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ALTERNATIVE LMI FORMULATIONS APPLIED TO A DYNAMIC ANTI-WINDUP SYNTHESIS METHOD

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Para Priscila, minha amada.

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O otimista é um tolo. O pessimista, um chato. Bom mesmo é ser um realista esperançoso.

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RESUMO

A síntese de controladores para sistemas com saturação é um problema especialmente importante quando há requisitos de alto desempenho e garantias formais de estabilidade. Em face de tais objetivos rigorosos não é possível evitar o comportamento saturado do sistema e técnicas especiais precisam ser empregadas na análise e síntese de controladores para estes sistemas não-lineares.

Neste trabalho de mestrado é estudado uma técnica existente para a análise e síntese de uma classe de compensadores anti-windup modernos baseados na formulação de problemas LMI (Desigualdades Matriciais Lineares, em inglês). Esta técnica pode considerar a minimização de um critério de energia $\mathcal{L}_2$ ou a maximização de um domínio de estabilidade assintótica. Em ambos os casos o problema LMI formulado utiliza um critério de estabilidade baseado em Lyapunov bem como condições de setor modificadas para a representação das saturações. Adicionalmente, são utilizadas condições para a obtenção de controladores anti-windup com restrições de posicionamento dos polos do controlador.

Então, utilizando o Lema da Projeção, uma formulação alternativa para o problema LMI considerado é proposta. Esta formulação é uma generalização dos métodos existentes e os resultados de ambas formulações são comparados usando um exemplo numérico.

Por fim, as vantagens e desvantagens dessa formulação alternativa para o problema LMI são destacadas. O fato de que esta nova formulação fornece um grau de liberdade adicional ao problema LMI é provavelmente a principal contribuição para trabalhos futuros, visto que esta flexibilidade pode permitir a inclusão de objetivos adicionais ao problema de síntese do controlador anti-windup.

Palavras-chave: Controle, Não-linearidades, Saturação, Condições de Setor, LMI, Lema da Projeção, Lema de Finsler.

RESUMO EXPANDIDO

Sistemas de controle tornam possíveis muitas das maravilhas tecnológicas que vemos hoje em dia. Para otimizar o desempenho e reduzir os custos, tais sistemas empregam leis de controle cada vez mais elaboradas.
Nesse cenário, fatores como: tamanho, consumo, custo e peso são decisivos para a viabilidade ou não de um sistema. Uma lei de controle que consegue extrair o melhor desempenho possível de um atuador enquanto fornece garantias formais de estabilidade é de crucial importância. Em face de tais objetivos rigorosos é necessário levar o sistema a operar nos seus limites e, portanto, o comportamento saturado do sistema precisa ser considerado.
Controladores chamados anti-windup modernos [Tarbouriech et al. 2011, Tarbouriech and Turner 2009] são especialmente desenvolvidos para estes casos. Ao contrário dos esquemas anti-windup clássicos que basicamente visam "não carregar" a ação integral de um controlador PID quando o atuador está saturado, os anti-windup modernos visam minimizar o efeito da saturação ao mesmo tempo que fornecem garantias formais de estabilidade regional ou global.
Esse trabalho tem três principais objetivos. O primeiro trata-se do estudo de uma técnica de síntese de controladores anti-windup modernos baseada na formulação de problemas LMI (Desigualdades Matriciais Lineares, em inglês). Esta técnica considera a maximização de uma região de estabilidade assintótica ou então a otimização do desempenho através da minimização da energia de um sinal de comparação com uma dada referência. Em ambos os casos a síntese do controlador anti-windup é estudada com e sem considerar restrições de posicionamento de polos do controlador [Roos and Biannic 2008].
O segundo e principal objetivo é propor uma formulação LMI alternativa para a síntese de tais controladores. Essa formulação alternativa apoia-se principalmente na utilização do lema da projeção para a adição de um grau de liberdade no problema LMI. Tal formulação alternativa pode potencialmente reduzir o conservadorismo inerente na utilização das técnicas de síntese de anti-windup estudadas ou então possibilitar a inclusão de novos objetivos de projeto.
Por fim o terceiro objetivo é a comparação da técnica de síntese proposta com a já existente através do uso de um exemplo numérico que considera o controle longitudinal de uma aeronave caça.
O estudo do método de síntese de controladores anti-windup modernos

compreende primeiramente a representação do sistema a ser estudado. Este sistema em malha fechada compreende uma planta com um atuador com limitações de posição e/ou taxa de variação, um controlador linear e um sistema autônomo gerador de sinais de referência e/ou perturbação. Além disso, há também um sistema linear (sem saturação) nominal cuja saída, utilizada para fins de desempenho, representa o comportamento desejado do sistema não linear e, por fim, um controlador anti-windup dinâmico.

Para a formulação do problema LMI de análise ou síntese é feita a substituição, sem perda de generalidade, das saturações por funções do tipo zona-morta e em seguida a adoção de uma representação da zona-morta através de uma condição modificada de setor, tal como em [Gomes da Silva Jr. and Tarbouriech 2005]. Além disso, são utilizados conceitos de estabilidade baseada em Lyapunov e um critério de energia $\mathcal{L}_2$ para um sinal de comparação entre o sistema linear nominal e o sistema não linear, no caso de síntese com foco em desempenho, ou alternativamente um critério de maximização de uma região de estabilidade assintótica.

Para a formulação alternativa do problema LMI de síntese de controlador anti-windup foi utilizado o conceito de formulações LMI estendidas tal como em [Pipeleers et al. 2009]. Tais formulações estendidas são baseadas no uso do lema da projeção de forma a adicionar uma nova variável ao problema LMI.

Em linhas gerais, esse processo consiste em utilizar a equivalência entre as duas condições dadas pelo lema da projeção. Para isso é preciso primeiramente tomar a inequação que representa os objetivos de síntese do controlador anti-windup (i.e., a inequação que contempla as condições de estabilidade, de representação da saturação e de desempenho) juntamente com uma inequação adicional, selecionada a fim de acrescentar o menos possível de conservadorismo, e reescrevê-las conforme um "um lado" da equivalência dada pelo lema da projeção. Então, usando-se o "outro lado" da equivalência dada pelo lema da projeção é possível obter as condições alternativas de síntese. Ao se fazer isso aparece um fator multiplicador do lema da projeção e surge também uma nova variável no problema LMI. Esta variável pode ser vista como um grau de liberdade adicional no problema e pode ainda ser explorada para outros objetivos.

As condições LMI alternativas obtidas seguindo tal abordagem são formalizadas por meio de teoremas e algoritmos para a análise e síntese de controladores anti-windup modernos.

Um exemplo numérico foi usado para ilustrar os resultados obtidos com o controlador anti-windup calculado através do método de síntese pro-

posto e também para comparar estes resultados com resultados de controladores obtidos por uso do método de síntese já existente. O exemplo numérico escolhido trata-se do controle de atitude no eixo longitudinal de um avião caça com base na posição do profundor. Este exemplo contempla um atuador não linear para o profundor com saturações de posição e também de taxa de variação. Além disso, o modelo utilizado contempla um ponto crítico do envelope de voo relacionado à baixa pressão aerodinâmica onde a dinâmica é instável. Isto torna o exemplo bastante interessante para este estudo, visto que apenas estabilidade local pode ser obtida para uma planta instável com um atuador com limitações.

Foram estudados os casos de síntese com foco em maximização de um domínio de estabilidade e também com foco em otimização de desempenho. Em ambos os casos foi também estudada a variação de um parâmetro multiplicador que aparece devido à utilização do lema da projeção e também se considerou os casos com e sem restrição de polos no controlador anti-windup.

Os resultados de simulação do sistema com os controladores obtidos através do método de síntese proposto se assemelham muito com os resultados obtidos com controladores calculados através do método já existente.

O fato de que esta nova formulação fornece um grau de liberdade adicional ao problema LMI é provavelmente a principal contribuição para trabalhos futuros, visto que esta flexibilidade pode permitir a inclusão de objetivos adicionais ao problema de síntese do controlador anti-windup.

Palavras-chave: Controle, Não-linearidades, Saturação, Condições de Setor, LMI, Lema da Projeção, Lema de Finsler.

ABSTRACT

The synthesis of modern anti-windup controllers for saturated systems is a very important problem specially when high performance and guarantee of stability are required. For such objectives it is not possible to avoid the system saturated behavior and special techniques must be used for analysis and synthesis of controllers for linear systems subject to control saturation.
In this work the existing results on analysis and synthesis of a class of anti-windup compensators based on the formulation of LMI (Linear Matrix Inequalities) problems for minimization of a $\mathcal{L}_2$ energy criteria or enlargement of the domain of asymptotic stability are studied. These LMI problems use Lyapunov stability conditions as well as a representation of the saturation through modified sector conditions. Additionally, conditions for pole placement on the anti-windup controller are used.
Then, considering the Projection Lemma, an alternative LMI formulation is proposed for the considered problem. This formulation is shown to be a generalization of existing methods and the results of both formulations are compared using a numerical example.
Finally the advantages and disadvantages of the proposed LMI formulation are highlighted. The additional degree of freedom in the LMI problem is probably the main contribution for future research since the resulting flexibility can be exploited to include additional objectives to the synthesis problem.
Keywords: Control Systems, Nonlinearity, Saturation, Sector Conditions, LMI, Projection Lemma, Finsler Lemma.

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1 INTRODUCTION

Control Systems are becoming more and more present in our daily activities, whether it is noticed or not. Cars, electricity production, satellites and airplanes are some examples of systems that surround us and which are only made possible by means of control laws that ensure that these systems behave as expected. I.e., remain stable, follow desired reference inputs and/or reject disturbances.

Analysis of these systems and synthesis of controllers are therefore of increasing importance. Analysis and synthesis tasks are widely supported by linear control theory and linearization techniques. These concepts are powerful tools and for a great number of applications are proved to be efficient even when the system present nonlinearities, which is the case for many practical applications.

However, when operating far from the linearization point, usually due to high performance goals, the nonlinearity effects become more pronounced and nonlinear behaviors must be taken into account. This is the case for actuator saturations, which are probably one of the most common examples of such nonlinearities.

Unless actuator size, weight, cost and consumption are not critical to the system, saturation can be avoided by over-dimensioning the actuator such that it will virtually never operate in the saturation regime. Another way of doing this is by slowing the closed-loop system expected response such that the required control signal amplitude / or energy to drive the system to its desired operation point remains inside the actuator saturating limits. These approaches are called saturation avoidance. Typically this is not the case in most applications, where engineers strive to deliver systems with higher performance at lower costs, smaller size and lighter weight. These challenging goals require the control engineer to seek for the optimum performance a given actuator can provide. This means that the actuator will be pushed to operate on its limit, in other words, it will saturate.

Additionally, a very important fact must be noted about the open-loop system for which controller is being designed/analyzed which is the stability of the open-loop plant. An unstable plant operation relies on the fact that the controller will continuously act, by means of a control input, in order to bring the closed-loop system to stability. It is an intuitive, however very important, conclusion that if the control energy is bounded the closed-loop system will no longer be stable for all initial conditions, i.e., global stability can no longer be pursued.

Knowing the boundaries of stability for a closed-loop nonlinear system is not usually a simple task, however it is a key safety aspect since it establishes the allowable boundary within which the system will safely operate.

To further highlight the importance of knowing such stability boundary some famous examples of the possibly catastrophic effects of unstable plants under control saturation pushed to operate outside its stability boundaries are highlighted in [Stein 2003], the most remarkable ones being the crashes of the Gripen prototype fighter aircraft in 1989 and 1993 and the Chernobyl nuclear power-plant disaster in 1986.

1.1 OBJECTIVES

This work is aimed at analysis of closed-loop systems under saturations and synthesis of "modern" anti-windup controllers [Tarbouriech et al. 2011, Tarbouriech and Turner 2009] (see also [Grimm et al. 2003] and [Sajjadi-Kia and Jabbari 2009]) optimizing some criteria such as $\mathcal{L}_2$ performance or maximization of the stability domain. Such optimization criteria may be considered with or without pole placement restrictions on the anti-windup controller.

The analysis method provides an approximation of the stability region of the closed-loop system[1] and also highlights indicators of system performance such as the energy of the error signal between chosen outputs of the nonlinear saturated model and a linear reference system.

On the other hand, the synthesis methods of such modern anti-windup controllers provide means for computing an anti-windup compensator that satisfy the aforementioned performance criterion (by reducing the nonlinearity effect) while providing guaranteed regions of stability.

Some of these analysis and synthesis objectives are already covered by other works such as [Biannic and Tarbouriech 2009] (performance $\mathcal{L}_2$ and maximization of the stability domain) and [Roos and Biannic 2008] (maximization of the stability domain and constraining the anti-windup controller poles). However combination of these three objectives (i.e. performance $\mathcal{L}_2$, maximization of the stability domain and anti-windup pole constraints) in the same framework, although not

[1] Further in the document it is shown that by using some techniques to represent the reference/disturbance input signals as additional closed-loop states, the boundaries of the stability region in the direction of these states indicate the maximum reference/disturbance signal amplitude for which the system is guaranteed to be stable.

very difficult to obtain from these cited works, is still not presented as far as the author is aware.

Nevertheless, the main objective of the present work is not only the unification of the results from [Biannic and Tarbouriech 2009] and [Roos and Biannic 2008] but also an attempt to expand and recast them in a more general form by using the Projection Lemma [Gahinet and Apkarian 1994,Pipeleers et al. 2009]. Such alternative formulation may allow reduction of the conservatism that naturally arises when using approximate representations of the nonlinearities and assuming a particular geometry for the stability region. Additionally, this alternative conditions may stress out hidden additional degrees of freedom in the optimization problem, and hence make possible to incorporate additional goals/restrictions in the synthesis procedure.

1.2 ANALYSIS AND SYNTHESIS OF SATURATED SYSTEMS

Complete analysis of the phase portrait of a nonlinear system and how it changes behavior when one or more parameter vary can be very involved, for higher order systems sometimes intractable. A very powerful tool for studying qualitative changes in the system behavior is the bifurcation analysis. The use of bifurcation analysis provide understanding of how the characteristics around an equilibrium point change for some values of a chosen parameter, however it is not a systematic approach and involves a heavy mathematical background. With the bifurcation analysis it is possible, for example, to observe when limit cycles appear and how they can bound the attraction basin of the system.

However, this is not a systematic method and its use seems only suitable for closed-loop systems with not more than two states. For higher order systems more systematic and generalized approaches are preferred such as for example the Linear Matrix Inequalities (LMI) approach.

Linear Matrix Inequalities, as defined in [Boyd et al. 1994], are extensively used to formulate the analysis and synthesis problems in terms of finding matrix variables that satisfy LMI constraints while minimizing a chosen criteria. The problem of minimizing a criteria under LMI constraints is a convex optimization problem and solvers for this type of problem can be considered as a relatively mature research field and be used as a *technology*. The tricky part remains most of the time in the ability to formulate or express the problem as an LMI problem, once this is done the numerical solvers will deal with it and present a solution. In this work the "LMI Lab" solver from the MATLAB® "Robust Control Toolbox" is used to specify, solve and validate LMIs.

To use the LMI framework the representation of the nonlinearities such as the saturation block (or equivalently the dead-zone operator[2]) through sector conditions is required. This usually adds conservatism to the analysis and synthesis procedures, since these representations are often too general, i.e. they represent a wide class of nonlinear systems. In this work, the saturation (or dead-zone) nonlinearity is represented through modified sector conditions, as in [Gomes da Silva Jr. and Tarbouriech 2005]. The modified sector condition is an enhancement of the "conventional" sector condition focused on the dead-zone representation. It is a less general representation, i.e., more suited for the dead-zone function, adding less conservatism to the analysis/synthesis procedure.

The cornerstone of the analysis and synthesis procedures using LMI framework is stability in the Lyapunov sense. A Lyapunov function is chosen to represent a generalized concept of energy of the closed-loop system, this function needing to be always positive definite. Stability is then ensured by forcing/verifying the time-derivative of this function to be negative definite (or at least not positive) along the closed-loop trajectories, which means that the energy of the system decreases (or at least does not increase) with time.

From the Lyapunov function it is also obtained an expression for the approximated region of attraction of the closed-loop system.

Another important tool which is useful to concatenate many scalar inequality restrictions in a single one is the S-procedure [Boyd et al. 1994, Jönsson 2006]. The S-procedure is used in this work to put together stability conditions, modified sector conditions and performance conditions under the same LMI condition.

Additionally, when the obtained conditions are not presented directly in the LMI form, which is usually the case for most control problems, some matrix manipulations may allow the conditions to be suitably reformulated in equivalent LMI conditions. The Schur Complement [Boyd et al. 1994, Tarbouriech et al. 2011] and the pre-/post-multiplication by a matrix and its transposed are useful tools to get around quadratic terms and eliminate undesired variable multiplications.

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[2]The equivalency between the saturation and the dead-zone operators is shown later in the document.

The main drawback of the LMI based analysis and synthesis procedures is probably the conservatism that arises from a particular choice of the Lyapunov function and its associated region of attraction as well as the choice of the criteria to maximize it. Also, conservatism is resultant from the use of sector conditions to represent the nonlinearity. An example of the estimation of the region of attraction of a second order closed-loop system is shown on [Bombieri, Pagano and Ponce 2011] comparing the LMI based analysis and the bifurcation analysis procedures.

The reduction of the conservatism on the LMI based analysis and synthesis procedures is a very interesting research theme. In this sense the current work envisions to formulate, by means of the Projection lemma, more general conditions which may allow reduction of conservatism or even other design objectives (new constraints) to be included in the optimization problem.

1.3 CONTROL OF LINEAR SYSTEMS UNDER SATURATIONS

In this section we present some results concerning linear systems with control signal under magnitude constraints and representation of the saturation operator by modified sector conditions.

Consider a linear time-invariant (LTI) system represented by

$$\dot{x} = Ax + Bu \tag{1.1}$$

where $x \in \Re^n$ and $u \in \Re^m$ are, respectively, the state and control input and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. For simplicity of the discussion consider a linear state-feedback control law $u = Fx(t)$, with $F \in \Re^{m\times n}$, such that the closed-loop system is given by

$$\dot{x} = (A + BF)x. \tag{1.2}$$

The stability of such LTI closed-loop system is characterized by the eigenvalues of the closed-loop state matrix $(A + BF)$. If the open-loop system is stabilizable the resulting closed-loop system can be globally asymptotic stable, i.e. $(A + BF)$ can have all eigenvalues in the negative semi-half of the complex plane even when the open-loop system is unstable.

Supposing now that the control signal is subject to magnitude constraints, that is, each component of the control signal $u(t)$ is replaced by $sat(u_i(t))$, defined by:

$$sat(u_i(t)) = \begin{cases} -\mu_i & \text{if} \quad u_i(t) < -\mu_i \\ u_i(t) & \text{if} \quad -\mu_i \leq u_i(t) \leq \mu_i \\ \mu_i & \text{if} \quad u_i(t) > \mu_i \end{cases} \tag{1.3}$$

with $\mu_i > 0$, $i = 1, \cdots, m$, then the system is said to be under control saturations and may operate in two different modes:

- Avoiding the saturation behavior, i.e. remaining always in the *linearity region* where the control $sat(u_i(t)) = u_i(t)$, such as in [Gutman and Hagander 1985].

- Allowing the saturated operation, and therefore taking the non-linear behavior into account, such as in [Tarbouriech and Garcia 1997], [Hu and Lin 2001], [Kapila and Grigoriadis 2002], [Gomes da Silva Jr. and Tarbouriech 2005] and [Biannic and Tarbouriech 2009] among others.

Note that the first approach usually is interesting when the saturation limits are big enough such that they do not interfere with the design goals for the closed-loop system (which is usually an indication of an over-dimensionalized actuator). However, when dealing with performance constraints, decoupling of modes and/or disturbance rejection it is desirable to “extract” the maximum performance of the system and the saturated condition can no longer be avoided.

Moreover, when the open-loop plant is unstable, it is possible to see that for initial conditions sufficiently far from the equilibrium point, the control signal required to bring the closed-loop system to the equilibrium will be higher than the saturation limits. This implies that only local stability can be guaranteed.

In this sense, the Analysis Problem consists of determining the region of attraction (or a region of asymptotic stability) for the closed-loop system and to verify if this region contains all the states where the system can be initialized or be taken to due to temporary disturbances.

The region of attraction of the origin of the saturated system is usually very hard to be determined, sometimes even impossible. Instead, regions of asymptotic stability, which are approximations of the region of attraction, are usually pursued.

The other type of problem that can be formulated is the synthesis of a control law taking into account the constraints on the control signal. In this case the region of the state space where the system must be stable is considered as given data of the problem.

1.3.1 The Anti-windup Approach

The anti-windup technique is used to tackle the problems of stability and performance degradation for linear systems with saturated inputs.

The anti-windup technique is a *two-step* procedure. First, a linear controller which does not explicitly take into account the saturation constraints is designed, usually using standard linear design tools. Then, after this controller has been designed, a so-called anti-windup compensator is designed to handle the saturation constraints in order to recover, as much as possible, the performance induced by the previous design carried out on the basis of the unsaturated system and to ensure that stability is maintained (at least in some region near the origin).

A characteristic that makes the anti-windup compensator attractive in practice is that the anti-windup compensator becomes active and acts to modify the closed-loop behavior only when saturation is encountered.

Many authors note that the term "windup" was a phenomenon associated with saturation in systems with integral controllers and alluded to the build up of charge on the integrator capacitor during saturation. The subsequent dissipation of this charge would then cause long settling times and excessive overshoot, thereby degrading the system performance. Modifications to the controller which avoided this charge build-up were often termed "anti-windup" modifications and hence the term anti-windup was born. Since then however, the term "anti-windup" has evolved and it now means the generic two-step procedure for controller design which was described earlier. The "modern" anti-windup technique can be seen as a systematic method used to design an anti-windup compensator which provides rigorous guarantees of stability and or performance [Tarbouriech and Turner 2009,Tarbouriech et al. 2011, Zaccarian and Teel 2011].

The general principle of the anti-windup scheme is depicted in Figure 1. In this figure, we can notice the separation of the so called "unconstrained controller" and the "anti-windup controller" which is driven by the difference between the unconstrained signal u and the signal that is actually fed into the plant, the constrained signal u_c.

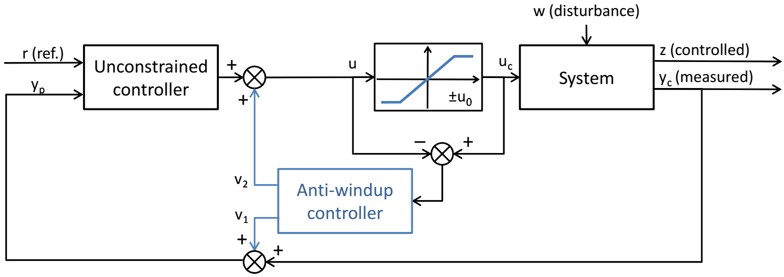

Figure 1: Principle of anti-windup (Adapted from [Tarbouriech and Turner 2009]).

1.3.2 Representation of Saturations

The exact representations of nonlinearities are in general not directly tractable in the analysis and synthesis frameworks. In that sense, representations less complex are very useful by control engineers in order to support analysis and synthesis techniques which require specific characteristics such as linearity, convexity, static properties, etc. The research field for better representations of nonlinearities is vast since each representation may present benefits either generalizing results for a wider class of nonlinearities or, on the other way around, by taking advantage of particular characteristics of a nonlinearity.

In this work, the modified sector conditions, as in [Gomes da Silva Jr. and Tarbouriech 2005], are used to represent the saturation function, or more precisely its associated dead-zone function $\phi(v)$ defined by the equivalence condition $\phi(v) \triangleq v - sat(v)$ showed in Figure 2.

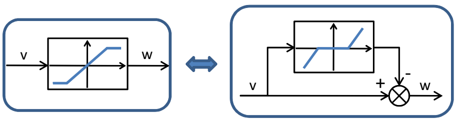

Figure 2: Equivalency relation of saturation and dead-zone functions.

The dead-zone operator can be seen as an indicator of the saturation presence, i.e. when its output is zero it means that system is not in the saturated regime while an output different than zero represent the “amount” of the signal that is saturated. The dead-zone operator is defined for each component $\phi(v_i)$ of $\Phi(v)$ as

$$\phi(v_i) = \begin{cases} v_i + \mu_i & \text{if} \quad v_i < -\mu_i \\ 0 & \text{if} \quad -\mu_i \leq v_i \leq \mu_i \\ v_i - \mu_i & \text{if} \quad v_i > \mu_i \end{cases} \tag{1.4}$$

where v_i, $i = 1, \cdots, m$ is each component of the vector v and $\mu_i \in \Re$, $\mu_i > 0$.

Note that this representation considers symmetric saturation/dead-zones nonlinearities. Moreover, we can assume $\mu_i = 1$ and simply multiply each i-th input and output component of the dead-zone block respectively by $1/\mu_i$ and μ_i.

The great advantage of using modified sector conditions to represent the nonlinearity is that tools from the absolute stability theory can be applied to evaluate the closed-loop stability and the resulting conditions for anti-windup purposes are directly in LMI form and can be recast into a convex optimization problem under LMI constraints.

To understand how the modified sector conditions are applied to represent the dead-zone function let us first define the following polyhedral set

$$\mathcal{S}(\mu) \triangleq \{v, w \in \Re^m; |v_i + w_i| \leq \mu_i, \ i = 1, \cdots, m\}. \tag{1.5}$$

Lemma 1. *(Adapted from [Gomes da Silva Jr. and Tarbouriech 2005]) If v and w are elements of $\mathcal{S}(\mu)$ then the nonlinearity $\Phi(v)$ satisfies the following inequality*

$$\phi(v_i)' T_{i,i} \left(\phi(v_i) + w_i\right) \leq 0, \ i = 1, \cdots, m \tag{1.6}$$

for any diagonal positive definite matrix $T \in \Re^{m \times m}$, $T_{i,i}$ is the element in the i-th row and i-th column of T.

Proof. Consider the following three cases:

1. For $-\mu_i \leq v_i \leq \mu_i$:

   In this case $\phi(v_i) = 0$ and therefore condition (1.6) holds $\forall T_{i,i}$ as $\phi(v_i)' T_{i,i} \left(\phi(v_i) + w_i\right) = 0, \ i = 1, \cdots, m$.

2. For $v_i > \mu_i$:

   In this case $\phi(v_i) = v_i - \mu_i$ and $\phi(v_i) > 0$. Since $v_i \in \mathcal{S}$ it implies that $v_i + w_i - \mu_i \leq 0$. Then, we have $\phi(v_i) + w_i = v_i + w_i - \mu_i \leq 0$ which implies condition (1.6) $\forall T_{i,i} > 0$.

3. For $v_i < -\mu_i$:

In this case $\phi(v_i) = v_i + \mu_i$ and $\phi(v_i) < 0$. Since $v_i \in \mathcal{S}$ it implies that $v_i + w_i + \mu_i \geq 0$. Then, we have $\phi(v_i) + w_i = v_i + w_i + \mu_i \geq 0$ which implies condition (1.6) $\forall T_{i,i} > 0$.

□

Combining the three cases above we can verify that the modified sector condition represents not only the dead-zone function $\phi(v_i)$ but also a class of nonlinearities which are in the grey region and black line in Figure 3.

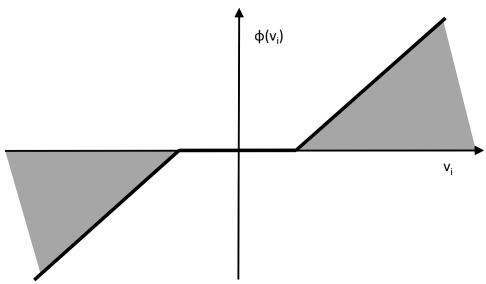

Figure 3: Modified Sector Conditions - Graphic Interpretation.

1.4 STRUCTURE OF THE DOCUMENT

This master thesis proposes to review the dynamic anti-windup analysis and synthesis techniques based on LMI conditions and using modified sector conditions to represent the saturation; then we use these concepts to propose alternative LMI conditions and compare them by means of a numerical example. The organization of the document follows:

Chapter 2 is where we present the analysis and synthesis problems and the results of [Biannic and Tarbouriech 2009] and [Roos and Biannic 2008] which consider at the same time the $\mathcal{L}_2$ performance objective, maximization of the stability domain and constraining the anti-windup controller poles as well.

Chapter 3 The purpose of this chapter is to present the alternative LMI formulations obtained from application of the Projection Lemma on the results of the previous chapter.

Chapter 4 In this chapter it is presented a numerical example based on the longitudinal control of a fighter aircraft on an critical point of the flight envelope and comparisons between the results with controllers obtained by the synthesis methods presented on Chapter 2 and Chapter 3 are presented.

Chapter 5 Presents the final conclusions of this work as well as the perspectives for future works.

2 ANALYSIS AND SYNTHESIS OF DYNAMIC ANTI-WINDUP CONTROLLERS FOR SATURATED SYSTEMS

In this chapter we present linear systems with the control signal under magnitude constraints and the closed-loop system that is object of study of this work. It is also presented the formulation of the the analysis and synthesis problems. The main results for analysis and synthesis conditions are then presented as well as algorithms for addressing each one of the problems formulated. Finally, conditions for pole restrictions on the anti-windup controller are presented as well as the modification of the synthesis algorithms to include such constraints.

2.1 PROBLEM STATEMENT

In this section the closed-loop system structure is presented, adapted from [Biannic and Tarbouriech 2009, Roos and Biannic 2008] for which the analysis and sythesis of anti-windup controllers is studied in this work.

2.1.1 The Closed-Loop System

This work considers the closed-loop system depicted in Figure 4 which is composed by state space descriptions of a linear plant $G(s)$, a linear controller $K(s)$, a reference and disturbance generator $R(s)$, a low-pass filter $F(s)$, a saturation block, nominal (linear) reference closed-loop system $L(s)$ and an anti-windup compensator $J(s)$.

It is assumed that the controller $K(s)$ is preliminarily designed to ensure stability and good performance properties to the closed-loop system in absence of saturations. Then, in a second step, the signals v_1 and $\bar{v}_2$ are added at the input and the output of the controller $K(s)$, respectively, in order to mitigate the adverse effects of the saturation. Note from Figure 4 that $\bar{v}_2 = v_2$ if the low-pass filter $F(s)$ is omitted. Such signals v_1 and v_2 are the outputs of the dynamic anti-windup compensator $J(s)$ to be determined:

$$J(s) : \begin{cases} \dot{x}_J = A_J x_J + B_J w \in \Re^{n_J} \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_J x_J + D_J w \in \Re^{p_J} \end{cases} \tag{2.1}$$

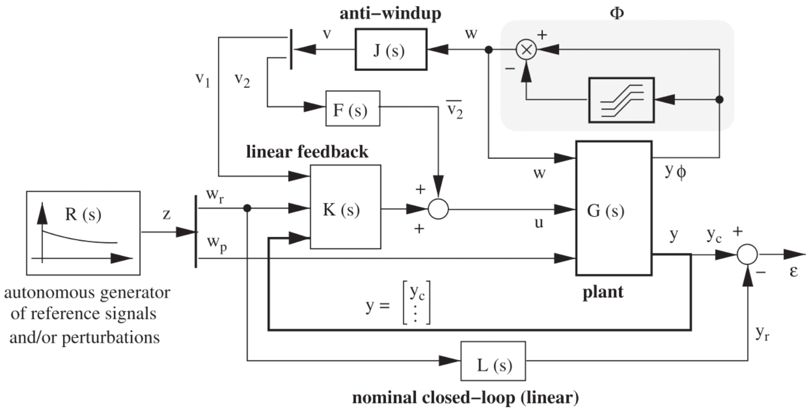

Figure 4: Standard form for dynamic anti-windup synthesis. Adapted from [Biannic and Tarbouriech 2009]

where the components of the input vector $w = \Phi(y_\phi)$ are dead-zone type nonlinearities[1]. It is also assumed, without loss of generality, that the inputs and outputs of plant $G(s)$ are correctly rescaled so that the dead-zone functions are normalized, i.e., the dead-zone limits are $\mu_i = 1,\ \forall i = 1, \cdots, m$.

To avoid the presence of an exogenous input signal, a linear *autonomous* reference and disturbance signal generator $R(s)$, as per [Biannic, Tarbouriech and Farret 2006], is employed:

$$R(s) : \left\{ \begin{array}{l} \dot{x}_R = A_R x_R,\ x_R(0) \in \mathcal{W}_{n_R,p_R}(\rho) \subset \Re^{n_R} \\ z = \begin{bmatrix} w_r \\ w_p \end{bmatrix} = [x_{R_1} \ \cdots \ x_{R_{p_R}}]' \in \Re^{p_R} \end{array} \right. \tag{2.2}$$

The outputs $z = [w_r' \ w_p']'$, where w_r and w_p are respectively the reference and disturbance signals, are assumed, without loss of generality, to be the first p_R states of x_R and the compact set of initial conditions $\mathcal{W}_{n_R,p_R}(\rho)$ is defined by

$$\mathcal{W}_{n_R,p_R}(\rho) = \{x_R \in \Re^{n_R};\ \forall i \le p_R,\ |x_{Ri}| \le \rho,\ \forall j > p_R,\ x_{Rj} = 0\}$$

However, in order to have a convex representation of this set, it is assumed that $\mathcal{W}_{n_R,p_R}(\rho)$ can be described by the following convex

[1]Note from the equivalence condition shown in Figure 2 that this representation is suitable for saturated systems.

hull:

$$\mathcal{W}_{n_R,p_R}(\rho) = \mathrm{conv}(\rho x_{R1}, \cdots, \rho x_{Rq}), \; x_{Ri} \in \Re^{n_R}. \tag{2.3}$$

Remark 1. *[Biannic and Tarbouriech 2009] The above description is a straightforward generalization of the first-order model which was used in [Biannic, Tarbouriech and Farret 2006] and [Biannic and Tarbouriech 2007] to approximate step signals. Set for example $A_R = -\lambda$, $z = x_R$, and $\mathcal{W}_{1,1}(\rho) = \{x_R \in \Re; |x_R| \leq \rho\} = [-\rho, \rho]$. It is then readily checked that the output z is bounded by ρ for any positive scalar λ and converges towards a step signal on any finite horizon when $\lambda \to 0$.*

Although throughout this work a single reference input w_r will be used, this approach allows to approximate step signals for references and disturbances bounded by ρ on any finite horizon. This representation of the reference input signal has particular properties that are further exploited in the forthcoming algorithms for defining, estimating and enlarging the stability region of the closed-loop system.

Finally, in order to establish a measure for performance of the anti-windup compensator, a linear model $L(s)$ is introduced as a reference for the nominal behavior of the closed-loop system without saturations. The performance characterization is obtained by the tracking error ε between the output y_r of this reference model and the controlled output y_c of $G(s)$.

Let us now redraw the nonlinear interconnection of Figure 4 as shown in Figure 5 involving a stable augmented closed-loop system $M(s)$. Note that the stability of $M(s)$ is a non-restrictive assumption since the dynamics of this system include the linear stabilizing controller $K(s)$. It is also assumed, that the transfer functions $G_{w \to y_\phi}(s)$ e $G_{w \to y}(s)$ are strictly proper, so that $M(s)$ can be described by

$$M(s) : \begin{cases} \dot{\xi} = A\xi + B_\phi w + B_v v \\ y_\phi = C_\phi \xi, \quad \varepsilon = y_c - y_r = C_\varepsilon \xi \\ \xi(0) \in \mathcal{X}(\rho) = \mathrm{conv}(\rho\mathcal{X}_1, \cdots, \rho\mathcal{X}_q) \subset \Re^{n_M} \end{cases} \tag{2.4}$$

where the augmented state vector ξ is constructed in a way such that its first n_R components coincide with those of $R(s)$ and $\mathcal{X}_i' = [x_{R_i}' \; \mathbf{0}]$. Note that the remaining states that compose the augmented vector ξ can be organized as desired, they will encompass, besides the states x_R of the reference generator $R(s)$, the states of the plant $G(s)$, the filter $F(s)$ (if exists), the controller $K(s)$ and the linear nominal system $L(s)$. An example of how this augmented state ξ is obtained can be found in chapter 4 (more specifically in section 4.1.3).

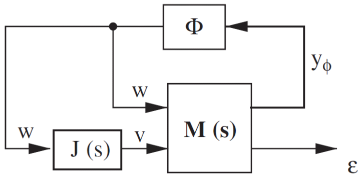

Figure 5: Simplified interconnection of Figure 4. Extracted from [Biannic and Tarbouriech 2009]

Combining (2.1) and (2.4), the nonlinear interconnection of Figure 5 can be represented as:

$$\Sigma(s) : \begin{cases} \dot{x} = \mathbb{A}x + \mathbb{B}\Phi(y_\phi) \\ y_\phi = \mathbb{C}_\phi x \in \Re^m, \quad \varepsilon = \mathbb{C}_\varepsilon x \in \Re^p \\ x(0) \in \overline{\mathcal{X}}(\rho) = \text{conv}(\rho\overline{\mathcal{X}}_1, \cdots, \rho\overline{\mathcal{X}}_q) \subset \Re^n \end{cases} \tag{2.5}$$

with $x = [\xi' \ x_J']'$, $n = n_M + n_J$, $\overline{\mathcal{X}}_i' = [\mathcal{X}_i' \ \mathbf{0}]$ and

$$\mathbb{A} = \begin{bmatrix} A & B_v C_J \\ \mathbf{0} & A_J \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B_\phi + B_v D_J \\ B_J \end{bmatrix},$$
$$\mathbb{C}_\phi = \begin{bmatrix} C_\phi & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbb{C}_\varepsilon = \begin{bmatrix} C_\varepsilon & \mathbf{0} \end{bmatrix}.$$

2.1.2 Design Objectives

Now, with respect to the nonlinear closed-loop system defined in the previous paragraph, the following Analysis and Synthesis Problems can be stated.

Problem 1. *(Analysis: Stability Region) Given a dynamic anti-windup compensator $J(s)$ determine what is the maximum value of ρ for which the closed-loop system $\Sigma(s)$ remains stable.*

Problem 2. *(Analysis: Performance Characterization) Given a dynamic anti-windup compensator $J(s)$ and the maximum admissible value of ρ, determine an indication of the energy of the tracking error signal ε.*

Note that by the definition of the autonomous reference generator system $R(s)$, the parameter ρ is associated with the boundaries of its initial conditions $x_R(0)$. This means that ρ is strictly related to the size of the stability region. Moreover, by determining the maximum value of ρ for which the closed-loop is stable we also obtain the maximum amplitude of a reference (or disturbance) signal that the closed-loop system can admit without becoming unstable.

The other problems that can be formulated are related to the synthesis of anti-windup controllers.

Problem 3. *(Synthesis: Enlargement of the Stability Region) Determine a dynamic anti-windup compensator $J(s)$ which maximizes the positive scalar ρ for which the nonlinear closed-loop system $\Sigma(s)$ remains stable.*

Problem 4. *(Synthesis: Performance Optimization) Given a positive scalar ρ, the problem is to compute a dynamic anti-windup compensator $J(s)$ such that the nonlinear closed-loop system $\Sigma(s)$ remains stable and the energy of the tracking-error signal ε is minimized.*

As noted before, the value of ρ is associated to the size of the attraction domain in the direction corresponding to the reference (or disturbance) signal magnitude. Hence, synthesis procedures which aim for big values of ρ usually result in poor performance characteristics and, on the other way, controllers that provide very good performance response (i.e. very small energy on ε) usually are not stable for reference or disturbance signals with larger amplitude. Therefore, a good controller must represent a tradeoff between reference tracking (big ρ) and linear behavior (small ε).

Remark 2. *(Pole Restrictions) Additionally to the synthesis Problems 3 and 4, restrictions to the anti-windup controller poles may be added. This pole restrictions may be interesting for practical purposes.*

2.2 ANALYSIS OF ANTI-WINDUP CONTROLLERS USING LMI FORMULATION

In this section the Anti-Windup Controller analysis method used in [Biannic and Tarbouriech 2009] is presented which uses the Lyapunov and absolute stability concepts, the modified sector conditions to take saturations into account and the characterization of the desired performance objectives as main tools.

First, the analysis aimed at the performance characterization is presented (Problem 2). Then, the determination of the attraction region (Problem 1) is stated based on simple modifications in the previous algorithm.

2.2.1 Representation of objectives as LMI constraints

As a starting point the conditions for the closed-loop system stability must be stated. In this work a Quadratic Lyapunov function $V(P) = x'Px$ is used to represent a generalization of the energy of the system and stability is enforced by its time derivative being negative definite $\dot{V}(x(t)) < 0,\ \forall x \neq 0$ [Khalil 2002, Tarbouriech et al. 2011].

Moreover, from the Quadratic Lyapunov function it is also defined an ellipsoidal region $\mathcal{E}$ of asymptotic stability for the initial conditions x_0 of the closed-loop system (2.5).

$$\mathcal{E} = \{x \in \Re^n;\ x'Px \leq 1\}. \tag{2.6}$$

To deal with the nonlinearities in a LMI framework it is used a characterization of the dead-zone operator through modified sector conditions such as presented in section 1.3.2 above. Hence, in this context, consider a matrix $\bar{G} \in \Re^{m\times n}$ and let us re-define the polyhedral set (1.5) as

$$\mathcal{S} = \{x \in \Re^n; |(\mathbb{C}_{\phi_i} + \bar{G}_i)x| \leq 1,\ i = 1, \cdots, m\} \tag{2.7}$$

where $\mathbb{C}_{\phi_i}$ and $\bar{G}_i$ are the notations for the i-th rows of $\mathbb{C}_\phi$ and $\bar{G}$, respectively. With the above definition the Sector Condition (1.6) now becomes

$$\Phi(\mathbb{C}_\phi x)'T(\Phi(\mathbb{C}_\phi x) + \bar{G}x) \leq 0 \tag{2.8}$$

with T diagonal, $T > 0,\ T \in \Re^{m\times m}$.

The reduction of the nonlinear effects of the saturation on the closed-loop system is addressed by minimizing the energy of the output $\varepsilon(t)$ which represents the difference between the closed-loop nonlinear system and the linear nominal model $L(s)$.

For a stable closed-loop system, the energy of the tracking error signal $\varepsilon(t)$ can be bounded by a scalar factor γ from

$$\gamma\dot{V} + \varepsilon'\varepsilon < 0 \tag{2.9}$$

which corresponds (by integration) to

$$\forall \tau \geq 0, \int_0^\tau \varepsilon(t)'\varepsilon(t)\mathrm{d}t \leq \gamma(V(x(0)) - V(x(\tau))). \tag{2.10}$$

Using the definition of the ellipsoid $\mathcal{E}$ we can verify that $\forall x(0) \in \mathcal{E}$, restriction (2.9) enforces

$$\forall \tau \geq 0, \int_0^\tau \varepsilon(t)'\varepsilon(t)\mathrm{d}t \leq \gamma. \tag{2.11}$$

Now, to guarantee the desired performance, the modified sector condition is considered by using the S-procedure [Boyd et al. 1994, Jönsson 2006, Tarbouriech et al. 2011]

$$\dot{V} - 2\Phi(\mathbb{C}_\phi x)'T(\Phi(\mathbb{C}_\phi x) + \bar{G}x) + \frac{\varepsilon'\varepsilon}{\gamma} < 0 \tag{2.12}$$

or equivalently as[2]

$$\begin{bmatrix} x \\ \Phi \end{bmatrix}' \begin{bmatrix} \mathbb{A}'P + P\mathbb{A} + \frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon & P\mathbb{B} - \bar{G}'T \\ \mathbb{B}'P - T\bar{G} & -2T \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} < 0. \tag{2.13}$$

Notice that (2.12) means that (2.9) is satisfied for any nonlinearity satisfying (2.8), which includes the dead-zone nonlinearity. Furthermore, inequality (2.13) is verified for any $[x \quad \Phi]' \neq 0$ if and only if

$$\begin{bmatrix} \mathbb{A}'P + P\mathbb{A} + \frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon & P\mathbb{B} - \bar{G}'T \\ \mathbb{B}'P - T\bar{G} & -2T \end{bmatrix} < 0. \tag{2.14}$$

2.2.2 Analysis main results

Using the definitions obtained in the previous subsection we can enunciate the main results for the Analysis Problems.

Theorem 1 (Analysis: Performance characterization, adapted from [Biannic and Tarbouriech 2009])**.** *Consider the nonlinear closed-loop system described by (2.5) with a given anti-windup controller $J(s)$ and the polyhedral set $\overline{\mathcal{X}} = conv(\overline{\mathcal{X}}_1, \cdots, \overline{\mathcal{X}}_q) \subset \Re^n$ where $\overline{\mathcal{X}}_i = [x'_{R_i}\ \mathbf{0}_{n-n_R}]'$. If there exist matrices:*

- $Q = Q' = P^{-1} \in \Re^{n\times n}$

[2]For the sake of readability, the term $\Phi(\mathbb{C}_\phi x)$ is often represented simply by Φ.

- $S = T^{-1} = diag(s_1, \cdots, s_m) > \mathbf{0}$
- $Z \in \Re^{m \times n}$

and positive scalars γ and $\beta = 1/\rho^2$ such that

$$\begin{pmatrix} Q & \overline{\mathcal{X}}_i \\ \overline{\mathcal{X}}_i' & \beta \end{pmatrix} > \mathbf{0}, \; i = 1, \cdots, q, \tag{2.15}$$

$$\begin{pmatrix} Q & \star \\ Z_i + \mathbb{C}_{\phi_i} Q & 1 \end{pmatrix} > \mathbf{0}, \; i = 1, \cdots, m, \tag{2.16}$$

$$\begin{pmatrix} \mathbb{A}Q + Q\mathbb{A}' & \star & \star \\ S\mathbb{B}' - Z & -2S & \star \\ \mathbb{C}_\varepsilon Q & \mathbf{0} & -\gamma I_p \end{pmatrix} < \mathbf{0}, \tag{2.17}$$

then, the nonlinear closed-loop system (2.5) remains stable for all initial condition x_0 contained in the ellipsoid $\mathcal{E} \supset \rho\overline{\mathcal{X}}$ and the ε output energy satisfies condition (2.11)[3].

Proof. Considering $Z = \bar{G}Q$ and using standard matrix manipulations such as Schur's complement on the quadratic term $\frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon$ and pre-/post-multiplication of (2.13) by $\text{diag}(Q, S, I_p)$ we can readily verify the correspondence between (2.13) and (2.17).

Condition (2.16) is, after using pre-/post-multiplication by $\text{diag}(Q, 1)$ and the aforementioned definition of Z, the matrix representation of

$$x'(\mathbb{C}_{\phi_i} + \bar{G}_i)'(\mathbb{C}_{\phi_i} + \bar{G}_i)x \leq x'Px.$$

This relation represents the inclusion of the ellipsoid $\mathcal{E}$ is in the polyhedral set $\mathcal{S}$ of validity of the sector conditions.

In a similar manner, condition (2.15) enforces the inclusion of $\overline{\mathcal{X}}(\rho)$ in $\mathcal{E}$. □

2.2.3 Analysis Algorithms

Theorem 1 above allows us to address the Analysis Problem of Performance Characterization (Problem 2) since it provides, for a given anti-windup controller $J(s)$ and set of admissible initial states $\mathcal{E} \supset \rho\overline{\mathcal{X}}$, conditions to determine the minimum value of γ for which is the upper limit for the energy of the tracking error signal ε. This process is detailed in the following Algorithm:

[3] The symmetric terms in matrix inequalities are represented by "$\star$"

Algorithm 1. *(Analysis: Performance Characterization)*

1. *Define the vertices* $\overline{\mathcal{X}}_i$ *of* $\overline{\mathcal{X}}(\rho)$.
2. *Initialize* ρ *according to the design objectives (maximum admissible reference/disturbance input).*
3. *Minimize* γ *under the LMI constraints of Theorem 1 w.r.t. the matrix variables* Q, S *and* Z.
4. *If the above problem is infeasible, decrease* ρ *and repeat previous step.*

To address Problem 1 (Analysis: Stability Region) we are interested in maximizing the parameter ρ which, based on the inclusion condition (2.15), will therefore enlarge the ellipsoid $\mathcal{E}$ which is an estimate of the region of attraction of the closed-loop system. We can also obtain conditions to maximize ρ from Theorem 1 with only small modifications as presented below in Algorithm 2.

Algorithm 2. *(Analysis: Stabilily Region)*

1. *Define the vertices* $\overline{\mathcal{X}}_i$ *of* $\overline{\mathcal{X}}(\rho)$.
2. *Minimize* $\beta = 1/\rho^2$ *(maximize* ρ*) under the LMI constraints (2.16), (2.15) and*

$$\begin{pmatrix} \mathbb{A}Q + Q\mathbb{A}' & \star \\ S\mathbb{B}' - Z & -2S \end{pmatrix} < \mathbf{0} \tag{2.18}$$

w.r.t. the matrix variables Q, S *and* Z.

Note that the vertices of $\overline{\mathcal{X}}(\rho)$ chosen in *step 1.* represent the directions on the space-state where maximization of the ellipsoid of asymptotic stability $\mathcal{E}$ is performed.

It is natural in the Analysis problems to first perform Stability Region Analysis (Algorithm 2) to obtain the maximum value ρ_{max} for which stability is assured, this will help choosing a suitable value for ρ in step 2 of Algorithm 1.

2.3 SYNTHESIS OF ANTI-WINDUP CONTROLLERS USING LMI FORMULATION

The results presented in Theorem 1 characterize performance and stability of a saturated system, however this result is only applicable to the Analysis Problem, since the the anti-windup compensator

$J(s)$ needs to be known in order to have LMI conditions. If $J(s)$ is to be determined, condition (2.17) present matrix variable multiplication since the matrices A_J, B_J, C_J and D_J (which appear inside the matrices $\mathbb{A}$ and $\mathbb{B}$) are also variables to be determined and the synthesis problem become no more convex.

Nevertheless, there exist a technique that allows us to remove (or add) variables to an LMI problem which can be used to obtain synthesis conditions. This result is the Projection Lemma [Gahinet and Apkarian 1994, Pipeleers et al. 2009] (see also [Oliveira and Skelton 2001]).

Lemma 2. *(Projection Lemma) Given a symmetric matriz $\Psi \in \Re^{m\times m}$ and two matrices $\mathcal{U}$ and $\mathcal{V}$ of column dimension m; there exists an unstructured matrix Ω that satisfies*

$$\Psi + \mathcal{U}'\Omega\mathcal{V} + \mathcal{V}'\Omega'\mathcal{U} < 0, \tag{2.19}$$

if and only if the following projection inequalities with respect to Ω are satisfied:

$$\begin{cases} N_U'\Psi N_U < 0, \\ N_V'\Psi N_V < 0, \end{cases} \tag{2.20}$$

where N_U and N_V are arbitrary matrices whose columns form a basis to the nullspaces of $\mathcal{U}$ and $\mathcal{V}$, respectively.

This result can be used either to add variables, i.e. to go from (2.20) to (2.19), as in [Pipeleers et al. 2009], or conversely, removing variables by going from (2.19) to (2.20), as in [Gahinet and Apkarian 1994, Biannic and Tarbouriech 2009].

2.3.1 Aplication of the Projection Lemma to obtain Synthesis Conditions

The method to obtain synthesis conditions in [Biannic and Tarbouriech 2009] follows a scheme proposed in [Gahinet and Apkarian 1994] in which the first step is to represent condition (2.17) as (2.19) and then, using the Projection Lemma, determine the equivalent conditions (2.20) which do not rely directly on the anti-windup controller matrices. This will enable synthesis conditions in LMI format. The anti-windup controller is finally reconstructed in a second step which will be exploited later including also some modified conditions to add restrictions to its poles.

Let us define a matrix Ω which gathers all the parameters of the anti-windup controller $J(s)$

$$\Omega = \begin{bmatrix} A_J & B_J \\ C_J & D_J \end{bmatrix} \tag{2.21}$$

and some shorthand matrices

$$\begin{gathered} \mathbb{A}_0 = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathbf{0} & B_v \\ I_{n_M} & \mathbf{0} \end{bmatrix} \\ \mathcal{C} = \begin{bmatrix} \mathbf{0} & I_{n_M} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbb{B}_0 = \begin{bmatrix} B_\phi \\ \mathbf{0} \end{bmatrix}, \mathcal{D} = \begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix} \end{gathered} \tag{2.22}$$

such that we can rewrite $\mathbb{A}$ and $\mathbb{B}$ as $\mathbb{A} = \mathbb{A}_0 + \mathcal{B}\Omega\mathcal{C}$ and $\mathbb{B} = \mathbb{B}_0 + \mathcal{B}\Omega\mathcal{D}$.

With the above notation, we can rewrite condition (2.17) as (2.19) where

$$\begin{gathered} \Psi = \begin{pmatrix} \mathbb{A}_0 Q + Q\mathbb{A}_0' & \star & \star \\ S\mathbb{B}_0' - Z & -2S & \star \\ \mathbb{C}_\varepsilon Q & \mathbf{0} & -\gamma I_p \end{pmatrix}, \\ \mathcal{U} = \begin{bmatrix} \mathcal{B}' & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \mathcal{C}Q & \mathcal{D}S & \mathbf{0} \end{bmatrix}. \end{gathered} \tag{2.23}$$

These definitions enable us to finally apply the Projection Lemma and obtain conditions equivalent to (2.20) which are used for the synthesis procedure.

2.3.2 Synthesis main results

This subsection presents Theorem 2 which summarizes the synthesis conditions that allow us to address the problem of obtaining an anti-windup controller that minimizes the the tracking-error energy criteria (Problem 4).

Theorem 2. *(Synthesis: Peformance Optimizition, adapted from [Biannic and Tarbouriech 2009]) Consider the nonlinear closed-loop system (2.5). There exists an anti-windup controller $J(s)$ such that the conditions of Theorem 1 are satisfied* iff *there exist:*

- $\bar{X} = \bar{X}' = X^{-1}, \ Y = Y' \in \Re^{n_M \times n_M}$,
- $S = T^{-1} = diag(s_1, \cdots, s_m) > \mathbf{0}$
- *U and V* $\in \Re^{m \times n_M}$

and a positive scalars γ and $\beta = 1/\rho^2$ such that

$$\begin{pmatrix} Y & \star & \star \\ \bar{X} & \bar{X} & \star \\ \mathcal{X}_i' & \mathcal{X}_i' & \beta \end{pmatrix} > \mathbf{0}, \ i = 1, \cdots, q \tag{2.24}$$

$$\begin{pmatrix} Y & \star & \star \\ \bar{X} & \bar{X} & \star \\ V_i + C_{\phi_i} Y & U_i & 1 \end{pmatrix} > \mathbf{0}, \ i = 1, \cdots, m \tag{2.25}$$

$$\begin{pmatrix} A\bar{X} + \bar{X}A' & \star \\ C_\varepsilon \bar{X} & -\gamma I_p \end{pmatrix} < 0 \tag{2.26}$$

$$\begin{pmatrix} N_v'(AY + YA')N_v & \star & \star \\ (SB_\phi' - V)N_v & -2S & \star \\ C_\varepsilon Y N_v & \mathbf{0} & -\gamma I_p \end{pmatrix} < \mathbf{0} \tag{2.27}$$

where N_v is any matrix whose columns are a basis to the nullspace of B_v'.

Proof. It was already showed in the previous subsection that using the definitions in (2.23) we can verify the correspondence between condition (2.17) from Theorem 1 and inequality (2.19) from the Projection Lemma. The proof of Theorem 2 consists basically of showing that conditions (2.26) and (2.27) are traced to the inequalities (2.20) in the Projection Lemma.

First, let us partition matrices Q and P as

$$Q = \begin{bmatrix} Y & N' \\ N & \hat{Y} \end{bmatrix}, \quad P = \begin{bmatrix} X & M' \\ M & \hat{X} \end{bmatrix} \tag{2.28}$$

and define auxiliar matrices

$$\theta_1 = \begin{bmatrix} Y & I_{n_M} \\ N & \mathbf{0} \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} I_{n_M} & X \\ \mathbf{0} & M \end{bmatrix} \tag{2.29}$$

where the nonsingular square matrices M and N are the solutions of $M'N = I_{n_M} - XY$. Note that using the fact that $Q = P^{-1}$, we can rewrite matrix Q as

$$Q = \theta_1 \theta_2^{-1}. \tag{2.30}$$

From the definition of $\mathcal{U}$ and $\mathcal{V}$ (and also $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{B}$) suitable choices for basis of the nullspace of these matrices are:

$$N_U = \text{diag}\left(\begin{bmatrix} N_v \\ \mathbf{0} \end{bmatrix}, I_m, I_p\right)$$

$$N_V = \text{diag}(Q^{-1}, S^{-1}, I_p)\left[\begin{bmatrix} I_{n_M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ I_p \end{bmatrix}\right]'.$$

With these definitions in mind and also the partitions (2.28) and the definitions of Ψ, $\mathbb{A}_0$, $\mathbb{B}_0$ and $\mathbb{C}_\varepsilon$ it is now possible to check that condition $N_V'\Psi N_V < 0$ results, after pre- and post-multiplication by $\text{diag}(\bar{X}, I_p)$, with $\bar{X} = X^{-1}$, in condition (2.26).

In the same manner condition $N_U'\Psi N_U < 0$, after pre- and post-multiplication by $\text{diag}(\bar{X}, I_m, I_p)$ and considering the partition $Z = [V\ \tilde{U}]$, results in condition (2.27).

Using the partition (2.28) and the previous definition of Z, the equivalence between (2.16) and (2.25) can be demonstrated by pre- and post-multiplying inequality (2.16) by $\text{diag}(\theta_2', I_m)$ and its transposed, then changing the variable X by $\bar{X}$ (which is accomplished through pre-/post-multiplication by $\text{diag}(I_{n_M}, \bar{X}, I_p)$) and considering $U = \bar{U}\bar{X}$ with $\bar{U} = VX + C_\phi + \tilde{U}M$.

Finally, following a similar procedure and considering the definition of $\overline{\mathcal{X}}'(\rho) = [\mathcal{X}'(\rho)\ \mathbf{0}]$, the equivalence between (2.15) and (2.24) can be readily verified. □

2.3.3 Synthesis Algorithms

Theorem 1 above provides conditions to find the tracking error signal energy bound γ and matrices $X = \bar{X}^{-1}$, Y, V, $\tilde{U}$ and S which allow us to reconstruct matrices Q and Z. Nevertheless, the anti-windup controller matrices still need to be determined. The Algorithm below presents the steps to compute an anti-windup controller focused on Problem 4:

Algorithm 3. *(Synthesis: Performance Optimization)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$ and initialize ρ according to the design objectives (maximum admissible reference/disturbance input).*

2. *Minimize γ under the LMI constraints of Theorem 2 w.r.t. the matrix variables $\bar{X}$, Y, V, U and S.*

3. *If the above problem is infeasible, decrease* ρ *and repeat previous step.*

4. *Compute* M *and* N *as the solution of* $M'N = I_{n_M} - XY$ *and compute* Q *as per (2.30).*

5. *Fix* Q *in inequality (2.17) and solve the convex feasibility problem w.r.t. the variables* A_J, B_J, C_J *and* D_J *(and optionally* γ, S, Z, $\tilde{B_J} = B_J S$ *and* $\tilde{D_J} = D_J S$*)*[4].

Note that sometimes it may be numerically interesting (due to less matrices reconstructions) to replace steps 4 and 5 in the above algorithm by

5a. *Compute* $\bar{\Psi} = diag(\theta_2', I_m, I_p)\Psi\, diag(\theta_2, I_m, I_p)$, $\bar{\mathcal{U}} = \left[\begin{array}{ccc} \mathcal{B}'\theta_2 & \mathbf{0} & \mathbf{0} \end{array}\right]$ *and* $\bar{\mathcal{V}} = \left[\begin{array}{ccc} \mathcal{C}\theta_1 & \mathcal{D}S & \mathbf{0} \end{array}\right]$.

6a. *Solve the feasibility problem* $\bar{\Psi} + \bar{\mathcal{U}}'\Omega\bar{\mathcal{V}} + \bar{\mathcal{V}}'\Omega'\bar{\mathcal{U}} < 0$ *w.r.t. the variable* Ω *and determine* A_J, B_J, C_J *and* D_J *from (2.21).*

Similarly to the Analysis Algorithms, it is natural that before using Algorithm 3 we want to determine what is the biggest stability region that can be obtained regardless of performance constraints, i.e. we want to obtain a controller $J(s)$ which provides the maximum value ρ_{max} for which stability is assured (Problem 3). The conditions for synthesis with optimization of the stability domain can also be obtained from Theorem 2 by considering $\gamma = \infty$ as pointed on Remark 3.3 in [Biannic and Tarbouriech 2009].

It is important to remind that there is always a compromise between performance and stability, and therefore it is noted that the controller that maximize the stability region usually does not present good performance response. That is why the parameter ρ must be reasonably initialized in Algorithm 3, i.e. according to the reference following and/or disturbance rejection design objectives (values close to ρ_{max} shall be avoided if possible).

Algorithm 4. *(Synthesis: Enlargement of the Stability Region*

1. *Define the vertices* $\mathcal{X}_i$ *of* $\mathcal{X}(\rho)$.

2. *Minimize* $\beta = 1/\rho^2$ *(maximize* ρ*) under the LMI constraints (2.24), (2.25),*

$$A\bar{X} + \bar{X}A' < 0 \tag{2.31}$$

[4]Matrices S, Z and scalar γ do not need to be fixed when solving the feasibility problem. In this case the following change of variable needs to be considered $\tilde{B_J} = B_J S$ and $\tilde{D_J} = D_J S$.

and

$$\begin{pmatrix} N_v'(AY+YA')N_v & \star \\ (SB_\phi' - V)N_v & -2S \end{pmatrix} < 0 \qquad (2.32)$$

w.r.t. the matrix variables $\bar{X}$, Y, V, U *and* S.

3. *Compute* M *and* N *as the solution of* $M'N = I_{n_M} - XY$ *and compute* Q *as per (2.30).*

4. *Fix* Q *in inequality (2.17) and solve the convex feasibility problem w.r.t. the variables* A_J, B_J, C_J *and* D_J *(and optionally* γ, S, Z, $\tilde{B_J} = B_J S$ *and* $\tilde{D_J} = D_J S$*).*

2.4 POLE CONSTRAINTS ON THE ANTI-WINDUP CONTROLLER

As is observed in [Roos and Biannic 2008], the synthesis of dynamic anti-windup based on the procedures described in the previous section is possible due to the choice of an anti-windup controller $J(s)$ with the same order of the plant $M(s)$. This is called *full-order* synthesis. However, such anti-windup controllers usually exhibit slow dynamics that remains visible even when the saturations are no longer active.

To tackle this issue, modifications of Algorithms 3 and 4 are presented in this section which impose restrictions to the real part of the anti-windup controller poles and therefore avoid particularly slow behaviors.

2.4.1 Pole Restriction Conditions

It is a classical result that if there exist a positive definite symmetric matrix $\Delta \in \Re^{n_M \times n_M}$ such that $A_J\Delta + \Delta A_J' < 0$, then all the eigenvalues (or poles) of A_J are located in the left semi-plane of complex plane. Similarly, the poles $\lambda_1, \ldots, \lambda_{n_J}$ of A_J verify

$$\Re(\lambda_j) < -\lambda, \; j = 1, \cdots, n_J \qquad (2.33)$$

iff there exist a positive definite symmetric matrix $\Delta \in \Re^{n_M \times n_M}$ such that:

$$(A_J + \lambda I_{n_M})\Delta + \Delta(A_J + \lambda I_{n_M})' < 0. \qquad (2.34)$$

To cope with anti windup pole restrictions in the optimization portion of Algorithms 3 and 4 we consider a pole restriction as in [Roos and Biannic 2008], where we impose condition (2.34), with $\Delta = \hat{Y}$, on condition (2.17), which will now read:

$$\begin{pmatrix} \mathbb{A}Q + Q\mathbb{A}' + \begin{bmatrix} 0 & 0 \\ 0 & 2\lambda\hat{Y} \end{bmatrix} & \star & \star \\ S\mathbb{B}' - Z & -2S & \star \\ \mathbb{C}_\varepsilon Q & \mathbf{0} & -\gamma I_p \end{pmatrix} < \mathbf{0}. \tag{2.35}$$

With the above definition we can now use the Projection Lemma following the same procedure used to obtain synthesis conditions in Theorem 2 to obtain modified synthesis conditions with pole constraints.

Theorem 3. *(Synthesis: Peformance Optimizition with anti-windup pole constraints) Consider the nonlinear closed-loop system (2.5). There exists an anti-windup controller $J(s)$ such that the conditions of Theorem 1 are satisfied* iff *there exist:*

- $\bar{X} = \bar{X}' = X^{-1},\ Y = Y' \in \Re^{n_M \times n_M}$,
- $S = T^{-1} = diag(s_1, \cdots, s_m) > \mathbf{0}$
- U *and* $V \in \Re^{m \times n_M}$

and a positive scalar γ such that

$$\begin{pmatrix} A\bar{X} + \bar{X}A' - 2\lambda\bar{X} & \star & \star \\ 2\lambda Y & -2\lambda Y & \star \\ C_\varepsilon\bar{X} & 0 & -\gamma I_p \end{pmatrix} < 0 \tag{2.36}$$

and conditions (2.24), (2.25) and (2.27) hold.

Moreover, the poles of the anti-windup controller $J(s)$ satisfy (2.33).

Proof. The proof of this Theorem is similar to the proof of Theorem 2 and consists only in showing that inequalities (2.36) and (2.27) are equivalent to inequalities (2.20) in the Projection Lemma while the inequality (2.19) from the Projection Lemma comes from the modified condition (2.35).

Similar to Theorem 2, the first step is to represent condition (2.35) as $\Psi_\lambda + \mathcal{U}'\Omega\mathcal{V} + \mathcal{V}'\Omega'\mathcal{U} < 0$ where

$$\Psi_\lambda = \begin{pmatrix} \mathbb{A}_0 Q + Q\mathbb{A}_0' + \begin{bmatrix} 0 & 0 \\ 0 & -2\lambda\hat{Y} \end{bmatrix} & \star & \star \\ S\mathbb{B}_0' - Z & -2S & \star \\ \mathbb{C}_\varepsilon Q & \mathbf{0} & -\gamma I_p \end{pmatrix} \tag{2.37}$$

and $\mathcal{U}$ and $\mathcal{V}$ are defined as (2.23).

Using the same matrix manipulations and variable changes as in the proof of Theorem 2 we can easily see that $N_U' \Psi_\lambda N_V < 0$ is equivalent to condition (2.27) and condition $N_V' \Psi_\lambda N_V < 0$ result in a condition that consists of adding the term $2\lambda M' \hat{Y} M$ to the first element of (2.26). Now, from the definitions of Q and P we use the fact that $M'\hat{Y}M = -M'NX = -X + XYX$, as well as the change of variables $\bar{X} = X^{-1}$ and a Schur's complement argument to finally obtain condition (2.36). □

The algorithms for performance optimization (Algorithm 3) and for stability domain enlargement (Algorithm 4) may now be modified to include pole restriction conditions on the anti-windup controller.

Algorithm 5. *(Synthesis: Performance Optimization with Anti-windup pole restrictions)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$ and initialize ρ according to the design objectives (maximum admissible reference/disturbance input).*

2. *Choose λ according to the design objectives.*

3. *Minimize γ under the LMI constraints of Theorem 3 w.r.t. the matrix variables $\bar{X}$, Y, V, U and S.*

4. *If the above problem is infeasible, decrease ρ and repeat the previous step.*

5. *Compute M and N as the solution of $M'N = I_{n_M} - XY$ and compute Q as per (2.30).*

6. *Fix Q in inequality (2.35) and solve the convex feasibility problem w.r.t. the variables A_J, B_J, C_J and D_J (and optionally γ, S, Z, $\tilde{B_J} = B_J S$ and $\tilde{D_J} = D_J S$).*

Algorithm 6. *(Synthesis: Enlargement of the Stability Region with Anti-windup pole restrictions)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$.*

2. *Choose λ according to the design objectives.*

3. *Minimize* $\beta = 1/\rho^2$ *(maximize* ρ*) under the LMI constraints (2.24), (2.25), (2.32) and*

$$\begin{pmatrix} A\bar{X} + \bar{X}A' - 2\lambda\bar{X} & \star \\ 2\lambda Y & -2\lambda Y \end{pmatrix} < 0 \tag{2.38}$$

   w.r.t. the matrix variables $\bar{X}$, Y, V, U *and* S.

4. *Compute* M *and* N *as the solution of* $M'N = I_{n_M} - XY$ *and compute* Q *as per (2.30).*

5. *Fix* Q *in inequality (2.35) and solve the convex feasibility problem w.r.t. the variables* A_J, B_J, C_J *and* D_J *(and optionally* γ, S, Z, $\tilde{B}_J = B_J S$ *and* $\tilde{D}_J = D_J S$*).*

3 ALTERNATIVE CONDITIONS FOR ANALYSIS AND SYNTHESIS OF DYNAMIC ANTI-WINDUP CONTROLLERS

In this chapter we propose an alternative formulation for analysis and synthesis of dynamic anti-windup compensators based on the use of the Projection Lemma.

The main idea follows [Pipeleers et al. 2009], where the goal is to explore the LMI problem, highlighting general characteristics to avoid unnecessary constraints, explore degrees of freedom and/or reduce the conservatism. We show that the alternative conditions derived in this chapter are a generalization of the results from chapter 2 and may present some advantages in the synthesis procedure.

In a way similar to the previous chapter it will be shown how to use these alternative conditions for analysis and synthesis as well as how to add restrictions on the anti-windup poles in this new formulation. The differences and possible advantages and disadvantages of the alternative conditions are pointed out along the chapter.

3.1 ANALYSIS CONDITIONS USING ALTERNATIVE LMI FORMULATION

Using a methodology such as in [Pipeleers et al. 2009] (and also used before in [Peaucelle et al. 2000] and [Peaucelle and Arzelier 2001]) the idea of this work is to use the Projection Lemma to add new variables to the LMI problem. This is done by rewriting the analysis condition (2.12) and an additional condition related to the Lyapunov matrix P under the form of (2.20) and then, using the Projection Lemma, replacing it by its equivalent condition (2.19).

To do so, the first step is to select the desired conditions equivalent to (2.20). The first inequality comes from the design objective (2.12), which from (2.14), can be equivalently described by:

$$\begin{bmatrix} I_{n_M} & 0 \\ \mathbb{A} & \mathbb{B} \\ 0 & I_m \end{bmatrix}' \begin{bmatrix} \frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon & P & -\bar{G}'T \\ P & 0 & 0 \\ -T\bar{G} & 0 & -2T \end{bmatrix} \begin{bmatrix} I_{n_M} & 0 \\ \mathbb{A} & \mathbb{B} \\ 0 & I_m \end{bmatrix} < 0. \tag{3.1}$$

This condition can be seen as $N_{U_0}'\Psi_0 N_{U_0} < 0$, from which the choices of $\mathcal{U}_0$ and its respective nullspace N_{U_0} came from the closed-loop relations (2.5):

$$\mathcal{U}_0 = \begin{bmatrix} \mathbb{A} & -I_{n_M} & \mathbb{B} \end{bmatrix}, N_{U_0} = \begin{bmatrix} I_{n_M} & 0 \\ \mathbb{A} & \mathbb{B} \\ 0 & I_m \end{bmatrix}. \tag{3.2}$$

Now, before we can use the Projection Lemma, we need to choose the second inequality in (2.20), i.e. we need to choose N_{V_0}, and hence $\mathcal{V}_0$, in a way to avoid additional conservatism. This means that the condition $N'_{V_0}\Psi_0 N_{V_0} < 0$ shall be designed such that it does not impose additional unwanted constraints. In [Pipeleers et al. 2009] possible choices for N_{V_0} are presented in a way such that $N'_{V_0}\Psi_0 N_{V_0} < 0$ yields trivial inequalities (such as $I > 0$), conditions related to the Lyapunov Matrix (such as $P > 0$) or combinations of both strategies ($P > 0$ and $I > 0$). The choice for N_{V_0} in this work is similar to Extension II in [Pipeleers et al. 2009]:

$$\mathcal{V}_0 = \begin{bmatrix} I & \alpha I & \Xi \\ 0 & 0 & I \end{bmatrix}, N_{V_0} = \begin{bmatrix} \alpha I \\ -I \\ 0 \end{bmatrix}. \tag{3.3}$$

with $\alpha > 0$ an arbitrary scalar referred as the Projection Lemma multiplier factor. The use of this scalar α avoids using unnecessary extra variables without introducing conservatism as shown in [Pipeleers et al. 2009].

This choice of N_{V_0} does not result in a non conservative condition (such as $P > 0$ or/and $I > 0$) since the sector conditions as well as performance conditions contained in matrix Ψ_0 could not be eliminated when applying the Projection Lemma, hence the resulting condition $N'_{V_0}\Psi_0 N_{V_0} < 0$ represents:

$$P > \frac{\alpha}{2\gamma}\mathbb{C}'_{\varepsilon}\mathbb{C}_{\varepsilon}. \tag{3.4}$$

Note that as $\alpha \to 0$ the above condition becomes equivalent to $P > 0$ which suggests that with $\alpha \neq 0$ some additional conservatism may exist.

With the above definitions for $\mathcal{U}_0$ and $\mathcal{V}_0$ we can finally use the Projection Lemma to replace conditions (3.1) (equivalent to (2.13)) and (3.4) by the equivalent condition $\Psi_0 + \mathcal{U}'_0\mathbb{F}_0\mathcal{V}_0 + \mathcal{V}'_0\mathbb{F}'_0\mathcal{U}_0 < 0$ where the new matrix variable $\mathbb{F}_0 = [\mathbb{F}_1 \;\; \mathbb{F}_2]$ is added:

$$\begin{bmatrix} \frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon & P & -G'\bar{T} \\ P & 0 & 0 \\ -T\bar{G} & 0 & -2T \end{bmatrix} + \\ + \begin{bmatrix} \mathbb{A}' \\ -I \\ \mathbb{B}' \end{bmatrix} \begin{bmatrix} \mathbb{F}_1 & \alpha\mathbb{F}_1 & \mathbb{F}_1\Xi + \mathbb{F}_2 \end{bmatrix} + \\ + \begin{bmatrix} \mathbb{F}_1' \\ \alpha\mathbb{F}_1' \\ \Xi'\mathbb{F}_1' + \mathbb{F}_2' \end{bmatrix} \begin{bmatrix} \mathbb{A} & -I & \mathbb{B} \end{bmatrix} < 0. \tag{3.5}$$

For synthesis purposes, and since the variable Ξ does not appear in (3.4), we set $\Xi = -\mathbb{F}_1^{-1}\mathbb{F}_2$ on the above condition. This choice makes the term $\mathbb{F}_1\Xi + \mathbb{F}_2$ equal to zero. Using this choice of Ξ we can write condition (3.5) above more compactly as:

$$\begin{bmatrix} \frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon + \mathbb{A}'\mathbb{F} + \mathbb{F}'\mathbb{A} & \star & \star \\ P + \alpha\mathbb{F}'\mathbb{A} - \mathbb{F} & -\alpha(\mathbb{F} + \mathbb{F}') & \star \\ -TG + \mathbb{B}'\mathbb{F} & \alpha\mathbb{B}'\mathbb{F} & -2T \end{bmatrix} < 0 \tag{3.6}$$

where $\mathbb{F} = \mathbb{F}_1$.

3.1.1 Analysis main results - Alternative LMIs

Similarly to Section 2.2 we now can use the above conditions to state Theorem 1 below which provides LMI conditions to verify stability and performance characterization for a given anti-windup controller.

Theorem 4 (Performance characterization)**.** *Consider the nonlinear closed-loop system described by (2.5) with a given anti-windup controller $J(s)$. If there exist matrices:*

- $\mathbb{M} = \mathbb{F}^{-1}$
- $\mathbb{W} = \mathbb{W}' = \mathbb{M}'P\mathbb{M} \in \Re^{n\times n}$
- $S = T^{-1} = diag(s_1, \cdots, s_m) > 0$
- $Z = [Z_1' \cdots Z_m']' \in \Re^{m\times n}$

and positive scalars γ *and* $\beta = 1/\rho^2$ *such that*

$$\begin{bmatrix} \mathbb{M} + \mathbb{M}' - \mathbb{W} & \overline{\mathcal{X}}_i \\ \overline{\mathcal{X}}_i' & \beta \end{bmatrix} > 0, \ i = 1, \cdots, q \tag{3.7}$$

$$\begin{bmatrix} \mathbb{W} & \star \\ Z_i + \mathbb{C}_{\phi_i}\mathbb{M} & 1 \end{bmatrix} > 0, \ i = 1, \cdots, m \tag{3.8}$$

$$\begin{bmatrix} \mathbb{M}'\mathbb{A}' + \mathbb{A}\mathbb{M} & \star & \star & \star \\ \mathbb{W} + \alpha\mathbb{A}\mathbb{M} - \mathbb{M}' & -\alpha(\mathbb{M} + \mathbb{M}') & \star & \star \\ S\mathbb{B}' - Z & \alpha S\mathbb{B}' & -2S & \star \\ \mathbb{C}_\varepsilon\mathbb{M} & 0 & 0 & -\gamma I_p \end{bmatrix} < 0 \tag{3.9}$$

where $\alpha > 0$ *is an arbitrarily chosen scalar, then, the nonlinear closed-loop system (2.5) remains stable for all initial condition* x_0 *contained in the ellipsoid* $\mathcal{E} = \{x \in \Re^{n_M}, x'Px \leq 1\}$, $\mathcal{E} \supset \rho\overline{\mathcal{X}}$, *the* ε *output energy satisfies condition (2.11) and* P *is limited by (3.4).*

Proof. Considering $Z = \bar{G}\mathbb{M}$ and with some standard matrix manipulation such as Schur's complement in the term $\frac{1}{\gamma}\mathbb{C}_\varepsilon'\mathbb{C}_\varepsilon$ and pre-/post-multiplication by $\text{diag}(\mathbb{M}', \mathbb{M}', S, I_p)$ and its transposed, we can verify the correspondence between (3.6) and (3.9).

Using the same matrix manipulation tools, we can see that condition (3.8) is equivalent to condition (2.16) in Theorem 1 which ensures the inclusion of the ellipsoid $\mathcal{E}$ in the polyhedral set of validity of the sector conditions $\mathcal{S}$.

Finally, in a similar manner as in [Castelan et al. 2006], we use the the fact that $(P^{-1} - \mathbb{M})'P(P^{-1} - \mathbb{M}) > 0$ (which from the definition of $\mathbb{W} = \mathbb{M}'P\mathbb{M}$ can also be written as $P^{-1} > \mathbb{M} + \mathbb{M}' - \mathbb{W}$) to indirectly write the inclusion of $\overline{\mathcal{X}}(\rho)$ in $\mathcal{E}$ as $\mathbb{M} + \mathbb{M}' - \mathbb{W} > \rho^2\overline{\mathcal{X}}_i\overline{\mathcal{X}}_i'$ which is equivalent to (3.7) using a Schur complement argument. □

Theorem 4 above is very similar to Theorem 1 presented in the previous chapter. Moreover, we can see that by forcing $\alpha = 0$ and $\mathbb{W} = \mathbb{M} = Q$ in Theorem 4 we eliminate the second row and the second column in condition (3.9) therefore obtaining the conditions of Theorem 1.

Nevertheless Theorem 4 cannot be seen as an extension of Theorem 1 since, as per definition of (3.4) we can see that conditions in Theorem 1 (which implies $\alpha = 0$) tend to be less conservative than the new conditions in Theorem 4 and therefore we cannot claim that any solution of Theorem 1 is also a solution of Theorem 4 when $\alpha \neq 0$.

The advantages of this new theorem rely in the fact that the

Lyapunov Matrix do not appear multiplied by the state matrices, hence providing an additional degree of freedom which may allow additional objectives to be included in the synthesis problems.

3.1.2 Analysis Algorithms - Alternative LMIs

The algorithms for performance characterization or stability domain determination considering the Alternative LMI conditions are very similar to the ones presented in section 2.2.2 being the only difference the need to initialize the parameter α in the beginning and the need to reduce it in case the problem is not feasible with current choice.

Since in the Analysis problems we are interested in characterizing an already existing anti-windup controller, the possible advantage of the additional degree of freedom provided by this formulation is not taken into account. The alternative formulation become more interesting in the Synthesis conditions presented in the next section.

3.2 SYNTHESIS CONDITIONS USING ALTERNATIVE LMI FORMULATION

In this section we shown, in a very similar way to section 2.3, how to use the Projection Lemma to eliminate the anti-windup matrices from condition (3.9). These matrices appear in $\mathbb{A}$ and $\mathbb{B}$, making the synthesis problem no longer an LMI problem. The following sections show how to explore particular structure of the problem for full order anti-windup controller.

3.2.1 Synthesis main results - Alternative LMIs

The results for synthesis procedures using alternative LMI conditions are summarized in Theorem 5 below.

Theorem 5 (Peformance Optimizition - Alternative Formulation)**.** *Consider the nonlinear closed-loop system of Figure 5. There exists an anti-windup controller $J(s)$ such that the conditions of Theorem 1 are satisfied* iff *there exist:*

- $\bar{X} = X^{-1}$, Y *and* $\bar{T}_\theta \in \Re^{n_M \times n_M}$,
- $S = T^{-1} = diag(s_1, \cdots, s_m) > 0$,

- *V and $U \in \Re^{m\times n_M}$*
- *$W_{11} = W_{11}'$, $\overline{W}_{22} = \overline{W}_{22}'$ and $\overline{W}_{12} \in \Re^{n_M\times n_M}$*

and positive scalars γ and $\beta = 1/\rho^2$ such that

$$\begin{bmatrix} Y + Y' - W_{11} & \star & \star \\ \bar{X}' + \bar{T}_\theta - \overline{W}_{12}' & \bar{X} + \bar{X}' - \overline{W}_{22} & \star \\ \mathcal{X}_i' & \mathcal{X}_i' & \beta \end{bmatrix} > 0,\ i = 1, \cdots, q \quad (3.10)$$

$$\begin{bmatrix} W_{11} & \star & \star \\ \overline{W}_{12}' & \overline{W}_{22} & \star \\ C_{\phi_i} Y + V_i & U_i & 1 \end{bmatrix} > 0,\ i = 1, \cdots, m \quad (3.11)$$

$$\begin{bmatrix} -2\alpha W_{11} & \star & (\alpha Y'A' + Y' - W_{11})N_v & \star & \star \\ -2\alpha \overline{W}_{12}' & -2\alpha \overline{W}_{22} & (\alpha \bar{X}'A' + \bar{X}' - \overline{W}_{12}')N_v & \star & \star \\ \star & \star & N_v'(Y'A' + AY)N_v & \star & \star \\ -\alpha V & \alpha(C_\phi \bar{X} - U) & (SB_\phi' - V)N_v & -2S & \star \\ \alpha C_\varepsilon Y & \alpha C_\varepsilon \bar{X} & C_\varepsilon Y N_v & 0 & -\gamma I_p \end{bmatrix} < 0 \quad (3.12)$$

$$\begin{bmatrix} \bar{X}'A' + A\bar{X} & \star & \star & \star \\ \overline{W}_{12} + \alpha A\bar{X} - \bar{T}_\theta' & -\alpha(Y + Y') & \star & \star \\ \overline{W}_{22} + \alpha A\bar{X} - \bar{X}' & -\alpha(\bar{X}' + \bar{T}_\theta) & -\alpha(\bar{X} + \bar{X}') & \star \\ C_\varepsilon \bar{X} & 0 & 0 & -\gamma I_p \end{bmatrix} < 0 \quad (3.13)$$

where N_v denotes any basis of the null-space of B_v' and $\alpha > 0$ is a arbitrarily chosen scalar.

Proof. Using the definition of Ω from (2.21) and the shorthand matrices (2.22) we can rewrite $\mathbb{A}$ and $\mathbb{B}$ as $\mathbb{A} = \mathbb{A}_0 + \mathcal{B}\Omega\mathcal{C}$ and $\mathbb{B} = \mathbb{B}_0 + \mathcal{B}\Omega\mathcal{D}$ allowing us to express condition (3.9) as the first inequality in the projection lemma $\Psi + \mathcal{U}'\Omega\mathcal{V} + \mathcal{V}'\Omega'\mathcal{U} < 0$ where

$$\Psi = \begin{bmatrix} \mathbb{M}'\mathbb{A}_0' + \mathbb{A}_0\mathbb{M} & \star & \star & \star \\ \mathbb{W} + \alpha\mathbb{A}_0\mathbb{M} - \mathbb{M}' & -\alpha(\mathbb{M} + \mathbb{M}') & \star & \star \\ S\mathbb{B}_0' - Z & \alpha S\mathbb{B}_0' & -2S & \star \\ \mathbb{C}_\varepsilon\mathbb{M} & 0 & 0 & -\gamma I_p \end{bmatrix}, \quad (3.14)$$

$$\mathcal{U} = \begin{bmatrix} \mathcal{B}' & \alpha\mathcal{B}' & 0 & 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} \mathcal{C}\mathbb{M} & 0 & \mathcal{D}S & 0 \end{bmatrix}.$$

Using the projection lemma we can now remove the variable Ω from (2.19) by using its equivalency to (2.20) where N_V and N_U are, respectively, bases for the null space of $\mathcal{V}$ and $\mathcal{U}$ and can be chosen from the definitions of $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{B}$ as

$$N_V = \text{diag}\left(\mathbb{F}\begin{bmatrix} I_{n_M} \\ 0 \end{bmatrix}, \theta_2, 0, I_p\right) \tag{3.15}$$

$$N_U = \begin{bmatrix} \alpha\theta_2 & \begin{bmatrix} N_v \\ 0 \end{bmatrix} & 0 & 0 \\ -\theta_2 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix} \tag{3.16}$$

where θ_1 and θ_2 now come from the following partitions:

$$\mathbb{M} = \begin{bmatrix} Y & M \\ H & \hat{Y} \end{bmatrix}, \quad \mathbb{F} = \begin{bmatrix} X & N \\ E & \hat{X} \end{bmatrix}, \tag{3.17}$$

$$\theta_1 = \begin{bmatrix} Y & I_{n_M} \\ H & 0 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} I_{n_M} & X \\ 0 & E \end{bmatrix}, \tag{3.18}$$

$$\mathbb{M} = \theta_1\theta_2^{-1}. \tag{3.19}$$

Considering the above choice of N_V, we can see that $N_V'\Psi N_V < 0$ is traced to condition (3.13) by pre- and post-multiplicating it by $\text{diag}(\bar{X}', I_{n_M}, \bar{X}', I_{n_M})$ and its transposed and using the following definitions:

$$\mathbb{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & W_{22} \end{bmatrix},$$

$$\overline{\mathbb{W}} = \begin{bmatrix} W_{11} & \overline{W}_{12} \\ \overline{W}_{12}' & \overline{W}_{22} \end{bmatrix} = \begin{bmatrix} I_{n_M} & 0 \\ 0 & \bar{X}' \end{bmatrix} \theta_2' \mathbb{W} \theta_2 \begin{bmatrix} I_{n_M} & 0 \\ 0 & \bar{X} \end{bmatrix},$$

$$\bar{T}_\theta = \bar{X}' T_\theta \text{ and } T_\theta = E'H + X'Y.$$

In a very similar way, by using the definition of N_U as per (3.16) and the above mentioned variable definitions and pre-/post-multiplication by $\text{diag}(I_{n_M}, \bar{X}', I_{3n_M})$ and its transposed, we can see that (3.12) corresponds to $N_U'\Psi N_U < 0$.

Conditions (3.11) and (3.10) are linked to (2.16) and (2.15), respectively. This is shown by pre- and post-multiplying (3.11) and (3.10) by $\text{diag}(I_{n_M}, \bar{X}', 1)$ and $\text{diag}(\theta_2', 1)$ and their transposes and also considering the partition $Z = [V \;\; \tilde{U}]$ with $U = \bar{U}\bar{X}$ and $\bar{U} = VX + C_\phi + \tilde{U}E$. □

Theorem 5 above presents conditions to find a Lyapunov Matrix

$P = \mathbb{M}^{-T}\mathbb{W}\mathbb{M}^{-1}$, a scalar γ and auxiliary matrices that allow us to recover an anti-windup controller that satisfy conditions of Theorem 4 and address the concerns of Problem 4.

3.2.2 Synthesis Algorithms - Alternative LMIs

From the alternative LMI conditions presented in Theorem 5 we can now propose the following algorithm containing the steps to compute an anti-windup controller aimed on Problem 4:

Algorithm 7. *(Synthesis: Performance Optimization - Alternative Formulation)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$ and initialize ρ according to the design objectives (maximum admissible reference/disturbance input).*

2. *Choose an initial value for α ($\alpha = 1$ is usually chosen as the first guess).*

3. *Minimize γ under the LMI constraints of Theorem 5 w.r.t. the matrix variables $\bar{X}$, Y, $\bar{T}_\theta$, V, U, S and W_{11}, $\overline{W}_{12}$ and $\overline{W}_{22}$.*

4. *If the above problem is infeasible, decrease α and repeat previous step.*

5. *If $\alpha \approx 0$ and problem is infeasible, decrease ρ and restart from step 2.*

6. *Compute E and H as the solution of $E'H = T_\theta - X'Y$ and compute $\mathbb{M}$ as per (3.19).*

7. *Fix $\mathbb{M}$ and $\mathbb{W}$ in inequality (3.9) and solve the convex feasibility problem w.r.t. the variables A_J, B_J, C_J, D_J[1].*

If the focus is on Problem 3 we are interested in obtaining an anti-windup controller that maximize the range of acceptable reference inputs (given by the maximum ρ). For this purpose, conditions in Theorem 5 can be slightly modified in a way similar to what was done in the previous chapter (and following remark 3.3 in [Biannic and Tarbouriech 2009]) and the following algorithm can be stated:

[1]The matrices Z, S and the scalar γ do not need to be fixed when solving conditions in Theorem 4.

Algorithm 8. *(Synthesis: Enlargement of the Stability Region - Alternative Formulation)*

1. *Define the vertices* $\mathcal{X}_i$ *of* $\mathcal{X}(\rho)$.

2. *Choose an initial value for* α *(*$\alpha = 1$ *is usually chosen as the first guess).*

3. *Minimize* $\beta = 1/\rho^2$ *(maximize* ρ*) under the LMI constraints (3.10), (3.11),*

$$\begin{bmatrix} -2\alpha W_{11} & \star & (\alpha Y'A' + Y' - W_{11})N_v & \star \\ -2\alpha \overline{W}'_{12} & -2\alpha \overline{W}_{22} & (\alpha \bar{X}'A' + \bar{X}' - \overline{W}'_{12})N_v & \star \\ \star & \star & N'_v(Y'A' + AY)N_v & \star \\ -\alpha V & \alpha(C_\phi \bar{X} - U) & (SB'_\phi - V)N_v & -2S \end{bmatrix} < 0, \tag{3.20}$$

and

$$\begin{bmatrix} \bar{X}'A' + A\bar{X} & \star & \star \\ \overline{W}_{12} + \alpha A\bar{X} - \bar{T}'_\theta & -\alpha(Y + Y') & \star \\ \overline{W}_{22} + \alpha A\bar{X} - \bar{X}' & -\alpha(\bar{X}' + \bar{T}_\theta) & -\alpha(\bar{X} + \bar{X}') \end{bmatrix} < 0 \tag{3.21}$$

w.r.t. the matrix variables $\bar{X}$, Y, $\bar{T}_\theta$, V, U, S *and* W_{11}, $\overline{W}_{12}$ *and* $\overline{W}_{22}$.

4. *Decrease the value of* α *and repeat the previous step until no improvement in the value of* ρ *is obtained.*

5. *Compute* E *and* H *as the solution of* $E'H = T_\theta - X'Y$ *and compute* $\mathbb{M}$ *as per (3.19).*

6. *Fix* $\mathbb{M}$ *and* $\mathbb{W}$ *in inequality (3.9) and solve the convex feasibility problem w.r.t. the variables* A_J, B_J, C_J, D_J *(and optionally* γ, S, Z, $\tilde{B}_J = B_J S$ *and* $\tilde{D}_J = D_J S$*).*

3.3 POLE CONSTRAINTS ON THE ANTI-WINDUP CONTROLLER USING ALTERNATIVE LMI FORMULATION

Following the same rationale used in section 2.4, we present in this section the Alternative LMI condition for performance optimization or stability domain enlargement using anti-windup pole restrictions.

The starting point for this development is to re-define condition (3.9) in a way such that it incorporates as well the anti-windup controller pole restrictions. The condition obtained is similar to condition (2.35) in the previous chapter and is defined as

$$\begin{bmatrix} \mathbb{M}'\mathbb{A}' + \mathbb{A}\mathbb{M} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda(\hat{Y} + \hat{Y}') \end{bmatrix} & \star & \star & \star \\ \mathbb{W} + \alpha\mathbb{A}\mathbb{M} - \mathbb{M}' + \alpha \begin{bmatrix} 0 & 0 \\ 0 & \lambda\hat{Y} \end{bmatrix} & -\alpha(\mathbb{M} + \mathbb{M}') & \star & \star \\ S\mathbb{B}' - Z & \alpha S\mathbb{B}' & -2S & \star \\ \mathbb{C}_\varepsilon\mathbb{M} & 0 & 0 & -\gamma I_p \end{bmatrix} < 0. \tag{3.22}$$

Based in the above matrix we now use the Projection Lemma as we did previously to obtain the synthesis conditions in Theorem 5. This results in the following Theorem:

Theorem 6 (Peformance Optimizition with anti-windup pole constraints - Alternative Formulation)**.** *Consider the nonlinear closed-loop system of Figure 5. There exists an anti-windup controller $J(s)$ such that the conditions of Theorem 1 are satisfied* iff *there exist:*

- $\bar{X} = X^{-1}$, Y *and* $\bar{T}_\theta \in \Re^{n_M \times n_M}$,
- $S = T^{-1} = diag(s_1, \cdots, s_m) > 0$,
- V *and* $U \in \Re^{m \times n_M}$
- $W_{11} = W_{11}'$, $\overline{W}_{22} = \overline{W}_{22}'$ *and* $\overline{W}_{12} \in \Re^{n_M \times n_M}$

and positive scalar γ such that (3.10), (3.11), (3.12) and

$$\begin{bmatrix} A\bar{X} + \lambda(Y - \bar{T}_\theta) & 0 & \alpha\lambda(Y' - \bar{T}_\theta') & 0 \\ \overline{W}_{12} + \alpha A\bar{X} - \bar{T}_\theta' & -\alpha Y & -\alpha\bar{X} & 0 \\ \overline{W}_{22} + \alpha A\bar{X} - \bar{X}' & -\alpha\bar{T}_\theta & -\alpha\bar{X} & 0 \\ C_\varepsilon\bar{X} & 0 & 0 & -\frac{\gamma}{2}I_p \end{bmatrix} + [\star] < 0. \tag{3.23}$$

Moreover, the poles of the anti-windup controller $J(s)$ satisfy (2.33).

Proof. The proof of this Theorem is similar to the proof of Theorem 5 and Theorem 3 and consists only in showing that inequalities (3.12) and (3.23) can be written under the form of inequalities (2.20) in the Projection Lemma while the inequality (2.19) from the Projection Lemma corresponds to the modified condition (3.22).

Similar to Theorem 3, the first step is to represent condition (3.22) as $\Psi_\lambda + \mathcal{U}'\Omega\mathcal{V} + \mathcal{V}'\Omega'\mathcal{U} < 0$ where

$$\Psi_\lambda = \Psi + \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \lambda(\hat{Y} + \hat{Y}') \end{bmatrix} & \star & \star & \star \\ \alpha \begin{bmatrix} 0 & 0 \\ 0 & \lambda\hat{Y} \end{bmatrix} & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.24}$$

with $\mathcal{U}$, $\mathcal{V}$ and Ψ defined by (3.14).

Using the same matrix manipulations and variable changes as in the proof of Theorem 5 we can easily see that $N_U'\Psi_\lambda N_V < 0$ is equivalent to condition (3.12) and condition $N_V'\Psi_\lambda N_V < 0$ result in a condition that is equivalent to (3.23) by considering the fact that $E'\hat{Y}E = -E'HX = -T_\theta X + X'YX$, as well as the change of variables $\bar{X} = X^{-1}$ and $\bar{T}_\theta = \bar{X}'T_\theta$. □

The algorithms for performance optimization (Algorithm 7) and for stability domain enlargement (Algorithm 8) using the Alternative LMI formulation may now be modified to include pole restriction conditions in the anti-windup controller.

Algorithm 9. *(Synthesis: Performance Optimization with Anti-windup pole restrictions - Alternative Forumulation)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$ and initialize ρ according to the design objectives (maximum admissible reference/disturbance input).*

2. *Choose λ according to the design objectives.*

3. *Choose an initial value for α ($\alpha = 1$ is usually chosen as the first guess).*

4. *Minimize γ under the LMI constraints of Theorem 6 w.r.t. the matrix variables $\bar{X}$, Y, $\bar{T}_\theta$, V, U, S and W_{11}, $\overline{W}_{12}$ and $\overline{W}_{22}$.*

5. *If the above problem is infeasible, decrease α and repeat previous step.*

6. *If $\alpha \approx 0$ and problem is infeasible, decrease ρ and restart from step 2.*

7. *Compute E and H as the solution of $E'H = T_\theta - X'Y$ and compute $\mathbb{M}$ as per (3.19).*

8. *Fix $\mathbb{M}$ and $\mathbb{W}$ in inequality (3.22) and solve the convex feasibility problem w.r.t. the variables A_J, B_J, C_J, D_J (and optionally γ, S, Z, $\tilde{B_J} = B_J S$ and $\tilde{D_J} = D_J S$)..*

Algorithm 10. *(Synthesis: Enlargement of the Stability Region with Anti-windup pole restrictions - Alternative Formulation)*

1. *Define the vertices $\mathcal{X}_i$ of $\mathcal{X}(\rho)$.*

2. *Choose λ according to the design objectives.*

3. *Choose an initial value for α ($\alpha = 1$ is usually chosen as the first guess).*

4. *Minimize $\beta = 1/\rho^2$ (maximize ρ) under the LMI constraints (3.10), (3.11), (3.20) and*

$$\begin{bmatrix} A\bar{X} + \lambda(Y - \bar{T}_\theta) & 0 & \alpha\lambda(Y' - \bar{T}_\theta') \\ \overline{W}_{12} + \alpha A\bar{X} - \bar{T}_\theta' & -\alpha Y & -\alpha\bar{X} \\ \overline{W}_{22} + \alpha A\bar{X} - \bar{X}' & -\alpha\bar{T}_\theta & -\alpha\bar{X} \end{bmatrix} + [\star] < 0. \quad (3.25)$$

   w.r.t. the matrix variables $\bar{X}$, Y, $\bar{T}_\theta$, V, U, S and W_{11}, $\overline{W}_{12}$ and $\overline{W}_{22}$.

5. *Decrease the value of α and repeat the previous step until no improvement on the value of ρ is obtained.*

6. *Compute E and H as the solution of $E'H = T_\theta - X'Y$ and compute $\mathbb{M}$ as per (3.19).*

7. *Fix $\mathbb{M}$ and $\mathbb{W}$ in inequality (3.22) and solve the convex feasibility problem w.r.t. the variables A_J, B_J, C_J, D_J (and optionally γ, $\mathbb{W}$, S, Z, $\tilde{B_J} = B_J S$ and $\tilde{D_J} = D_J S$).*

4 RESULTS AND COMPARISONS

In this section the concepts and algorithms presented in chapters 2 and 3 are applied to a numerical example extracted from [Biannic and Tarbouriech 2009]. This example, issued from a realistic flight control problem, is used to discuss the results of the already existing synthesis method for performance optimization or stability region enlargement as well as the new alternative synthesis method presented in chapter 3. The effects of pole placement restrictions on the anti-windup controller is also studied in both stability and performance problems.

4.1 THE NUMERICAL EXAMPLE

4.1.1 Introduction to the flight control problem

To illustrate the techniques developed in this work we use a practical example extracted from [Biannic and Tarbouriech 2009] which is focused in the longitudinal control design of a fighter aircraft. In this example the focus is to control the aircraft angle-of-attack α_a based on the elevator position δ_{e_r}. It is considered a particular point of the flight envelope associated to low dynamic pressure for which the aircraft exhibits unstable dynamics. At this point the linearized short-term dynamics is given by the following state-space equations:

$$\begin{pmatrix} \dot{\alpha}_a \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -0.5 & 1 \\ 0.8 & -0.4 \end{pmatrix} \begin{pmatrix} \alpha_a \\ q \end{pmatrix} + \begin{pmatrix} -0.2 \\ -5 \end{pmatrix} \delta_{e_r} \qquad (4.1)$$

where α_a, q and δ_{e_r} denote, respectively, the angle-of-attack, the pitch-rate and the elevator deflection.

The elevator actuator is the focus of our attention in this work since it has magnitude and rate limitations. The nonlinear actuator is represented by the block diagram of Figure 6, where we note the presence of two limited integrators to represent the rate limitation $L_r = 80°/\text{s}$ and the magnitude limitation[1] $L_p = 15°$.

The Limited Integrator is a dynamic nonlinearity, i.e., its output

[1] As noted in [Biannic and Tarbouriech 2009], the magnitude limitation is not usually symmetric. In this example, due to trimming conditions the magnitude limits of the actuator are $[-25,\ 15]$, but it is considered the interval $[-15,\ 15]$ to enforce symmetry.

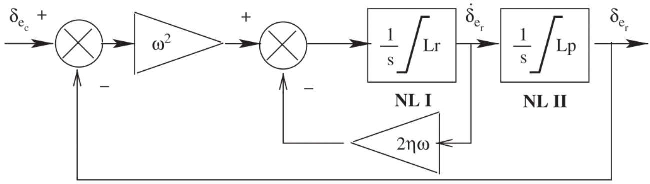

Figure 6: Nonlinear actuator model. Extracted from [Biannic and Tarbouriech 2009]

cannot be determined only with the instantaneous value of the input, it depends on the *state*. The block diagram representing a Limited Integrator is depicted in Figure 7 below:

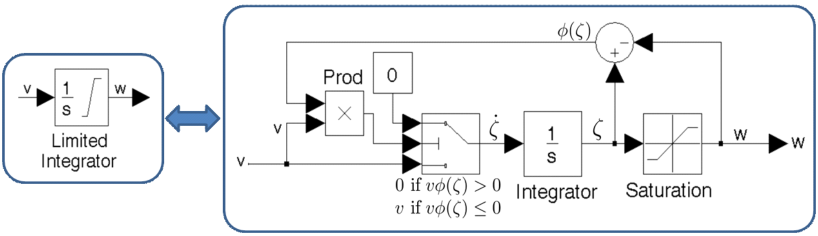

Figure 7: Block diagram of the Limited Integrator.

Following the anti-windup approach, first we compute a linear controller which provides desired reference tracking and robustness properties disregarding the nonlinear (saturated) behavior. For this, we considered a linearized version of the actuator of Figure 6 where the limited integrators are replaced by standard integrators. This linear actuator model is given by the following transfer function:

$$\frac{\delta_{e_r}(s)}{\delta_{e_c}(s)} = \frac{\omega^2}{s^2 + 2\eta\omega s + \omega^2} \tag{4.2}$$

where δ_{e_c} denotes the commanded elevator deflection and $\omega = 60$ rad/s is the natural frequency and $\eta = 0.6$ is the damping ratio.

The next step is to design a controller for the closed-loop plant composed by the plant (4.1) and the actuator (4.2). Again, for comparison purposes the chosen controller follows [Biannic and Tarbouriech 2009] which suggests implementation of a PID-structured linear controller to track the angle-of-attack as fast as possible without steady-state error:

$$\left\{ \begin{array}{l} \dot{x}_K = \alpha_c - \alpha_a \\ \delta_{e_c} = K[\ \alpha_c \quad x_K \quad \alpha_a \quad q\ ]' \end{array} \right. \tag{4.3}$$

where α_c is the desired angle-of-attack (reference) and the PID gains in vector K are selected as

$$K = [\ -1 \quad -30 \quad 15 \quad 2.5\ ].$$

For these choices of controller and actuator model, the resulting fifth-order system (two states from plant, two from actuator and one from controller) can be approximated by a second-order system $L(s)$, which is referred as the linear nominal model

$$L(s) = \frac{1}{(0.25s + 1)^2}. \tag{4.4}$$

4.1.2 Nonlinear actuator modeling

Since the limited integrator is a *dynamic nonlinearity* it cannot be directly handled by the anti-windup approaches discussed in this work. However, in [Biannic and Tarbouriech 2009] the authors propose an approximation of this nonlinearity such as depicted in Figure 8. In this representation, the dynamic nonlinearities are replaced by a scheme that involves saturation blocks.

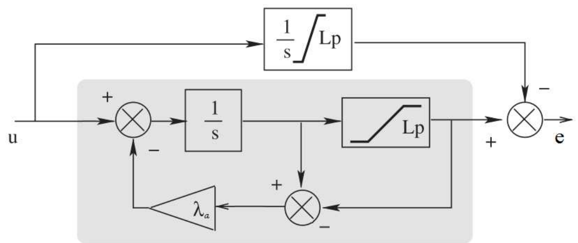

Figure 8: Approximation of a Limited Integrator. Adapted from [Biannic and Tarbouriech 2009]

In this scheme we can observe that, for non saturated initial conditions and an input signal u with bounded derivatives, the approximation error e tends to zero as λ_a increases. Similar to [Biannic and Tarbouriech 2009] we use in this work an empirical choice $\lambda_a = 100$ which shows good accuracy for the desired approximation.

With the representation of the actuator from Figure 6 using the approximation of the limited integrator given by Figure 8, we can now present the open-loop system $G(s)$ (following the notation of Figure 4) by combining the actuator to the plant (4.1):

$$G(s): \begin{cases} \dot{x}_G = A_G x_G + B_{G_\phi}\Phi(y_\phi) + B_{G_\delta}\delta_{e_c} \\ y_c = \alpha_a \\ y_\phi = [\zeta_1 \quad \zeta_2]' \end{cases} \tag{4.5}$$

with

$$x_G = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \alpha_a \\ q \end{bmatrix} \qquad A_G = \begin{bmatrix} -72 & -675 & 0 & 0 \\ 5.33 & 0 & 0 & 0 \\ 0 & -3 & -0.5 & 1 \\ 0 & -75 & 0.8 & -0.4 \end{bmatrix}$$

$$B_{G_\phi} = \begin{bmatrix} -28 & 675 \\ -5.33 & -100 \\ 0 & 3 \\ 0 & 75 \end{bmatrix} \qquad B_{G_\delta} = \begin{bmatrix} 45 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the saturation type nonlinearities where normalized and replaced by dead-zones $\phi(\cdot)$ which depends only on the actuator states ζ_1 and ζ_2, thus no nested saturations are involved.

4.1.3 Closed-loop system

The stability of the closed-loop system composed by system $G(s)$, given by (4.5), and the PID controller (4.3) is analyzed in [Biannic and Tarbouriech 2009] and numerical experiments showed that for $\alpha_c \approx 8°$ stability is lost. Introducing a first-order reference filter $F_R(s)$ with time constant $\tau = 0.1$ s this value was increased up to $\alpha_c \approx 13°$. However this is still too small, since desired angle-of-attack would be as high as $\alpha_c = 25°$. To meet such constraint an anti-windup scheme is proposed as per notation in Figure 4.

To apply the anti-windup synthesis algorithms presented in chapters 2 and 3 first we need to obtain a representation of the closed-loop system $M(s)$ which is formed by the interconnection of system $G(s)$ as per (4.5), the PID controller $K(s)$, the reference generator $R(s)$ and its reference filter $F_R(s)$, as well as the filter $F(s)$ for the anti-windup output signal v_2 and, finally, the linear nominal system $L(s)$. This representation is given by the the following relations:

$$
\begin{aligned}
R(s):&\quad \begin{cases} \dot{x}_R = -0.01x_R,\ x_R(0) = \rho \in \Re \\ \alpha_c = x_R \end{cases} \\
F_R(s):&\quad \left\{\ \dot{\overline{\alpha}}_c = 10(\alpha_c - \overline{\alpha}_c) \right. \\
F(s):&\quad \left\{\ \dot{\overline{v}}_2 = 20(v_2 - \overline{v}_2) \right. \\
K(s):&\quad \begin{cases} \dot{x}_K = \overline{\alpha}_c - \alpha_a + v_1 \\ \delta_{e_c} = K[\overline{\alpha}_c\ x_K\ \alpha_a\ q]' + \overline{v}_2 \end{cases} \\
L(s):&\quad \begin{cases} \dot{x}_{L_1} = x_{L_1} \\ \dot{x}_{L_2} = -16x_{L_1} - 8x_{L_2} + \overline{\alpha}_c \\ y_r = 16x_{L_1} \end{cases}
\end{aligned}
\tag{4.6}
$$

where $\overline{\alpha}_c$ and $\overline{v}_2$ are the filtered reference signal and filtered anti-windup output v_2, respectively; x_{L_1} and x_{L_2} are the states of the linear nominal model. These interconnections are depicted in Figure 9 below.

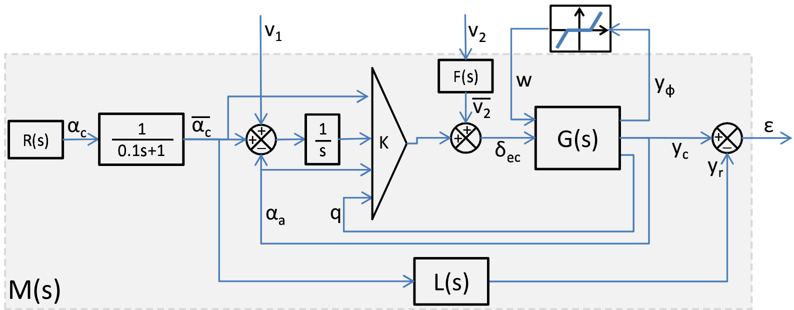

Figure 9: Closed-loop system $M(s)$. Adapted from [Biannic and Tarbouriech 2009].

Considering the above definitions the closed-loop system $M(s)$ is now represented in terms of equations (2.4), with the following matrix definitions:

$$A = \begin{bmatrix} -0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -72 & -675 & 675 & 112.5 & 0 & 0 & -45 & -1350 & 45 \\ 0 & 5.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -0.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -75 & 0.8 & -0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -16 & -8 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B_v' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix}$$

$$B_\phi' = \begin{bmatrix} 0 & -28 & -5.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 675 & -100 & 3 & 75 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_\phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_\varepsilon = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -16 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.7}$$

where the augmented state vector ξ is constructed as

$$\xi = \begin{bmatrix} x_R & \zeta_1 & \zeta_2 & \alpha & q & x_{L1} & x_{L2} & \overline{\alpha}_c & x_K & \overline{v}_2 \end{bmatrix}'. \tag{4.8}$$

4.2 ANTI-WINDUP SYNTHESIS - STABILITY REGION ENLARGEMENT

In this section we are interested in applying Algorithms 4 and 8 in order to obtain an anti-windup controller that maximizes the region of stability.

When considering the stability region enlargement problem, we are particularly interested in maximizing the ellipsoid $\mathcal{E}$, which represents the approximated region of attraction, in some particular directions. From the definition of the reference generator $R(s)$ we observe that the initial state $x_R(0) = \rho$ represents the desired angle-of-attack α_c for which guaranteed stability is sought. Therefore, in order to maximize the admissible reference, in this problem we choose to maximize $\mathcal{E}$ in the direction corresponding to the state x_R. Since we defined the states as (4.8), we can choose the vertices of $\mathcal{X}(\rho)$ as $\mathcal{X}_1 = [1 \;\; 0 \cdots 0]'$ in Algorithms 4 and 8.

For sake of presentation we will refer to the anti-windup controllers obtained by the methods presented in chapter 2 as DAW (Dynamic Anti-Windup) and the anti-windup controllers obtained from the methods presented in chapter 3 as ADAW (Alternative Dynamic Anti-Windup).

We present as well in this section the result of the Algorithms 6 and 10 which include pole constraints on the anti-windup controller and we discuss the effect of these constraints on the region of stability and as well on the performance.

4.2.1 Results using DAW and ADAW synthesis methods - Stability

To define and solve the LMI problems we use MATLAB® and its toolbox "LMI Lab". We used the AWAST toolbox [Biannic and Roos 2009] for the DAW synthesis and analysis procedures. For the ADAW method, we developed a MATLAB® routine which represents conditions from Theorems 4, 5 and 6 and perform Algorithms 7, 8, 9 and 10. This routine is available upon request.

In Algorithms 8 we started with $\alpha = 1$ in step 2, however this choice resulted infeasible conditions in step 3. We then progressively reduced α until feasibility was reached, this occurred around $\alpha = 0.05$. Alternatively, we also used Algorithms 6 and 10 including anti-windup controller pole restrictions.

Table 1 and Figures 10, 11 and 12 show results for selected values of the pole restriction λ and the Projection Lemma multiplier α.

Method	$\boldsymbol{\lambda = 0}$	$\boldsymbol{\lambda = 1}$	$\boldsymbol{\lambda = 5}$
	ρ_{max}	ρ_{max}	ρ_{max}
DAW	34.02	24.95	20.02
$ADAW_{\alpha=0.05}$	29.11	22.54	18.44
$ADAW_{\alpha=0.01}$	33.84	26.36	20.69
$ADAW_{\alpha=0.001}$	34.02	26.48	20.76

Table 1: DAW and ADAW stability synthesis results.

We present on Table 2 below an example of the anti-windup controllers poles obtained using the DAW and ADAW methods focused on stability optimization for $\lambda = 1$ and using $\alpha = 0.01$ in the ADAW method.

Poles of $J(s)$, $\lambda = 1$

DAW	***ADAW***$_{\boldsymbol{\alpha=0.01}}$
$\lambda_1 = -1.81$	$\lambda_1 = -1.79$
$\lambda_2 = -2.28$	$\lambda_2 = -1.91$
$\lambda_3 = -3.43$	$\lambda_3 = -5.25$
$\lambda_4 = -3.96$	$\lambda_4 = -9.24$
$\lambda_5 = -27.74$	$\lambda_{5,6} = -11.33 \pm 1.93i$
$\lambda_6 = -28.95$	$\lambda_7 = -102.87$
$\lambda_{7,8} = -140.58 \pm 12.30i$	$\lambda_8 = -126.92$
$\lambda_9 = -216.53$	$\lambda_{9,10} = -128.98 \pm 89.30i$
$\lambda_{10} = -306.83$	

Table 2: Stability Synthesis - Poles of $J(s)$ for $\lambda = 1$.

We can observe from the values shown in Table 1 as well as in Figures 10, 11 and 12 which were obtained by simulating the DAW and ADAW controller using the nonlinear model (using magnitude and rate limitations represented by limited integrators) that both methods show very similar results in terms of size of the stability domain (parameter ρ_{max}).

We can note from Table 1 that the value of ρ obtained using the ADAW method varies with the choice of the Projection Lemma multiplier α, smaller values for α resulting bigger values of the optimized criteria ρ.

Comparing both DAW and ADAW methods (considering $\alpha = 0.001$), we observe that for $\lambda = 0$ both methods presented the same value for ρ, however, as we increased λ the ADAW method resulted bigger values of ρ.

Also, with the increase of λ, which represents the constraints to the real part of the anti-windup controller poles, we notice that the stability domain reduced for both methods, but the simulations showed better time response results. We can observe that for $\lambda = 0$ the stability enlargement algorithms provide a slow time response and steady state error. From Figures 10, 11 and 12 we can observe that as λ increases the steady state error is no longer observed and the time response become faster, however, after some point, further increases of

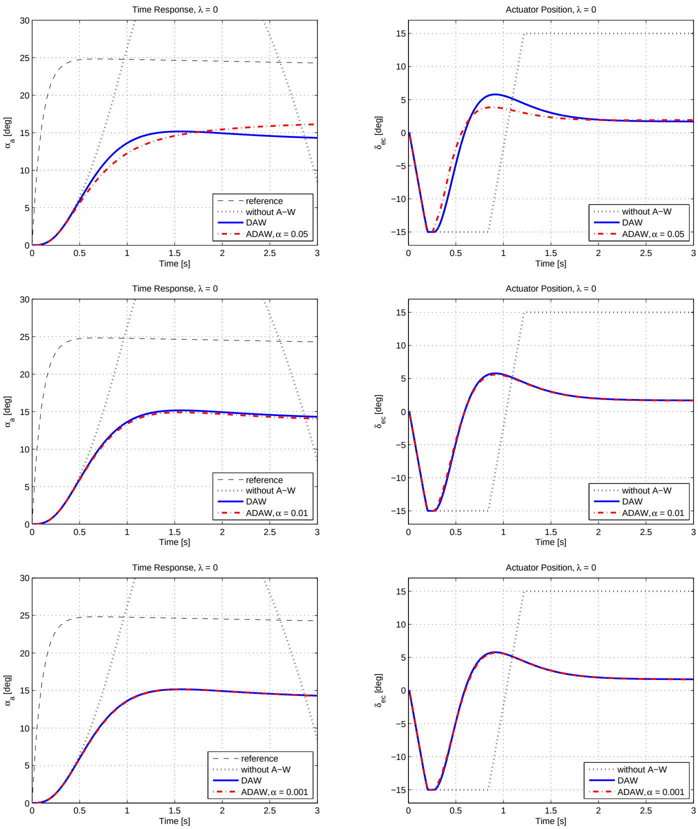

Figure 10: Stability Synthesis - Time response for $\lambda = 0$ and variations of α.

λ do not improve time response anymore. Much bigger values of λ are not desirable since they may cause implementation difficulties due to its very fast dynamics.

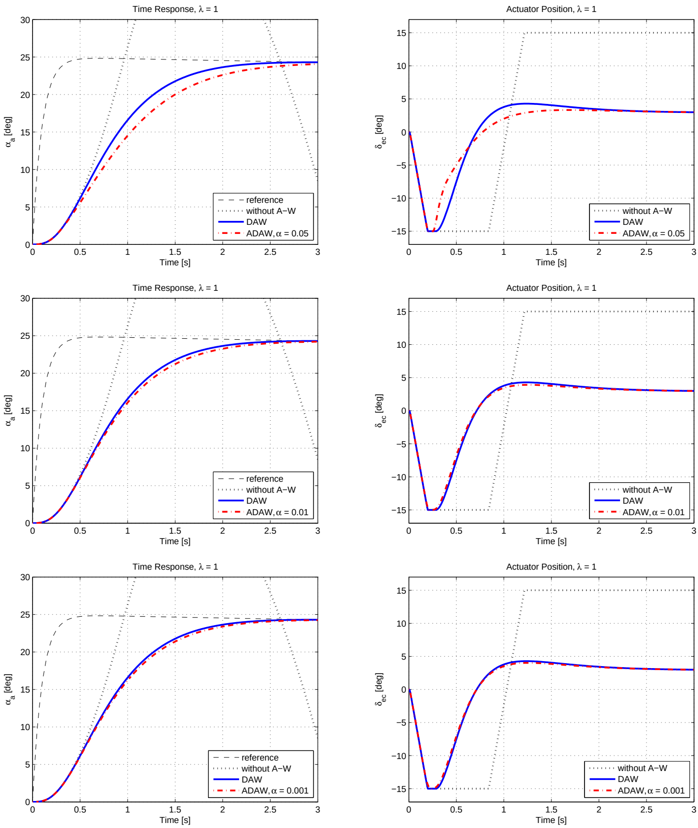

Figure 11: Stability Synthesis - Time response for $\lambda = 1$ and variations of α.

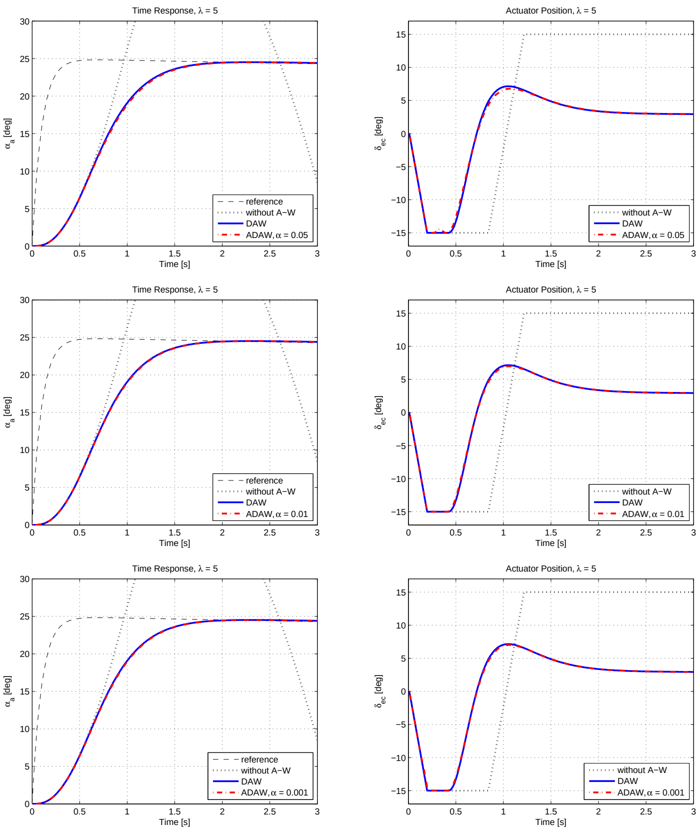

Figure 12: Stability Synthesis - Time response for $\lambda = 5$ and variations of α.

4.3 ANTI-WINDUP SYNTHESIS - PERFORMANCE OPTIMIZATION

In this section we exemplify the application of Algorithms 3 and 7 on the aircraft longitudinal control problem presented in section 4.1. These algorithms are aimed at minimizing a $\mathcal{L}_2$ energy criteria for the tracking-error signal between the nonlinear system and a linear nominal system.

As a initial step, we need to select a suitable value of ρ, which in this example represents the angle-of-attack reference α_c for which the closed-loop system has guaranteed stability. As showed in [Biannic and Tarbouriech 2009] the increase of parameter ρ in the performance optimization affects the optimized parameter γ, i.e. as ρ increases the performance indicator γ decreases. Similarly to [Biannic and Tarbouriech 2009], we fix $\rho = 11$ in this section and we focus only on the performance criteria γ for both the DAW (Dynamic Anti-Windup - as presented in chapter 2) and ADAW (Alternative Dynamic Anti-Windup - as presented in chapter 3) methods. Note that this choice of ρ is below the ρ_{max} values presented in Table 1, nevertheless the numerical simulations showed that even fixing $\rho = 11$ we can get the closed-loop system to converge to the reference for much bigger values.

We present as well in this section the results of the Algorithms 5 and 9 which including pole constraints on the anti-windup controller and we discuss the effect of parameter λ on the performance.

4.3.1 Results using DAW and ADAW synthesis methods - Performance

Similarly to the previous section we use MATLAB® and its toolbox "LMI Lab" to define and solve the LMI problems. We used the AWAST toolbox [Biannic and Roos 2009] for the DAW synthesis and analysis procedures and a MATLAB® routine developed to implement the ADAW method conditions.

In Algorithm 7 we started by using $\alpha = 1$ in step 2, however this choice resulted infeasible conditions in step 3. We then progressively reduced α until feasibility was reached, this occurred around $\alpha = 0.05$. Alternatively, we also used the Algorithms 5 and 9 incluiding anti-windup controller pole restrictions. Table 3 summarizes all results obtained in terms of the optimized criteria γ, while Figures 13, 14 and 15 show the time-domain evaluation as well as the actuator position obtained by simulating the nonlinear closed-loop system from Figure

4 (using magnitude and rate limitations represented by limited integrators) with the obtained anti-windup controller and angle-of-attack reference $\alpha_c = 25°$ for some choices of λ and α.

Method	$\boldsymbol{\lambda = 0}$ γ	$\boldsymbol{\lambda = 0.004}$ γ	$\boldsymbol{\lambda = 1}$ γ	$\boldsymbol{\lambda = 5}$ γ
DAW	2.40	0.40	0.42	0.50
$ADAW_{\alpha=0.05}$	0.72	0.73	0.79	0.92
$ADAW_{\alpha=0.01}$	0.47	0.48	0.45	0.63
$ADAW_{\alpha=0.001}$	0.48	0.59	0.55	0.93

Table 3: DAW and ADAW performance synthesis results for $\rho = 11$.

One important thing to notice is that the anti-windup controllers obtained with the DAW method for $\lambda = 0$, i.e. when no anti-windup controller pole restrictions are imposed, usually present at very "slow" and very "fast" poles as we can see in Table 4 below. This imposes numerical difficulties for implementation and simulations do not complete on reasonable time. For this case ($\lambda = 0$) the ADAW method presented much better numerical results as can be seen on Tables 3 and 4. To overcome this issue in the DAW method, we perform as in the AWAST toolbox and instead of using $\lambda = 0$ we select λ as the slowest pole of the plant $M(s)$ (disregarding the reference generator state) arbitrarily factored by 1000. In the numerical example used in this work this resulted $\lambda = 0.004$.

Poles of $J(s)$, $\lambda = 0$

DAW	$ADAW_{\alpha=0.01}$
$\lambda_1 = -0.08$	$\lambda_1 = -0.03$
$\lambda_2 = -0.95$	$\lambda_2 = -1.30$
$\lambda_3 = -6.02$	$\lambda_3 = -5.01$
$\lambda_{4,5} = -9.18 \pm 5.00i$	$\lambda_{4,5} = -8.31 \pm 6.16i$
$\lambda_6 = -2.69 \times 10^3$	$\lambda_6 = -27.92$
$\lambda_{7,8} = (-5.47 \pm 7.32i) \times 10^4$	$\lambda_7 = -91.93$
$\lambda_9 = -1.62 \times 10^6$	$\lambda_8 = -141.68$
$\lambda_10 = -5.65 \times 10^7$	$\lambda_{9,10} = -249.26 \pm 159.86i$

Table 4: Performance Synthesis - Poles of $J(s)$ for $\lambda = 0$.

We also present on Table 5 another example of the anti-windup controllers poles obtained using the DAW and ADAW methods focused on performance optimization for $\lambda = 1$ and using $\alpha = 0.01$ in the ADAW method.

Poles of $J(s)$, $\lambda = 1$

DAW	$ADAW_{\alpha=0.01}$
$\lambda_1 = -2.56$	$\lambda_1 = -2.32$
$\lambda_2 = -3.22$	$\lambda_2 = -3.31$
$\lambda_3 = -6.37$	$\lambda_3 = -4.38$
$\lambda_{4,5} = -10.24 \pm 3.01i$	$\lambda_{4,5} = -9.72 \pm 5.32i$
$\lambda_{6,7} = -100.32 \pm 160.72i$	$\lambda_6 = -42.29$
$\lambda_8 = -259.32$	$\lambda_{7,8} = -127.61 \pm 31.67i$
$\lambda_{9,10} = -374.24 \pm 34.29i$	$\lambda_{9,10} = -132.85 \pm 90.60i$

Table 5: Performance Synthesis - Poles of $J(s)$ for $\lambda = 1$.

From Figures 13, 13 and 13 and Table 3 we can observe that results for the DAW and ADAW synthesis methods are very similar. As showed before on Tables 3 and 4, for $\lambda = 0$ the ADAW method presents better numerical results, i.e the ADAW results an anti-windup with poles in smaller frequency range and return smaller values for the minimized criteria γ. However, as λ grows we can see that the DAW and ADAW methods provide very similar results in terms of γ as well as on time response, with the DAW method providing slightly better results. This is expected since from condition (3.4) we observe that the ADAW method impose an additional restriction to the Lyapunov matrix P which potentially increases the conservatism as α increases and as γ reduces.

On the ADAW method we also evaluated the anti-windup controller obtained for three different values of the Projection Lemma multiplier α. Although we expected that as α decreases the ADAW results would get closer to the DAW results, we observed that for $\alpha = 0.01$ the minimized criteria γ was slightly better than for $\alpha = 0.05$ and $\alpha = 0.001$.

Observing the actuator position for the DAW and ADAW anti-windup controllers on Figures 13, 14 and 15 we note that both have very similar behavior, but the ADAW controllers remain a little less operating on the saturation regime than the DAW method.

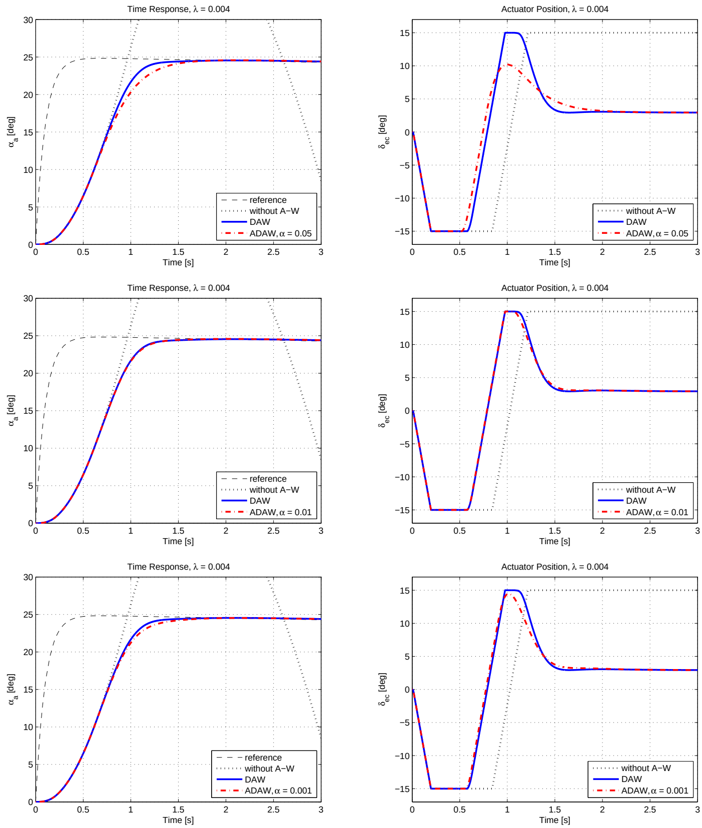

Figure 13: Performance Synthesis - Time response for $\lambda = 0.004$ and variations of α.

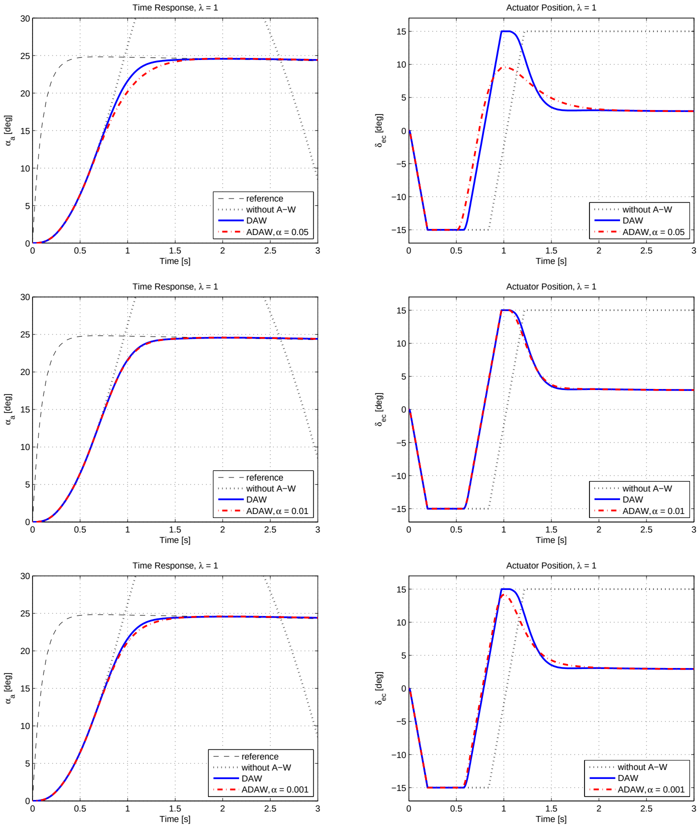

Figure 14: Performance Synthesis - Time response for $\lambda = 1$ and variations of α.

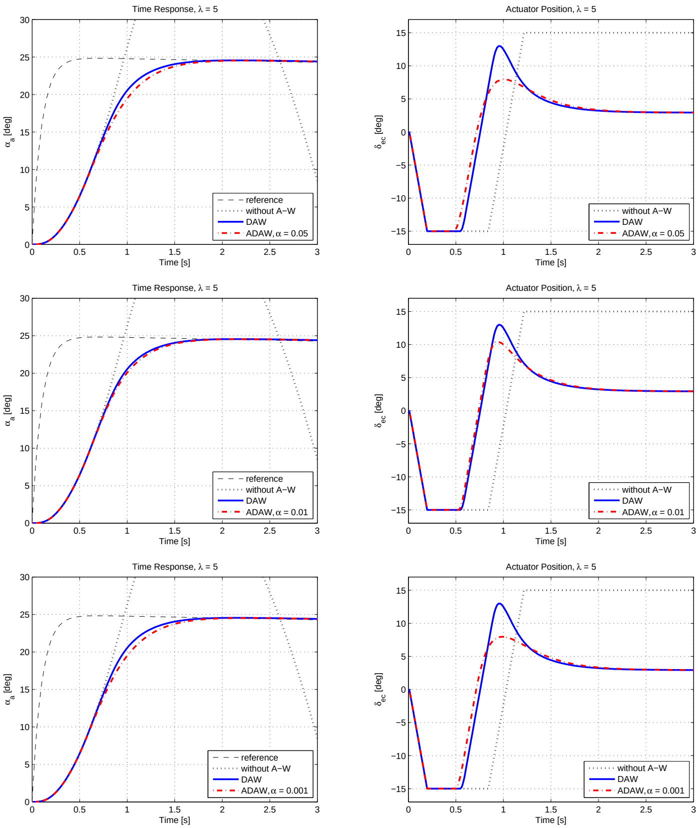

Figure 15: Performance Synthesis - Time response for $\lambda = 5$ and variations of α.

5 CONCLUSIONS AND PERSPECTIVES

In this work we studied systems under saturation control. The focus was on two classes of problems: the analysis problems, in which we are interested either in determining a stability region or obtain a measure of the closed-loop system performance; and the synthesis problems, where the focus is to obtain an anti-windup controller which either maximize the stability region or minimize a tracking-error energy criteria based on a linear nominal model.

On chapter 2 we studied the existing analysis and synthesis methods proposed in [Biannic and Tarbouriech 2009] and [Roos and Biannic 2008] which are based on the formulation of an LMI problem using Lyapunov Stability criteria and a representation of the saturations (replaced by dead-zones) through modified sector conditions as [Gomes da Silva Jr. and Tarbouriech 2005], these methods are referred in this work as DAW (Dynamic Anti-Windup).

While in [Biannic and Tarbouriech 2009] the authors are focused on optimization of an $\mathcal{L}_2$ energy criteria providing as well conditions for the stability enlargement problem, the focus of [Roos and Biannic 2008] is on maximization of the stability region considering pole restrictions applied to the anti-windup controller. In this work we present both these results combined, i.e. we provide conditions for anti-windup synthesis based on a performance criteria or stability region enlargement criteria while at same time considering anti-windup pole placement restrictions.

The main contribution of this work is on chapter 3 where we used the Projection Lemma in a way similar to [Pipeleers et al. 2009] and [Peaucelle and Arzelier 2001] to propose an alternative LMI formulation for the analysis and synthesis problems discussed on chapter 2. We showed that this alternative formulation, referred as ADAW (Alternative Dynamic Anti-Windup), can be seen as a generalization of the DAW method. The main difference of ADAW is the use of an additional variable on the LMI problem. This additional variable allow us to avoid multiplications of the Lyapunov matrix P (or its inverse Q) by the state space matrices $\mathbb{A}$ and $\mathbb{B}$. This represents an additional degree-of-freedom which can be exploited for reduction of conservatism or inclusion of new design objectives (constraints). On the ADAW method we used a Projection Lemma multiplier scalar α and we observed that by fixing this parameter to zero and forcing the additional variable $\mathbb{M}$ to be equal to the Lyapunov Matrix (i.e. $\mathbb{M} = \mathbb{W} = Q$) we showed that we obtain the DAW conditions.

The Analysis and Synthesis Problems as well as the associated Theorems and Algorithms studied/developed in chapters 3 and 2 are summarized in Table 6 below.

Method	**Analysis**		**Synthesis**	
	Stab. Prob. 1	**Perf. Prob. 2**	**Stab. Prob. 3**	**Perf. Prob. 4**
DAW	Thm. 1 Alg. 2	Thm. 1 Alg. 1	Thm. 2 Alg. 4	Thm. 2 Alg. 3
DAW, pole	- -	- -	Thm. 3 Alg. 6	Thm. 3 Alg. 5
ADAW	Thm. 4 -	Thm. 4 -	Thm. 5 Alg. 8	Thm. 5 Alg. 7
ADAW, pole	- -	- -	Thm. 6 Alg. 10	Thm. 6 Alg. 9

Table 6: Summary of Theorems and Algorithms for Analysis and Synthesis Problems.

In chapter 4 we used the concepts studied on chapter 2 and developed on chapter 3 on a numerical example which represents the longitudinal behavior of a fighter aircraft. This example, extracted from [Biannic and Tarbouriech 2009], is very interesting to illustrate the concepts of application of modern anti-windup techniques since it contains both magnitude and rate limitations on the actuator and it considers a particular point of the flight envelope where the dynamics are unstable.

We applied the synthesis techniques focused on the stability region enlargement problem (Problem 3) as well synthesis based on performance criteria (Problem 4) in the numerical example presented. The results obtained allowed us to verify that the synthesis algorithms using alternative LMI conditions successfully computed anti-windup controllers which satisfy the imposed restrictions. Additionally we studied the effects of pole placement restrictions on the anti-windup controller in both stability and performance problems and using both the DAW and ADAW methods.

In general, the ADAW method resulted slightly better results for the stability domain enlargement problem, i.e. the value of ρ obtained with the alternative conditions is slightly bigger than the obtained with the DAW method. In this case, the anti-windup pole restrictions resulted better time response results than the case without pole restrictions (when $\lambda = 0$), which presented slower response and steady state error. The value of ρ was affected by the choice of the pole restriction parameter λ. As λ increased the size of the stability domain reduced.

For the performance optimization synthesis procedures the DAW and ADAW methods provided very similar results. The DAW method however, provided slightly better results, i.e. the value of the parameter γ which represents the bound for the energy of the tracking-error signal ε was slightly smaller then the ones obtained with the anti-windup controllers computed using the ADAW method. The effect of the parameter λ on the performance synthesis is similar to the stability enlargement case, as we increased the value of λ the minimized criteria became bigger. An interesting fact comparing the DAW and ADAW anti-windup controllers is that the DAW usually resulted anti-windup controllers with poles in a wider range of frequencies which sometimes difficult the implementation, this characteristic is highlighted for the case with $\lambda = 0$ where the anti-windup controller obtained with the DAW method was not able simulate in reasonable time.

One of the objectives of this work was to explore the fact due to the additional variable included by the Projection Lemma the Lyapunov matrix does not appear multiplied by the system state matrices which means that different Lyapunov functions could be used at in the same LMI problem with different purposes. As a possibility for future works we can try using one Lyapunov function to comply with stability, performance characterization and modified sector conditions restrictions and a second Lyapunov function to enforce anti-windup pole placement restriction. In this work this approach was not presented since we could not obtain feasible conditions for the controller reconstruction using the "LMI lab" solver. As a next step the use of other LMI solvers can be tested to verify if feasible results are obtained.

Additionally, this degree-of-freedom can be exploited for other design objectives such as considering bounded parametric uncertainties in the plant or actuator parameters for example. The extension of the ADAW method to the discrete-time case and for systems with transport delay are also open tasks for future works.

As pointed in paragraph 1.3.2 a very important step in this work is the use of modified sector conditions to represent the dead-zone nonlinearity as well as the approximation of the Limited Integrator by means of saturations (or dead-zones) as showed on paragraph 4.1.2. These representations/approximations are used to allow the application of the LMI framework to perform the analysis and synthesis procedures, however they add conservatism as they are generalizations to represent a certain class of nonlinearities. The use of more specialized representation for these nonlinearities may reduce the conservatism and the study of such representations is a very important and promising research field.

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