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ANÁLISE CONCEITUAL DE ESTRUTURAS CINEMÁTICAS PLANAS E ESPACIAIS

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Esta dissertação foi julgada adequada para a obtenção do título de

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n	number of links or vertices
g	number of joints or edges
M	Mobility
λ	order of the screw system
ν	Number of fundamental circuits
K_{ij}	degrees of control between two links i and j
C_{ij}	Connectivity between two links i and j
R_{ij}	Redundancy between two links i and j
G	Grafo
T	Tree
L	number of independent loops
A_j	Adjacency matrix
A_i	Incidence matrix
B	Circuit matrix
B_f	Fundamental circuit matrix
D_{min}	Minimum distance matrix

Resumo

Em geral a escolha da estrutura cinemática de um mecanismo é baseada na experiência e na habilidade do projetista. É possível escolher a estrutura topológica através de uma forma mais sistemática: a enumeração de cadeias cinemáticas. A enumeração de cadeias cinemáticas é uma metodologia reconhecida para encontrar os melhores mecanismos que satisfazem um conjunto de especificações. Na prática, existem dificuldades para implementar essa metodologia, já que o número de cadeias cinemáticas geradas é geralmente muito grande para considerar manualmente os méritos individuais de cada cadeia. Os conceitos de redundância, conectividade e variedade podem ser usados para classificar as cadeias cinemáticas de acordo com as especificações requeridas.

Esse trabalho apresenta uma nova metodologia para o cálculo dos graus de controle, conectividade, redundância e variedade de uma cadeia cinemática, permitindo a classificação das cadeias cinemáticas geradas pela metodologia de enumeração de acordo com as especificações do projeto. Para ilustrar sua utilidade, alguns exemplos de aplicação do algoritmos propostos são apresentados.

Abstract

Usually the designer depends upon intuition to select the best possible kinematic topology of a mechanism for the specified task; nevertheless, this procedure may not always lead to optimum results. The enumeration of kinematic chains, also known as *number synthesis*, has been used for at least the past four decades as a means of finding better mechanisms for some predefined purpose. In practice, however, enumeration can be difficult to implement since the number of kinematic chains generated is often too large to manually consider the individual merits of each chain. For this reason, the concepts of and *connectivity*, *redundancy* and *variety* can be used to classify kinematic chains according to the constraints required.

This work presents a new methodology for the computation of degrees of control, connectivity, redundancy and variety of kinematic chains; allowing the classification of the kinematic chains generated with respect to the constraints required. To illustrates its usefulness, some examples of application of the algorithm proposed are presented.

1 Introduction

Design is the creation of solutions in the form of products or systems that satisfy customer's requirements [Dieter 1991, Pahl and Beitz 1992, Suh 1990, Ullman 1992]. Given a design problem, as many feasible solutions as possible are generated based on the knowledge and on the available information of the problem. Then, these concepts are evaluated against the customer's requirements and a most promising concept is selected for design analysis and design optimisation. Design can be regarded as a mapping of the customer's requirements into a physical embodiment. The better the problem associated with the customer's requirements is understood, the better design can be achieved.

This text will concentrate on mechanism design. Traditionally, mechanisms are created by the designer's intuition, ingenuity, and experience. This ad hoc approach, however, cannot ensure the identification of all feasible design alternatives, nor does it necessarily lead to an optimum design. Two approaches have been developed to alleviate the problem. The first involves the development of atlases of mechanisms grouped according to function for use as a primary source of ideas. The second makes use of a symbolic representation of the kinematic structure and the combinatorial analysis as a tool for enumeration of mechanisms.

The last methodology is very attractive, because a complete set of solutions is produced. This approach is partly analytical and partly algorithmic. It is based on the idea that, during the conceptual design phase, some of the functional requirements of a desired mechanism can be transformed into structural characteristics that can be employed for systematic enumeration of mechanisms. The kinematic structure of a mechanism contains the essential information about which link is connected to which other link by what type of joint. Using graph theory, combinatorial analysis, and computer algorithms, kinematic structures of the same nature, i.e., the same the number of degrees of freedom, type of motion (planar or spatial), and complexity can be enumerated in an essentially systematic and unbiased manner.

However, the number of solutions generated is generally large, and analysis and classification of the kinematic structures generated are not feasible by manual inspection. Consequently, new tools of conceptual analysis and algorithms are needed to automatic classify and evaluate the solutions generated.

The goal of this work is to allow automatic evaluation and classification of kinematic structures, with respect to a set of functional requirements of the design. New theoretical concepts of analysis are introduced, and novel algorithmic tools are proposed and implemented, which allow the designer to analyse and evaluate the solutions generated by enumeration methodologies. This process eventually results in a class of feasible mechanisms that can be subject to dimensional synthesis, kinematic and dynamic analysis, design optimisation, and design detailing.

1.1 Mechanism design

Mechanism design may be regarded as a process of *product design*. Product design is defined as the idea generation, concept development, testing and manufacturing or implementation of a physical object or service. An integrated methodology of product design is described in [Back et al. 2006]. It is divided in eight different phases: *project planning*, *product specifications*, *conceptual design*, *preliminary design*, *detailed design*, *pilot production*, *product marketing* and *product validation*.

This work will focus specifically on mechanism design, and three interrelated phases, as proposed by Tsai [Tsai 2001], are considered:

1. *specification and planning phase*: in this phase the customer's requirements are identified and translated into engineering specifications, in terms of the functional requirements and the time and money available for the development. Finally the project is planned accordingly.

2. *conceptual design phase*: during this phase, as many design alternatives as possible are generated and evaluated against the functional requirements; the most promising concept is selected for design detailing. A rough idea of how the product will function and what it will look like is developed.

3. *product design phase*: in the last phase, a design analysis and optimisation are performed, together with a simulation of the selected concept. Function, shape, material, and production methods are considered.

Design is a continuous process of refining customer's requirements into a final product.

The process is iterative in nature and the solutions are usually not unique. It involves a process of decision making. A talented and experienced engineer can often make sound engineering decisions to arrive at a fine product. Although the third phase is usually the most time consuming phase, most of the manufacturing cost of a product is committed by the end of conceptual design phase. According to a survey, 75% of the manufacturing cost of a typical product is committed during the first two phases. Decisions made after the conceptual design phase only have 25% influence on the manufacturing cost. Therefore, it is critical to pay sufficient attention to the product specification and conceptual design phases. One approach for the generation of concepts is to identify the overall function of a device based on the customer's requirements, and decompose it into sub-functions. Then, various concepts that satisfy each of the functions are generated and combined into a complete design. Techniques for generation of concepts include literature and patent search, imitation of natural systems, analysis of competitor products, brainstorming, etc.

This work focus on the conceptual design phase of mechanisms. During this phase, the designer usually depends upon intuition, experience and capability to select the best possible kinematic topology of a mechanism for the specified task. In practice, some fundamental properties of the kinematic chains, such as number of links, number of kinematic pairs, type of joints, and end-effector mobility, are parameters fixed at the earliest stage of the design. Nevertheless, this procedure may not always lead to optimum results, since new promissory topologies may not be considered.

An alternate approach is to generate an atlas of mechanisms classified according to functional characteristics for use as the sources of ideas for mechanism designers [Artobolevsky 1975, Horton 1951, Jensen 1991, Jensen 1930, Jensen 1936, Newell and Horton 1967]. This approach, however, cannot guarantee the identification of all feasible mechanisms, nor does it necessarily lead to an optimum design.

In Section 1.1.2, a different systematic procedure for the conceptual design phase of mechanisms is presented.

1.1.1 Kinematics of mechanisms

A rigid body is said to be under motion when it is instantaneously changing its position and/or orientation. Since the change of position can only be observed with respect to another body, the motion of a rigid body is a relative measure. Kinematics of a mechanism is the study of relative motion among the several links of a mechanism or machine by neglecting the inertia effects and the forces that cause the motion. In studying the kinematics of a mechanism, the motion of a link is often measured with respect to a fixed link or a reference frame, which may not necessarily be at rest.

There are two branches of kinematics known as *kinematic analysis* and *kinematic synthesis*.

Kinematic analysis

Kinematic analysis is the study of relative motions associated with the links of a mechanism or machine and is a critical step toward proper design of a mechanism. Specifically, given a mechanism and the motion of its input link(s), the relative displacement, velocity, acceleration, etc., of the other links are to be found. These characteristics can be derived by considering the constraints imposed by the joints.

Kinematic synthesis

Kinematic synthesis is the reverse problem of kinematic analysis. In this case, the designer is challenged to devise a new mechanism that satisfies certain desired motion characteristics of an output link. The kinematic synthesis problem can be further divided into three interrelated phases:

1. *Type synthesis* refers to the selection of a specific type of mechanism for product development. During the conceptual design phase, the designer considers as many types of mechanism as possible and decides what type has the best potential of meeting the design objectives. The type of mechanism – cam, linkage, gear train, and so on – is determined. The selection depends to a great extent on the functional requirements of a machine and other considerations such as materials, manufacturing processes, and cost.

2. *Structural synthesis of kinematic chains* deals with the determination of the number of links, type of joint, and number of joints needed to achieve a given number of degrees of freedom of a desired mechanism. Number synthesis also involves the enumeration of all feasible kinematic structures or linkage topologies for a given number of degrees of freedom, number of links, and type of joints. For this reason structural synthesis is sometimes called *number synthesis* or topological synthesis. Several methodologies have been developed for systematic enumeration of kinematic structures [Freudenstein and Maki 1979, Mruthyunjaya 2003] . A thorough understanding of the structural characteristics of a given type of mechanism is critical for the development of an efficient algorithm.

3. *Dimensional synthesis* deals with the determination of the dimensions or proportions of the links of a mechanism. Laying out a cam profile to meet a desired lift specification is a dimensional synthesis problem. Determination of the centre distance between two pivots of a link in a bar-linkage is also a dimensional synthesis problem. Both geometric and analytical methods of synthesis may be used in dimensional synthesis. Typical problems in dimensional synthesis include function generation, coupler-point curve synthesis, and rigid body guidance.

1.1.2 A new systematic approach to mechanism design

The kinematic topology of a mechanism can be chosen through a more systematic approach by taking into account all the constraints that derive from the desired characteristics, such as the kind of task required, the environment, the number of degrees of freedom, the possible redundancy, and so on.

This methodology is based on the application of graph theory and combinatorial analysis. First, the functional requirements of a class of mechanisms are identified. Second, kinematic structures of the same nature are enumerated systematically using graph theory and combinatorial analysis. Third, each kinematic structure is sketched and qualitatively evaluated according to its potential to satisfy the functional requirements. Finally, a promising concept is chosen for dimensional synthesis, design optimization, computer simulation, and prototype demonstration. The process may be iterated several times until a final product is achieved.

The methodology may be summarised as follows:

1. Identify the functional requirements, based on customer's requirements, of a class of mechanisms of interest.

2. Determine the nature of motion (i.e., planar, spherical, or spatial mechanism), degrees of freedom or mobility, type, and complexity of the mechanisms.

3. Identify the set of structural characteristics associated with some of the functional requirements.

4. Enumerate *all* possible kinematic chains that satisfy the largest subset of structural characteristics using graph theory and combinatorial analysis.

5. Select kinematic chains satisfying the complete set of structural characteristics

6. Sketch the corresponding mechanisms and evaluate them qualitatively in terms of satisfaction the remaining functional requirements, resulting in a set of feasible mechanisms.

7. Select the most promising mechanism for dimensional synthesis, design optimisation, computer simulation, prototype demonstration, and documentation.

8. Enter the production phase.

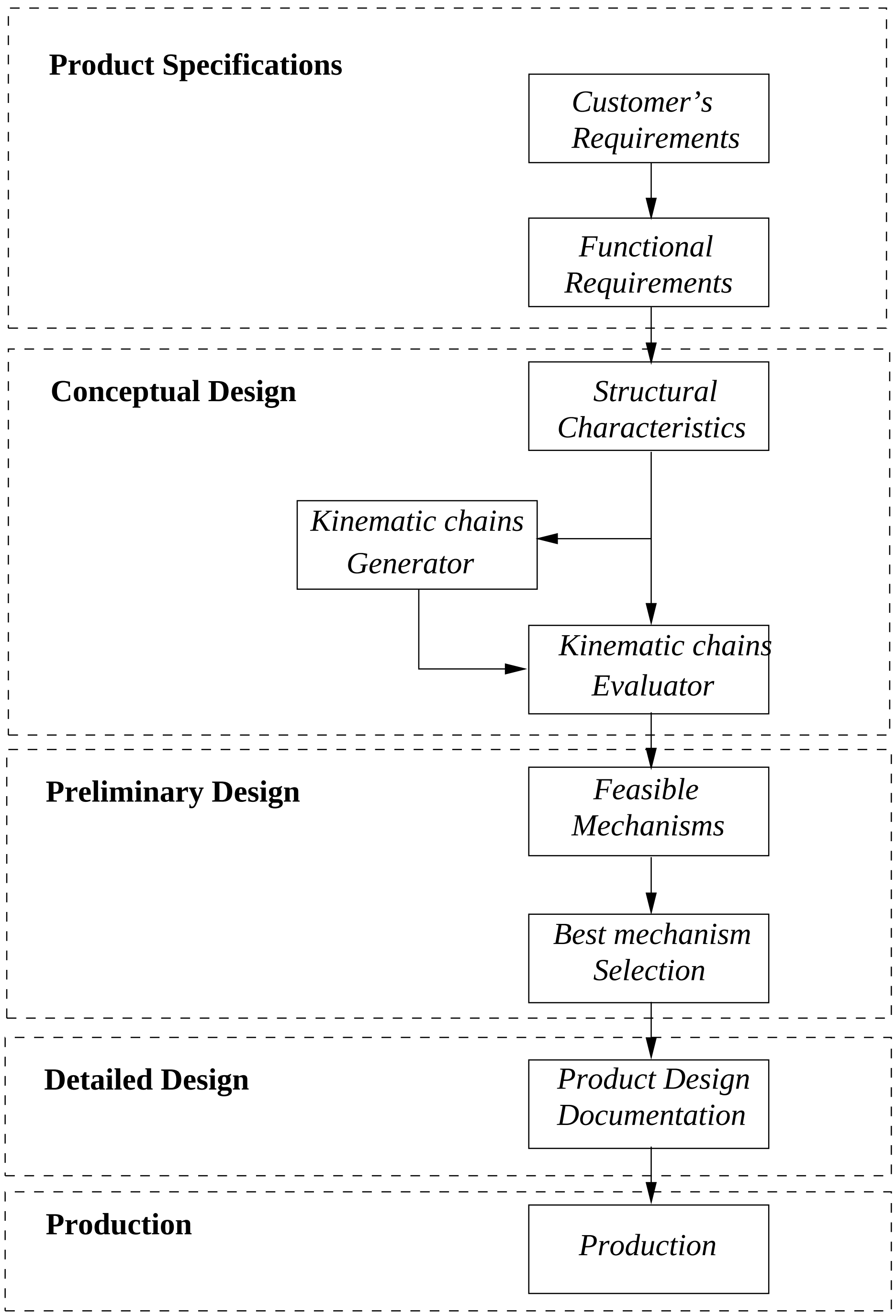


Figure 1.1: A systematic mechanism design procedure

Figure 1.1 shows a block diagram of this methodology; the steps of the methodology are related with the correspondent product design phases [Back et al. 2006].

Thus the methodology consists of two engines: a generator and an evaluator as shown in Figure 1.1. Some of the functional requirements are transformed into the structural characteristics and incorporated in the generator as rules of enumeration, such as number of links, mobility, number of loops. The generator enumerates all possible solutions using graph theory and combinatorial analysis.

The enumeration of kinematic chains, also known as number synthesis, introduced in Section 1.1.1, has been used for at least the past four decades, *e.g.* [Davies and Crossley 1966], as a generator for finding better mechanisms for some predefined purpose. Our approach is new and based on an abstract representation of the kinematic structure. The kinematic structure contains the essential information about which link is connected to which other links by what types of joint. It can be conveniently represented by a graph and the graph can be enumerated systematically using combinatorial analysis and computer algorithms [Crossley 1964, Crossley 1964, Davies 1968, Dobrjanskyj and Freudenstein 1967, Freudenstein and Maki 1979, Freudensten and Woo 1974, Woo 1967].

In practice, however, enumeration can be difficult to implement since the number of kinematic chains generated is often too large to manually consider the individual merits of each chain. Consequently the remaining structural requirements are incorporated in the evaluator as evaluation criteria for the selection of kinematic chains.

The concepts of *connectivity* and *variety* can be used to classify kinematic chains according to the constraints required [Tischler et al. 1995] [Tischler et al. 1998] [Tischler et al. 2001]. Other concepts, created and adapted in [Belfiore and Benedetto 2000], such as *degrees-of-control* and *redundancy*, are also important to this individuation process. In Chapter 3 at page 23 structural characteristics of kinematic chains are introduced.

An interesting example of selection of kinematic chains by means of variety is presented in Appendix C at Page 109.

This results in a class of feasible mechanisms. Finally, a most promising candidate is chosen for the product design. The process may be iterated several times until a final product is achieved. This methodology has been successfully applied in the structure synthesis of planar linkages [Crossley 1964, Freudenstein and Dobrjanskyj 1965], epicyclic gear trains [Buchbaum and Freudenstein 1970, Chatterjee and Tsai 1994, Tsai and Lin 1972], automotive transmission mechanisms [Sohn and Freudenstein 1986], variable-stroke engine mechanisms [Freudenstein and Maki 1983], robotic wrist mechanisms [Lin and Tsai 1989], etc .

Algorithms to automatic calculate the main structural parameters in order to classify kinematic chains are needed as a complementary step to the process of kinematic chains enumeration. This work concerns primarily with automatic calculation of connectivity, variety, redundancy and degrees of control.

1.2 Overview

Chapter 2 introduces the basic concepts of mechanism and machine theory. Different ways of representing mechanisms are described, and the convenience of adopting graph representation is explained.

Chapter 3 examines the structural characteristics of kinematic chain and mechanism. The concepts of mobility, degrees of control, connectivity, redundancy and variety are defined and analysed. The Tischler-Samuel-Hunt conjectures, introduced by Tischler *et al.* in 1995 (formally proved in this work in Section 5.7), stating the relation between connectivity and variety are herein presented. Improper kinematic chains are defined in Section 3.10, and Baranov chains and Assur groups are also briefly presented.

Chapter 4 critically reviews the past contributions to the connectivity calculation, and the limits of the various methods proposed are analysed. Counterexamples are presented for the each algorithm found in literature.

Chapter 5 presents new definitions of the concepts of degrees of control, connectivity and variety. These new definitions, which are one of the main contributions of this work, are not conflicting with the previous ones found in literature. By the new definitions of connectivity and variety, the Tischler-Samuel-Hunt conjectures stating the relation between connectivity and variety, are formally proved in Section 5.7.

Chapter 6 describes a novel methodology for calculating the main parameters of a kinematic chain, *i.e.* degrees of control, connectivity, redundancy and variety. The algorithm, described in this section, is one of the major contributes of this work. Example of application of the algorithm are given at the end of this chapter.

Appendix A introduces some fundamental concepts of graph theory. They are essential for topological analysis and number synthesis of mechanisms. It is important to remember that the topology of a mechanism can be uniquely identified by its graph representation, where links and joints of the mechanism are represented, respectively, by the vertices and edges of the graph.

Appendix B introduces some fundamental concepts of screw theory. The screw systems commonly used in mechanism design are also examined.

Appendix C presents an application [Tischler et al. 2001] of the methodology of mechanism design introduced in Section 1.1.2. A feasible kinematic chain for robot's finger design is selected by means of variety, between a set of enumerated kinematic chains.

Appendix D presents a detailed description of the methodology proposed by Liberati and Belfiore [Liberati and Belfiore 2006] for connectivity calculation. The steps of the algorithm are presented, and an example of application is also described. A counterexample to this algorithm is presented in this work at Section 4.3.

Appendix E documents the implementation of the algorithm proposed in this work. The algorithm has been implemented in C++ language, and the class structure is detailed described here.

2 Concepts of mechanism theory

This chapter introduces the basic concepts of mechanism and machine theory in Sections 2.1 and 2.2. Different ways of representing mechanisms are described in Section 2.3, and the convenience of adopting graph representation is explained.

2.1 Links and joints

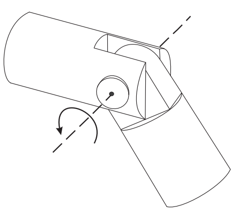
A *material body* is defined as a rigid body if the distance between any two points of the body remains constant. In reality, rigid bodies do not exist, since all known materials deform under stress. However, a body can be considered rigid if its deformation under stress is negligibly small. The use of rigid bodies makes the study of kinematics of mechanisms easier. However, for light-weight and high-speed mechanisms, the elastic effects of a material body may become significant and must be taken into consideration. In this text, unless otherwise stated, all bodies will be considered as being rigid. Moreover, a rigid body may be considered as being infinitely large for study of the kinematics of mechanisms.

The individual rigid bodies making up a machine or mechanism are called *members* or *links*. From the kinematics point of view, two or more members connected together such that no relative motion can occur between them will be considered as one link.

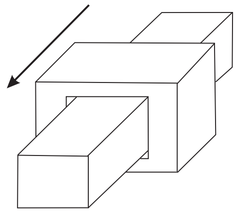
The links in a machine or mechanism are connected in pairs. The connection between two links is called a *joint*. A joint physically adds some constraint(s) to the relative motion between the two members. The kind of relative motion allowed by a joint is governed by the form of the surfaces of contact between the two members. The degrees of freedom of a joint is the number of independent parameters that uniquely determine the orientation of the joint with respect to the joint reference frame. The surface of contact of a link is called a *pair element*. Two such paired elements form a *kinematic pair*.

Kinematic pairs are classified into *lower pairs* and *higher pairs* according to type of the contact between the paired elements. A lower pair is a kinematic pair that is formed by surface contact between its elements. A higher pair is a kinematic pair that is formed by point or line contact between its elements.

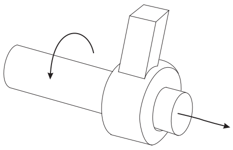
There are only six lower pairs, as showed in Figure 2.1.



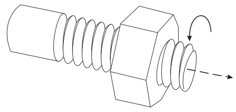
(a) Revolute joint



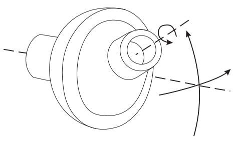
(b) Prismatic joint



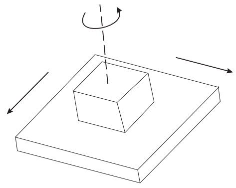
(c) Cylindric joint



(d) Helical joint



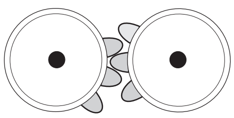
(e) Spherical joint



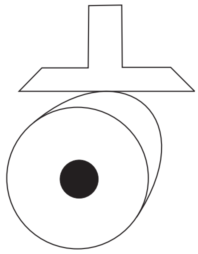
(f) Plane joint

Figure 2.1: Lower kinematic pairs

Two higher pairs are also frequently used in mechanisms as shown in Figure 2.2.



(a) Gear pair



(b) Cam pair

Figure 2.2: Frequently used higher kinematic pairs

A brief description of the kinematic pairs showed in Figures 2.1 and 2.2 is presented in the following paragraphs.

A *revolute joint R* allows two paired elements to rotate with respect to one another about an axis that is defined by the geometry of the joint. Therefore, the revolute joint is a one degree of freedom joint; that is, it imposes five constraints on the paired elements. The revolute joint is sometimes called a *turning pair*, a *hinge*, or a *pin joint*.

A *prismatic joint P* allows two paired elements to slide with respect to each other along an axis defined by the geometry of the joint. Similar to a revolute joint, the prismatic joint is a one-degree of freedom joint. It imposes five constraints on the paired elements. The prismatic joint is also called a *sliding pair*.

A *cylindric joint C* allows a rotation about and an independent translation along an axis defined by the geometry of the joint. Therefore, the cylindric joint is a two-degrees of freedom joint. It imposes four constraints on the paired elements. A cylindric joint is kinematically equivalent to a revolute joint in series with a prismatic joint with their joint axes parallel to or coincident with each other.

A *helical joint H* allows the paired elements to rotate and translate along an axis defined by the geometry of the joint. However, the translation is related to the rotation by the pitch of the joint. Hence, the helical joint is a one-degree of freedom joint. It imposes five constraints on the paired elements. The helical joint is sometimes called a *screw pair*.

A *spherical joint S* allows one element to rotate freely with respect to the other about the center of a sphere. It is a ball-and-socket joint that allows no translations between the paired elements. Hence, the spherical joint is a three-degrees of freedom joint; that is, it imposes three constraints on the paired elements. A spherical joint is kinematically equivalent to three intersecting revolute joints.

A *plane pair E* allows two translational degrees of freedom on a plane and a rotational degree of freedom about an axis that is normal to the plane of contact. Hence, the plane pair is a three-degrees of freedom joint; that is, it imposes three constraints on the paired elements.

A *gear pair G* allows one gear to roll and slide with respect to the other at the point of contact between two meshing teeth. In addition, the motion space of each gear is constrained on a plane perpendicular to its central axis of rotation. Therefore, the gear pair is a two-degrees of freedom joint. It imposes four constraints on the paired elements.

Similar to a gear pair, a *cam pair* Cp allows a follower to roll and slide with respect to the cam at the point of contact. Hence, the cam pair is also a two-degrees of freedom joint

Further, there is a commonly used composite joint called the *universal joint* as shown in Figure 2.3. A universal joint is made up of two intersecting revolute joints. Therefore, it is a two-degrees of freedom joint. The universal joint is sometimes referred to as the *Hooke joint* or *Cardan joint*.

A link is called a binary link if it is connected to only two other links, a ternary link if it is connected to three other links, a quaternary link if it is connected to four other links, and so on. A joint is called a binary joint, if it connects only two links, and a multiple joint, if it connects

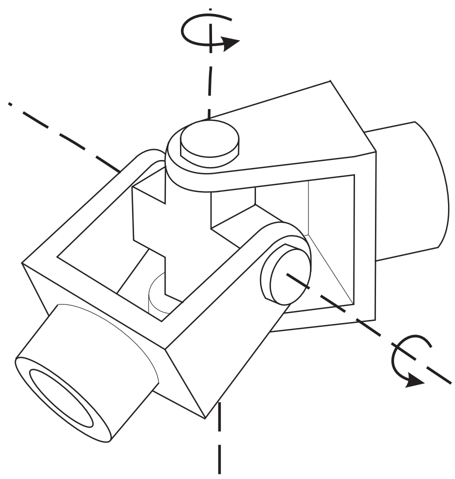


Figure 2.3: Cardan joint or Hook joint

more than two links.

2.2 Kinematic chains, mechanisms and machines

A *kinematic chain* is an assemblage of links, or rigid bodies, that are connected by joints. A kinematic chain in which there is at least one link which carries only one kinematic pairing element is called an *open-loop chain*. In other words an open-loop chain has every link connected to every other link by one and only one path.

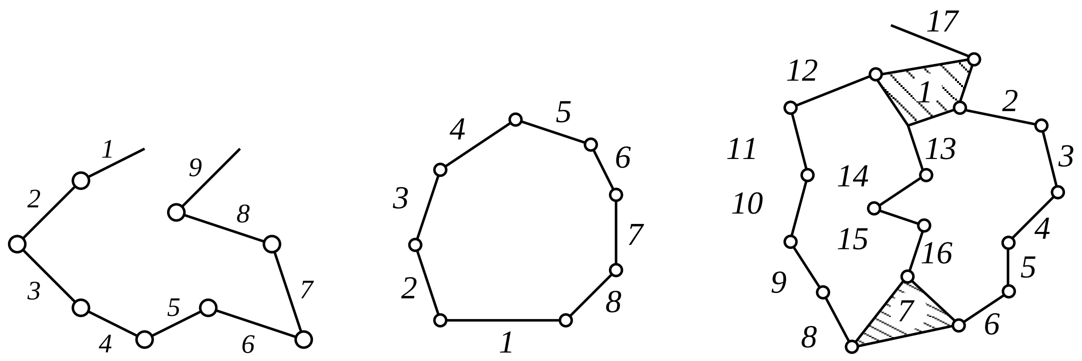
On the other hand, a kinematic chain in which each link is connected with at least two other links is called a *closed-loop chain*. Alternatively a closed-loop chains has every link connected to every other link by at least two distinct paths.

Clearly, it is possible for a kinematic chain to contain both closed- and open-loop chains: such a kinematic chain is called a *hybrid kinematic chain*. Figure 2.4 shows an example of open, closed and hybrid kinematic chain.

Given a kinematic chain H, a *subchain* of H is a kinematic chain having all links and joints contained in H.

A *mechanism* is defined as a system of bodies designed to convert motions of, and forces on, one or several bodies into constrained motions of, and forces on, other bodies. Alternatively a mechanism is a kinematic chain with one of its components links taken as a frame.

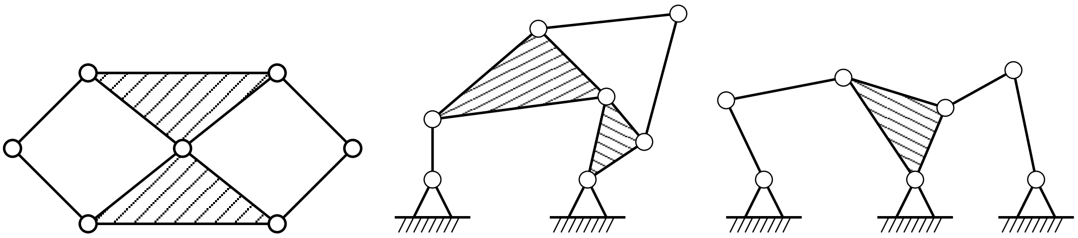
The link taken as a frame is called the *fixed link*. As the input link(s) move with respect to the frame, all other links perform constrained motions. From a given kinematic chain, different mechanisms, or *inversions*, may be derived, with different choice of the fixed link.



(a) (b) (c)

Figure 2.4: Open chain (a), closed chain (b) and hybrid chain (c)

For example, Figure 2.5 shows the Watt chain and the two mechanisms derived: Watt I and Watt II (the two mechanisms are represented by the structural representation described in Section 2.3).



(a) Watt chain (b) Watt I (c) Watt II

Figure 2.5: Watt chain and its derived mechanisms

When one or more mechanisms are assembled together with other hydraulic, pneumatic, and electrical components such that mechanical forces of nature can be compelled to do work, such an assembly is called a *machine*. That is, a machine is an assemblage of several components for the purpose of transforming external energy into useful work.

Although the terms mechanism and machine are often interchangeable, in reality there is a definite difference. When actuators, sensors, spindles, loading/unloading mechanisms, and controllers are incorporated to one ore more mechanisms, the system becomes a machine. It may be observed that a machine may consist of several mechanisms. However, a mechanism is not necessarily a machine since it may be part of a machine to serve as a motion transformation device.

2.3 Structural representation of kinematic chain and mechanism

The kinematic structure of a kinematic chain or of a mechanism contains the essential information about which link is connected to which other link by what type of joint. The kinematic structure can be represented in several different ways. Some methods of representation are fairly straightforward, whereas others may be rather abstract and do not necessarily have a one-to-one correspondence. In this section several methods of representation of the kinematic structure of a mechanism or kinematic chain are described. For convenience, the following assumptions are made for all methods of representation.

1. For simplicity, all parallel redundant paths in a mechanism will be illustrated by a single path. Parallel paths are usually employed for increasing load capacity and achieving better dynamic balance of a mechanism.

2. All joints are assumed to be binary. A multiple joint will be substituted by a set of equivalent binary joints. In this regard, a ternary joint will be replaced by two coaxial binary joints, a quaternary joint will be replaced by three coaxial binary joints, and so on.

3. Two mechanical components rigidly connected for the ease of manufacturing or assembling will be considered and shown as one link. For example, two gears keyed together on a common shaft to form a compound gear set are considered as one link.

2.3.1 Functional schematic representation

Functional schematic representation refers to the most familiar cross-sectional drawing of a mechanism. Shafts, gears, and other mechanical elements are drawn as such. For clarity and simplicity, only those functional elements that are essential to the structural topology of a mechanism are shown.

Figure 2.6 shows the model of a machine: Watt engine, built in 1784. The elements 1, 2, 3, 4, 5 and 6, and their joints a, b, c, d, e, f and g form a well known mechanism: the so called Watt I. The functional schematic is represented in Figure 2.8a.

Two functional schematics representing different physical embodiments might sometimes share the same structural topology.

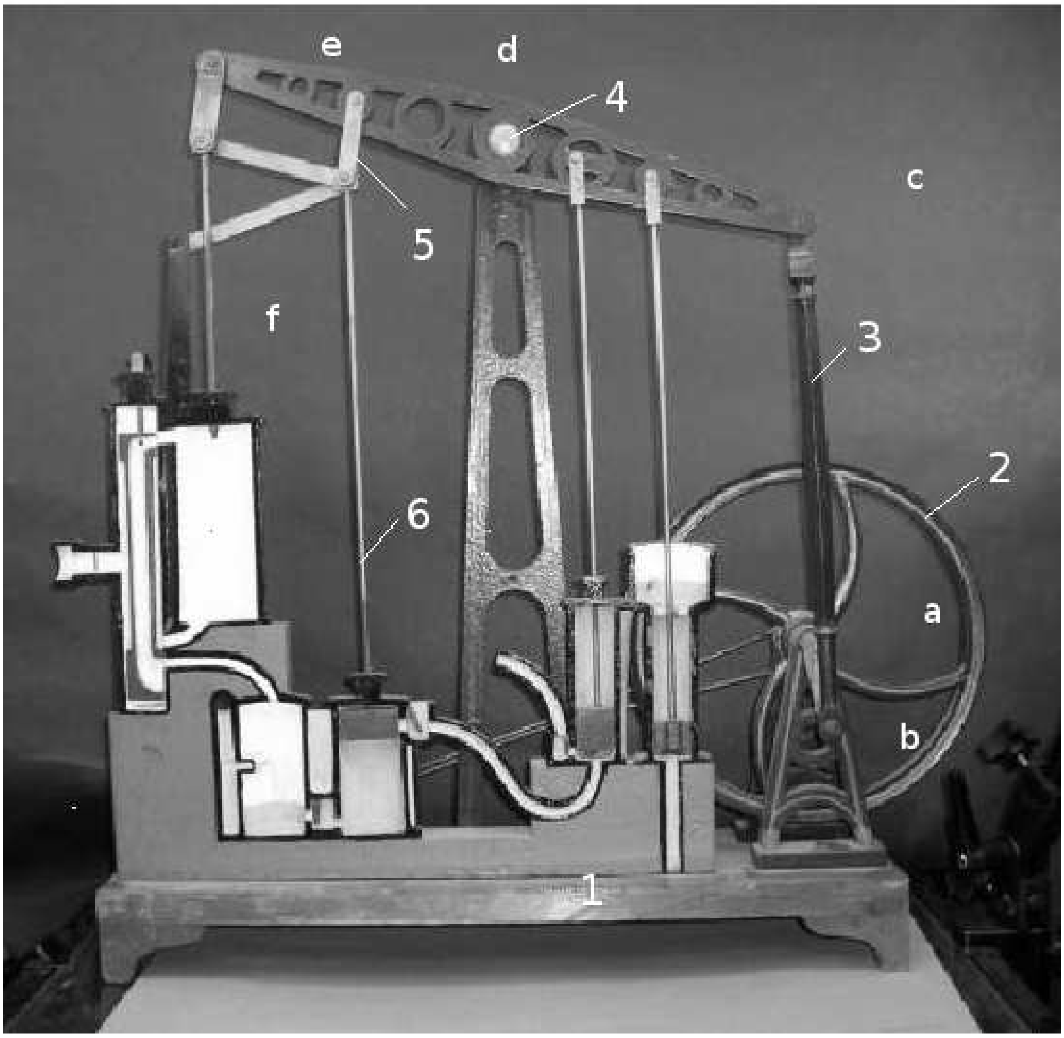


Figure 2.6: Model of Watt engine, 1784

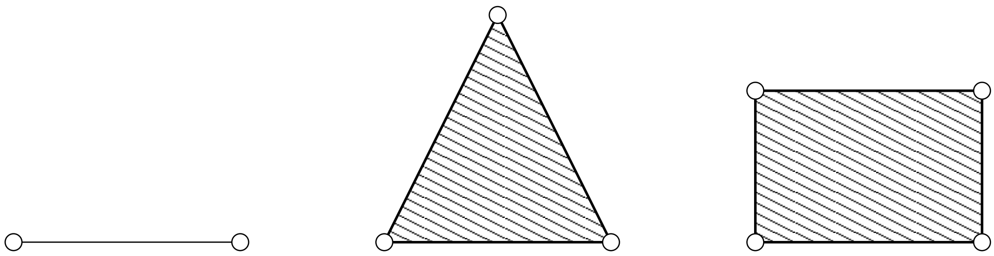


Figure 2.7: Structural representation of links

2.3.2 Structural representation

In a *structural representation*, each link of a mechanism is denoted by a filled polygon whose vertices represent the kinematic pairs. Specifically, a binary link is represented by a line with two end vertices, a ternary link is represented by a cross-hatched triangle with three vertices, a quaternary link is represented by a cross-hatched quadrilateral with four vertices, and so on. Figure 2.7 shows the structural representation of a binary, ternary, and quaternary link. The vertices of a structural representation can labeled for the identification of pair connections.

The structural representation of a mechanism is defined similarly, except that the polygon denoting the fixed link is labeled accordingly. Unlike the functional schematic representation, the dimensions of a mechanism, such as the offset distance and twist angle between two adjacent links, are not shown in the structural representation.

Figure 2.8b shows the structural representation of the Watt engine of Figure 2.6, where the link number and the vertex letter identify the correspondent part of Figure 2.6. Link 1 is marked as fixed.

2.3.3 Graph representation

Since a kinematic chain is a collection of links connected by joints, this link and joint assemblage can be represented in a more abstract form called the *graph representation*. In a graph representation, the vertices denote links and the edges denote joints of a mechanism. The edge connection between vertices corresponds to the pair connection between links. To distinguish different pair connections, the edges can be labeled. Figure 2.8c shows the graph representation of the Watt engine.

The advantages of using a graph representations [Tsai 2001] are:

1. Many network properties of graphs are directly applicable. For example, it is possible to apply Euler's equation to obtain the *loop mobility criterion* of mechanisms directly.
2. The structural topology of a mechanism can be uniquely identified. Using graph representation, the similarity and difference between two different mechanism embodiments can be recognised.
3. Graphs may be used as an aid for the development of computer-aided kinematic and dynamic analysis of mechanisms.
4. Graph theory may be employed for systematic enumeration of mechanisms.
5. Graphs can be used for systematic classification of mechanism.
6. Graphs can be used as an aid in automated sketching of mechanisms.

Basic concepts of graph theory are presented in Appendix A at Page 91.

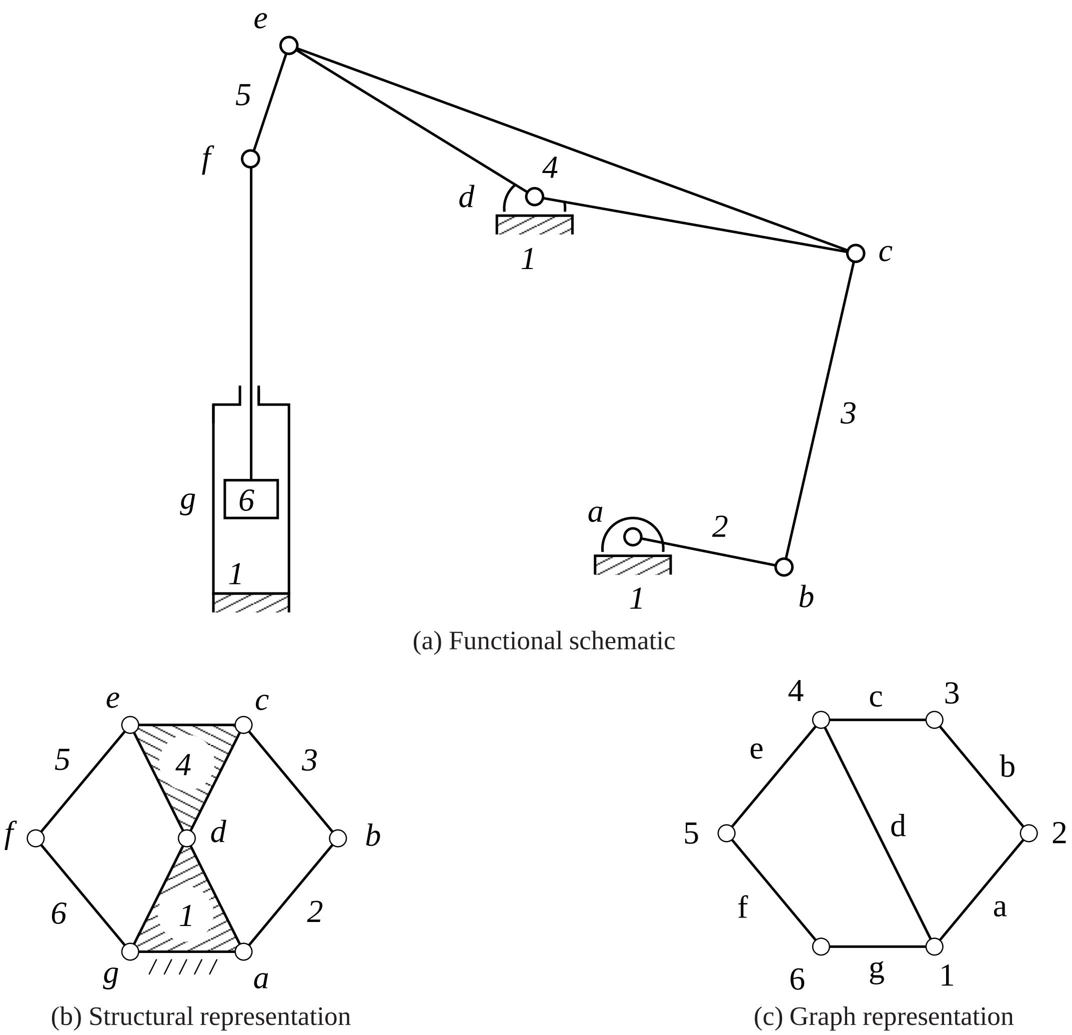


Figure 2.8: Watt mechanisms and its kinematic representations

2.3.4 Matrix representation

For computer programming, the kinematic structure of a kinematic chain is represented by a graph and the graph conveniently is expressed in matrix form. There are several methods of matrix representation as described in Appendix A at Page 91. Perhaps, the most frequently used method is the link-to-link form of adjacency matrix. Other methods of representation, such as the incidence matrix, circuit matrix, and path matrix, are also useful for the identification and classification of mechanisms. Matrix representations are particularly useful for computer aided enumeration of kinematic structures of mechanisms. In the following, the adjacency and incidence matrix representations of kinematic chains are briefly described.

Adjacency matrix

The links of a kinematic chain are numbered sequentially from 1 to n. Since in the graph, representation vertices correspond to links and edges correspond to joints, the link-to-link adjacency matrix, A_j, is defined as follows:

$$A_j[i,j] = \begin{cases} 1, & \text{if link } i \text{ is connected to link } j \text{ by a joint} \\ 0, & \text{otherwise (including } i = j\text{)} \end{cases} \tag{2.1}$$

By definition, the adjacency matrix is an $n \times n$ symmetric matrix with zero diagonal elements. The matrix determines the structural topology of a kinematic chain up to structural isomorphism. For example, the link-to-link adjacency matrix of the Watt mechanism graph shown in Figure 2.8c is given by

$$A_j = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \tag{2.2}$$

The matrix representation given by Equation (2.2) provides no distinction for the types of joint used in a mechanism. The $(4,5)$ element in Equation (2.2) simply provides the information that link 4 is connected to link 5 by a joint. It does not give information about the type of joint. In this work only one-degree of freedom joints are considered.

Incidence matrix

Another useful matrix representation is the *incidence matrix*, A_i. In addition to labelling the links, the joints are labeled as well. In an incidence matrix each row represents a link, whereas each column denotes a joint as outlined below.

$$A_i[i,j] = \begin{cases} 1, & \text{if link } i \text{ contains joint } j \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The incidence matrix also determines the structural topology of a kinematic chain up to structural isomorphism. Equation (2.4) shows the incidence matrix of the Watt mechanism graph of Figure 2.8c.

$$A_i = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \tag{2.4}$$

3 Conceptual analysis of kinematic chains and mechanism

Conceptual analysis is the study of the nature of connection among the links of a kinematic chain and its mobility. It is concerned primarily with the fundamental relationships among the number of degrees of freedom, the number of links, the number of joints, and the type of joints used in a kinematic chain. It should be noted that conceptual analysis only deals with the general functional characteristics of a kinematic chain and not with the physical dimensions of the links.

This work focuses on the topological characteristics of kinematic chains and mechanisms. In general, topological characteristics of a mechanism are equivalent to the topological characteristics of the kinematic chain from which the mechanism is derived.

3.1 Correspondence between mechanisms and graphs

Since the topological structure of a kinematic chain can be represented by a graph, many useful characteristics of graphs can be translated into the corresponding characteristics of a kinematic chain. Table 3.1 describes the correspondence between the elements of a kinematic chain and that of a graph.

Graph	*Symbol*	*Mechanism*	*Symbol*
Number of vertices	V	Number of links	n
Number of edges	E	Number of joints	g
Number of vertex of degree i	v_i	Number of links having i joints	n_i
Degree of vertex i	d_i	Number of joints on link i	g_i
Number of independent loops	ν	Number of independent loops	ν

Table 3.1: Correspondence between mechanisms and graphs

In this work, a complete correspondence between graphs and kinematic chains has been adopted. Consequently, in order to avoid any possible confusion, the elements of a kinematic chain and of the correspondent graph are indicated with the same kinematic chain symbols, *i.e.* n and g represent respectively the number of links and joints of the kinematic chain considered and the number of vertices and edges of the correspondent graph. A brief review of the concepts of graph theory is presented in Appendix A at Page 91.

3.2 Mobility or number of degrees of freedom

The *mobility M*, or number of *degrees of freedom* of a kinematic chain is perhaps the first concern in the study of kinematics and dynamics of kinematic chains. The number of degrees of freedom of a kinematic chain refers to the number of independent parameters required to completely specify the configuration of the kinematic chain in space. Except for some special cases, it is possible to derive a general expression for the number of degrees of freedom of a kinematic chain in terms of the number of links, number of joints, and types of joints incorporated in the kinematic chain.

Definition 1. *The* mobility*, or number of* degrees of freedom *of a kinematic chain is the number of independent parameters required to completely specify the configuration of the kinematic chain in the space, with respect to one link chosen as the reference.*

Intuitively, the mobility of a kinematic chain is equal to the degrees of freedom of all the moving links diminished by the degrees of constraint imposed by the joints. If all the links are free from constraint, the degrees of freedom of an n-link kinematic chain with respect to one link chosen as the reference would be equal to $\lambda(n-1)$, where λ is the order of the screw system to which all the joints screw belong. A brief review of screw theory is presented in Appendix B at Page 105.

Since the total number of constraints imposed by the joints are given by $\sum_i c_i$, where c_i is the degrees of constraint on relative motion imposed by joint i, the net number of degrees of freedom of a mechanism is

$$M = \lambda(n-1) - \sum_{i=1}^{j} c_i \tag{3.1}$$

The constraints imposed by a joint and the number of degrees of freedom allowed by the joint are related by

$$c_i = \lambda - f_i \tag{3.2}$$

where f_i is degrees of relative motion allowed by joint i. Substituting Equation (3.2) into Equation (3.1) yields:

$$M = \lambda(n - g - 1) + \sum_{i=1}^{j} f_i \tag{3.3}$$

Equation (3.3) is known as the Grübler or Kutzbach criterion.

Considering only single degree of freedom joints, the mobility of a kinematic chain, with n links and g single degree of freedom joints, may be calculated by the general mobility criterion [Hunt 1978] applied to a set of n links and g single degree of freedom joints:

$$M = \lambda(n - g - 1) + g \tag{3.4}$$

In this work, all joints are assumed to be single degree of freedom joints, since it can be demonstrated that multiple degree of freedom joints can be substituted by a set of equivalent binary joints.

For instance, the mobility of the planar closed-loop kinematic chain shown in Figure 3.1 is, by equation (3.4), $M = 5$.

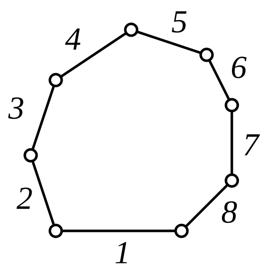


Figure 3.1: Closed-loop kinematic chain with $M = 5$

3.2.1 Full mobilty, partial mobility and fractionated mobility kinematic chains

Broadly, a kinematic chain can posses the following types of mobility:

1. *Fractionated mobility*: A kinematic chain has fractionated mobility if it has a separation link or joint, when cut into two, splits the chain into separate (closed) kinematic chains. Hence, the graph of a non-fractionated kinematic chain is a biconnected graph.

2. *Partial mobility*: A kinematic chain with $M > 0$ degrees of freedom, has partial mobility

if it has at least one closed subchain with M' number of degrees of freedom, such that $0 \leq M' < M$

3. *Total mobility*: A kinematic chain with $M > 0$ degrees of freedom, has total mobility if all its closed subchains have $M' \geq M$ number of degrees of freedom.

There is a close relationship between these types of mobility and the concept of Variety, as discussed in Section 3.8.

3.3 Loop mobility criterion

In the previous section, an equation that relates the degrees of freedom of a kinematic chain to the number of links, number of joints, and type of joints is derived. It is also possible to establish an equation that relates the number of independent loops to the number of links and number of joints in a kinematic chain.

The number of loops ν of a mechanism can be calculated with the *Euler's equation*:

$$\nu = g - n + 1 \tag{3.5}$$

Substituting Equation (3.5) into Equation (3.4) yields:

$$M = g - \lambda \nu \tag{3.6}$$

Let us consider Figure 3.2, which shows the structural representation of the Watt mechanism.

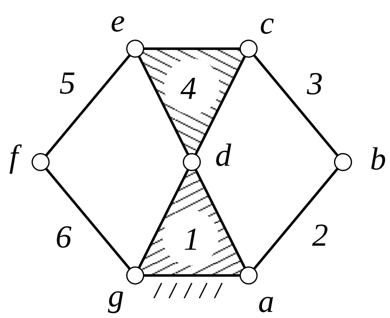


Figure 3.2: Structural representation of Watt mechanism

Applying Equation (3.6) to calculate the mobility of the Watt mechanism, the mobility is $M = 1$, because $\lambda = 3$ (planar mechanism), $g = 7$ and $\nu = 2$.

3.4 Degrees of control

Belfiore and Di Benedetto in [Belfiore and Benedetto 2000] introduced another important concept: *degrees of control*.

Definition 2 (Belfiore and Di Benedetto [Belfiore and Benedetto 2000])**.** *The* degrees of control K_{ij} *between two links i and j of a kinematic chain is the minimum number of independent actuating pairs needed to determine the relative position between the two links i and j, possibly leaving some other link-relative position undetermined as when* K_{ij} *is less than the mobility M.*

In other words, the relative positions between two links cannot be determined by a number of independent parameters less than their degrees of control. Let us consider, for example, two links (say, i and j) of a kinematic chain having M number of degrees of freedom. If the total number of actuating pairs is simultaneously equal to K_{ij} and less than M, then there must be a subchain that is uncontrolled since there are in the kinematic chain more number of degrees of freedom than actuators. In this case, the actuating pairs may be assigned in such a way that the two links' relative positions are controlled, but a different assignment may lead to their indeterminacy.

Consider links 1 and 3 of the kinematic chain represented in Figure 3.1. Their degrees of freedom is $K = 2$, because two independent actuators determine the relative position between links 1 and 3, leaving a part of the kinematic chain undetermined.

3.5 Connectivity

Definition 3 (Hunt [Hunt 1978])**.** *The* connectivity C_{ij} *between two links i and j of a kinematic chain is the relative mobility between links i and j.*

In other words, the connectivity can be defined as the number of degrees of freedom between two specific links in a kinematic chain. The concept of *joint in the bag equivalence*, introduced by Phillips [Phillips 1984], is also useful for the conceptual definition of connectivity. According to such equivalence, all the interposing links and joints between two links i and j may be considered as hidden inside a flexible *black bag*. This bag can be regarded as an equivalent unknown joint between links i and j, and the number of degrees of freedom of this equivalent joint is a measure of the connectivity between the two joints.

Figure 3.3 illustrates the joint in the bag equivalence: considering the connectivity C_{15} between links 1 and 5, the interposing links and joints between links 1 and 5 may be considered as an equivalent unknown joint between links 1 and 5, whose number of degrees of freedom is a measure of the connectivity between the two joints $C_{15} = 4$.

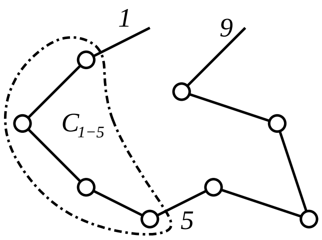


Figure 3.3: Joint in the bag equivalence

It should be remembered that the number of degrees of freedom of any single joint cannot be greater than the maximum number of degrees of freedom of a rigid body in the system considered, usually referred to as the *dimension of the screw system* λ. Consequently, the connectivity is upper-bounded by the value of λ. Therefore, it will be less than or equal to 3 in the case of planar or spherical screw systems ($\lambda = 3$) and it will be less than or equal to 6 in the general spatial motion system ($\lambda = 6$).

For a better understanding of the importance of the concept of connectivity let us consider Figure 3.4. Figure 3.4a represents an open kinematic chain with mobility $M = 8$, but the connectivity between any two links does not exceed 2. Consequently the relative mobility between any two links i and j cannot be greater than 2. Figure 3.4b represents a closed kinematic chain with mobility $M = 3$, but the connectivity between any two links does not exceed 2. From these two simple examples, and as already outlined in previous papers [Shoham and Roth 1997] [Belfiore and Benedetto 2000] [Liberati and Belfiore 2006], it is evident that connectivity, not mobility, determines the ability of an output link to perform a task relative to a frame.

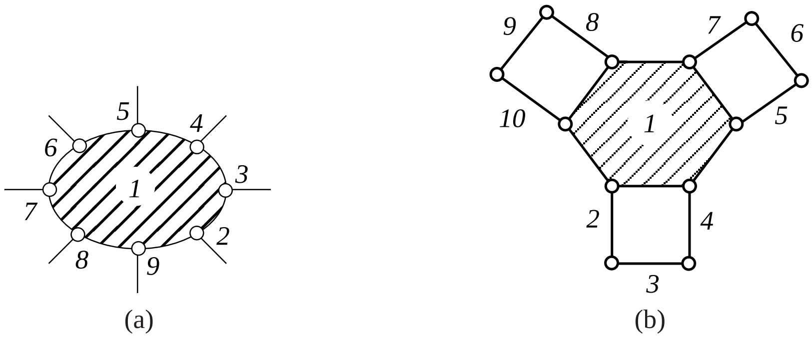


Figure 3.4: Kinematic chains with maximum connectivity between links of 2 *i.e.* $C_{ij} \leq 2 \;\; \forall i, j$

The connectivity can be derived, once the order of the screw system has been established, by applying the following mathematical axiomatic definition [Belfiore and Benedetto 2000]:

Proposition. *If $K[i, j]$ is greater than λ, than $C[i, j] = \lambda$, otherwise, $C[i, j]$ will be equal to $K[i, j]$.*

3.6 Redundancy

Based on the definition of degrees of control and connectivity, the definition of *redundancy* may now be introduced.

Definition 4. *The* redundancy R_{ij} *between two links i and j of a kinematic chain is the difference between the number of degrees of control* K_{ij} *and the connectivity* C_{ij} *between these links.*

From these definitions the parameters R_{ij}, C_{ij} and K_{ij} do not have to be independently evaluated. It is important to note that the concept of degrees of control introduced by Belfiore and Di Benedetto [Belfiore and Benedetto 2000] allows the calculation of the redundancy directly from connectivity and degrees of control, as stated in the following lemmas [Belfiore and Benedetto 2000]

Lemma 1. *The redundancy* R_{ij} *is given as the difference between* K_{ij} *and* C_{ij}*:* $R = K - C$.

Redundancy is one of the most important parameters of a kinematic chain. In the field of parallel robots for machine-tools, redundancy has been used to increase the workspace of the robot (such as in the Eclipse parallel robot [Ryu et al. 1998]) and to deal with singularities. Another form of redundancy is the concept of modular robots [Yang et al. 1999] where additional actuators allow the adaption of the geometry of the robot according to the task to be performed.

Redundant robots are used in confined spaces, in order to avoid collisions [Simas et al. 2003] and redundancy is an important parameter in cooperative robots [Dourado 2005], with the application of virtual chains [Campos et al. 2005] [Campos et al. 2003].

3.7 Application of the concepts of connectivity, degrees of control and redundancy

The above properties are invariantly relative to the permutation of indices: $ij \leftrightarrow ji$; therefore, a convenient way of representing the full set of degrees of control, connectivities and redundancies of a kinematic chain is by symmetric matrices.

As an example, the concepts of connectivity, degrees of control, and redundancy are applied to a planar, closed-loop kinematic chain with eight links and eight simple 1-degrees of freedom kinematic pairs as shown in Figure 3.5. Let us consider links 1 and 4: their degrees of control is $K_{1,4} = 3$, *i.e.* three independent actuators must be used in order to determine the relative position between the two links. The connectivity between the same pair of links is $C_{1,4} = 3$, *i.e.* the two links have full mobility (the relative mobility is equal to the order of the screw system λ where all the joint screws belong). Finally, the redundancy between links 1 and 4 is $R_{1,4} = 0$. Choosing link 1 as the frame and link 4 as the end-effector, the parallel manipulator derived from the kinematic chain has no degree of redundancy.

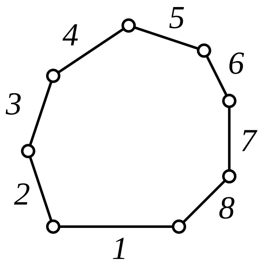


Figure 3.5: Closed-loop kinematic chain with $M = 5$. For links 1 and 4: $K_{14} = 3$, $C_{14} = 3$, $R_{14} = 0$. For links 1 and 5: $K_{15} = 4$, $C_{15} = 3$, $R_{15} = 1$.

Consider now links 1 and 5 of the kinematic chain in Figure 3.5. The degrees of control between these two links is $K_{1,5} = 4$ and their connectivity is $C_{1,5} = 3$, because it is upper-bounded by the value of λ; therefore, the redundancy is $R_{1,5} = 1$. One conclusion is that choosing link 1 as the frame and link 5 as the end-effector, or vice-versa, a redundant parallel manipulator is obtained from the kinematic chain.

As a sample case, the concepts of connectivity, degrees of controls, and redundancy are applied to an open-loop spatial kinematic chain represented in Figure 3.6a, having nine links and eight simple 1-number of degrees of freedom kinematic pairs. The matrixes of degrees of control, connectivity and redundancy are:

$$K = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1&2&3&4&5&6&7&8&9 \end{array} \\ \begin{bmatrix} 0&1&2&3&4&5&6&7&8\\ 1&0&1&2&3&4&5&6&7\\ 2&1&0&1&2&3&4&5&6\\ 3&2&1&0&1&2&3&4&5\\ 4&3&2&1&0&1&2&3&4\\ 5&4&3&2&1&0&1&2&3\\ 6&5&4&3&2&1&0&1&2\\ 7&6&5&4&3&2&1&0&1\\ 8&7&6&5&4&3&2&1&0 \end{bmatrix} \end{array} \quad (3.7)$$

$$C = \begin{array}{c|ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 6 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 8 & 6 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 \\ 9 & 6 & 6 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \tag{3.8}$$

$$R = \begin{array}{c|ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{3.9}$$

There are eight degrees of control between the end links 1 and 9, as there are eighth independent actuators needed to define the positions of one link to respect to the other. On the other hand, a body in space can be positioned through the identification of only six parameters. The connectivity is therefore limited to 6. Hence, the redundancy between link 1 and 9 is equal to 2. Let us now consider one of the two end links (*e.g.* 1) and the median link 5: their degrees of control are 4, which is less then the mobility number, and their connectivity is also 4. Hence, their redundancy is equal to 0. One possible choice of the end-effector and of the frame is represented in Figure 3.6b, which shows an open-loop redundant manipulator. On the other end, Figure 3.6c shows one functional representation of a 4-degrees of freedom cooperating arm system, obtained by choosing link 5 as the frame link.

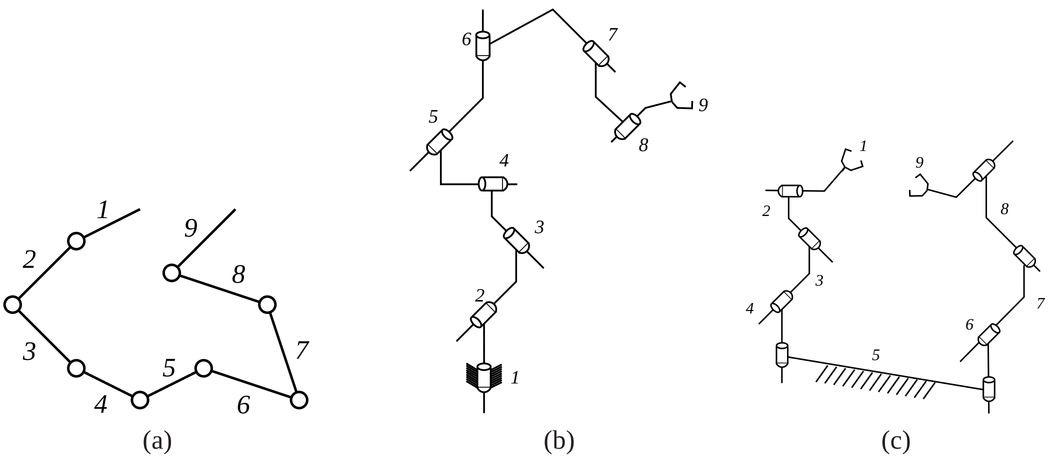


Figure 3.6: *Open loop kinematic chain and two manipulators derived*

3.8 Variety

Variety is a useful property for determining the relative connectivities within a chain and also for selecting actuated pairs. Variety may also be used to classify kinematic chains according to the constraints required [Tischler et al. 2001].

The definition of variety was proposed by Tischler *et al.* [Tischler et al. 1995]:

Definition 5. *[Tischler* et al. *] A kinematic chain is* Variety V *if it does not contain any loop, or subset of loops, with a mobility less than $M-V$, but does contain at least one loop, or subset of loops, which has a mobility of $M-V$.*

Remembering the general mobility criterion [Hunt 1978] applied to a set of n links and g single degrees-of-freedom joints:

$$M = \lambda(n-g-1)+g \tag{3.10}$$

where λ is the order of the screw system to which all the joint screws belong, the constraint equation for the relationship between the variety of a kinematic chain and the number of joints and links in a given subset k of loops can be obtained:

$$g_k = \lambda v_k + (M-V) \tag{3.11}$$

where v_k is the number of independent loops of the subset and g_k is the number of joints in the subset k.

Consequently, a kinematic chain is variety $V=0$ if it contains no loop or subsets of loops with a mobility less than the mobility of the whole chain M. If the variety V of a kinematic

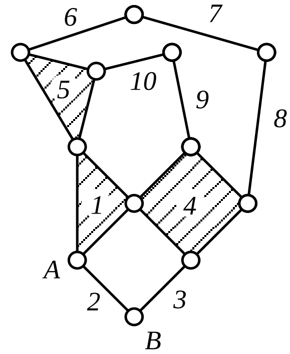


Figure 3.7: Closed-loop kinematic chain with $\lambda = 3$, mobility $M = 3$ and variety $V = 2$

chain with mobility M exceeds $M-1$, then the chain must contain a loop with mobility $M-V = M-M = 0$, and this implies that it is improper and should be discarded, see reference [Tischler et al. 1995] and Section 3.10.

Remembering the concepts of full mobility and fractionated mobility, introduced in Section 3.2.1, a kinematic chain with variety $V = 0$ has full mobility, while a kinematic chain with variety $V > 0$ has partial mobility.

As an example, the concept of variety is applied to a planar, closed-loop kinematic chain with ten links and twelve one-degree of freedom kinematic pairs as shown in Figure 3.7. The mobility of the full chain is $M = 3$. Let us now consider the subchain $1-2-3-4$: it is a four-bar linkage, so the mobility of this subchain is $M' = 1$. As a consequence any pair of links belonging to that subchain has a relative mobility equal to 1. By Definition 5, the variety of the kinematic chain is $V = 2$.

A useful interpretation of variety is as a relationship between inputs and outputs of a kinematic chain (namely between actuated kinematic pairs and passive kinematic pairs). A chain with variety $V = 0$ presents no manifest hierarchy of some joints in relation to the others; every input is capable of contributing to every output. As variety increases, the influence of the inputs on the outputs becomes more restricted. Considering the planar chain of Figure 3.7, if A is actuated, then B is completely controlled by A; even though two further joints must be selected as actuators to control the rest of the linkage, they have no effect on B.

An interesting example of selection of kinematic chains by means of variety is presented in Appendix C at Page 109.

3.8.1 Tischler-Samuel-Hunt conjectures

Tischler *et al.* [Tischler et al. 1995] summarize the relationship between variety and connectivity through a series of propositions. These propositions, which in fact are conjectures, were considered true therein in the absence of counter-examples despite lacking formal proofs, as asserted by this quote:

> *"We assert the following proposition, and its subsequent corollaries, without formal proof. To date we see no reason why these statements should not be true in absence of counter-examples of them. At some later date these statements may yield to formal proof."* Tischler *et al.* [Tischler et al. 1995]

The formal proof of the Tischler-Samuel-Hunt conjectures is one of the main contributions of this work. This result has been published in [Martins and Piga Carboni 2006] and it is presented in Section 5.7 at Page 59.

Conjecture 1 (Tischler *et al.* [Tischler et al. 1995])**.** *If a variety V kinematic chain has a mobility less than, or equal to, the order of the screw system,* i.e. *if $M \leq \lambda$, any two links of the chain, separated by at least $M-V$ joints, have a relative connectivity $C \geq M-V$.*

Tischler *et al.* [Tischler et al. 1995] stated two more conjectures, which they considered corollaries derived from conjecture 1.

Conjecture 2. *If a variety V kinematic chain has a mobility greater than the order of the screw system that generally prevails,* i.e. *if $M > \lambda$, then any two links, separated by at least $\lambda - V$ joints, have relative connectivity $C \geq \lambda - V$.*

Conjecture 3. *Two links separated by a minimum of g single-freedom joints, where $g < M-V$ and $g < \lambda - V$, have a relative connectivity $C = g$.*

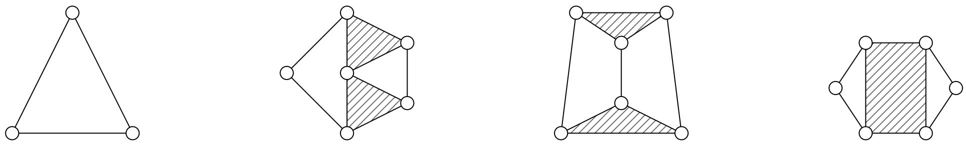
An intriguing feature of these conjectures is that, contrary to the assertion of Tischler *et al.* , the author proved independently these conjectures as theorems, but could not find a way to prove conjectures 2 and 3 as derived from conjecture 1. So, in Section 5.7 three independent theorems are presented and not a theorem followed by a pair of corollaries.

The above statements set lower bounds for the connectivity of two links in a kinematic chain; however, the exact bounds are only found by identifying subsets of links with mobility $M-V$, and by checking the position of the two links relative to the corresponding subset of loops. If both links belong to the subset, then the relative connectivity will be equal to $M-V$.

The upper bound for the relative connectivity is the minimum number of joints which separates the two bodies.

3.9 Minimal sets and variety

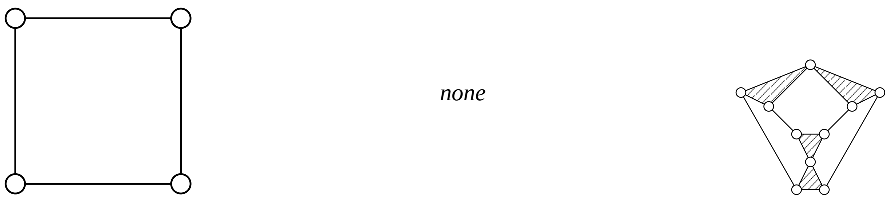
The *minimal set* is another important concept introduced in [Tischler et al. 1995] which has a close relationship with the concept of improper kinematic chains. An improper kinematic chain is a kinematic chain where some joints in the chain do not allow any relative displacement between the two links they connect. Let us consider all the improper chains with $\lambda = 3$ and $\nu = \{1, 2\}$, represented in Figure 3.8.



(a) (b) (c) (d)

Figure 3.8: Improper kinematic chains with $\lambda = 3$ and $\nu \leq 2$

Through visual inspection, it is possible to verify that the chains 3.8b and 3.8d contain at least one subset of links isomorphic to kinematic chain 3.8a. The kinematic chain 3.8a and the kinematic chain 3.8c constitute a *minimal set* of improper kinematic chains with $\lambda = 3$ and $\nu = \{1, 2\}$. Every $\lambda = 3$ improper kinematic chain can be represented as a simple graph that contains at least one of these two chains as a subset, provided that the kinematic chain has $\nu \leq 2$. In Section 3.10 improper kinematic chains are further examined. The definition of *minimal set* can be extended to chains with mobility $M > 0$



(a) $\nu = 1$ (b) $\nu = 2$ (c) $\nu = 3$

Figure 3.9: The minimal set of kinematic chains with $\lambda = 3$, $M = 1$ and $\nu \leq 3$

Figure 3.9 shows the minimal set of kinematic chains with $\lambda = 3$, $M = 1$ and $\nu \leq 3$; note that there are no chains for the case $\nu = 2$.

The relation between minimal sets and variety is straightforward and unidirectional. Whilst all members of the minimal sets are variety $V = 0$ kinematic chains, not all kinematic chains

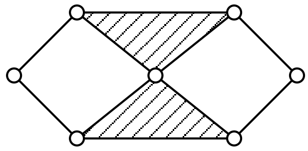
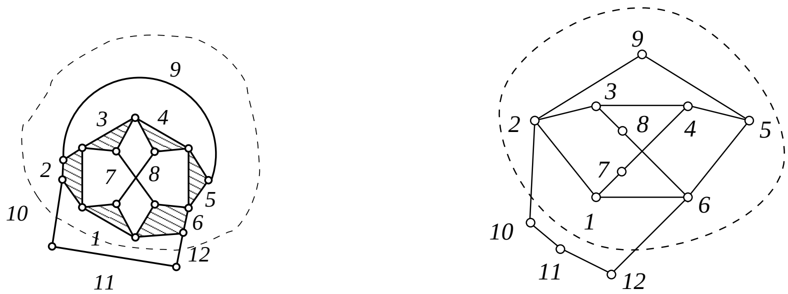


Figure 3.10: Kinematic chain with $\lambda = 3$, mobility $M = 1$ and variety $V = 0$, which does not belong to the minimal set of kinematic chain with $M = 1$ and $\nu \leq 3$ shown in Figure 3.9.

with $V = 0$ belong to minimal sets. For example, consider the kinematic chain represented in Figure 3.10: it has mobility $M = 1$ and variety $V = 0$, but two subchains equal to the kinematic chain of Figure 3.9a may be identified. Hence the kinematic chain of Figure 3.10 does not belong to the minimal set of kinematic chains with $M = 1$ and $\nu \leq 3$.

3.10 Improper kinematic chains

An improper kinematic chain is a kinematic chain with $M > 0$, where at least one biconnected subchain has mobility $M' \leq 0$. As an example of an improper kinematic chain, consider the kinematic chain in Figure 3.11a and its corresponding graph in Figure 3.11b. The subchain formed by links 1-2-3-4-5-6-7-8-9, has mobility $M' = 0$ and its links act as a rigid body. A further inspection identify of the subchain as a Baranov chain. Generally, improper chains are of no interest in pure kinematic analysis.



(a) (b)

Figure 3.11: Improper planar kinematic chain with $M = 1$ and partial mobility ($V = 1$) because it contains a Baranov subchain 1-2-3-4-5-6-7-8-9: structural representation (a) and its graph (b)

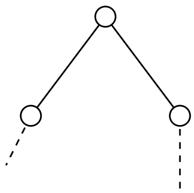
3.10.1 Baranov chains and Assur groups

From the minimal sets of improper chains it is possible to identify Assur groups. Manolescu [Manolescu 1968, Manolescu and Manafu 1963] defines an Assur group as an open subset of links which can be added to a kinematic chain without affecting the mobility of the chain.

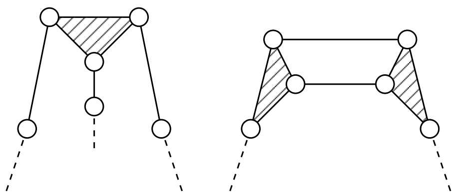
Baranov [Baranov 1985] defines Assur group as a minimal group with mobility zero; minimal in the sense that no simpler Assur group could be find as a subchain of a complex Assur group.

Baranov [Baranov 1985] defines the closed kinematic chain which is a Assur group connect to a single link as the base kinematic chain of Assur groups. He shows also that these chains can be considered the source of all Assur groups. Later, Manolescu [Mruthyunjaya 1979, Manolescu 1979, Manolescu 1973, Manolescu 1968, Manolescu and Tempa 1967] named these base kinematic chain as Baranov trusses. Afterward Tischler *et al.* [Tischler et al. 1995, Tischler et al. 1995, Tischler 1995] presents some lists of Baranov trusses, without mentioning this name, in the lists of $M = 0$, $V = 0$, $\lambda = 3$ kinematic chains for $\nu = 1,2,3,4$.

The simplest $\lambda = 3$ Assur group is a binary dyad. If the unconnected joints of an Assur group were attached directly to a single body, a kinematic chain with a mobility $M = 0$ would result. It is now possible to define Assur groups for a specified ν number of loops, generated by removing one link from the minimal set of improper chains with ν loops and $M = 0$.



(a) $\nu = 1$



(b) $\nu = 2$

Figure 3.12: Assur groups with $\lambda = 3$ and $\nu \leq 2$

Figure 3.12 shows Assur groups with $\lambda = 3$ and $\nu \leq 2$. There are more groups with $\nu \leq 2$, *i.e.* derived from chains with mobility $M = 0$, but they can be obtained by joining two or more links belonging to Assur groups, and consequently these groups can be thought of as non-minimal Assur groups.

All variety $V = 0$ and mobility $M = 0$ kinematic chains, which do not belong to Assur groups with mobility M, can be obtained by joining one or more of Assur groups to a member of the minimal set for that mobility. For example, the Watt and Stephenson kinematic chains can be obtained by adding to the $\lambda = 3$ four-bar chain (which is a member of the minimal set of chains with $\lambda = 3$ and $M = 1$) a binary dyad (which is a member of Assur groups with $\lambda = 3$). Verho [Verho 1973] introduced the concept of non-Assur groups, similar to Assur groups but with mobility $M > 0$. A minimal set of non-Assur groups, with mobility M can be obtained from the minimal set of the kinematic chains with the same mobility M, and the same value of λ, by removing one link from the chain. Kinematic chains of variety $V \neq 0$ and mobility M can be obtained from a chain belonging to the minimal set of mobility $M - V$ kinematic chains.

Assur groups set and non-Assur groups from the minimal set can be added, provided that the sum of the mobilities of the non-Assur groups is equal to V. Then the synthesised kinematic chain has the required mobility M.

The basic building-blocks of all kinematic chains within a given value of λ include the minimal set of improper chains together with the minimal set of kinematic chains with mobility $M > 0$. These building-blocks are the basis for the process of structural synthesis and analysis.

Tischler suggests that variety of a kinematic chain may be determined by inspection, in order to find at least one subset within the chain that is a member of a minimal set. The subset with the smallest mobility M' can be used to calculate the variety (*i.e.* $V = M - M'$). Knowing the minimal sets for several values of M in a given screw system allows us to classify chains according to their variety and to identify improper kinematic chains.

4 *Critical review of connectivity calculation*

The importance of the connectivity is emphasised by [Hunt 1978, Tischler et al. 2001, Tischler et al. 1995, Liberati and Belfiore 2006, Belfiore and Benedetto 2000] and others, which drives the efforts to find an algorithm for the numerical calculation of connectivity. In this section, a critical review of the past contributions to the connectivity calculation is presented, and the limits of the various methods proposed are analysed. Counterexamples are presented for the algorithms found in literature.

The critical analysis of the following contribution to connectivity calculation was a basis for the development of a novel methodology for degrees of control, connectivity, redundancy and variety calculation, presented in Section 6 at Page 63, which is one of the contribution of this work.

4.1 Contribution of Tischler *et al.*

A fundamental previous contribution to the calculation of the connectivity of a kinematic chain is found in Tischler *et al.* [Tischler et al. 1995]. Two important new concepts are introduced there [Tischler et al. 1995]: the *variety* of a kinematic chain and the *minimal sets* of kinematic chains (Section 3.8 at Page 32 and Section 3.9 at Page 35).

The relation between variety and connectivity was originally presented through a series of conjectures, which are referred to in [Tischler et al. 1995] as propositions and corollaries (presented in Section 3.8 at Page 32). A formal proof of the Tischler-Samuel-Hunt conjectures is one of the major contributions of this work, and it is introduced in Section 5.7 at Page 59.

Although these propositions are theoretically relevant to the study of the mobility of a given kinematic chain, they are rather difficult to be adopted in order to build a procedure that automatically computes the connectivity between any two links of a given kinematic chain. Consequently different methodologies are proposed in the literature for connectivity calculation, as presented in the following sections.

4.2 Contribution of Shoham and Roth

An important contribution to the automatic calculation of connectivity in a kinematic chain is found in [Shoham and Roth 1997]. In this work, a correspondence between kinematic chains and graphs is adopted and the connectivity matrix is introduced. The connectivity matrix is defined in a different way from the loop connectivity matrix, proposed by Agrawal and Rao [Agrawal and Rao 1987]. The loop connectivity matrix, later slightly modified by Liu and Yu [Liu and Yu 1995], is a matrix where the elements are the number of common joints between each pair of loops in the mechanism.

In [Shoham and Roth 1997] the connectivity matrix is defined as the symmetric matrix C where each element $C[i, j]$ is equal to the connectivity between links i and j. Open loop chains with only one degree of freedom joints are first analysed. In this case, the number of joints between two links i and j, namely their connectivity, equals the distance between the corresponding vertices in the mechanism's graph representation. The same concept can be easily extended in the presence of joints with $f > 1$ degrees-of-freedom, by representing these joints in the graph by an edge with weight f.

The method above is valid for all open kinematic chains; however, the same method does not hold in general for closed kinematic chains because, in this case, the distance between two links is not a measure of their connectivity. Shoham and Roth [Shoham and Roth 1997] propose changes to the graph representation of a closed loop kinematic chain in order to analyse connectivity by the well developed mathematical tools and algorithms of graph theory.

The differences with respect to the connectivity, between graphs of closed and open kinematic chains are first analysed. The graph of a closed kinematic chain is no longer a tree, since it contains loops. Hence, there is more than one path between two vertices and, consequently, more than a single distance. A loop in a kinematic chain introduces constraints, *i.e.* reduces the mobility of the mechanism, consequently mobility must be an upper-bound on the set of connectivities of the mechanism. In order to use the distance as a measurement of connectivity even in the presence of loops, the followings steps are suggested [Shoham and Roth 1997] to modify the graph representation:

1. Since each loop which is a structure (*i.e.* it has mobility $M = 0$) behaves like a single rigid body, such a loop is shrunk to a single vertex in the graph.

2. Virtual edges (joints) are added so that the same procedure used for open kinematic chains, namely taking the distance between vertices as a measure of connectivity, is applicable also for a general mechanism.

In a single-loop mechanism, the connectivity is upper bounded by the loop mobility, and it is given by:

$$C_{ij} = \min(D_{min}[i,j], M) \tag{4.1}$$

where $D_{min}[i,j]$ is the shortest distance between two vertices (links) i and j. The general equation for mobility is:

$$M = \sum_{i=1}^{g} f_i - \lambda = \bar{F} - \lambda \tag{4.2}$$

where f_i is the freedom in the i^{th} joint, g is the total number of joints, and

$$\bar{F} \triangleq \sum_{i=1}^{g} f_i \tag{4.3}$$

is the gross degree of freedom of the chain, *i.e.* the sum of all joint freedoms of the kinematic chain.

It is important to know when mobility, not distance, determines connectivity. As in a closed loop any two links are connected by two different sides of the chain, the distance between these links is upper-bounded by:

$$D_{min}[i,j] \leq \frac{1}{2}\bar{F} \tag{4.4}$$

Connectivity is determined by mobility when the mobility is smaller than distance. As $\bar{F} \in \mathbb{N}$, this case is described by the following inequality:

$$\bar{F} - \lambda < \lfloor \frac{1}{2}\bar{F} \rfloor \tag{4.5}$$

which relates eq. (4.2) and (4.4). For the general spatial case ($\lambda = 6$) the sum of the joint freedoms is less than six, inequality (4.5) yields:

$$6 < \bar{F} < 11 \tag{4.6}$$

which implies that the only loops where mobility, and not the distance, determines the upper-bound for connectivity are those having joint-freedom sum of 7, 8, 9 or 10. Hence for those cases where connectivity is determined by loop mobility rather than by distance, virtual edges can be added to the graph with weight equal to the mobility of the loop. The connectivity matrix elements are obtained simply by the distances between vertices. This method can be applied to planar kinematic chains (screw system $\lambda = 3$). In this case, the inequality (4.5) becomes

$$3 < \bar{F} < 5 \Longrightarrow \bar{F} = 4 \tag{4.7}$$

From relation (4.7) the only one-loop planar mechanism where the graph need to be modified adding virtual edges is the four-bar linkage. Once the above algorithm is applied to the shortest independent loops (or, as in graph theory, to a set of fundamental cycles) the connectivity matrix is obtained simply as the distance between vertices. The simple case of the planar four-bar linkage showed in Fig 4.1a is analysed.

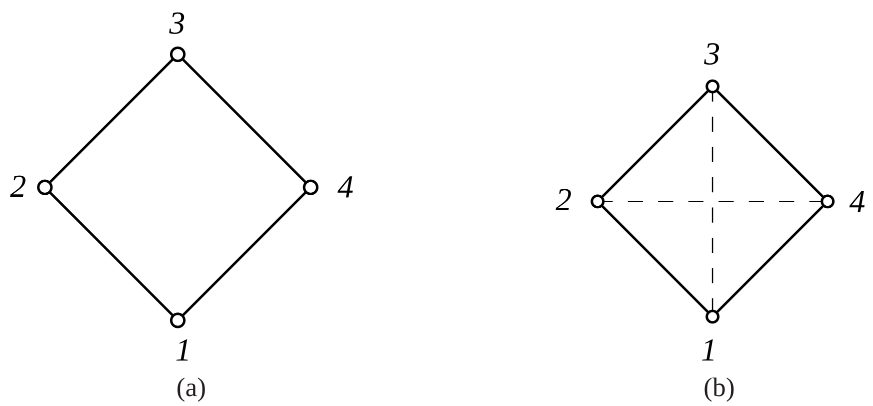


Figure 4.1: The graph of a four bar linkage (a) and the same graph with virtual edges (b)

The mobility of the chain is $M = 1$, so it is necessary to add two virtual edges with unitary weight to the graph, as in Figure 4.1b

The steps of the algorithm presented in [Shoham and Roth 1997] are:

1. Select a set of fundamental cycles
2. For each circuit k:
   (a) Evaluate the loop mobility M_k from the equation

$$\bar{F} - \lambda \tag{4.8}$$

   (b) for each pair $i - j$ of vertices of the circuit, perform the following point:
   (c) Evaluate the distance $D_{\min}[i, j]$ from vertices i and j

(d) If M_k is less than $D_{\min}[i, j]$ then the loop mobility, not the distance, determines connectivity between i and j; therefore, a weighted virtual edge $i - j$ must be added to the original graph. According to the reported examples, the weight is equal to M_K.

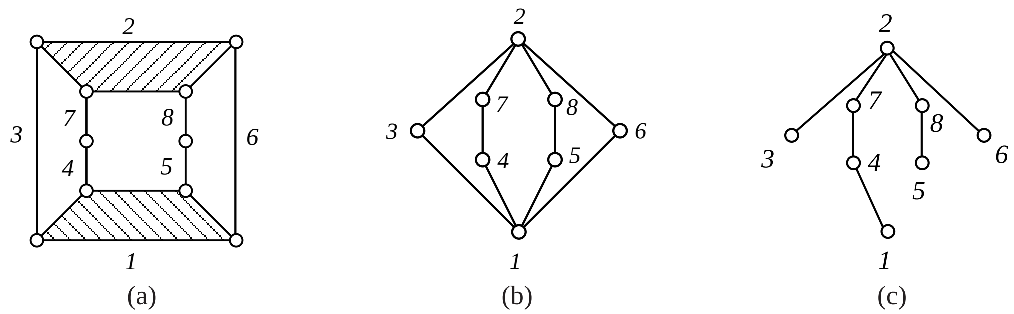


Figure 4.2: *One degree of freedom chain: structural representation (a), its graph (b) and one minimum spanning tree (c)*

Consider now the chain represented in Figure 4.2a, and the corresponding graph in Figure 4.2b.

Applying the algorithm to the set of fundamental cycles generated by the minimum spanning tree of Figure 4.2c (*i.e.* the circuits *1-4-7-2-3-1*, *1-5-8-2-7-4-1* and *1-6-2-7-4-1*), the connectivity matrix in evaluated as:

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 & 3 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 1 & 2 & 1 & 3 & 2 & 0 & 2 \\ 2 & 1 & 2 & 3 & 1 & 2 & 2 & 0 \end{bmatrix} \end{array} \tag{4.9}$$

This result is not coherent with the definition of connectivity. The whole chain has mobility $M = 1$, which is an upper bound to the connectivity for each pair of links of the chain. In this case, the algorithm was not able to find the reduced mobility induced by multi-loop subchains (a multi-loop subgraphs) of the chain. This limit has been pointed out by Belfiore and Di Benedetto in [Belfiore and Benedetto 2000], where a new algorithm (discussed in Section 4.3) is proposed.

Consider now a spatial hybrid kinematic chain, with two biconnected components: one two-closed-loop subchain and one single-link open-loop subchain as shown in Figure 4.3. This chain was analyzed by Shoham and Roth (Figure 14 in [Shoham and Roth 1997]) and the connectivity between links 2 and 8 was found to be $C[2,8] = 6$. This value contradicts the connectivity definition as shown in [Belfiore and Benedetto 2000]. In fact, both links 2 and 8 belong to the subchain 1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-16, whose mobility is $M' = 5$. Hence, the connectivity between any pair of links in that subchain is upper-limited by the mobility of this subchain; in this case $M' = 5$.

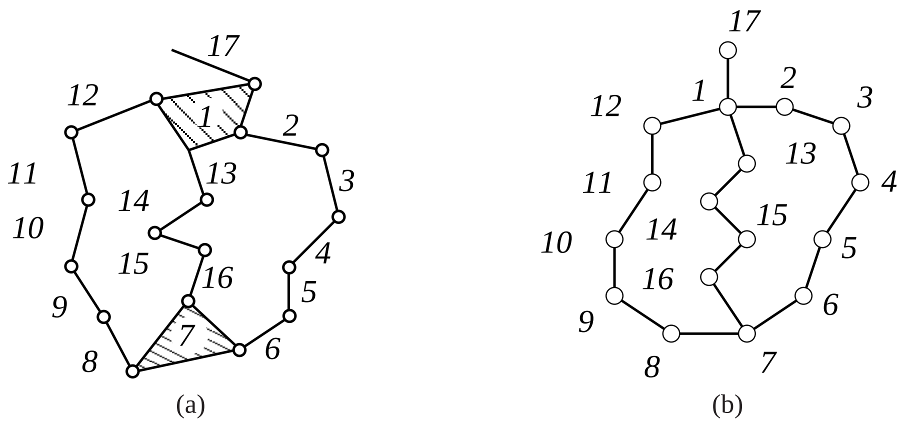


Figure 4.3: Hybrid kinematic chain with mobility $M = 6$ and connectivity between links *2* and *8* $C[2,8] = 5$: structural representation[(a) and its graph (b)

4.3 Contribution of Belfiore and Di Benedetto

In [Belfiore and Benedetto 2000] a pure topological treatment of the problem of connectivity calculation is presented. A new method, the *Topological Method* distinguished from the *Variety Method* [Tischler et al. 1995] is introduced. The *topological method* relies on the following assumptions [Belfiore and Benedetto 2000]:

1. No attention is paid to the nature of the loci traced instantaneously or over a full cycle of a movement, by any point of any body of the mechanism.
2. Infinitesimal movements are not analysed,
3. The nature and the order of the screw system such as full-cycle mobility, critically over-constrained linkages, and stationary and uncertainty configurations are not investigated.
4. The order of the screw system corresponds to that of planar, spherical or general spatial motion.

5. Mechanisms having special proportions, for which the number of degrees of freedom is temporally or permanently greater than F, are excluded.

In [Belfiore and Benedetto 2000] the important concept of degrees of control (as defined in Section 3.4) is introduced. Taking in account only the topological properties of the kinematic chains, the one-to-one correspondence between mechanisms and graphs is adopted. The methodology proposed for connectivity calculation is based on a series of propositions presented below.

The connectivity can be derived, once the order of the screw system has been established, by applying the following mathematical axiomatic definition:

Proposition 1. *(Theorem 1 in [Belfiore and Benedetto 2000]) If degrees of control $K[i,j]$ is greater than λ, than the connectivity $C[i,j] = \lambda$, otherwise, $C[i,j]$ will be equal to $K[i,j]$.*

Finally, the redundancy matrix R will be given as the difference between K and C:

$$R = K - C \tag{4.10}$$

Considering an open loop kinematic chain, the calculation of the matrix K of degrees of control is straightforward, since it is simply the sum of the number of degrees of freedom of the joints that are interposed between the two links considered. If, with no loss of generality, only single degrees of freedom joints are assumed, the degrees of control $K[i,j]$ between two links i and j is equal to the distance $D_{min}[i,j]$ between the links.

Proposition 2. *(Theorem 2 in [Belfiore and Benedetto 2000]) In an open kinematic chain, the degrees of control $K[i,j]$ between two links i and j is equal to the distance $D_{min}[i,j]$ between these two links. The connectivity $C[i,j]$ is derived from Proposition 1.*

Considering closed-loop structures, the computation of the degrees of control between two links as the distance between the two vertices is no longer effective, since generally more than one path connect the two vertices. Taking into account just the shortest path between two vertices fails when the vertices belong to a chain for which:

$$\bar{F} - \lambda \nu = M < D_{min}[i,j] \tag{4.11}$$

In other words the mobility M of the chain affects the relative mobility between the two links, as already stated in [Tischler et al. 1995]. It is also necessary to take into account the mobility M_k of the subchain where both links belong, this value is an upper bound to the relative mobility between the links. Considering the chain in Figure 4.4a and the corresponding graph in Figure 4.4b and applying Equation (4.11) to links 1 and 8, the mobility is $M = 3 < D_{\min}[1,8] = 4$.

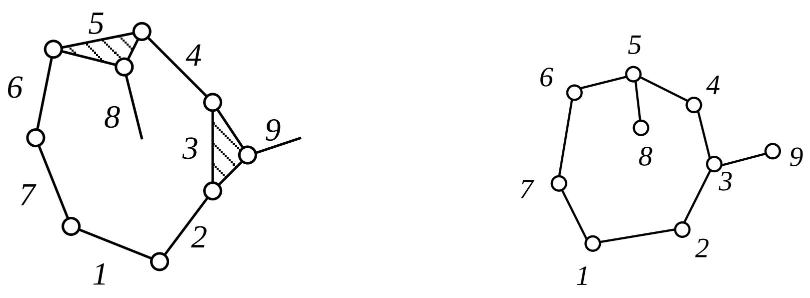


Figure 4.4: *A spatial one-loop $M = 3$ kinematic chain: structural representation (a) and its graph (b)*

However, the connectivity $C[1,8] = 3$ is not correct because links 1 and 5 belong to a subchain with mobility $M' = 1$. Also the connectivity between links 5 and 8 must be 1 since they are directly connected. Hence the connectivity $C[1,8] = 3$ between links 1 and 8 must be 2. In [Belfiore and Benedetto 2000], kinematic chains with partial mobility are excluded (this limitation is removed in a later algorithm presented in [Liberati and Belfiore 2006]). The next propositions are then focused on the connectivity calculation for chains with total and fractionated mobility.

Proposition 3. *(Theorem 3 in [Belfiore and Benedetto 2000]) In a biconnected subchain with total mobility (variety $V = 0$), the degrees of control $K[i, j]$ between two links i and j is equal to the lowest value among the minimum distance $D_{\min}[i, j]$ between the two links and the mobility M_k of the biconnected component. The connectivity $C[i, j]$ is derived from Proposition 1.*

Requiring the biconnectivity property permits a correct application of (4.11). In order to consider chains with fractionated mobility, it is necessary the last proposition:

Proposition 4. *(Theorem 4 in [Belfiore and Benedetto 2000]) In a kinematic chain whose biconnected subchains have total mobility, the degrees of control $K[i, j]$ between two links i and j is equal to the length of the minimum path between the vertices i and j of the graph obtained by adding to the graph corresponding to the kinematic chain (having all the edges unitary weighted) new edges in such a way that the biconnected components become complete graphs, the new edges having as a weight the value of the mobility M_k of the biconnected component to which they belong. The connectivity $C[i, j]$ is derived from Proposition 1.*

The *topological method* in [Belfiore and Benedetto 2000] is derived from the method in [Shoham and Roth 1997]. Introducing the concept of biconnectivity components, the connectivity calculation is extended to fractioned mobility (which were not correctly analysed in [Shoham and Roth 1997]), and excluding the chains with partial mobility. Based on the propositions 1, 2, 3, 4 the automated procedure can be resumed:

1. Build the graph G corresponding to the kinematic chain to be analysed.

2. Copy G into a graph G'.

3. For each pair of vertices i and j of G, evaluate their mutual distance $D_{\min}[i,j]$.

4. Build a matrix D_{min} whose element $D_{min}[i,j]$ is equal to $D_{\min}[i,j]$.

5. Build a set B of subgraphs composed of the biconnected components of G.

6. For each k^{th} member of G_k of B, perform the following Step 7.

7. Perform the partial mobility test. If the component does not have partial mobility, then perform the following steps (from 8 to 13):

   8 Evaluate the number of independent loops ν_k by means of the Euler polyhedron formula $\nu_k = g_k - n_k + 1$, where g_k and n_k denote, respectively, the total number of pairs and links in the k^{th} component.

   9 Evaluate the mobility M_k of the k^{th} biconnected component G_k by means of the relation $M_k = \sum_{i=1}^{g_k} f_i - \lambda \nu_k = g_k - \lambda \nu_k$

   10 Build the complete graph KG_k of G_k.

   11 For each edge $t-h$ of KG_k, perform the following steps (12 and 13).

   12 Find the pair of vertices r and s of G that corresponds to the end of the edge $t-h$ of KG_k.

   13 If $M_k < D_{min}[r,s]$ then add to G' a virtual edge $r-s$ with a weight equal to M_k.

14. For each pair of vertices i and j of G', evaluate their mutual distance $D'_{\min}[i,j]$.

15. Build a matrix D'_{min} whose element $i-j$ is equal to $D'_{\min}[i,j]$.

16. Build the degrees of control matrix K in such a way that the element $K[i,j]$ is equal to $D'_{min}[i,j]$.

17. Build the connectivity matrix C in such a way that for each element, $C[i, j]$ is equal to $D'_{min}[i, j]$ or λ, depending on whether $D'_{min}[i, j]$ is less than λ or not.

18. Build the redundancy matrix R in such a way that for each element, $R[i, j] = K[i, j] - C[i, j]$.

The mobility test reported in step 7 of the algorithm is implemented through the following steps:

1. Evaluate the number v_k of degrees of freedom of the biconnected component.

2. Build the graph corresponding to the biconnected component.

3. Evaluate a set of independent circuits of the component based on a minimum weight spanning tree, having assigned a unitary value to the edge weights.

4. Detect the cycle having the lowest length q.

5. If $(q < M_k + \lambda)$ then the biconnected component has partial mobility.

In [Belfiore and Benedetto 2000] the exclusion of the chains with partial mobility (with variety $V \neq 0$) is performed through the mobility test of Step 7 reported above.

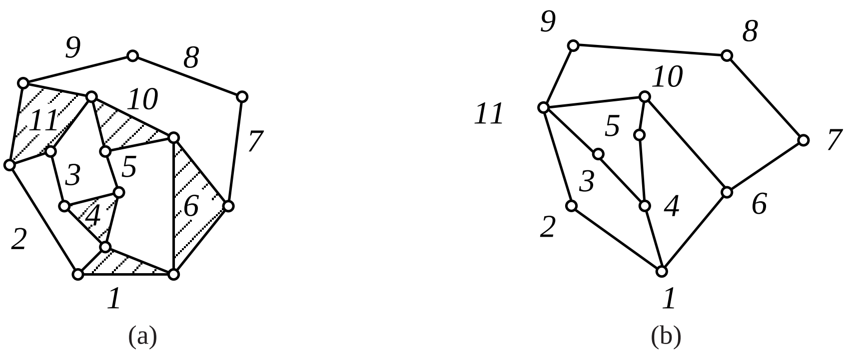


Figure 4.5: *Kinematic chain of variety $V = 1$, $\lambda = 3$, $M = 2$: structural representation (a) and its graph (b)*

Applying the topological method to the chain in Figure 4.5a, whose corresponding graph is represented in Figure 4.5b. The mobility of the chain is $M = 2$, and the chain also has partial mobility (variety $V = 1$), because the subchain formed by links 1-2-3-4-5-6-10-11 has mobility $M = 1$. The partial mobility of the chain is not detected by the mobility test; consequently, the algorithm calculates a set of connectivities not coherent with the connectivity definition.

In fact, given a kinematic chain with mobility M, the mobility test presented by Belfiori recognises the partial mobility in relation to the existence of at least one circuit whose length q be less then the sum of M_k and λ. This condition, *i.e.* the presence of single loops of length q is not sufficient to state that the kinematic chain under analysis has partial mobility.

4.4 Contribution of Liberati and Belfiore

In [Liberati and Belfiore 2006] Liberati and Belfiore present a new algorithm, aiming at correctly detecting partial mobility chain and calculating their mobility. The method is based on the concept of *gradual freezing of the circuits of a kinematic chain*, first introduced in [Mruthyunjaya and Raghavan 1984]. The method of *gradual freezing of the circuits of a kinematic chain* was originally developed to determinate if a link of a chain is a separation link. The separation link in a chain represents a "cut vertex" in the corresponding graph.

Let A be the matrix representing the graph of the kinematic chain and A_k the matrix representing the graph resulting from deletion of vertex k from the graph of A. The vertex k represents a cut vertex if the graph of A_k is disconnected. In order to find whether the graph of the matrix A_k is disconnected, start with any vertex i in the graph of A_k and "fuse" with it, one at a time, the vertices adjacent to it. Each fusion reduces the number of vertices and edges by one as the edge between the fused vertices disappears. The fusion of adjacent vertices in a graph is analogous to "freezing"of the kinematic pair between the corresponding links in the kinematic chain with the result that the two links coalesce into a single rigid link. In terms of the matrix, fusing a vertex j with the vertex i can be accomplished by adding j^{th} row to i^{th} row and j^{th} column to i^{th} column and then discarding the j^{th} row and the j^{th} column, the addition being done as per Modulo-2 algebra: $1+0=0+1=1$, $0+0=0$ and $1+1=0$. The process of fusion continues until the vertex i has no vertex adjacent to it.

When the graph is reduced to a single isolated vertex i it indicates that the graph of A_k is not disconnected. If, however, there are more then one vertex remaining in the graph, then the graph of A_k is disconnected. In this case, k indicates a cut vertex representing a separation link.

For a better understanding of the method proposed in [Liberati and Belfiore 2006], consider Figure 4.6 which represents a kinematic chain with mobility $M=2$ and the corresponding graph.

The subgraph composed by the vertices $1,2,3,4$ is a circuit of length $q=4$. Considering a unitary weight for any edge, the mobility for the mentioned subgraph is $M=q-\lambda=1$; then only one actuator is necessary to control it. By adding the actuator and blocking the circuit, a

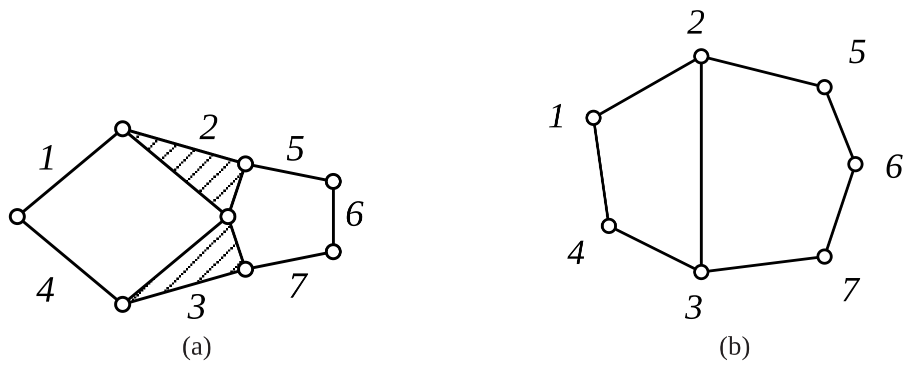


Figure 4.6: *Kinematic chain with partial mobility: structural representation (a) and its graph (b)*

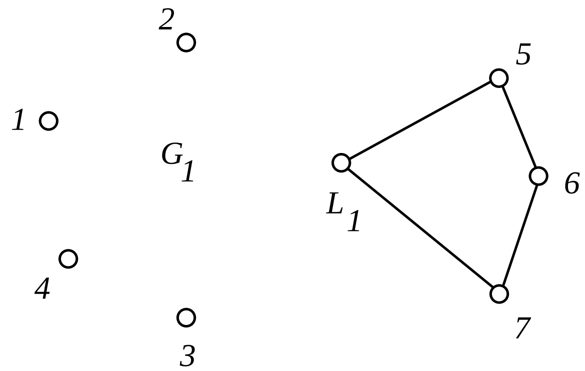


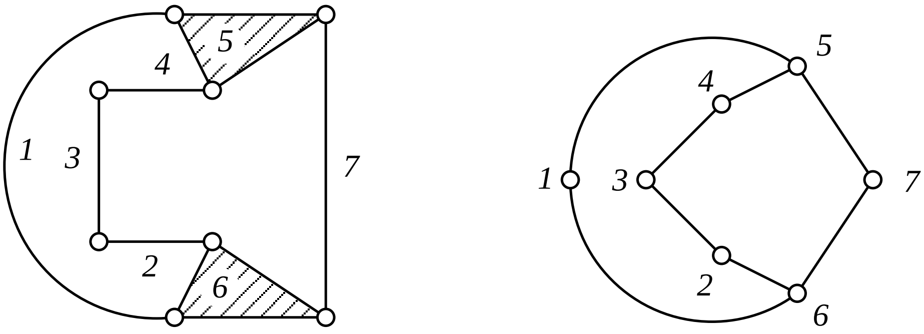
Figure 4.7: Reduced graph

frozen loop results. A frozen loop can be seen as a unique new member that can be represented as a vertex belonging to a novel reduced graph, as presented in Figure 4.7.

Now, the reduced graph needs another actuator in order to be fully controlled, since the mobility of the whole kinematic chain is $M = 2$, therefore, it is possible to affirm that the initial kinematic chain has partial mobility. In fact, the subchain composed by the links $1, 2, 3, 4$, which has mobility less than two, is controlled by only one actuator. Hence, the freezing of one circuit has helped in recognising the partial mobility and in identifying a subchain with mobility $M_k < M$.

The method proposed by Liberati and Belfiore [Liberati and Belfiore 2006] is quite complex and, for a complete description and details of the algorithm, the interested reader should go to Appendix D, where a detailed description is presented with an application example.

Applying this algorithm to the kinematic chain represented in Figure 4.8 (the steps of the algorithm are referenced herein as originally numbered in Liberati and Belfiore [Liberati and Belfiore 2006] - Section 6 and the notation is as in the original paper). This kinematic chain is an original counterexample to the algorithm of [Liberati and Belfiore 2006].



(a) (b)

Figure 4.8: Kinematic chain with partial mobility (variety $V = 1$), $\lambda = 3$ and $M = 2$: structural representation (a) and its graph (b)

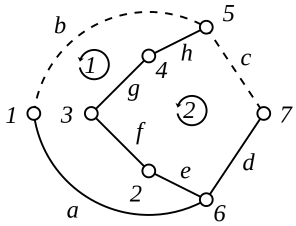


Figure 4.9: Minimum-weight spanning tree of the graph of Figure 4.8b

The corresponding graph of the kinematic chain with $N = 7$ links and $P = 8$ joints (*step 1*) is presented in Figure 4.8b. The matrix D of the minimal distance between vertices is calculated (*step 3*, *4*) as in Equation (4.12).

$$D_{\min} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 2 & 3 & 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 0 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 3 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 & 1 & 1 & 0 \end{array}\right] \end{array} \tag{4.12}$$

The number of independent circuits (*step 5*) is $L_{ind} = 2$ and $L^{”}_{ind} = L_{ind} = 2$. The mobility of the whole kinematic chain is (*step 6*) $M = 2$. The minimum-weight spanning tree for the graph (*step 7*) is represented in Figure 4.9 and circuits 1-5-4-3-2-6-1 and 7-6-2-3-4-5-7 (indicated with dashed line) are a set of fundamental circuits, each one with length $q = 6$.

Starting from the circuit $c_1 = 1-5-4-3-2-6-1$, apply the main recursive procedure (*steps 12 to 19*) to freeze iteratively all the subchains. The mobility of the circuit c_1 is $M_1 =$

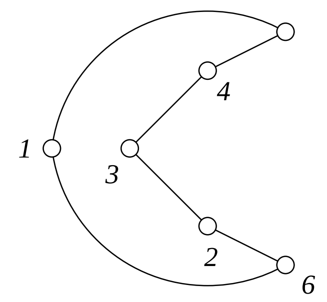


Figure 4.10: Graph G_1 of the circuit c_1

Figure 4.11: Graph of matrix A', after the freezing of the circuit c_1

$q-\lambda = 6-3 = 3$ and $N^{'} = 7$ (*step 13.1*). The graph G_1 of the circuit c_1 is represented in Figure 4.10 and $c_1^{'} = c_1 = 1-5-4-3-2-6-1$. The graph corresponding to matrix $A^{'}$, after the process of freezing, is represented in Figure 4.11 (*steps 13.3, 13.4, 13.4, 13.5*). Analyzing this graph, the number of independent circuit is $L_{ind}^{'} = 0$, $c_1^{'} = \{\}$ and $L_{ind}^{'} \neq L_{ind}^{''} - 1$, as depicted in the algorithm (*case step 13.7*). Set $M_1 = 2$ and $L_{ind}^{''} = 1$ (*step 13.7*). The next step of the algorithm is the case $L_{ind}^{'} = L_{ind}^{''} - 1 = 0$ (*step 13.9*), set $M_1^{'} = M = 3$ and perform the next section of the algorithm (*steps 14, 15, 16, 17, 18 and 19*). It is easy to verify that, each element being $D[r,s] \leq M_1^{'}$, no modification is introduced either in the graph $G^{'}$ or in the matrix D.

Repeating the same procedure for the other circuit $c_2 = 7-6-2-3-4-5-7$, no modification is introduced either in the graph $G^{'}$ nor in the matrix D. Performing the last part of the algorithm (*steps 20, 21, 22, 23*), a new matrix $D_{\min}$ of minimum distance is calculated from the graph $G^{'}$ and the connectivity is set equal to $\min:\{C_{ij},\lambda\}$. For links 1 and 7, the algorithm evaluates their connectivity $C_{1,7} = 2$, which contradicts the connectivity definition. In fact, both links belong to the subchain $1-5-7-6$, which is a four-bar chain with mobility $M = 1$. Consequently, the connectivity between links 1 and 7 must be equal to 1.

In this case, the algorithm was not able to detect the closed-loop subchain formed by links 1-5-7-6. All connectivities between these links are upper-bounded by the mobility of the subchain, in this case 1. Consequently, the connectivity between the links 1 and 7 must not be higher than 1 *i.e.* $C_{1,7} \leq 1$ and cannot be 2 as the algorithm predicted.

A deeper look into the algorithm shows that the calculation of connectivities of a kinematic chain is strongly dependent upon the chosen set of fundamental circuits. The step 7 from the algorithm [Liberati and Belfiore 2006] states:

7. Evaluate a set of independent circuits of the kinematic chain based on a minimum weight-

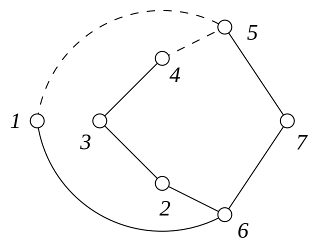


Figure 4.12: Different choice of minimum spanning tree of the graph of Figure 4.8b, which permits a correct evaluation of the connectivity

spanning tree, having assigned a unitary value to the edge weights of the graph. Detect the cycles having the lowest length qmin;

Consider a different set of fundamental circuits for the graph in Figure 4.8b, represented in Figure 4.12 with a dashed line. Applying the algorithm with this set of fundamental circuits, the connectivity between any pair of links is correctly evaluated.

Different set of independent circuits may lead to different evaluation of the connectivity, as highlighted in the counterexample presented in this work. However, there is no indication in [Liberati and Belfiore 2006] that the selection of the minimum spanning tree is not arbitrary. Furthermore, there is no guarantee that such a “best” minimum spanning tree exists for all cases, particularly for complex kinematic chains.

5 A new approach to degrees of control, connectivity and variety

In this chapter, new definitions of the concepts of degrees of control, connectivity and variety are presented (Section 5.2, 5.3 and 5.5). These new definitions, which are one of the main contributions of this work, do not conflict with the previous found in the literature. The Tischler-Samuel-Hunt conjectures stating the relation between connectivity and variety, introduced by Tischler *et al.* in [Tischler 1995], are formally proved in Section 5.7.

5.1 Introduction

The definitions of degrees of control, connectivity and variety, presented in 3.4, Section 3.5 and 3.8, respectively at Pages 27, 27, and 32, are relatively easy to understand, however, these definitions do not provide a systematic procedure to obtain their specific values.

The need for a constructive method to obtain the main parameters of kinematic chain prompted the author to redefine degrees of control, connectivity, and variety in an algorithmically orientated form.

The new definitions, presented in the next sections and in [Piga Carboni and Martins 2007, Martins and Piga Carboni 2006], do not conflict with the previous definitions found in the literature, such as Definitions 3.4, Section 3.5 and 3.8. Instead alternative ways of defining degrees of control, connectivity and variety are presented in this work, which identify a systematic procedure for calculating these parameters.

Degrees of control, connectivity and variety, as defined in the literature [Belfiore and Benedetto 2000, Hunt 1990, Tischler et al. 1995], are influenced by the order of the screw system λ, the mobility of the subchains M_k and the minimum distance between links. Moreover, the values of degrees of control, connectivity, and variety of a kinematic chain are completely and univocally determined by the order of the screw system λ, the mobility of the biconnected subchains M_k, and the minimum distance between links. In the following definitions formal relations between respectively degrees of control, connectivity, variety and the order of the screw system λ, the mobility of the biconnected subchains M_k and the minimum distance between links are stated.

5.2 New definition of degrees of control

Definition 6. *In a kinematic chain represented by a graph G, the degrees of control between two links i and j is*

$$K_{ij} = \min : \{D_{\min}[i,j], M'_{\min}\} \tag{5.1}$$

where $D_{\min}[i,j]$ *is the minimum distance between vertices i and j of G,* $M'_{\min}$ *is the minimum biconnected subgraph mobility of G containing vertices i and j,* i.e. $M'_{\min} = \{min : M(G'_k)\ \forall G'_k \in B_s\}$ *with* $M(G'_k)$ *the mobility of the* k^{th} *biconnected subgraph, and* B_s *is the set of biconnected subgraphs of graph G.*

In Definition 6 $M'_{\min}$ is the mobility of the biconnected subgraph having the lowest value of mobility and containing the two vertices. Such subgraph may coincide with the whole graph (representing the kinematic chain).

5.3 New definition of connectivity

Definition 7. *In a kinematic chain represented by a graph G, the connectivity between two links i and j is*

$$C_{ij} = \min : \{D_{\min}[i,j], M'_{\min}, \lambda\} \tag{5.2}$$

where the symbols of Equation (5.2) *are the same of Equation* (5.1)

5.4 Redundancy calculation

The definition of *redundancy* is based on the concepts of degrees of control and connectivity, as previously introduced by Belfiore and Di Benedetto in [Belfiore and Benedetto 2000].

Definition 8. *In a kinematic chain represented by a graph G, the redundancy between two links i and j is the difference between* K_{ij} *and* C_{ij}

$$R_{ij} = K_{ij} - C_{ij} \tag{5.3}$$

A direct consequence of Definitions 7 and 6 is

$$C_{ij} = \min : \{K_{ij}, \lambda\} \tag{5.4}$$

Some of the most important parameters in a kinematic chain are connectivity and redundancy. The following lemma [Belfiore and Benedetto 2000], easily proved based on the previous definitions, may now be introduced in order to directly calculate redundancy from degrees of control.

Lemma 2. *(Theorem 1 in [Belfiore and Benedetto 2000]) Let a kinematic chain in a screw system of order λ be represented by a graph G. Consider two vertices i and j of G (representing two links of the kinematic chain) and and K_{ij} the degrees of control between links i and j.*

Then, the redundancy R_{ij} between links i and j is calculated as:

$$R_{ij} = \begin{cases} 0, & \text{if } K_{ij} \leq \lambda \\ K_{ij} - \lambda, & \text{if } K_{ij} > \lambda \end{cases} \tag{5.5}$$

5.5 New definition of variety

Definition 9. *Let a kinematic chain of mobility M be represented by a graph G, the variety of the kinematic chain is:*

$$V = M - \min : \{M(G'_k) \ \ \forall G'{}_k \in B_s\} \tag{5.6}$$

where M is the mobility of the chain, $M(G'_k)$ is the mobility of the k^{th} biconnected subgraph and the other terms are the same as in Definition 6.

5.6 Application of the new definitions

For a better understanding of the new definitions introduced in the following sections, consider Figure 5.1, which shows a $\lambda = 3$, mobility $M = 3$ and variety $V = 2$ kinematic chain.

Consider the pairs of links (2, 4), (2, 9) and (2, 7). The order of the screw-system to which all joints of the kinematic chain belong is $\lambda = 3$ (planar kinematic chain). The minimum

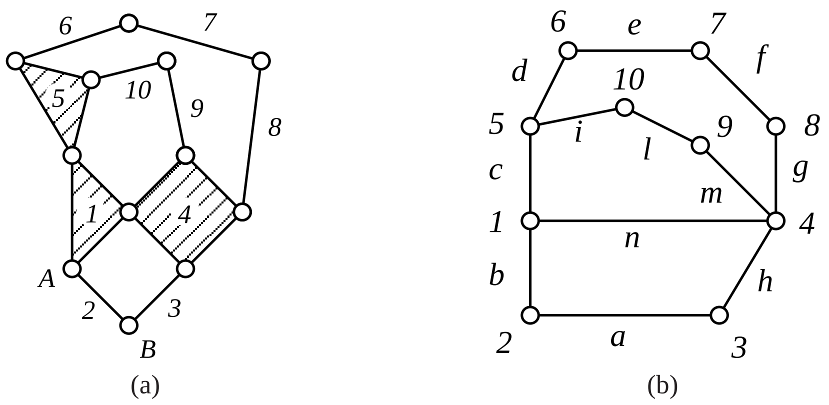


Figure 5.1: Closed-loop kinematic chain with $\lambda = 3$, mobility $M = 3$ and variety $V = 2$: kinematic structure (a) and its graph (b)

distance between links 2 and 4 is $D_{min}[2,4] = 2$, and both links belong to the subchain $1-2-3-4$, which has mobility $M_k = 1$. Consequently, by Definition 6 and 7 the degrees of control between links 2 and 4 is $K_{2,4} = 1$ and the connectivity between the same pair of links is $C_{2,4} = 1$.

The minimum distance between links 2 and 9 is $D_{min}[2,9] = 3$, and both links belong to the subchain $1-2-3-4-5-9-10$ (including edge n), which has mobility $M_k = 2$. Consequently, by Definition 6 and 7 the degrees of control between links 2 and 9 is $K_{2,9} = 2$ and the connectivity between the same pair of links is $C_{2,9} = 2$.

The minimum distance between links 2 and 7 is $D_{min}[2,7] = 4$, and both links belong to the subchain $1-2-3-4-5-6-7-8-9-10$ (which is the whole chain), which has mobility $M_k = 3$. Consequently, by Definition 6 and 7 the degrees of control between links 2 and 7 is $K_{2,7} = 3$ and the connectivity between the same pair of links is $C_{2,7} = 3$. By Definition 8 redundancy between all pairs of links is zero.

In order to calculate variety by Definition 9, it is necessary to identify the minimum mobility subchain: the subchain $1-2-3-4$ has mobility $M_k = 1$ and it is the minimum mobility subchain. Consequently the variety of the kinematic chain is $V = 2$.

Consider now Figure 5.2, which shows a $\lambda = 3$, mobility $M = 5$ and variety $V = 0$ kinematic chain. Consider the pairs of links (1, 5). The order of the screw-system to which all joints of the kinematic chain belong is $\lambda = 3$ (planar kinematic chain). The minimum distance between links 1 and 5 is $D_{min}[2,5] = 4$, and both links belong to the subchain $1-2-3-4-5-6-7-8$ (the whole chain), which has mobility $M_k = 5$. Consequently, by Definition 6 and 7 the degrees of control between links 2 and 4 is $K_{2,4} = 4$ and the connectivity between the same pair of links is $C_{2,4} = 3$. By Definition 8 redundancy between links 1 and 5 is $R_{14} = 1$. The variety of the kinematic chain is $V = 0$.

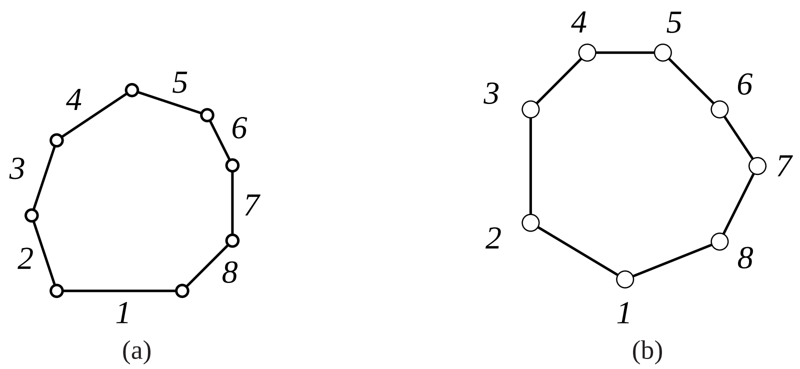


Figure 5.2: Closed-loop planar kinematic chain with $\lambda = 3$, mobility $M = 5$ and variety $V = 0$: kinematic structure (a) and its graph (b)

5.7 New theorems

Based on Definitions 7-9, the Tischler-Samuel-Hunt conjectures proposed by Tischler *et al.* [Tischler et al. 1995] in 1995 stating the relation between connectivity and variety, and presented in this work in Section 3.8.1 at Page 34, are herein demonstrated. This result has been published in [Martins and Piga Carboni 2006] and it is one of the major contributions of this work.

The Tischler-Samuel-Hunt conjectures are now stated as a series of theorems which are demonstrated in sequence. No evidence was found by the author that Conjectures 2 and 3 may be considered as corollaries of Conjecture 1, as originally stated by Tischler *et al.* in [Tischler et al. 1995].

Theorem 1. *If a variety V kinematic chain has a mobility less than, or equal to, the order of the screw system,* i.e. *if $M \leq \lambda$, than any two links i and j of the chain, separated by at least $M - V$ joints, have a relative connectivity $C_{ij} \geq M - V$.*

Proof: The proof of this theorem is by contradiction. Consider a kinematic chain G with variety V and mobility $M \leq \lambda$ in a screw system with order λ. Assume the existence in this kinematic chain of two links i and j separated by at least $M - V$ joints whose connectivity is $C_{ij} = M' < M - V$. Then, according to Definition 9, a biconnected subchain with mobility M' or lower and containing links i and j must exist. But if such a subchain exists, the variety of G is $V \geq M - M'$. As $V < M - M'$ in the hypothesis, a contradiction was encountered which demonstrates the theorem. ■

Consider now the first corollary introduced by Tischler *et al.* in [Tischler et al. 1995], which is now stated as an other theorem.

Theorem 2. *If a variety V kinematic chain has a mobility M greater than the order of the screw* *system that generally prevails* λ , i.e. *if* $M > \lambda$*, then any two links i and j, separated by at least* $\lambda - V$ *joints, have connectivity* $C_{ij} \geq \lambda - V$.

Proof: The proof of this theorem is also by contradiction. Consider a kinematic chain with variety V and mobility $M > \lambda$ in a screw system with order λ. Suppose the existence in this kinematic chain of two links i and j separated by at least $\lambda - V$ joints whose connectivity is $C_{ij} = M' < \lambda - V$. Then, according to Definition 9, a subchain with mobility M' or lower and containing links i and j must exist. If a subchain with mobility M' exists then, again according to the Definition 9, the variety of the kinematic chain is $V \geq M - M'$; thus:

$$M' < \lambda - V \;\Rightarrow\; V \geq M - M' \Rightarrow\; V \geq M - \lambda + V \Rightarrow M \leq \lambda$$

Therefore, a contradiction was encountered which demonstrates the theorem. ■

Consider now the third conjecture, Conjecture 3, proposed by Tischler *et al.* in [Tischler et al. 1995]. Using graph theory, the authors gave to the sentence *two links separated by a minimum of g single-freedom joints* in Conjecture 3, the meaning of two links whose minimum distance is equal to g. This sentence may lead to other, incorrect, interpretations.

In order to avoid any possible misunderstanding, a reformulation of Conjecture 3 using graph theory is proposed by the authors, herein stated as an other theorem:

Theorem 3. *(reformulation of Conjecture 3) Consider a kinematic chain represented by a graph G. Two links i and j of the kinematic chain, whose corresponding vertices in graph G have a minimum distance equal to g, where* $g < M - V$ *and* $g < \lambda - V$*, have a relative connectivity* $C_{ij} = g$.

Proof: The proof of this conjecture is by contradiction. The proof is divided in two parts: first it is proved that the connectivity C_{ij} cannot be $C_{ij} < g$, and then it is proved that the connectivity C_{ij} cannot be $C_{ij} > g$. Consider a kinematic chain with variety V and mobility M represented by a graph G. Assume the existence in this kinematic chain of two vertices i and j (represented by two vertices i and j in graph G), whose minimum distance is g, where $g < M - V$ and $g < \lambda - V$. Suppose that the connectivity between links i and j is $C_{ij} = M' < g$. Then, according to Definition 9, a subchain with mobility M' containing links i and j must exist. If a subchain with mobility M' exists, then according to Definition 9 the variety of the kinematic chain is $V \geq M - M'$. Hence $M' \geq M - V$ and $g > M'$ imply that $g \geq M - V$, which contradicts the hypothesis and consequently it is possible to state that C_{ij} cannot be $C_{ij} < g$. On the other hand, according to Equation (5.2), connectivity is upper-bounded by the minimum distance between links, so $C \leq g$. Thus we can conclude that $C_{ij} = g$. ■

5.7.1 Applications of the Theorems

As described in [Tischler et al. 1995], the above theorems set lower bounds for the connectivity of two links in a kinematic chain; however, the exact bounds are only found by identifying subsets of links with mobility $M-V$, and by checking the position of the two links relative to the corresponding subset of loops. If both links belong to the subset, then the connectivity will be equal to $M-V$. The upper bound for the relative connectivity is the minimum number of joints which separates the two bodies. Two corollaries of the theorems presented above may now be stated.

Corollary 1. *Given a proper kinematic chain with mobility* $M=1$ *and variety* $V=0$*, any pairs of links* i *and* j *have* $C_{ij}=1$.

Consider a kinematic chain with variety $V=0$ and mobility $M=1$. By definition, a proper kinematic chain with mobility $M=1$ must be variety $V=0$. Therefore, according to Theorem 1 presented, any two links separated by at least $M-V=1-0=1$ joint have a connectivity of at least $M-V=1$. Since it is not possible to have any two links with a connectivity greater than the mobility of the whole chain, $C_{ij}=1$ for all $i\neq j$. This result has long been recognised.

Corollary 2. *Given a kinematic chain with variety* $V=0$ *and mobility* $M=2$*, any two links* i *and* j *that are not directly connected to one another have a relative connectivity of* $C_{ij}=2$ *for all* $i\neq j$.

This is because the minimum number of joints between two links not directly connected is two, which is equal to $M-V$. This relationship is consistent for all chains where the same screw system applies to the whole kinematic chain.

6 *Implementation and results*

In this chapter, a novel methodology for calculating the main parameters of a kinematic chain, *i.e.* degrees of control, connectivity, redundancy and variety, is presented. A general description of the new methodology is presented in Section 6.1, and few fundamental concepts of subgraphs and vector spaces, used in the methodology, are introduced in Section 6.2. The steps of the algorithm are detailed described in Section 6.3, and some examples of application are analysed in Section 6.5.

6.1 General description of the new methodology

Based on the new definitions of degrees of control, connectivity and variety presented in Section 5 at Page 55, a novel methodology for degrees of control, connectivity, redundancy and variety calculation of a kinematic chain is proposed in this section. This methodology is an original contribution of this work and it has been presented in [Martins and Piga Carboni 2006] and [Piga Carboni and Martins 2007]. The algorithm, based on the complete correspondence between kinematic chains and graphs, may be divided into three main parts.

In the first part, a graph representation of the kinematic chain is adopted and the incidence and adjacency matrix of the graph are built. The mobility and the number of fundamental circuits of the graph are evaluated. The minimum distance matrix D_{min} between each pair of links is calculated.

As stated in Definitions 6, 7 and 9 at Pages 56, 56 and 57, degrees of control, connectivity and variety values depend also on the mobility of the biconnected subchains of the kinematic chain examined. Hence, the second part of the algorithm is the enumeration of all possible closed-loop biconnected subchains. A brief review of definitions and theorems used in this step is presented in Section 6.2.

In the last part, each biconnected subchain (more precisely, each biconnected subgraph) is analyzed, and the mobility evaluated. Each biconnected subchain is checked for properness and the algorithm stops if an improper subchain is found (Improper kinematic chain are introduced in Section 3.10 at Page 36).

Otherwise, based on Definitions 6, 7, 9 and Lemma 2 degrees of control, connectivity, redundancy and variety are finally calculated.

6.2 Subgraphs and circuit vector spaces

In this section some fundamental definitions and equations [Seshu and Reed 1961] are introduced, in order to illustrate the steps of the algorithm proposed.

Definition 10. *The set of all the subgraphs of a given graph G has a structure of vector space* V_G.

Definition 11. *All linear combinations of the rows of the matrix of circuits* B_f *is a vector subspace* V_B *of the vector space* V_G*, over the field mod 2.*

Definition 12. *The set of* ν *fundamental circuits is a basis for the vector subspace* V_B.

Theorem 4. *There are* 2^ν *elements (including 0), in* V_B*, and each of these is a circuit or disjoint unions of circuits of G (Proof is found in [Seshu and Reed 1961]).*

In order to find the fundamental circuits of a graph G, the following method, proposed by Seshu and Reed [Seshu and Reed 1961], may be adopted:

- Consider the vertex, or incidence matrix A_a of graph G.
- Find a spanning tree T for graph G.
- Reorder matrix A_a in the form: $A^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where the columns of the matrices A_{12} and A_{22} correspond to the edges of the spanning tree, and A_{12} is a square non singular matrix.
- The fundamental basis B_f is the set of rows of

$$B_f = [I, (A_{12}^{-1} A_{11})'] \tag{6.1}$$

  where I is an identity matrix, A_{12}^{-1} is the modular inversion of A_{12} and the columns of B_f correspond to the same edges as the columns of A^*.

6.3 Proposed algorithm for connectivity calculation

Based on the previous definitions, lemmas and theorems, a complete algorithm for degrees of control, connectivity, redundancy and variety calculation of a kinematic chain is now described. A detailed description of the algorithm implementation is found in Appendix E.

Let a kinematic chain with g joints and n edges represented by its graph G. The following steps are applied to the kinematic chain:

1. Calculate the mobility M of the kinematic chain $M = g - \lambda\nu$ where λ is the order of the screw system and ν is the number of independent circuits; ν is obtained using Euler's equation or by inspection. Alternatively mobility may be calculated by Equation 3.4.

2. Build the minimum distance matrix $D_{\min}$, whose element $D_{\min}[r,s]$ is the minimum distance between vertices r and s. Graph algorithm for calculating all-pairs shortest paths may be employed to obtain the matrix $D_{\min}[r,s]$.

3. Set variety $V = 0$.

4. Build the incidence matrix A_a of the graph G.

5. Build the adjacency matrix A_j of the graph G.

6. Enumerate all the fundamental circuits of the graph G. The method implemented in this work is suggested in [Seshu and Reed 1961]), *i.e.* considering the vector space of the circuits of the graph generated by the basis B_f of fundamental circuits. Alternative algorithms are proposed in [Liu and Wang 2006], [Johnson 1975], [Gibbs 1969], [Honkanen 1978] which permit a faster execution, because the sets of disjoint circuits are not generated.

7. Enumerate all the circuits of the graph G. A matrix B is generated, where the columns are the edges of the graph, and the rows are all the circuits of graph G. Matrix B may be obtained by Definition 11 as linear combinations of rows of matrix B_f over the field mod 2 ($0+0 = 1+1 = 0$ and $0+1 = 1+0 = 1$, *i.e.* exclusive OR), removing all rows which represent disjoint union of circuits.

8. Enumerate all biconnected subgraphs of graph G (every biconnected subgraph corresponds to a closed-loop subchain), *i.e.* by linear combinations of the rows of matrix B, using Boolean algebra ($0+0 = 0$ and $0+1 = 1+0 = 1+1 = 1$). In this way, a large number of subgraphs are considered (*i.e.* non biconnected subgraphs are included). In

fact, for the connectivity determination only biconnected subgraphs must be considered; therefore, a biconnectivity test is useful to discard non biconnected subgraphs.

9. Copy graph G into graph G'.

10. Iterate steps 10.1-10.8 for each subgraph G_k of graph G:

    10.1 Identify all vertices which belong to the subgraph represented by the row of matrix B examined (use the incidence matrix A_a)

    10.2 Calculate the mobility of the subgraph M_k.

    10.3 If $M_k \leq 0$ then exit the algorithm because an improper subchain exists (Section 3.10).

    10.4 If $M_k \leq M$ continue from the following step 10.5, if $M_k > M$ consider a new subgraph (Step 10.1)

    10.5 If $V < M - M_k$ then set variety $V = M - M_k$

    10.6 Build the subgraph G_k, composed of the edges and vertices identified.

    10.7 Build the complete graph KG_k of G_k.

    10.8 For every edge $t - h$ of KG_k do the following steps:

        10.8.1 Find every pair of vertices r and s of G that corresponds to the end of the edge $t - h$ of KG_k.

        10.8.2 If $D_{\min}[r,s] > M_k$ then add to G' a virtual edge of weight equal to M_k.

11. The variety of the kinematic chain is V.

12. Calculate a new matrix $D'_{\min}$ of the minimum distance between the vertices of graph G'. The degrees of control matrix is, by Definition 6, $K = D'_{\min}$.

13. Build the connectivity matrix C and redundancy matrix R as:

$$C_{ij} = \begin{cases} D'_{\min}[i,j], & \text{if } D'_{\min}[i,j] \leq \lambda \\ \lambda, & \text{if } D'_{\min}[i,j] > \lambda \end{cases} \qquad R_{ij} = \begin{cases} 0, & \text{if } D'_{\min}[i,j] \leq \lambda \\ D'_{\min}[i,j] - \lambda, & \text{if } D'_{\min}[i,j] > \lambda \end{cases}$$

6.4 Algorithm complexity

In order to evaluate the complexity of the algorithm proposed in the previous section, consider Step 8. It may be verified that Step 8 determines completely the complexity of the algorithm. Consider a kinematic chains with ν loops: the number of fundamental circuits is ν and the number of all circuits is 2^{ν} (Step 7). In Step 8 all biconnected subgraphs are generated: first

2^{2^ν} subgraphs are generated, as linear combinations of the 2^ν circuits generated in Step 7 , then a biconnectivity test is applied to each subgraph. The running time of the biconnectivity test is $O(n+g)$. Consequently the total running time of the algorithm is $O((n+g)\cdot 2^{2^\nu})$.

The algorithm here proposed is a valid solution for kinematic chains with a small number of independent loops, otherwise the number of subchains may increase dramatically, and the computational time required to perform the analysis may be unacceptably long. However, to the author's knowledge, this is the first algorithm that accurately calculates connectivity and redundancy in all cases, without exception.

New algorithms for enumeration of biconnected subgraphs should be investigated as a further work, in order to reduce the complexity of the algorithm.

6.5 Application of the algorithm

Three examples of the application of the algorithm are given in this section: a kinematic chain with $\nu = 2$ loops, a kinematic chain with $\nu = 3$ loops, and an improper kinematic chain. Degrees of control, connectivity, redundancy and variety are obtained for these three chains.

6.5.1 Example 1: kinematic chain with two fundamental circuits

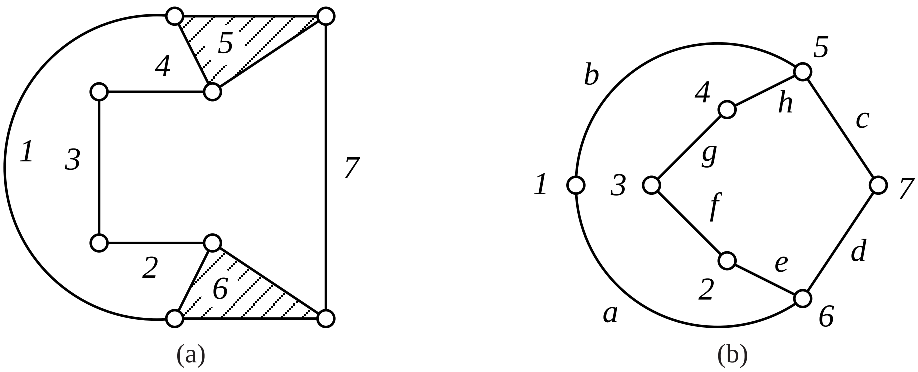


Figure 6.1: Kinematic chain with partial mobility (variety $V = 1$), $\lambda = 3$ and $M = 2$: structural representation (a) and its labelled graph (b)

Apply the algorithm to the kinematic chain of Figure 6.1, whose graph G is represented in Figure 6.1b. This example is the original counterexample to the algorithm of Liberati and Belfiore [Liberati and Belfiore 2006], presented in Section 4.4 at Page 49. This counterexample has been also presented in [Piga Carboni and Martins 2007].

The mobility of the kinematic chain, having $g = 7$ joints belonging to the screw system of order $\lambda = 3$, is $M = 2$ (Step 1). The minimum distance matrix $D_{\min}$ is evaluated as (Step 2):

$$D_{\min} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 2 & 3 & 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 0 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 3 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{6.2}$$

Considering the minimum spanning tree of Figure 6.2 and labelling the edges as in Figure 6.1b, the incidence matrix A_a is obtained (Step 4), as in Equation (6.3).

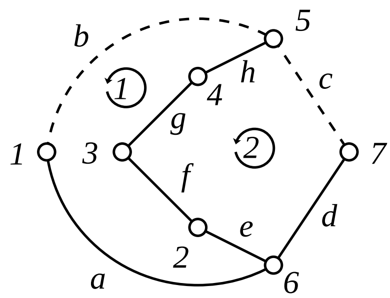


Figure 6.2: Minimum-weight spanning tree of the graph of Figure 6.1b

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{6.3}$$

Applying the method proposed by Seshu and Reed [Seshu and Reed 1961] and referred in Section 6.2, matrix A_a may be reordered in the form $A_a^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$A_a^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & d & e & f & g & h & & b \quad c \end{array} \\ \left[\begin{array}{cccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{6.4}$$

where the columns of the sub-matrix A_{11} and A_{21} are the edges of the minimum spanning tree of Figure 6.2, *i.e.* $a - d - e - f - g - h$.

By Equation (6.1), and reordering columns, matrix B_f is obtained (Step 6) as in Equation (6.5). Two fundamental circuits are found, corresponding to the circuits of Figure 6.2.

$$B_f = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{6.5}$$

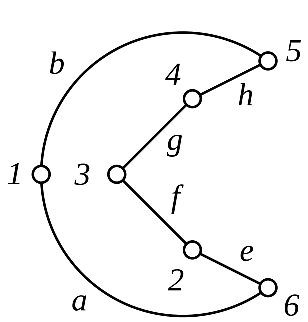
From the linear combinations of rows of matrix B_f over the field mod 2, *i.e.* $0+0 = 1+1 = 0$ and $0+1 = 1+0 = 1$, matrix B is obtained (Step 7), as in Equation (6.6), where each row represents a circuit of the graph.

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{6.6}$$

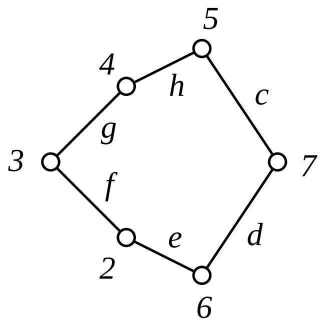
To enumerate all biconnected subgraphs, consider first all linear combinations of the rows of matrix B. The set of all linear combinations of rows of matrix B using Boolean Algebra, *i.e.* $0+0 = 0$ and $0+1 = 1+0 = 1+1 = 1$, include all biconnected subgraphs of G, are arranged in matrix B_s (Step 8). The string at the beginning of each row indicates the linear combination of the three rows of matrix B.

$$B_s = \begin{array}{c} \\ 0+0+0 \\ 1+0+0 \\ 0+1+0 \\ 0+0+1 \\ 1+1+0 \\ 1+0+1 \\ 0+1+1 \\ 1+1+1 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{6.7}$$

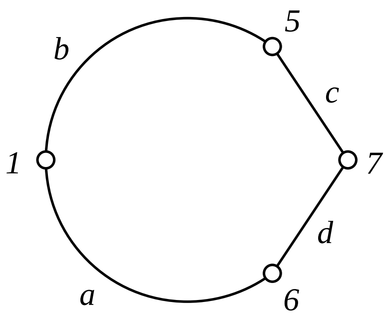
The rows of matrix B_s represent all possible subgraphs of graph G; in fact, some subgraphs appear more than once, but this does not affect the connectivity calculation. All of the independent biconnected subgraphs are represented by rows 2, 3, 4 and 5 of matrix B_s, and are represented in Figure 6.3.



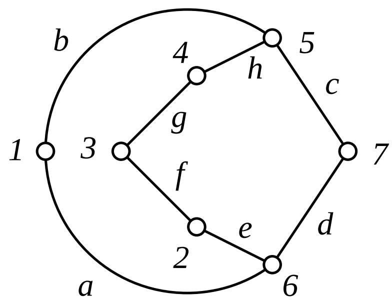
(a) row 2 ($M = 3$)



(b) row 3 ($M = 3$)



(c) row 4 ($M = 1$)



(d) row 5 ($M = 2$)

Figure 6.3: All subgraphs of graph G as identified by the rows 2, 3, 4 and 5 of matrix B_s

Considering the incidence matrix A_a of graph G (which relates vertices to edges), for each set of edges of a subgraph G_k (a row of matrix B_s) it is possible to identify the set of vertices which belong to the same subgraph G_k. Now it is possible to compute the mobility of the subgraph using the mobility equation. The mobility of each subgraph is indicated in Figure 6.3.

Consider now a copy $G^{'}$ of graph G. Applying to each subgraph the steps 10.1 - 10.8, the improperness of each biconnected subgraph is verified ($M_k \leq 0$). No improper subchains are found in the kinematic chain of Figure 6.1 (Step 10.3), consequently variety is calculated as in Step 10.5. For each pair of vertices r and s of the subgraph G_k, the mobility M_k of G_k is compared with the minimum distance $D_{min}[r,s]$ between vertices r and s. If mobility M_k is lower than the minimum distance $D_{min}[r,s]$, a virtual edge between the vertices r and s with weight M_k is added to graph $G^{'}$.

A useful representation for the graph $G^{'}$ is given by the adjacency matrix A_j; a virtual edge of weight W between vertices i and j of graph $G^{'}$ may be added simply by setting $A_j[i,j] = W$. The matrix A_j', representing graph G' with virtual edges added (in bold text) is:

$$A_j' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & \mathbf{2} & 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 0 & \mathbf{2} & 1 & 0 \\ \mathbf{2} & 1 & 0 & 1 & 0 & 0 & \mathbf{2} \\ 0 & 0 & 1 & 0 & 1 & \mathbf{2} & 0 \\ 1 & \mathbf{2} & 0 & 1 & 0 & \mathbf{1} & 1 \\ 1 & 1 & 0 & \mathbf{2} & \mathbf{1} & 0 & 1 \\ \mathbf{1} & 0 & \mathbf{2} & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{6.8}$$

We can now calculate the minimum distance matrix $D^{'}_{\mathrm{min}}$ of graph $G^{'}$ and applying Steps 12 - 13, the connectivity matrix is evaluated as:

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{6.9}$$

The degrees-of-control matrix K is equal to the connectivity matrix C and the redundancy matrix R is null. Variety of the kinematic chain is $V = 1$, because the mobility of the whole chain is $M = 2$ and the minimum subchain mobility is $M_k = 1$.

6.5.2 Example 2: kinematic chain with $\nu = 3$ loops

Consider now the kinematic chain of Figure 6.4, which represents a planar kinematic chain with $\lambda = 3$, mobility $M = 3$ and its corresponding graph G as shown in Figure 6.4b.

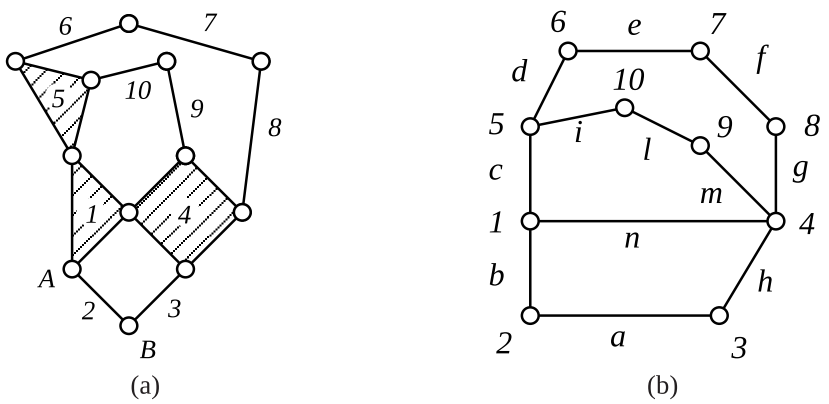


Figure 6.4: Planar kinematic chain with $\lambda = 3$ and mobility $M = 3$, $V = 2$ and no redundancy: structural representation (a) and its graph (b)

Applying Steps 2- 4 of the algorithm, the minimum distance matrix $D_{\min}$ is evaluated as:

$$
D_{\min} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left[\begin{array}{cccccccccc}
0 & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 2 & 2 \\
1 & 0 & 1 & 2 & 2 & 3 & 4 & 3 & 3 & 3 \\
2 & 1 & 0 & 1 & 3 & 4 & 3 & 2 & 2 & 3 \\
1 & 2 & 1 & 0 & 2 & 3 & 2 & 1 & 1 & 2 \\
1 & 2 & 3 & 2 & 0 & 1 & 2 & 3 & 2 & 1 \\
2 & 3 & 4 & 3 & 1 & 0 & 1 & 2 & 3 & 2 \\
3 & 4 & 3 & 2 & 2 & 1 & 0 & 1 & 3 & 3 \\
2 & 3 & 2 & 1 & 3 & 2 & 1 & 0 & 2 & 3 \\
2 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 0 & 1 \\
2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 1 & 0
\end{array}\right] \end{array} \qquad (6.10)
$$

and the incidence matrix is:

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \tag{6.11}$$

Considering the spanning tree of graph G (Step 6) represented in Figure 6.5, three fundamental circuits are found applying Equation 6.1.

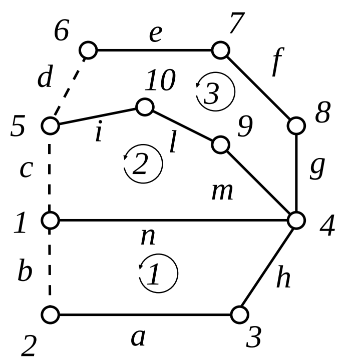


Figure 6.5: Spanning tree of graph G of Figure 6.4b with a set of fundamental circuits

The matrix B_f of fundamental circuits is now obtained by Equation (6.1):

$$B_f = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{6.12}$$

Matrix B, whose rows represent all the circuits of the graph, can now be evaluated as linear combinations of rows of matrix B_f (Step 7):

$$
B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\tag{6.13}
$$

In order to enumerate all biconnected subgraphs (Step 8), consider all linear combinations of rows of matrix B, using Boolean algebra. In this way, all possible combinations of the circuits of graph G (rows of matrix B) are considered. If all linear combinations of the rows of matrix B are enumerated, a new matrix of 64 rows is obtained. Checking and eliminating repeated rows, the matrix B_s is obtained:

$$
B_s = \begin{array}{c} \\ 000000 \\ 100000 \\ 010000 \\ 110000 \\ 001000 \\ 101000 \\ 011000 \\ 111000 \\ 000100 \\ 001100 \\ 000010 \\ 000110 \\ 000001 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\tag{6.14}
$$

which represent the set of different subgraphs of graph G, where the string at the beginning of each row indicates the linear combination of the six rows of matrix B. The first row is the null combination, and applying the biconnectivity test to the subgraphs represented by the other rows of matrix B_s, it is possible to verify that the subgraph represented by row 10 is not biconnected. Consequently the set of all biconnected subgraphs are represented by rows 2-3-4-5-6-7-8-9-11-12-13 of matrix B_s.

Applying Steps 10.1 - 10.8 of the algorithm, the adjacency matrix A_j of graph G' with all virtual edges added (in bold text) is:

$$A_j = \begin{array}{c|cccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 0 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{3} & 0 & \mathbf{2} & \mathbf{2} \\
3 & \mathbf{1} & 1 & 0 & 1 & \mathbf{2} & \mathbf{3} & 0 & 0 & 0 & \mathbf{2} \\
4 & 1 & \mathbf{1} & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
5 & 1 & 0 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
6 & 0 & 0 & \mathbf{3} & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & \mathbf{3} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
8 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
9 & 0 & \mathbf{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
10 & 0 & \mathbf{2} & \mathbf{2} & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{array} \tag{6.15}$$

It may be verified that the biconnected subgraph corresponding to row 9 of matrix B_s, has mobility $M_9 = 1$, and thus Step 10.5 calculate variety $V = M - M_9 = 3 - 1 = 2$.

Finally, the minimum distance matrix D'_{min} of graph G' (Steps 12 - 13) is calculated, and the connectivity matrix of the kinematic chain in Figure 3.7 is evaluated as:

$$C = \begin{array}{c|cccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 2 & 2 & 2 \\
2 & 1 & 0 & 1 & 1 & 2 & 3 & 3 & 3 & 2 & 2 \\
3 & 1 & 1 & 0 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\
4 & 1 & 1 & 1 & 0 & 2 & 3 & 2 & 1 & 1 & 2 \\
5 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 3 & 2 & 1 \\
6 & 2 & 3 & 3 & 3 & 1 & 0 & 1 & 2 & 3 & 2 \\
7 & 3 & 3 & 3 & 2 & 2 & 1 & 0 & 1 & 3 & 3 \\
8 & 2 & 3 & 2 & 1 & 3 & 2 & 1 & 0 & 2 & 3 \\
9 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 2 & 0 & 1 \\
10 & 2 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 1 & 0 \\
\end{array} \tag{6.16}$$

The degrees-of-control matrix K is equal to the connectivity matrix C and the redundancy matrix R is null.

6.5.3 Example 3: degenerated kinematic chain

A further important application of the algorithm presented in Section 6 is the detection of improper kinematic chains. An improper kinematic chain, as defined in Section 3.10 at Page 36, is a kinematic chain with at least one subchain with mobility $M_k \leq 0$.

Consider the planar kinematic chain represented in Figure 6.6 and its graph in Figure 6.7a.

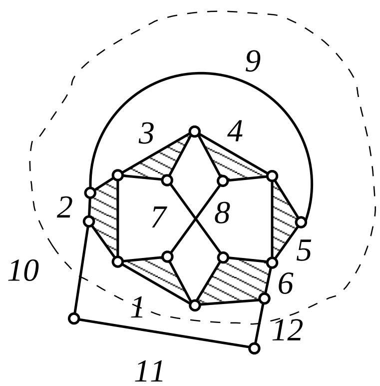


Figure 6.6: Improper planar kinematic chain with $M = 1$ and partial mobility ($V = 1$) because it contains a Baranov subchain 1-2-3-4-5-6-7-8-9 (dashed line)

The subchain 1-2-3-4-5-6-7-8-9 has been detected as a Baranov chain (mobility $M' = 0$). Each loop of this subchain has length $q_{min} \geq 5$. The whole chain has mobility $M = 1$ and variety $V = 1$. Detecting the Baranov chain is not trivial, and an exhaustive analysis of all subgraphs is necessary to correct evaluate connectivity.

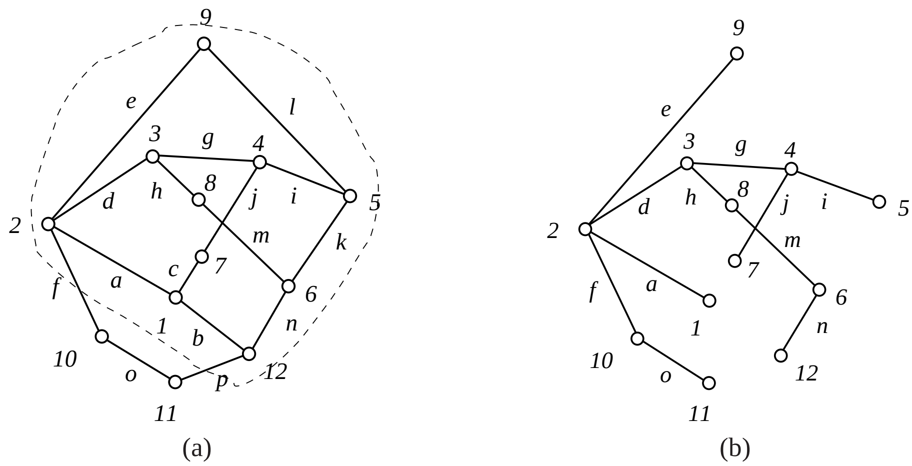


Figure 6.7: The graph and a minimum-weight spanning tree of the kinematic chain of Figure 6.6

The minimum distance matrix D_{min} is evaluated as ((Step 2):

$$D_{min} = \begin{array}{c|cccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 1 & 0 & 1 & 2 & 2 & 3 & 2 & 1 & 3 & 2 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 3 & 3 \\ 4 & 2 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 3 & 4 & 3 \\ 5 & 3 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 3 & 3 & 2 \\ 6 & 2 & 3 & 2 & 2 & 1 & 0 & 3 & 1 & 2 & 3 & 2 & 1 \\ 7 & 1 & 2 & 2 & 1 & 2 & 3 & 0 & 3 & 3 & 3 & 3 & 2 \\ 8 & 3 & 2 & 1 & 2 & 2 & 1 & 3 & 0 & 3 & 3 & 3 & 2 \\ 9 & 2 & 1 & 2 & 2 & 1 & 2 & 3 & 3 & 0 & 2 & 3 & 3 \\ 10 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 0 & 1 & 2 \\ 11 & 2 & 2 & 3 & 4 & 3 & 2 & 3 & 3 & 3 & 1 & 0 & 1 \\ 12 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 2 & 3 & 2 & 1 & 0 \end{array} \tag{6.17}$$

An arbitrary minimum-weight spanning tree is selected, as shown in Figure 6.7b. By Equation 6.1 the fundamental circuits are (Step 6):

$$B_f = \begin{array}{c|cccccccccccccccc} & a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \tag{6.18}$$

From the matrix B_f of fundamental circuits, the matrix B of all circuits of the graph is obtained (Step 7). From matrix B the matrix B_s of all subgraph may now be computed (Step 8). The complete set of subgraphs is composed by 124 different subgraphs. The subgraphs with mobility $M = 1$ are:

$$B_{M=1} = \begin{array}{c|cccccccccccccccc} & a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \tag{6.19}$$

The unique subchain with mobility $M = 0$ is:

$$B_{M=0} = \begin{array}{c} \\ 1 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p \end{array} \\ \left[\begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{6.20}$$

When the algorithm encounters a subchain with mobility $M_k \leq 0$, it stops and the chain analysed is marked as a degenerated chain. At this step the chain is of no interest for any mechanism design, and no further analysis is needed. Nevertheless, the algorithm is able to calculate the correct degrees of control, connectivity, redundancy and variety. The connectivity matrix is:

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array} \tag{6.21}$$

The degrees-of-control matrix K is equal to the connectivity matrix C and the redundancy matrix R is null. Variety of the kinematic chain is $V = 1$, because the minimum subchain mobility is $M_k = 0$.

Examining Equation (6.21), it is evident that the set of vertices 1-2-3-4-5-6-7-8-9 act as structure, and in term of functionality may be collapsed as suggested in [Shoham and Roth 1997]. The whole kinematic chain, when collapsed, becomes a four-bar chain, preserving the mobility $M = 1$ as expected.

6.6 Further application of the algorithm

The algorithm described in Section 6.3 has been implemented (see Appendix E) and applied [Simoni et al. 2007] to a set of enumerated kinematic chain in order to select, by means of connectivity, the best kinematic structures for robot's hand design.

The degeneracy test performed by the algorithm described in Section 6.3 has been intensively used in [Simoni and Martins 2007, Simoni et al. 2007] as a mean of finding and eliminating degenerated kinematic chains from sets of enumerated chains, in order to produce atlases of chains feasible for mechanisms design.

7 Conclusions

This thesis develops a novel methodology to perform *conceptual analysis* of kinematic chains. Conceptual analysis deals with the determination of the topological characteristics of kinematic chains, *i.e.*

- *degrees of control*
- *connectivity*
- *redundancy*
- *variety*

Conceptual analysis of kinematic chains is a complementary step to the enumeration of kinematic chains, also known as *number synthesis*, a methodology used for at least the past four decades as a means of finding better mechanisms for some predefined purpose. In practice, enumeration can be difficult to implement since the number of kinematic chains generated is often too large to manually consider the individual merits of each chain. For this reason, the concepts of *variety* and *connectivity* can be used to classify kinematic chains according to the constraints required as described in the literature.

In this regard, this work presents the following new results:

- Connectivity calculation methodologies presented in literature have been reviewed, and a counterexample is presented for each of them. For the methodology proposed in [Liberati and Belfiore 2006] an original counterexample is found.
- A redefinition of the concepts of *degrees of control*, *connectivity* and *variety* in an algorithmic form is introduced. These original definitions, which do not conflict with the previous ones found in literature, are built in an algorithmically oriented form and identify a systematic procedure for the calculation of these parameters.

- Based on these new definitions, a further important result has been obtained in this work. The Tischler-Samuel-Hunt conjectures, stating the relation between connectivity and variety, introduced by Tischler *et al.* in 1995 as conjectures lacking formal proof, are herein stated as theorems and formally proved.

- Finally, based on these definitions, a new methodology for the calculation of the main parameter of a kinematic chain, *i.e.* degrees of control, connectivity, redundancy and variety has been proposed and implemented. The new algorithm may be applied to kinematic chains with full mobility (variety $V = 0$) and partial mobility ($V \neq 0$). The full set of connectivities, degrees of control, redundancies and variety is calculated. The algorithm may be easily extended to partial mobility kinematic chains (chains with cut edges or cut vertices) applying the algorithm to their biconnected components. The algorithm here proposed is a valid solution for kinematic chains with a small number of independent loops, otherwise the number of subchains may increase dramatically, and the computational time required to perform the analysis may be excessively long. However, to the author' knowledge, this is the first algorithm that accurately calculates connectivity and redundancy in all cases, without exception.

7.1 Perspectives and further work

Other topics, related to this thesis, are worthy to mention. A list of possible new topics and enhancements of this work is presented herein.

- The improvement of the algorithm efficiency is an important step in order to perform automatic analysis of more complicated chains. In particular, the efficiency of the generation of a complete set of biconnected subgraph should be improved.

- A complete methodology for mechanism design, integrating structural synthesis, type synthesis and dimensional synthesis is still a great challenge. However, several topics could be investigated in order to provide a more general and systematic mechanism design methodology:

  – Define criteria in order to specify customer requirements in terms of structural characteristic of kinematic chains.

  – Define criteria in order to group single-freedom joints as multi-freedom joint, based on customer requirements.

- Publication of atlases of kinematic chains feasible for mechanism design, categorised by structural characteristics. These atlas would be a reference for mechanism designers.

- Extension of the conceptual analysis of kinematic chains to the molecular biology. Molecules present a strict analogy with kinematic chains, and some parameters of molecules, such as mobility and connectivity between components, are very important in the analysis and creation of new proteins. Conceptual analysis developed in this work could be adapted and extended to molecule analysis.

- This work suggested the existence of relation between Assur groups and the structural characteristics of kinematic chains, in particular connectivity and variety. New conceptual analysis methodologies based on Assur groups should be investigated.

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APPENDIX A – Graph Theory

In this section, some fundamental concepts of graph theory [Tsai 2001, Thomas et al. 2001] are introduced. They are essential for topological analysis and number synthesis of mechanisms. It is important to remember that the topology of a mechanism can be uniquely identified by its graph representation, where links and joints of the mechanism are represented, respectively, by the vertices and edges of the graph.

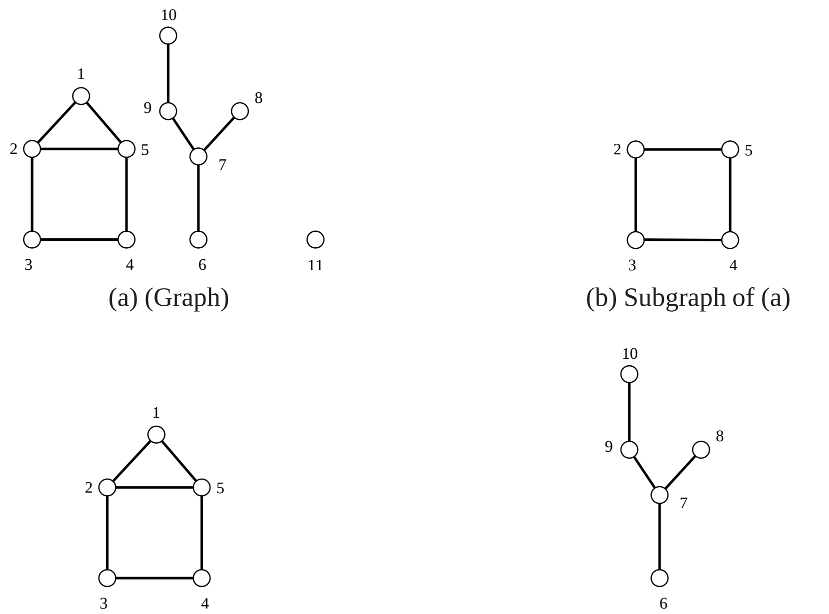
A.1 Definitions

A *graph* consists of a set of vertices (points) with a set of edges or lines. The set of vertices is connected by the set of edges. Let the graph be denoted by the symbol G, the vertex by set V, and the edge by set E. We call a graph with v vertices and e edges a (v, e) graph. Edges and vertices in a graph should be labeled or colored, otherwise they are indistinguishable.

Each edge of a graph connects two vertices called the end points. We specify an edge by its end points; that is, e_{ij} denotes the edge connecting vertices i and j. An edge is said to be incident with a vertex, if the vertex is an end point of that edge. The two end points of an edge are said to be adjacent. Two edges are adjacent if they are incident to a common vertex. For the (11, 10) graph shown in Figure A.1a, e_{23} is incident at vertices 2 and 3. Edges e_{12}, e_{23}, and e_{25} are adjacent.

A.1.1 Degree of a vertex

The *degree of a vertex* is defined as the number of edges incident with that vertex. A vertex of zero degree is called an *isolated vertex*. A vertex of degree two is called a binary vertex, a vertex of degree three a ternary vertex, and so on. For the graph shown in Figure A.1a, the degree of vertex 2 is three, the degree of vertex 10 is one, and vertex 11 is an isolated vertex.



(a) (Graph)

(b) Subgraph of (a)

(c) Component of (a)

(d) A tree that is also component of (a)

Figure A.1

A.1.2 Walks and circuits

A sequence of alternating vertices and edges, beginning and ending with a vertex, is call a *walk*. A walk is called a *trail* if all the edges are distinct and a path if all the vertices and, therefore the edges are distinct. In a path, no edge may be traversed more than once. The *length* of a path is defined as the number of edges between the beginning and ending vertices. If each vertex appears once, except that the beginning and ending vertices are the same, the path forms a *circuit* or *cycle*. For the graph shown in Figure A.1a, the sequence (2, e_{23}, 3, e_{34}, 4, e_{45} , 5) is a path, whereas the sequence (2, e_{23}, 3, e_{34}, 4, e_{45}, 5, e_{52}, 2) is a circuit.

A.1.3 Connected Graphs, subgraphs and components

Two vertices are said to be *connected*, if there exists a path from one vertex to the other. Note that two connected vertices are not necessarily adjacent. A graph G is said to be *connected* if every vertex in G is connected to every other vertex by at least one path. The minimum degree of any vertex in a connected graph is equal to one.

For example, the graph shown in Figure A.1b is connected, whereas the one shown in Figure A.1a is not.

A *subgraph* of G is a graph having all the vertices and edges contained in G. In other words, a subgraph of G is a graph obtained by removing a number of edges and/or vertices from G. The

(a) A multigraph

(b) A graph with a sling

Figure A.2

removal of a vertex from G implies the removal of all the edges incident at that vertex, whereas the removal of an edge does not necessarily imply the removal of its end points although it may result in one or two isolated vertices.

A graph G may contain several pieces, called *components*, each being a connected subgraph of G. By definition, a connected graph has only one component, otherwise it is disconnected. For example, the graph shown in Figure A.1a has three components; the graph shown in Figure A.1b is a subgraph, but not a component of Figure A.1a; whereas the graphs shown in Figures A.1c and A.1d are components of Figure A.1a.

A.1.4 Articulation points and bridges

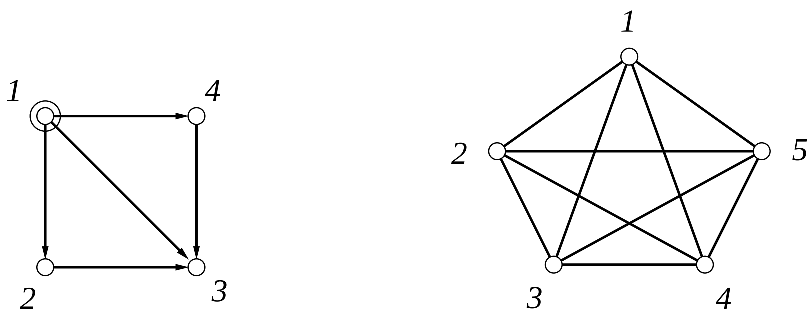
An *articulation point* or *cut point* of a graph is a vertex whose removal results in an increase of the number of components. Similarly, a *bridge* is an edge whose removal results in an increase of the number of components. For the graph shown in Figure A.1a, vertices 7 and 9 are cut points, whereas e_{67}, e_{78}, e_{79}, and $e_{9,10}$ are bridges.

A.1.5 Parallel edges, slings and multigraphs

Two edges are said to be *parallel*, if the end points of the two edges are identical. A graph is called a *multigraph* if it contains parallel edges. A *sling* or *self-loop* is an edge that connects a vertex to itself. Figure A.2a a multigraph, whereas Figure A.2b shows a graph with a sling. A graph that contains no slings or parallel edges is said to be a *simple graph*. In this text, we shall use the term graph to imply a simple graph unless it is otherwise stated.

A.1.6 Directed graph, undirected graph and rooted graph

When a direction is assigned to every edge of a graph, the graph is said to be a *directed graph*. In an *undirected graph* the edge set E consists of unordered pairs of vertices, rather than ordered pairs. A *rooted graph* is a graph in which one of the vertices is uniquely identified from the others. This unique vertex is called the *root*. The root is commonly used to denote the *fixed link* or *base* of a mechanism, and it is symbolically represented by two small concentric circles. Figure A.3a shows a directed graph in which vertex 1 is identified as the root.



(a) A directed graph (b) A complete graph K_5

Figure A.3

A.1.7 Complete graph and Bipartite

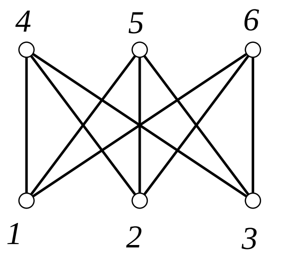
If every pair of distinct vertices in a graph are connected by one edge, the graph is called a *complete graph*. By definition, a complete graph has only one component. A complete graph of n vertices contains $n(n-1)/2$ edges and it is denoted as a K_n graph. Figure A.3b a K_5 graph.

A graph G is said to be a bipartite if its vertices can be partitioned into two subsets, V_1 and V_2, such that every edge of G connects a vertex in V_1 to a vertex in V_2. Furthermore, the graph G is said to be a complete bipartite if every vertex of V_1 is connected to every vertex of V_2 by one edge. A complete bipartite is denoted by $K_{i,j}$, where i is the number of vertices in V_1 and j the number of vertices in V_2. Figure A.4a shows a $K_{3,3}$ complete bipartite.

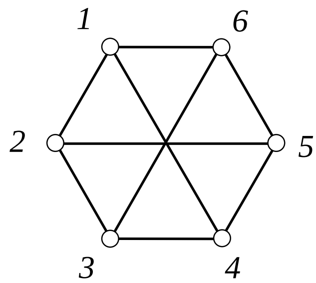
A.1.8 Graph isomorphism

Two graphs, G_1 and G_2 , are said to be isomorphic if there exists a one-to-one correspondence between their vertices and edges that preserve the incidence. It follows that two isomorphic graphs must have the same number of vertices and the same number of edges, and the degrees of the corresponding vertices must be equal to one another. Figure A.4b shows a $(6,9)$

graph that is isomorphic with the $K_{3,3}$ graph shown in Figure A.4a.



(a) A complete bipartite graph $K_{3,3}$



(b) A $(6,9)$ graph isomorphic to graph (a)

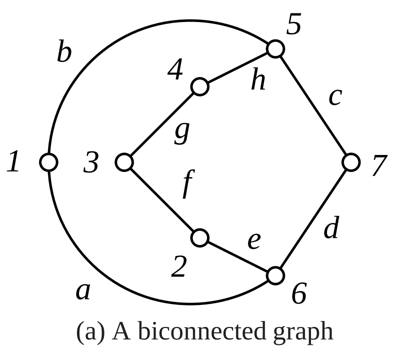
Figure A.4

A.1.9 Biconnected graph

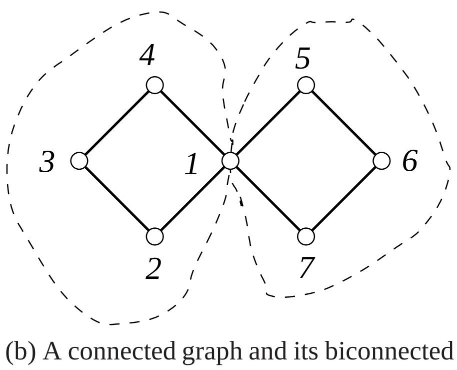
An undirected graph is called a block [Tsai 2001], or is said to be *biconnected* [Manber 1989] if there are at least two vertex disjoint paths from every vertex to every other vertex. A biconnected graph is connected and has no cut points. The minimal degree of a vertex in a biconnected graph is equal to two.

A *biconnected component* is defined as a maximal subset of edges such that its induced subgraph is biconnected (namely, there is no subset that contains it and induces a biconnected graph) [Manber 1989]. A connected graph can be partitioned into biconnected components (in [Manber 1989] an algorithm to find all biconnected components of an undirected subgraph is presented).

Figure A.5a shows a biconnected graph, while Figure A.5b shows a connected graph and its biconnected components with dashed boundaries.



(a) A biconnected graph



(b) A connected graph and its biconnected components with dashed boundaries

Figure A.5

A.2 Tree

A *tree* is a connected graph that contains no circuits. Let T be a tree with V vertices. T has the following properties:

1.Any two vertices of T are connected by one and precisely one path.

2.T contains $(V-1)$ edges.

3.Connecting any two nonadjacent vertices of a tree with an edge leads to a graph with one and only circuit.

Figure A.6 shows a family of trees with six vertices.

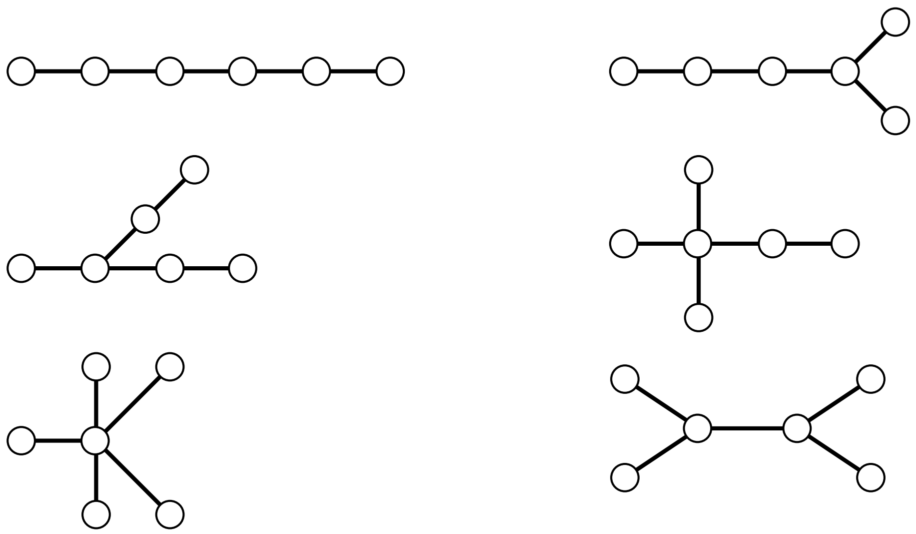


Figure A.6: A family of trees with six vertices

A.3 Planar graph

A graph is said to be *embedded* in a plane when it is drawn on a plane surface such that all edges are drawn as straight lines and no two edges intersect each other. A graph is *planar* if it can be embedded in a plane. Specifically, if G is a planar graph, there exists an isomorphic graph G' such that G' can be embedded in a plane. G' is said to be the planar representation of G. The graph shown in Figure A.7a a planar graph since it can be embedded in a plane as shown in Figure A.7a. However, the complete graph shown in Figure A.3b and the bipartite graph in Figure A.4a are not planar. Planar representation of a graph divides the plane into several connected regions, called *loops* or *circuits*. Each loop is bounded by several edges of the graph. The region external to the graph is called the *external loop* or *peripheral loop*. For example, Figure A.8 shows a planar graph with four loops (including the peripheral loop).

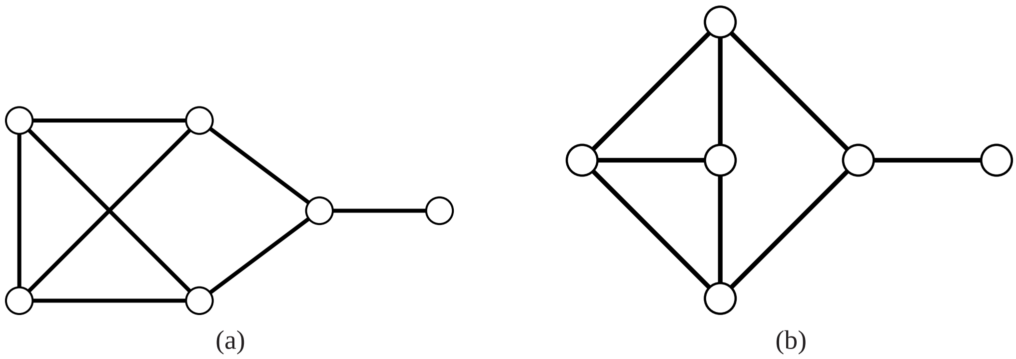


Figure A.7: A graph and its planar embedding

A.4 Spanning trees and fundamental circuits

A *spanning tree*, T, is a tree containing all the vertices of a connected graph G. Clearly, T is a subgraph of G. Corresponding to a spanning tree, the edge set E of G can be decomposed into two disjoint subsets, called the *arcs* and *chords*. The arcs of G consist of all the elements of E that form the spanning tree T , whereas the chords consist of all the elements of E that are not in T . The union of the arcs and chords constitutes the edge set E.

In general, the spanning tree of a connected graph is not unique. The addition of a chord to a spanning tree forms one and precisely one circuit. A collection of all the circuits with respect to a spanning tree forms a set of *independent loops* or *fundamental circuits*. The fundamental circuits constitute a basis for the circuit space. Any arbitrary circuit of the graph can be expressed as a linear combination of the fundamental circuits using the operation of modulo 2, i.e., $1+1=0$.

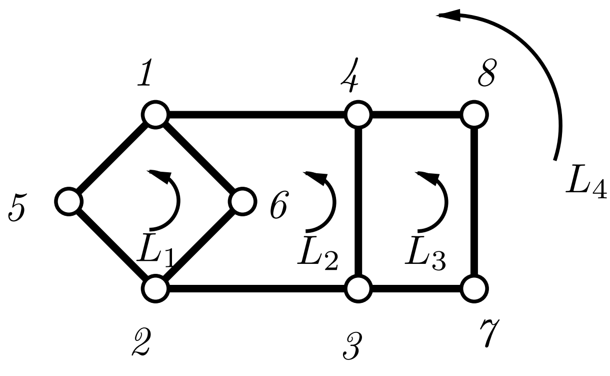
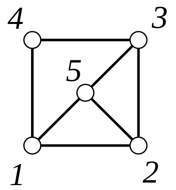
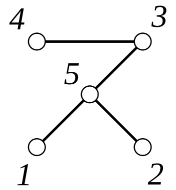


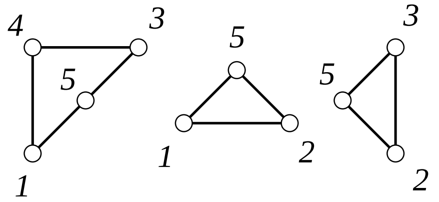
Figure A.8: A planar graph



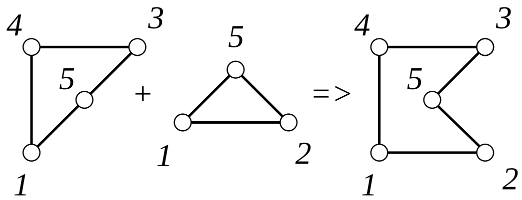
(a) A $(5,7)$ graph



(b) Spanning Tree T of (a)



(c) Fundamental set of circuits of (a) with respect to T (b)



(d) Linear combination of fundamental circuits

Figure A.9: A spanning tree and the corresponding fundamental circuits

Figure A.9a shows a $(5,7)$ graph G, Figure A.9b a spanning tree T, and Figure A.9c a set of fundamental circuits with respect to the spanning tree T . The arcs of G consist of edges e_{15}, e_{25}, e_{34}, and e_{35}. The chords of G consist of e_{12}, e_{23}, and e_{14} . Figure A.9d a circuit obtained by a linear combination of two fundamental circuits.

A.5 Euler's equation

Let L denote the number of independent loops of a planar connected graph and $\tilde{L}$ represent the total number of loops. Then

$$\tilde{L} = L + 1 \tag{A.1}$$

Euler's equation, which relates to the number of vertices, the number of edges, and the number of loops of a planar connected graph can be written as

$$\tilde{L} = E - V + 2 \tag{A.2}$$

In terms of the number of independent loops, we have

$$L = E - V + 1 \tag{A.3}$$

A.6 Matrix representation of graph

The topological structure of a graph can be conveniently represented in matrix form. In this section, we introduce a few frequently used matrix representations of graph. The matrix representation makes analytical manipulation of graphs on a digital computer feasible. It leads to the development of systematic methodologies for identification and enumeration of graphs.

A.6.1 Adjacency matrix

To facilitate the study, the vertices of a graph are labeled sequentially from 1 to V. A vertex-to-vertex adjacency matrix, A_j, is defined as follows:

$$A_j[i,j] = \begin{cases} 1, & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0, & \text{otherwise (including } i = j) \end{cases} \tag{A.4}$$

where $A_j[i,j]$ denotes the (i,j) element of A_j. It follows that A is a $V \times V$ symmetric matrix having zero diagonal elements. Each row (or column) sum of A_j corresponds to the degree of a vertex. Given a graph, the adjacency matrix is uniquely determined. On the other hand, given an adjacency matrix, one can construct the corresponding graph. Hence, the adjacency matrix identifies graphs up to graph isomorphism. For example, Figure A.10 shows a graph with both vertices and edges labeled sequentially. Further, vertex 1 is identified as the root. The adjacency matrix is

$$A_j = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \tag{A.5}$$

Clearly, the adjacency matrix depends on the labeling of vertices. If A_1 and A_2 are the adjacency matrices of a graph with two different labelings of the vertices, it can be shown that

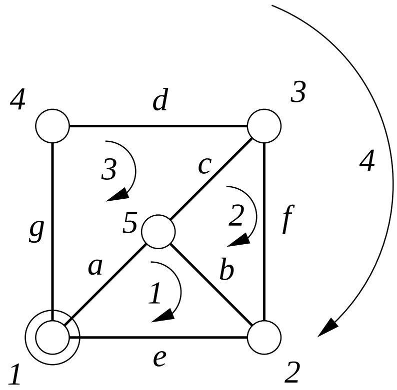


Figure A.10: A labeled graph with labeled circuits.

there exists a permutation matrix P such that

$$A_1 = P^{-1} A_2 P \tag{A.6}$$

A.6.2 Incidence Matrix

The vertices of a graph are labeled sequentially from 1 to V and the edges are labeled from 1 to E. An incidence matrix, A_i, is defined as a $V \times E$ matrix in which each row corresponds to a vertex and each column corresponds to an edge.

$$A_i = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ V \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & E \end{array} \\ \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,E} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,E} \\ \vdots & \vdots & \vdots & \vdots \\ a_{V,1} & a_{V,2} & \cdots & a_{V,E} \end{bmatrix} \end{array} \tag{A.7}$$

where

$$a_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is an end vertex of edge } j \\ 0, & \text{otherwise} \end{cases} \tag{A.8}$$

Since each edge has two end vertices, there are exactly two nonzero elements in each column. Hence, the sum of each column is always equal to 2, whereas the sum of each row is equal to the degree of a vertex. Similar to an adjacency matrix, the incidence matrix determines a graph up to graph isomorphism. For example incidence matrix of the labeled graph shown in Figure A.10 given by

$$A_i = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{A.9}$$

A.6.3 Circuit Matrix

The circuits of a graph are labeled sequentially from 1 to l and the edges are labeled from 1 to E. A circuit matrix, B is defined as an $l \times E$ matrix in which each row corresponds to a circuit and each column denotes an edge.

$$B = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ l \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & E \end{array} \\ \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,E} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,E} \\ \vdots & \vdots & \vdots & \vdots \\ b_{l,1} & b_{l,2} & \cdots & b_{l,E} \end{bmatrix} \end{array} \tag{A.10}$$

where

$$B[i,j] = \begin{cases} 1, & \text{if circuit } i \text{ contains edge } j \\ 0, & \text{otherwise} \end{cases} \tag{A.11}$$

Obviously, those edges that do not lie on any circuit do not appear in the circuit matrix. Hence, the circuit matrix does not provide complete information about a graph. Unlike the adjacency and incidence matrices, the circuit matrix does not determine a graph up to isomorphism. For example, Figure A.10 a graph with labeled circuits. Its circuit matrix is

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \tag{A.12}$$

The row vectors of B are not necessarily independent. For a connected graph G, the number of independent circuits is given by Euler's equation. Corresponding to a given spanning tree, each chord uniquely defines a fundamental circuit. The set of circuits determined from all the chords of G constitutes a basis for the circuit space. Any other circuits can be expressed as a linear combination of the base vectors with the arithmetic of modulo 2. For the above example, we observe that the last row of B is equal to the sum of the first three rows. The fundamental circuit matrix B_f contains represents only a set of fundamental circuits.

A.7 Graph algorithms

Graphs are a pervasive data structure in computer science, and algorithms for working with them are fundamental to the field. There are hundreds of interesting computational problems defined in terms of graphs. In this part, we touch on a few of the more significant ones [Cormen et al. 2001, Manber 1989], which have been used in the algorithm described in Chapter 6 at page 63.

A.7.1 Breadth-first Search

Breadth-first search is one of the simplest algorithms for searching a graph and the archetype for many important graph algorithms. Given a graph $G = (V,E)$ and a distinguished source vertex s, breadth-first search systematically explores the edges of G to "discover" every vertex that is reachable from s. It computes the distance (smallest number of edges) from s to each reachable vertex. It also produces a "breadth-first tree" with root s that contains all reachable vertices. For any vertex v reachable from s, the path in the breadth-first tree from s to v corresponds to a "shortest path" from s to v in G, that is, a path containing the smallest number of edges. The algorithm works on both directed and undirected graphs. The total running time of Breadth-first search is $O(V+E)$.

Shortest paths

The *shortest-path distance* (s,v) from s to v is defined as the minimum number of edges in any path from vertex s to vertex v; if there is no path from s to v, then $\delta(s,v) = \infty$. A path of length (s,v) from s to v is said to be a *shortest path* from s to v.

A.7.2 Depth-first Search

The strategy followed by *depth-first search* is, as its name implies, to search ”deeper” in the graph whenever possible. In depth-first search, edges are explored out of the most recently discovered vertex v that still has unexplored edges leaving it. When all of v's edges have been explored, the search ”backtracks” to explore edges leaving the vertex from which v was discovered. This process continues until we have discovered all the vertices that are reachable from the original source vertex. If any undiscovered vertices remain, then one of them is selected as a new source and the search is repeated from that source. This entire process is repeated until all vertices are discovered. The running time of Depth-first search is $O(V+E)$.

Biconnected components

A classic application of depth-first search is the decomposing a directed graph into its biconnected components, as defined in Section A.1.9. Many algorithms that work with directed graphs begin with such a decomposition. The same algorithm may be used as biconnectivity test: given a graph G verify if it has only one biconnected component, *i.e.* if it is biconnected. The running time of biconnected components algorithm search is $O(V+E)$.

A.7.3 All shortest paths algorithm

All shortest paths algorithm deals with the problem of finding shortest paths between all pairs of vertices in a graph. This problem might arise in making a table of distances between all pairs of cities for a road atlas. Given a weighted, directed graph $G=(V,E)$ with a weight function $w : E \rightarrow \mathbf{R}$ that maps edges to real-valued weights. We wish to find, for every pair of vertices u, v of V, a shortest (least-weight) path from u to v, where the weight of a path is the sum of the weights of its constituent edges. The output is typically arranged in a $V \times V$ matrix, the *minimum distance matrix* D_{min}, as

$$D_{min} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ V \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & V \end{array} \\ \begin{bmatrix} d_{1,1} & b_{1,2} & \cdots & d_{1,V} \\ d_{2,1} & d_{2,2} & \cdots & d_{2,V} \\ \vdots & \vdots & \vdots & \vdots \\ d_{V,1} & d_{V,2} & \cdots & d_{V,V} \end{bmatrix} \end{array} \tag{A.13}$$

where $d_{i,j}$ is the weight of a shortest path from u to v. All shortest paths problem may be solved by two algorithms: Floyd-Warshall algorithm and the Johnson algorithm. Floyd-Warshall algorithm runs in $O(V^3)$ time, while Johnson solves the all-pairs shortest paths problem in $O(V^2 lgV + VE)$ time, which makes it a good algorithm for large, sparse graphs.

APPENDIX B – Screw theory

Two theorems are fundamental in screw theory: Poinsot's theorem and Mozzi's theorem [Martins and Guenther 2003, Erthal 2007].

Theorem 5. *[Poinsot 1806] Given a set of forces and pure couples applied to a rigid body it is always possible to find a line along which the resultant force and the resultant moment vectors will be directed.*

The Poinsot theorem states that any load on a body can be represented by a force along a certain fixed axis plus a moment parallel to the same axis.

Theorem 6. *[Mozzi 1763] The velocities of the points of a rigid body at any instant are what they would be if the body were rotating about a certain fixed axis and simultaneously had a motion of translation along this axis.*

The Mozzi theorem states that the velocities of the points on a rigid body with respect to an inertial reference frame $O(X,Y,Z)$ may be represented by a differential rotation about a certain fixed axis and a simultaneous differential translation along the same axis.

Definition 13. *A screw \$ is a geometrical entity composed by a line and a number h called pitch which has length dimension.*

Any physical quantity that requires a line of action and a pitch can represented by a screw: consequently, by the theorems of Mozzi and Poinsot, movements and forces (or couples) can be represented by screws. A screw which represents a movement is called *twist*, a screw which represents a force (or a couple) is called a *wrench*.

A screw is completely described by Plücker line coordinates. A line can be expressed in Plücker coordinates as a vector:

$$\$^l = \begin{bmatrix} u \\ v \end{bmatrix} \tag{B.1}$$

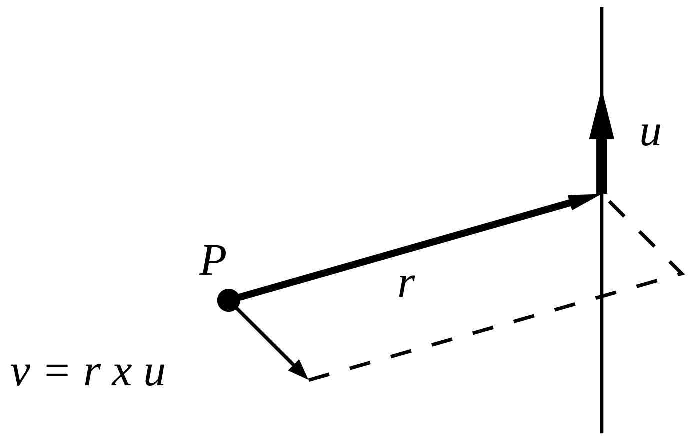


Figure B.1: Plücker line coordinates

where u represents the direction vector of the line and $v = r \times u$ is the "moment" of the line with respect to a point P, as shown in Figure B.1.

A screw in Plücker coordinates is :

$$\$ = \begin{bmatrix} s \\ s_0 \times s + hs \end{bmatrix} \tag{B.2}$$

where s a unit vector along the direction of the screw axis, s_0 is a position vector of any point located on the screw axis and h is the pitch of the screw, as in Figure B.2.

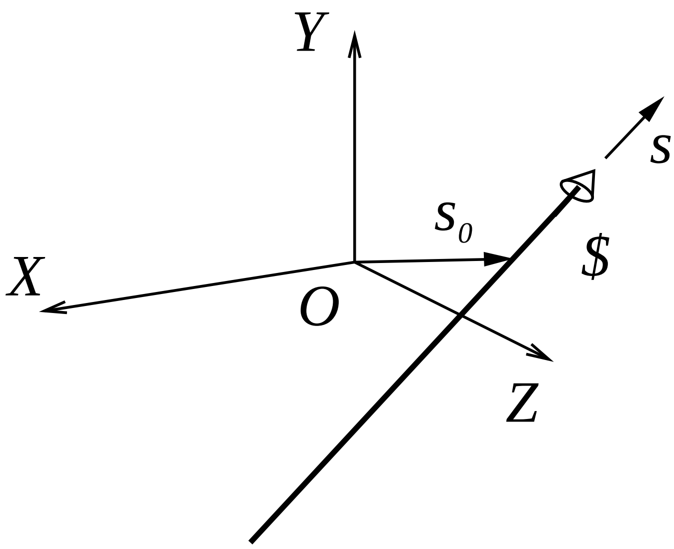


Figure B.2: A screw in Plücker coordinates

The notion of pitch h is associated with the relationship between both quantities along the screw axis. In kinematics, the pitch of the twist is given by

$$h = v_t / \omega$$

where v_t is the translational velocity and ω is the angular velocity. In statics, the pitch of the wrench is given by

$$h = C/F$$

where C is the couple and F is the force.

Twists can be easily calculated for certain common robotic joints. For a revolute joint ($v_t = 0$), the twist can be calculated as:

$$\$ = \begin{bmatrix} \omega \\ s_0 \times \omega \end{bmatrix} \tag{B.3}$$

For a prismatic joint ($\omega = 0$), the twist can be calculated as:

$$\$ = \begin{bmatrix} 0 \\ v_t \end{bmatrix} \tag{B.4}$$

By equation B.1, screws are represented by a vector with six independent coordinates The order λ of the screw system determines which coordinates are different from zero. In the general case ($\lambda = 6$), twists and wrenches are represented by the screws:

$$\$ = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{bmatrix} \qquad \textit{and} \qquad \$ = \begin{bmatrix} M_x \\ M_y \\ M_z \\ F_x \\ F_y \\ F_z \end{bmatrix}$$

In the planar case, or three-system ($\lambda = 3$), movements and forces (or couples) are planar. Consequently just the coordinates w_z, v_x, v_y, M_z, F_x and F_y are different from zero, and twists and wrenches are represented by the screws:

$$\$ = \begin{bmatrix} \omega_z \\ v_x \\ v_y \end{bmatrix} \qquad \textit{and} \qquad \$ = \begin{bmatrix} M_z \\ F_x \\ F_y \end{bmatrix}$$

The two system, $\lambda = 2$, is used to describe gear trains mechanism [Tsai 2001]. The three-system, $\lambda = 3$, is used for the general planar motion [Freudenstein and Maki 1984, Tsai 2001] and for spherical motion, *i.e.* robot wrist mechanism [Tsai 2001]. Tischler [Tischler 1995] enumerates kinematic chains belonging to the four-system $\lambda = 4$. The six-system $\lambda = 6$ is used for the general spatial motion [Tsai 2001, Tischler 1995]. Screw systems have been exhaustively treated in [Hunt 1978, Gibson and Hunt 1988]. Applications of different screw systems to robot design is described in [Davidson and Hunt 2004].

Figure B.1 briefly describes the main screw systems used in mechanisms.

λ	**Name**	**Application**
2	The *Two-system*	Gear trains
3	The *Three-system*	General planar motion
		Spherical motion
4	The *Four-system*	Schonflies motion
5	The *Five-system*	Constrained spatial motion
6	The *Six-system*	General spatial motion

1.5

Table B.1: Screw systems used in mechanisms

APPENDIX C – The selection of kinematic chains

In this section, an application [Tischler et al. 2001] of the methodology of mechanism design introduced in Section 1.1.2 at Page 5 is presented.

Tischler *et al.* [Tischler et al. 1998] present the design for a dextrous robot finger shown in Figure C.1. The movement of the finger-tip relative to the base is controlled by three actuated P-pairs as marked by the heavy arrows. The actuators govern the z-coordinates of points A', B', and C' above the xy-plane, namely z_a, z_b, and z_c. The position of the finger-tip is related to the actuator displacements by quartic polynomials [Tischler et al. 1998], but linear combinations of the z_i's, as set out in Equation C.1, gives quadratic polynomials relating the finger-tip position to the new finger inputs δ, ζ_c and ζ_0.

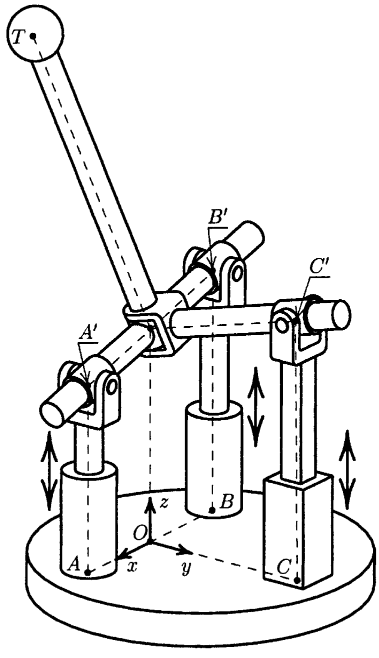


Figure C.1: A dextrous robot finger [Tischler et al. 2001]. The design of a differential transmission for driving the linear inputs of this finger exemplifies the application of variety for selecting appropriate kinematic forms

$$\left.\begin{aligned} \delta &= z_a - z_b \\ \zeta_c &= z_c - \tfrac{z_a+z_b}{2} \\ \zeta_0 &= \tfrac{z_a+z_b}{2} \end{aligned}\right\} \tag{C.1}$$

Now the path of the finger-tip can be planned more simply in increments of δ, ζ_c and ζ_0. However, to actuate the finger, these quantities must be transformed back into z_a, z_b, and z_c. Though this conversion is simple it has to be performed frequently as the finger moves. The simplicity of the conversion suggests that a mechanical transmission could do it, and the form this transmission must take is herein investigated. The inverted relations of Equation C.1, namely

$$\begin{bmatrix} z_a \\ z_b \\ z_c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta \\ \zeta_0 \\ \zeta_c \end{bmatrix} \tag{C.2}$$

(where δ, ζ_0 and ζ_c are the inputs and z_a, z_b, and z_c are the outputs) are to be achieved mechanically.

An epicyclic gear train seems one way of implementing this transmission because Equation C.2 is independent of the configuration of the transmission, (*i.e.* , the gear ratios need to be constant) and the rotations of the output shafts may be easily converted into linear displacements by connecting them to lead screws.

In order to enumerate kinematic chains by means of number synthesis, some properties of the kinematic chains must be specified: the nature of motion, *i.e.* the order of the screw system to which all joint belong, the mobility or number of degrees of freedom M and the number of loops v.

Gear trains using only spur or helical gears instantaneously satisfy the geometrical conditions of the IIB_2 two-system [Gibson and Hunt 1990]. Consequently, the order of the screw system is $\lambda = 2$. Since Equation C.2 has three independent input variables (i.e., δ, ζ_c, and ζ_0), the epicyclic gear train we seek must have mobility $M = 3$.

The fewer independent loops in the linkage, the simpler the kinematic chain, so small values of v are preferred. To fix the relative disposition of the three inputs and three outputs, one body of the chain must have at least six connections which will then be taken as the fixed body in the linkage because then the axes of the input and output shafts remain in fixed positions in the gear-box housing.

The degree of a body is equal to the number of incident joints and the maximum degree, d_{max}, of any single body in a closed kinematic chain is 2ν. Therefore no chain with fewer than three loops contains a body with six incident joints. However, chains with $(d_{max} - 2) > (\nu - 1)$ are necessarily fractionated [Tsai 2001]. A fractionated kinematic chain has two or more sections which are kinematically independent. Since Equation C.2 cannot be rearranged into independent parts, those sets of kinematic chains which produce only fractionated chains with bodies of degree six must be discarded. Therefore, no realistic solutions can be expected to the problem unless $\nu \geq 5$.

The simplest set of kinematic chains likely to contain a solution is one for which $\lambda = 2$, $M = 3$, and $\nu = 5$. With these values the general mobility criterion shows that each kinematic chain in the set consists of $n = 9$ links and $g = 13$ single-freedom joints. The enumeration of all kinematic chains satisfying the structural requirements produces a set of 2271 chains, a number too large to contemplate examination of each chain individually. Fortunately, variety is used to select a much smaller and manageable number.

C.1 Matching the function and variety in kinematic chain

To select a kinematic chain on the basis of its variety V, the desired motion of the mechanism, encapsulated within Equation C.2, must be studied in greater detail. The variety V of the kinematic chain may be only in the range $0 \leq V < M$; since $M = 3$, values of $V = 0, 1, 2$ need to be considered.

It has been stated above that the form of the input-output equations preclude the use of fractionated chains because of their interdependency. Equation C.2 also shows that no single output depends upon only one input, and hence an input-freedom and an output-freedom cannot belong to a subchain of joint freedoms which have mobility $M' = 1$. (The prime is to denote the mobility corresponding to a subchain of the kinematic chain.) Also, no two output-freedoms can be within a subchain of the chain with mobility $M' = 1$, since they cannot then be independent. Moreover, two input-freedoms cannot be placed within a subchain of mobility $M' = 1$, as they would then be unable to displace independently of one another. Accordingly, chains with subchains having mobility $M' = 1$ are inappropriate in all cases, and all kinematic chains with variety $V = 2$ can be safely discarded.

Equation C.2 shows that each output depends on no more than two of the three inputs. To prevent an input from influencing an output, the output together with its two inputs can be placed in a subchain of the chain with mobility $M' = 2$, and the third input somewhere outside this subchain. The two inputs inside the subchain fully govern the all movement of the subchain, including that of the output. Any chain containing a subchain of $M' = 2$ has variety $V = 1$. For chains of variety $V = 0$, every output is a function of every input because there are no subchains of loops with reduced mobility and the freedoms of the chain cannot be partitioned. But the freedoms are partitioned in Equation C.2; $V = 0$ can therefore be discounted. Only the kinematic chains of variety $V = 1$ remain to be enumerated, and hence the work of examining the chains generated is significantly reduced.

Above it was shown that all chains with $v < 5$ that contain a body of degree $d_{max} = 6$ are fractionated. Nevertheless, some chains with $v = 5$ will be fractionated too. Another theorem in Tischler *et al.* [Tischler et al. 1995] shows that a body-fractionated chain must be of variety $V \geq \frac{1}{2}M$ and, since $M = 3$ for this set of chains, all body-fractionated chains have $V = 2$ and so have already been discarded. Joint-fractionation is similarly prevented.

C.2 Identifying appropriate chains by means of variety

Among the 2271 kinematic chains enumerated in the set required, only five have variety $V = 1$, and the other 2266 can be discarded as completely unsuitable.

Figure C.2 shows schematically the five $V = 1$ chains in a form which assists in discussing how they work. Figure C.2 shows also a graph representation of each kinematic chain. In [Tischler et al. 2001] the kinematic chains were enumerated by means of a variation of Farrel's method [Tischler 1995], without graph representation. Consequently variety was determined by visual inspection for each one of the kinematic chain.

When the mechanism is given a physical form, each set of three adjacent freedoms must lie within an IIB_2 two-system, *i.e.* , parallel and coplanar, zero-pitched screws.

For chains (a), (c), (e), (g), (i) there is only one choice for the grounded body, namely the body of degree six; chain (i) has two bodies of degree six, the whole chain being bilaterally symmetric about a horizontal line, and so both choices for the grounded body are equivalent. The six joints on the grounded body must be divided into two groups, three for inputs and three for outputs. First, those subsets of one or more loops with mobility $M' = 2$ must be identified, including as much of the mechanism as possible in any subset. Candidates (a), (g) each have two sub-chains of mobility $M' = 2$, (i) has just one sub-chain; the extent of these sub-chains is marked by the dashed lines. In candidates (a) and (i) the two sub-chains overlap by one joint, while in (e) and (g). they are distinct.

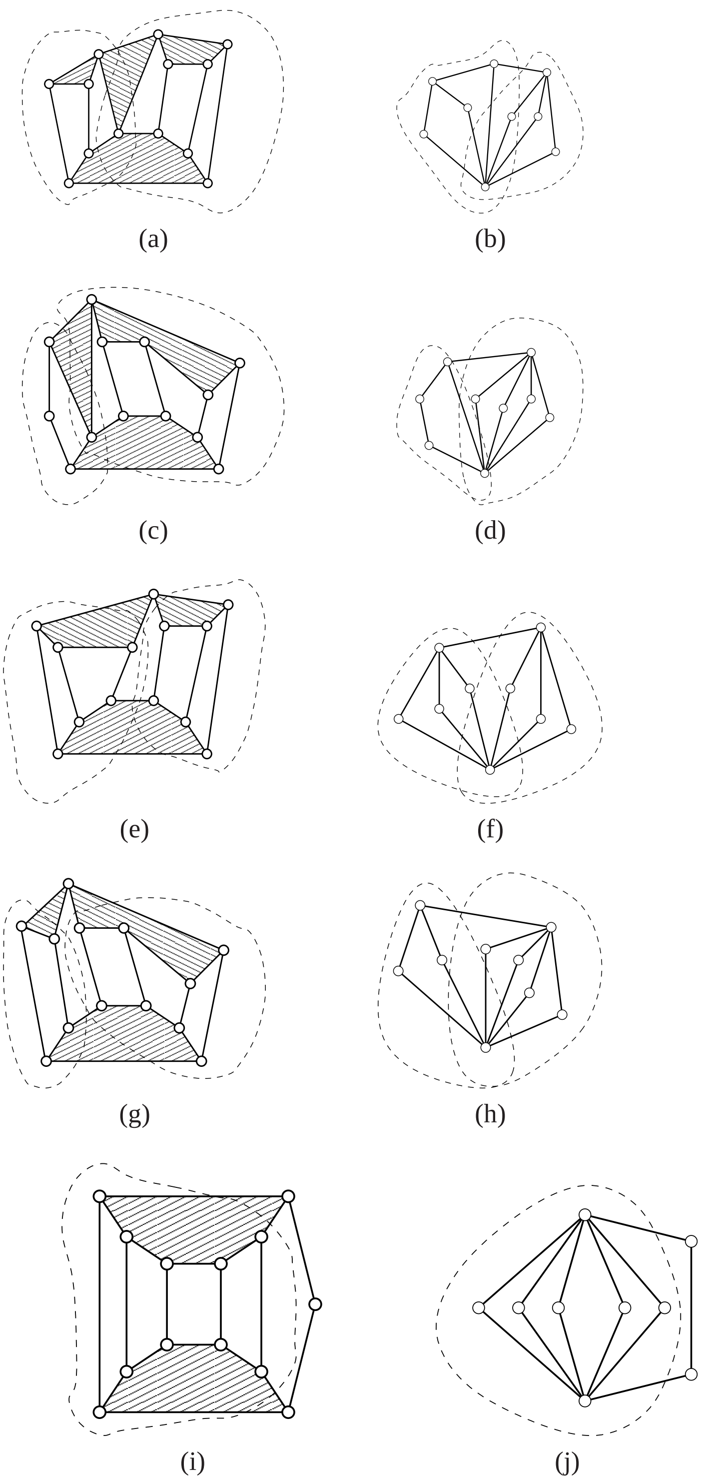


Figure C.2: The five kinematic chains and their correspondent graphs, enumerated with $\lambda = 2$, $M = 3$, $\nu = 5$, and $V = 1$. The component sub-chains within the dashed boundaries have mobility $M' = 2$.

Equation (C.2) leads to a 'best' choice from the five candidates. Each output is dependent on two inputs, and consequently each output must lie within a sub-chain of the linkage with $M' = 2$. While z_a and z_b depend on the same two inputs and can be placed in the same subchain,

z_c does not depend on δ and must be placed in a different subchain. Two sets of sub-chains are therefore required and chain (i) is discarded. Equation (C.2) shows that ζ_0 affects every output, and so its freedom must belong to every subchain of mobility $M' = 2$, and the two sub-chains need to overlap. Hence chains (e) and (g) are also discarded. Input ζ_c influences z_c only so, since ζ_0 must also be retained, one of the $M' = 2$ regions needs three connections to the base. Chain (a) is the only chain that has this feature. Since the input δ influences outputs z_a and z_b, these three must be placed in a mobility $M' = 2$ subchain with ζ_0, and four connections to the base are required. Fortunately candidate (a) contains such a subchain and each input and output has been matched to one of the freedoms of this chain, as shown in Figure C.3.

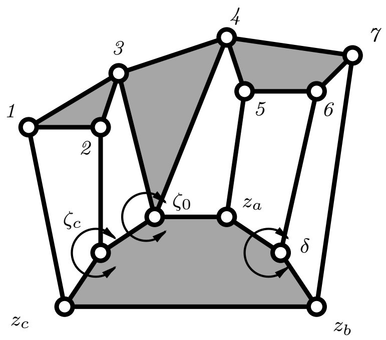


Figure C.3: The kinematic chain which best matches the predefined equations of motion

Figure C.3 is the only kinematic chain in this set of 2271 which is feasible. No other candidate matches the desired input/output equations, and the notion of variety has been the key to eliminating the bulk of them. Although the linkage still needs to be given a physical form and proper proportions, the kinematic skeleton for the design has been found. A physical representation of the kinematic chain of Figure C.3 as an epicyclic gear train is showed in Figure C.4.

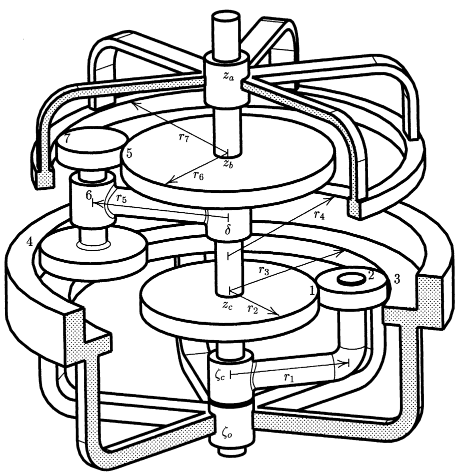


Figure C.4: A physical representation of the kinematic chain of Figure C.3 as an epicyclic gear train

APPENDIX D – Liberati and Belfiore algorithm [Liberati and Belfiore 2006]

The algorithm for automatic calculation of connectivity, proposed in [Liberati and Belfiore 2006], is based on the following steps :

Steps 1-11 : *Preliminary study of the given kinematic chain*

1. Build the graph G corresponding to the kinematic chain to be analysed with n members and g joints, let $N = n$
2. Build the adjacency matrix A of the graph G; copy A into a matrix A'
3. For each pair of vertices $i - j$ of G, evaluate their mutual distance $D_{\min}[i, j]$.
4. Build a matrix $D(n \times n)$ whose element $i - j$ is equal to $D_{\min}[i, j]$.
5. Calculate the number of independent circuits using the equation: $L_{ind} = 1 - n + g$; set $L''_{ind} = L_{ind}$
6. Calculate the degree of freedom, M, of the whole kinematic chain using the equation: $M = g - \lambda L_{ind}$, with $\lambda = 3$ in the plane and $\lambda = 6$ in the spatial motion.
7. Evaluate a set of independent circuits of the kinematic chain based on a minimum weight-spanning tree, having assigned a unitary value to the edge weights of the graph. Detect the cycles having the lowest length $q_{\min}$;
8. Copy G into a graph G'.
9. If $q_{\min} < M + \lambda$, the kinematic chain has partial mobility and go to step 11.
10. If $q_{\min} \geq M + \lambda$, then we do not know if the kinematic chain has partial mobility or not.

11.Build a set C of independent circuits of the kinematic chain.

Steps 12-19 : *Main recursive procedure, necessary to freeze iteratively all the subchains of the given kinematic chain*

12.For each element c_k of C, perform the following steps, starting from the independent circuits with length $q_{\min}$

12.1List all vertices of the circuit c_k: evaluate the mobility $M_k = q - \lambda$; let $N' = N$.

12.2Build the graph G_k corresponding to the circuit c_k; copy c_k into $c_k^{'}$, $c_k^{'} = c_k$.

12.3Add the lines and the columns of the matrix A' relative to the vertices of $c_k^{'}$, using the Modulo-2 algebra: $0+1 = 1+0 = 1$ and $0+0 = 1+1 = 0$.

12.4Add the resultant column and the resultant line at the matrix $A'[(N'+1)] \times (N'+1)]$ and set the element $[N'+1, N'+1] = 0$. Set to zero all the elements of the columns and lines corresponding to the vertices of $c_k^{'}$. (*This step help us in the implementation of the algorithm using an algebraic programming language.*

12.5Find the graph corresponding to the adjacency matrix A', where the column $N'+1$ corresponds to the vertex L_{MK}; Calculate the mobility M' of the new graph and if $M' > 0$, then the kinematic chain has partial mobility; calculate $N' = N'+1$.

12.6Calculate a set of independent circuits $L_{ind}^{'}$ of the new graph, and find the circuit $c_k^{'}$ with minimum length $q_{\min}^{'}$, and having as element the vertex L_{MK}. ($L_{ind}^{'}$ *is the number of independent circuits after each freezing. This must be compared with* $L_{ind}^{''}$ *that represents the number of independent circuits before freezing. Three possible cases can occur: 12.7 12.8 12.9*

12.7Case $L_{ind}^{'} \neq L_{ind}^{''} - 1$ then

$q_{\min}^{'} = 2$; add the vertex of $c_k^{'}$ to G_k and calculate $M_k = M_k - (\lambda - 2)$ and $L_{ind}^{''} = L_{ind}^{''} - 1$; go to step 12.3

12.8Case $L_{ind}^{'} = L_{ind}^{''} - 1 \geq 1$ then

12.8.1If $q_{\min}^{'} = \lambda$, then add the vertices to G_k; let $L_{ind}^{''} = L_{ind}^{'}$; go to step 12.3.

12.8.2If $q_{\min}^{'} > \lambda$:

12.8.2.1If $M_k \geq M$, then add the vertices to G_k: calculate $M_k = M_k + (q_{\min}^{'} - \lambda)$ and $L_{ind}^{''} = L_{ind}^{'}$; go to step 12.3

12.8.2.2If $M_k < M$, then lay $M_k^{'} = M_k$ and $L_{ind}^{''} = L_{ind}^{'}$; the kinematic chain has partial mobility; continue from step 13

12.8.3If $q^{'}_{\min} < \lambda$, add the vertices to G_k; calculate $M_k = M_k - (\lambda - q^{'}_{\min})$ and $L^{''}_{ind} = L^{'}_{ind}$; go to 12.3.

12.9Case $L^{'}_{ind} = L^{''}_{ind} - 1 = 0$, then $M^{'}_k = M_k$ and skip to the step 13

13.Build the complete graph KG_k of G_k.

14.For each edge $t-h$ of KG_k, perform the two following steps:

15.Find the pair of vertices $r-s$ of G that corresponds to the ends of the edge $t-h$ of KG_k.

16.If $M^{'}_k < D[r,s]$ then add to $G^{'}$ a virtual edge $r-s$ with a weight equal to $M^{'}_k$; In the matrix D, replace $D[r,s] = M^{'}_k$. *(In this step, the initial matrix D is updated with weights of virtual edges)*

17.If $L^{'}_{ind} \geq 1$, calculate $M_k = M_k + (q^{'}_{\min} - \lambda)$; add vertices to G_k and go to step 12.3. *(the given kinematic chain has not been completely frozen.)*

18.If $L^{'}_{ind} = 0$, then perform the following independent circuit of the initial set C, go to step 12.1, else continue. *(The given kinematic chain has been completely frozen.)*

Steps 20-23 : *After having frozen all the subchains of the given kinematic chain, the matrix of connectivity is finally computed*

19Copy the graph $G^{'}$ into a new graph $G^{''}$

20For each pair of vertices $i-j$ of $G^{''}$, evaluate their mutual distance $S_{\min}[i,j]$.

21Build the matrix S whose element $i-j$ is equal to $S_{\min}[i,j]$.

22Build the connectivity matrix C, in such a way that for each element, $C[i,j]$ is equal to $C[i,j] = S_{\min}[i,j]$ if $S_{\min}[i,j] \leq \lambda$, otherwise $C[i,j] = \lambda$ if $S_{\min}[i,j] > \lambda$.

As an example of application, consider the kinematic chain with mobility $M = 2$, shown in Figure D.1a, and its corresponding graph, Figure D.1b.

The subgraph composed of vertices $1,2,3,4$ is a circuit of length $q = 4$. Considering a unitary weight for any edge, the mobility of this subgraph is $M = q - \lambda = 1$; thus only one actuator is necessary to control the subchain. By adding the actuator and blocking the circuit a *frozen* loop results. A frozen loop can be seen as a single new member and, according to the algorithm, is represented by the vertex L_1 belonging to a novel reduced graph, Figure D.2.

The reduced graph now needs another actuator in order to be fully controlled, since the mobility of the whole kinematic chain is $M = 2$. At this point, it is possible to affirm that the

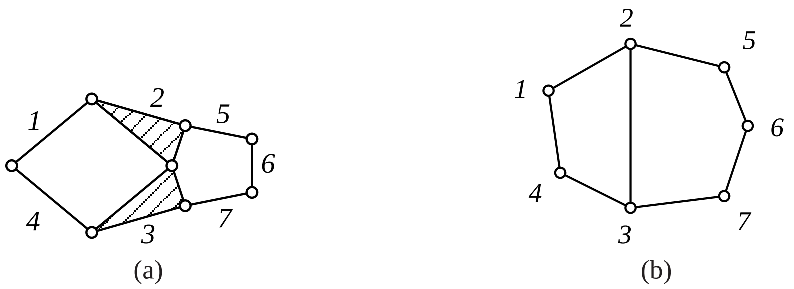


Figure D.1: Kinematic chain with partial mobility and corresponding graph: the mobility of the chain is $M = 2$, but subchain $1-2-3-4$ has mobility $M^{'} = 1$: structural representation (a) and its graph (b)

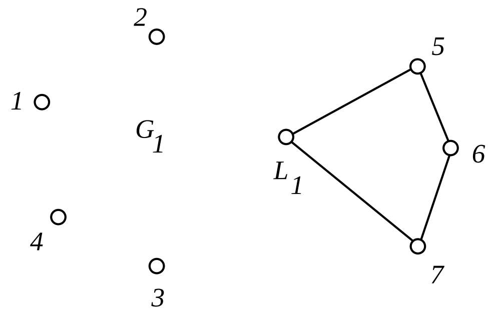


Figure D.2: The reduced graph obtained applying the *freezing of circuits*

initial kinematic chain has partial mobility. In fact, the subchain composed of links $1,2,3,4$ is a four-bar subchain with mobility $M_k = 1$, less than $M = 2$, controlled by only one actuator. Hence, the freezing of one circuit has helped to recognise the partial mobility and to identify a subchain with mobility $M_k < M$.

APPENDIX E – Algorithm implementation

The algorithm described in Section 6 has been implemented in C++. For graph structure implementation, Boost Graph Library [Siek et al. 2002] of Boost [Boost C++ Libraries] has been used. The implementation was based on completely free-software. A friendly graphical interface to the algorithm has been developed by two independent and trainee students: Luis Artur Cesar Portella and Marcelo Hisashi Mitsui. The graphical interface permits a complete integration of the present work with the algorithms of chains enumeration implemented by Roberto Simoni in [Simoni 2008]. As a result, a complete system of generation and evaluation of kinematic chains is available for mechanism design. Section E.1 presents the graphical interface to the algorithm.

E.1 Graphical interface description

Figure E.1 shows the main screen of the interface. Three main menus are available:

1. Síntese Estrutural de Cadeias Cínematicas
2. Análise Estrutural de Cadeias Cínematicas
3. Síntese Estrutural de Mecansimos

In this section, only the menu “Análise Estrutural de Cadeias Cínematicas” is described, which implements the algorithm described in Section 6. A complete description and reference for the menus ” Síntese Estrutural de Cadeias Cínematicas” and ”Síntese Estrutural de Mecansimos” is found in [Simoni 2008].

In order to analyse one ore more kinematic chains, the incidence matrix representation of the kinematic chain, as described in Section 2.3.4 at Page 19, must be provided in a text file.

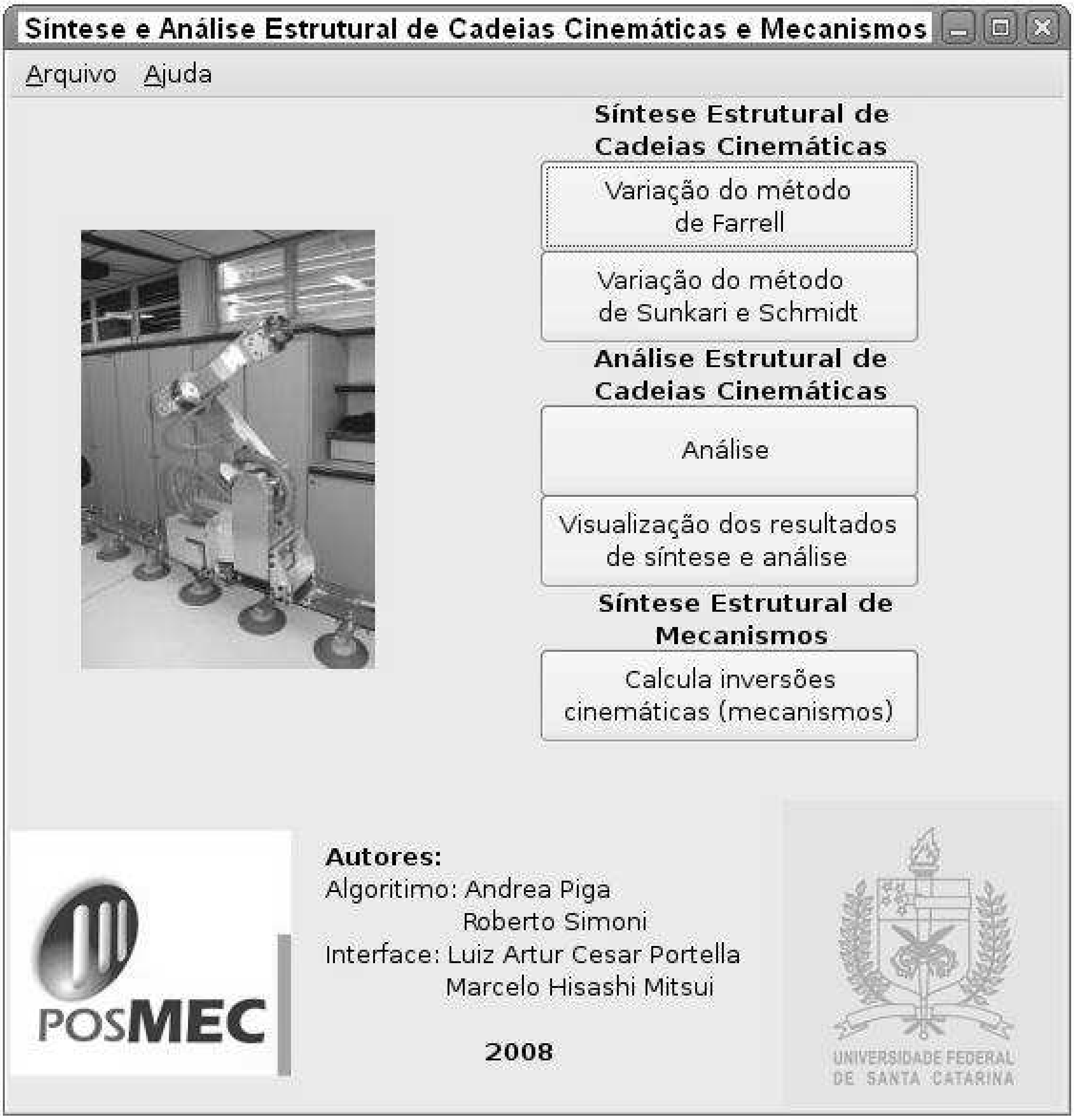


Figure E.1: The main screen of the interface

Figure E.2 shows screen "análise". The button *Arquivo* allows the user to choose the file containing the incidence matrix of the kinematic chains being examined. The result of the analysis is saved as a set of files:

- Adjacency matrix
- Redundancy matrix
- Connectivity matrix
- Degrees of control
- Chain properties (variety, minimal chain)

- Chain graph (.dot file)
- Chain graph (.txt file)

The buttons *Diretório* allows the choice of the folder where the output files are saved.

Figure E.2: The analise menu