alpha x_2,\ \alpha x_3)$,

we have a vector space. Now define a mapping from $R^3 \times R^3$ to R

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

We now have an inner product space. Let the norm be

$$\|(x_1, x_2, x_3)\| = \sqrt{(x_1, x_2, x_3) \cdot (x_1, x_2, x_3)} = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then the distance between any two points (x_1, x_2, x_3) and (y_1, y_2, y_3) becomes

$$\|(x_1, x_2, x_3) - (y_1, y_2, y_3)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

and we have a Euclidean vector space. Thus we started with an algebraic structure and arrived at a space for which we can give a geometric representation. This interpretation then provides an opportunity for the student to study Euclidean space from an algebraic point of view. In fact, we have algebraic geometry.

The foregoing example illustrates the power and spirit of mathematics in a contemporary setting. The emphasis on basic concepts such as sets, relations, functions, and operations and the study of fundamental structures such as groups and vector spaces unify all of mathematics. Contemporary mathematics--from the simplest arithmetic to the most abstract analysis can be described succinctly as the study of the ordered pairs.

$$(\mathcal{E}, S) = (Set,\ structure)$$

and all the derived activities. Elementary algebra is the study

of the set of real numbers, and its important subsets: cardinals, integers, and rationals. The structure is described by the field properties and relations of order and density. The activity consists of manipulating expressions, developing functions--such as polynomial, exponential, trigonometric--and the solution of sentences relating two functions. Geometry is the study of spaces--an $(\mathcal{E}, S)$ where the set is a collection of elements called points, with lines and planes as important subsets. The structure is described by relations such as betweenness, parallelism, perpendicularity; the action is provided by transformations, such as rotations, reflections, translations, shears, and their groups leading to isometries and similitudes. Probability is a study of a set--the outcome space and a measure structure imposed upon it. The activity consists of dependent and independent events, conditional relations, random variables (really functions), expectancy and distributions. The action is in its application to business, science, economics, game playing and the like.

The awakening to the realization that certain fundamental concepts are the structural backbone of all mathematical branches was stated by Lichnerowicz, who said:

> "Anyone who studies contemporary mathematics' view
> of itself will observe three major features. One
> is first struck, I think, by the absence of a
> privileged plan of mathematical beings. A set (or
> a category) is, I venture to say, a set of anything--
> numbers or functions certainly, but also a set of

sentences in a language, of elementary tasks in a
project, or of exchanges within an economy. Various
structures can be defined from these sets, the actual
concept of structure lending itself easily to a
technical definition, and is based on two fundamental
operations concerning sets: taking the product of
several sets, taking the set of subsets of a set.
Perfect dictionaries can exist between sets, relating or
transporting structures which leads us to the concept
of isomorphism between structures.

At the same time, ther is no idolatry of the thing
for itself, no charism, within the mathematical process.
The mathematician always works to the nearest perfect
dictionary and often unscrupulously identifies objects
of different nature when a perfect dictionary or
isomorphism assures him that he would only be saying
the same thing twice in two different languages.
Isomorphism takes the place of identity. The Being is
put between brackets and it is precisely this non-
ontological characteristic which gives mathematics its
power, its fidelity, and its polyvalence. In truth,
any fact can be regarded as mathematifiable so long
as it submits to this singular treatment of isomorphism
or rather insofar exactly what we overlook in this way
is not important to us. We can always weave a mathematical
net with an arbitarily close mesh but from which the
ontological wave will necessarily flow away.

A third feature of contemporary mathematics is its
<u>unity</u>. By making a common language and finding common
elementary structures it has cast aside the old
historical framework which would have broken it up
into disciplines evolving in different ways. That is
why we can speak of <u>The Mathematic</u>."

This is an admirable and eloquent description of the view-
point of mathematics today. It affords a sharp contrast between
mathematics in a traditional setting--isolated branches--and that
in the contempoary setting--a unified discipline. The traditional
setting--that which is still dominant in the schools--has held
sway for over seventy years--and even minor changes in this
curriculum have been notoriously slow in acceptance. In contrast,
the speed with which science and technology are transforming
conditions of modern life, as a theme, have been elaborated
upon so frequently as to become wearisome. As a matter of
fact, however, it is continuously filled with surprise and
wonder. Science fiction has become a reality within relatively
a few years.

I can now return to further the answer of the question posed
in the title of this paper. "Why should secondary school mathe-
matics be taught in a contemporary setting?" A satisfactory
curriculum must include provisions for a general education
essential for citizenship, as well as for the discovery and
development of individual talent. We can no longer think of our
subject as the secondary school levely merely as a preparation
for subsequent collegiate or university study of mathematics. It

should be a curriculum that provides a general liberal education in mathematics as it is conceived in the last quarter of the twentieth century. It should form the basis for entering any professional study whether it be in science, engineering, law, medicine, economics or the behavioral and humanistic endeavors. To see what mathematics is, and what its role in current society is, requires that it be conceived of as mathematicians know it today, that is, in the contemporary setting.

The amount of mathematical knowledge available for secondary school study has grown tremendously in the last 40 years. There is far too much of it to make a school program. So we must select. But on what basis? Certainly two criteria are acceptable, that of utility and that of greater generality of basic concepts. Under the first criterion, the writer believes that a large part of Euclid's synthetic development of geometry can be eliminated. A hundred years ago, it was the only subject available to teach axiomatic development, but today we have many smaller and simpler axiomatic systems in geometry, algebra, and probability for this purpose. Likewise, many of the special techniques and skills in algebra and geometric construction are not needed. Under the second criterion it is clear that the contemporary setting of sets and structures is certainly broader and more encompassing in its concepts and hence permits more efficient procedures for learning more and higher quality mathematical knowledge.

Any teacher or student who has plowed through a classical presentation before attacking the same material be the abstract

method will grant that by the latter the strain on the memory is greatly reduced and that insight is correspondingly increased. If the object is to learn mathematics with a minimum of impediments, the contemporary setting has no competitor. If we are to make progress, mathematics study cannot encumber itself with all the beautiful but less useful content it has gathered down through the ages.

It is quite natural that the modern conception of mathematics should descend from graduate research to undergraduate collegiate programs. After all, the same faculty serves or oversees both these levels of learning. However, a great gap appeared between secondary school mathematics and collegiate study of the subject. On the one hand, our students in the past decade have complained of entering an entire new world with a strange language on beginning collegiate study. On the other hand, the college professor complained of the ignorance of our students in the language and concepts necessary to grasp collegiate instruction. To bridge this gap, it is highly essential that at the earliest time possible we present our students with the content, concepts, language, symbolism and thinking that will enable them to go forward without shock, in further study of mathematics and its uses.

Some physicists complain of the abstractness of the new mathematics, and the lack of skill by students in using traditional procedures. This results from the lack of communication between mathematicians and scientists. Those scientists who know the contemporary aspect of mathematics find it far more

serviceable in solving problems in their field. Because of the use of mathematics as an <u>auxillary tool</u>, e.g., the use of probability and statistics in analysing reality, the contemporary structure of these topics is almost universally preferred by today's economists, psychologists, business executives and biologists. At the same time, structures are also used as <u>effective instruments of thinking</u>, that is, in constructing mathematical models of the behavior of situations in all sciences. In fact, whether admitted or not, the ambition of most scientists is to develop a mathematical model with as close an approximation as possible to all that he researches. The contemporary (set-structure) concept has become a powerful procedure in this pursuit.

Time after time down through the ages, new mathematical procedures outmoded the old as instruments in carrying out the affairs of men. Witness the electronic computer of today and its effect in applied science. The children in our schools today, at the end of their study over a period of 10 to 15 years will face a world which will be more thoroughly permeated by mathematics than ever before. If we desire that these young men and women shall not be anachronistic when they approach the adult age of responsibility, we must educate them today so that they will be up to the times tomorrow. It is the contemporary setting that can aid in doing this.