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Amenability and Følner C*-algebras

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Orientador: Prof. Alcides Buss, Dr.

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Rodrigo Samuel Roemig

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À minha esposa

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“Young man, in mathematics you don’t understand things.
You just get used to them.”
John von Neumann

“One reason we love doing mathematics is that we don’t know
what lies ahead that future research will uncover.”
Alain Connes

Resumo

Neste trabalho, estudamos mediabilidade em C*-álgebras através de aproximações de dimensão finita, traços mediáveis e redes de Følner. Primeiramente, apresentamos os conceitos basilares para a teoria: aplicações completamente positivas, produtos cruzados e as normas tipo-traço e de Hilbert-Schmidt. Posteriormente, caracterizamos mediabilidade de grupos e trabalhamos com traços mediáveis e redes de Følner. Além disso, mostramos que estes conceitos estão intimamente ligados entre si e com aproximações de dimensão finita. Então, definimos e caracterizamos as C*-álgebras de Følner. Finalmente, aplicamos a teoria desenvolvida à caracterização de produtos cruzados de Følner e estendemos alguns dos resultados para o caso de produtos cruzados parciais.

Palavras-chave: Mediabilidade; C*-álgebra de Følner; traços mediáveis; redes de Følner; produtos cruzados; ações parciais.

Abstract

In this work, we study amenability on C*-algebras through finite-dimensional approximations, amenable traces and Følner nets. Firstly, we present the basic concepts to the theory: completely positive maps, crossed products and trace-class and Hilbert-Schmidt norms. Subsequently, we characterize amenable groups and work with amenable traces and Følner nets. Furthermore, we show that these concepts are closely related to each other and with finite-dimensional approximations. Then, we define and characterize Følner C*-algebras. Finally, we apply the developed theory to the characterization of Følner crossed products and extend some results to the setting of partial crossed product.

Keywords: Amenability; Følner C*-algebras; amenable traces; Følner nets; crossed products; partial actions.

Resumo Expandido

Introdução

O conceito de grupo mediável (*amenable*) é profundo e abrangente na literatura matemática. Uma das suas características marcantes é a existência de várias caracterizações para grupos mediáveis. Em particular, algumas destas caracterizações envolvem as C*-álgebra de grupo $C^*(G)$ e $C_r^*(G)$. Sendo assim, é natural perguntar: quais propriedades de C*-álgebras melhor descrevem a noção de mediabilidade?

Existem várias respostas na literatura, a mais famosa delas é a noção de nuclearidade. Este trabalho tem como objetivo analisar outra possibilidade: o conceito de C*-álgebra de Følner.

Historicamente falando, [Bédos, 1995] definiu C*-álgebras fracamente hipertraciais como aquelas que admitem traços mediáveis, ou, equivalentemente, admitem redes de Følner. Posteriormente, provou-se que a existência de um traço mediável equivale à existência de uma certa rede de aplicações com contradomínio finito que são assintoticamente multiplicativas na norma do traço. Neste contexto, [Ara and Lledó, 2014] utilizaram esta aproximação de dimensão finita para definir C*-álgebras de Følner, cuja classe corresponde à classe das C*-álgebras fracamente hipertraciais.

Objetivos

- Estudar as C*-álgebras de Følner e sua relação com a existência de traços mediáveis e redes de Følner;
- Entender as várias caracterizações de grupos mediáveis;
- Demonstrar o teorema de caracterizarização para C*-álgebras de Følner;
- Aplicar a teoria desenvolvida ao estudo de produtos cruzados de Følner;
- Estender alguns resultados para o caso de produtos cruzados parciais.

Metodologia

Nesta dissertação, utilizamos uma abordagem teórica para o estudo de Mediabilidade e C*-álgebras de Følner. Inicialmente, realizamos uma revisão bibliográfica sobre os conceitos fundamentais e os principais resultados já estabelecidos na literatura. Em seguida, enunciamos os conceitos relevantes, apresentamos exemplos e provamos os teoremas centrais da teoria.

Resultados e Discussões

A partir do teorema de caracterização de C*-álgebras de Følner, analisamos suas relações com os conceitos de nuclearidade e quasidiagonalidade. Além disso, mostramos que se um quociente não-nulo de uma C*-álgebra é de Følner, então a C*-álgebra é de Følner. Vimos ainda que a propridade de ser Følner passa para C*-subálgebras que tem a mesma unidade.

No último capítulo, aplicamos a teoria desenvolvida ao estudo dos produtos cruzados. Inicialmente, mostramos condições necessárias e suficientes para que produtos cruzados reduzidos sejam de Følner. Posteriormente, introduzimos ações parciais e estudamos a construção de produtos cruzados parciais. Finalmente, na última seção apresentamos algumas contribuições originais, pois estendemos alguns dos resultados para produtos cruzados parciais reduzidos e apresentamos alguns contraexemplos para os casos que não se generalizam.

Considerações Finais

Neste trabalho mostramos que a noção de C*-álgebra de Følner é interessante e abrangente. Por outro lado, alguns resultados sobre C*-álgebras de Følner parecerem não usuais quanto comparados com resultados sobre C*-álgebras nucleares, por exemplo.

Além disso, obtemos resultados não muito satisfatórios na generalização de alguns resultados para o caso parcial, o que nos permitiu concluir que C*-álgebras de Følner não são tão bem comportadas em relação à produtos cruzados parciais quanto C*-álgebras nucleares.

Palavras-chave: Mediabilidade; C*-álgebra de Følner; traços mediáveis; redes de Følner; produtos cruzados; ações parciais.

Contents

Introduction

The main goal of this thesis is to investigate amenability on C*-algebras through Følner-type conditions, amenable traces and finite-dimensional approximations.

Amenability of groups is a consolidated notion in the literature. Furthermore, group C*-algebras, such as $C^*(G)$ and $C^*_r(G)$, act like bridges between group theory and C*-theory. Hence, it is natural to ask: which property on C*-algebras best describes the notion of amenability?

One of the most celebrated notions of amenability in C*-algebras is nuclearity. However, in this work, we explore an alternative answer: Følner C*-algebras.

A unital C*-algebra A is a Følner C*-algebra if there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that

$$\lim_i \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} = 0$$

for every $a, b \in A$. Here $\|x\|_{2,\mathrm{tr}} = \mathrm{tr}(x^*x)^{1/2}$ ($x \in M_{k(i)}(\mathbb{C})$), and tr is the normalized trace. However, this definition did not emerge out of nowhere, there is a historical development behind it. One of the underlying goals of this thesis is to present the motivation of this definition. For this reason, we postpone a proper definition of Følner C*-algebra until Section 2.5. For those interested in historical context, Følner C*-algebra first appeared under the name “weakly hypertracial algebras” in [Bédos, 1995] as (concrete) C*-algebras that admit an amenable trace.

In chapter 1, we present the preliminaries needed to develop the theory. To be more specific, we review the basic theory of C*- and von Neumann algebras, completely positive maps, nuclearity, crossed products, and trace-class and Hilbert-Schmidt operators. In particular, we state and give detailed proofs of fundamental theorems such as Stinespring Theorem and Arverson’s Extension Theorem.

In chapter 2, we define and characterize amenable groups. Later we work with the C*-analogues to invariant means and Følner nets of subsets: amenable traces and Følner nets of operators. We prove that these concepts are interrelated and connected to finite-dimensional approximations. Then, we present quasidiagonal C*-algebras, since they are linked with Følner type conditions. Then, we define Følner C*-algebras and prove Theorem 2.5.7. This result characterize Følner C*-algebras using Følner nets, amenable traces and approximations by u.c.p. maps.

Finally, in chapter 3 we apply the results to the study of “global” and partial crossed products. Specifically, we aim to prove the following theorem that characterizes Følner C*-algebras.

Theorem A. *Let G be a discrete group and let α be an action of G on a unital C*-algebra A.*

(i) If $A \rtimes_{\alpha,r} G$ is a Følner C-algebra, then A is a Følner C*-algebra;*

(ii) If $A \rtimes_{\alpha,r} G$ is a Følner C-algebra, then G is amenable;*

(iii) If G is amenable and A is a Følner C-algebra, then $A \rtimes_{\alpha,r} G$ is a Følner C*-algebra.*

Given this result, we ask: *Which of the statements above are true when we consider partial (reduced) crossed products?*

As we shall see, only item *(i)* holds in the partial setting. We provide counterexamples for the conditions that do not generalize and present some stronger hypotheses that allow for some generalization. This thesis contains original results, including Proposition 3.3.1 and Proposition 3.3.11, which have not appeared in the literature before.

Moreover, we prove the following theorem that can be viewed as a justification for the study of Følner C*-algebras as an amenability-like property.

Theorem B. *Let G be a discrete group. Then, the following are equivalent:*

(i) G is amenable;

(ii) $C_r^*(G)$ is nuclear;

(iii) $C_r^*(G)$ is a Følner C-algebra;*

Finally, we apply the theory developed to analyze which implications hold when considering the reduced partial group C*-algebra $C^*_{\mathrm{par},r}(G)$. In particular, the claims of Theorem 3.3.13 regarding Følner C*-algebras are original results.

Remarks on notation

All Hilbert spaces considered are complex and usually denoted by $\mathcal{H}$ or $\mathcal{K}$. Also, $\langle \cdot, \cdot \rangle$ will denote an inner product linear in the first entry. Elements of a Hilbert space are represented by Greek letters (e.g., ξ, η, etc.). The set of bounded operators is denoted by $B(\mathcal{H})$, and $K(\mathcal{H})$ is the set of compact operators. Furthermore, C*-algebras are denoted by capital letters (e.g., A, B, etc.), and their elements are represented by lowercase letters (e.g., a, b, etc.), even when $A = B(\mathcal{H})$. We use $\mathcal{U}(A)$ to denote the set of unitaries.

All groups considered are discrete groups.

We denote by Tr the canonical trace on $B(\mathcal{H})$ and tr the canonical tracial state on $M_n(\mathbb{C})$. Sometimes we use tr_n to make the dimension explicit.

We use $\subset$ when there is an injective morphism between two objects, and $\subseteq$ when one set is contained in another.

Finally, the end of some sections contain some historical and/or “advanced” notes and remarks. They are intended to motivate the reader to seek the connection between the theory presented here and other established concepts and results.

1 Preliminaries

In the pursuit of the unknown,
you are never sure what you
need to know.

Davidson, C*-algebras by example (1996)

This chapter contains the prerequisites to understand amenability and Følner C*-algebras: basic operator algebra theory, completely positive maps, and Hilbert-Schmidt operators. It also contains an introduction to crossed products, which will be used in the last chapter.

1.1 C*-algebras and von Neumann algebras

In this section, we give an overview on fundamental results about C*-algebras and von Neumann algebras. Since our purpose is not to "reinvent the wheel", we will not prove all the results. Also, there are many important theorems that will not be stated. All these results can be found in any standard book on operator algebras, for example, [Murphy, 2014] and [Blackadar, 2006].

Definition 1.1.1. A **C*-algebra** is a Banach $*$-algebra A that satisfies the **C*-axiom**

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in A. \tag{1.1}$$

This axiom is sometimes called the C*-identity and $\|\cdot\|$ is called a C*-norm.

Remark 1.1.2. Unless otherwise specified, by an **ideal** we will always mean a closed two-sided ideal. It is sometimes useful to consider left or right ideals, but we shall not have to do so in this work.

Example 1.1.3. Let X be a locally compact Hausdorff space and consider $C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and vanish at infinity}\}$[1]. Give $C_0(X)$ its usual pointwise operations and supremum norm

$$\|f\| := \sup_{x\in X} |f(x)| .$$

Define an involution by $f^*(x) = \overline{f(x)}$ for every $x \in X$. Then $C_0(X)$ is a commutative C*-algebra.

[1] This means that for every $\varepsilon > 0$ there is a compact $K \subseteq X$ such that $|f(x)| < \varepsilon$, for every $x \in X \setminus K$.

It is a well-established result in operator algebra theory that every commutative C*-algebra A is isomorphic to $C_0(X)$ for a unique (up to homeomorphism) locally compact Hausdorff space X. More precisely, this goes as follows: Recall that the spectrum of A is defined as $\widehat{A} := \{\tau : A \to \mathbb{C} \mid \tau \text{ is a nonzero homomorphism}\}$ and equipped with the $*$-weak topology, since $\widehat{A} \subseteq A^*$ (topological dual). As a consequence of the Banach-Alaoglu Theorem, $\widehat{A}$ is a locally compact Hausdorff space. Finally, for each $a \in A$, define the map $\hat{a} : \widehat{A} \to \mathbb{C}$ by $\hat{a}(\tau) := \tau(a)$. The proof of the following theorem can be found in [Murphy, 2014, Theorem 2.1.10].

Theorem 1.1.4 (Gelfand Representation). *Let A be a commutative C*-algebra. Then the Gelfand transform*

$$\begin{aligned} \Phi : A &\longrightarrow C_0(\widehat{A}) \\ a &\longmapsto \widehat{a} \end{aligned}$$

is a isometric $*$-isomorphism. □

Example 1.1.5. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ the Banach space of bounded operators with the sup norm. Then $B(\mathcal{H})$ is a C*-algebra with product as composition and involution as adjoint. Denote by $K(\mathcal{H}) \subseteq B(\mathcal{H})$ the space of compact operators. Since $K(\mathcal{H})$ is closed under linear combination, product and adjoint, it is a $*$-subalgebra. Since $K(\mathcal{H})$ is norm-closed, it is a C*-subalgebra.

Example 1.1.6. Let $(A_i)_{i\in I}$ be a family of C*-algebras. We define the **direct product** and the **direct sum** by

$$\prod_{i\in I} A_i = \{(a_i)_{i\in I} \mid \sup_{i\in I} \|a_i\| < \infty\} \quad \text{and}$$

$$\bigoplus_{i\in I} A_i = \{(a_i)_{i\in I} \mid \lim_{i\to\infty} \|a_i\| \to 0\}$$

where the limit means that for every $\varepsilon > 0$ there is only finitely many i such that $\|a_i\| > \varepsilon$. The direct product is a C*-algebra with respect to the norm $\|(a_i)_{i\in I}\| := \sup_{i\in I} \|a_i\|$, and the direct sum is a (closed two-sided) ideal of the direct product, hence also a C*-algebra on its own.

Example 1.1.7. If we take $\mathcal{H} = \mathbb{C}^n$ in Example 1.1.5 we would have $B(\mathbb{C}^n) \cong M_n(\mathbb{C})$. This is the most important example of finite dimensional C*-algebra. Every simple finite dimensional C*-algebra is of this form. Moreover, every finite dimensional C*-algebra is a (finite) direct sum of C*-algebras of the form $M_n(\mathbb{C})$ [Murphy, 2014, Theorem 6.3.8].

Example 1.1.8. Let $S \subset A$ be a subset of a C*-algebra, the **C*-algebra generated by** S is the smaller C*-subalgebra of A that contains S,

$$C^*(S) = \bigcap_{S\subseteq B} B \text{ where } B \subseteq A \text{ is a C*-subalgebra.}$$

Alternatively, we can see $C^*(S)$ in a more constructive way. First, we need to consider $S \cup S^*$ and take finite products $\hat{S} := \{a_1 \dots a_n \mid a_i \in S \cup S^*, n \in \mathbb{N}\}$, to get a set of elements closed by adjunction and products. Finally, we consider $\langle S \rangle := \overline{\text{span}(\hat{S})}$ to get set of elements closed by linear combinations and by limits of sequences. It is not hard to see that $\langle S \rangle \subseteq C^*(S)$ and that $\langle S \rangle$ is a C*-subalgebra of A, therefore $C^*(S) = \overline{\text{span}(\hat{S})}$.

Definition 1.1.9. Let A be a C*-algebra and $a \in A$. Then a is **self-adjoint** if $a = a^*$, **normal** if $a^*a = aa^*$, **idempotent** if $a^2 = a$ and a **projection** if $a^2 = a = a^*$. If $A \ni 1$, then a is an **isometry** if $a^*a = 1$ and **unitary** if $a^*a = 1 = aa^*$.

Remark 1.1.10. It is worth mentioning that when $A = B(\mathcal{H}) \ni p$, it is more common to say that p is a projection if $p^2 = p$ and p is an orthogonal projection if $p^2 = p = p^*$. In this work, we will work with orthogonal projection almost every time.

Definition 1.1.11. Let A and B be C*-algebras. An algebra homomorphism $\varphi : A \to B$ is called a **$*$-homomorphism** if it satisfies $\varphi(a^*) = \varphi(a)^*$, for all $a \in A$.

An interesting observation is that we do not explicitly require a $*$-homomorphism to be continuous, even though C*-algebras have a topological structure. This is because the C*-axiom and spectral theory ensure that every $*$-homomorphism is automatically contractive; that is, it satisfies $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.

Construction 1.1.12 (Unitization). Not every C*-algebra is unital. However, any non-unital C*-algebra A can be embedded into a unital C*-algebra $\tilde{A}$, called the *unitization* of A. Define $\tilde{A} := A \oplus \mathbb{C}$ with pointwise vector space operations and multiplication given by

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu),$$

for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. The involution is defined as

$$(a, \lambda)^* := (a^*, \overline{\lambda}),$$

and the norm is defined by $\|(a, \lambda)\| := \sup_{\|b\| \leq 1} \|ab + \lambda b\|$.

With these operations, $\tilde{A}$ becomes a unital C*-algebra. It is straightforward to verify that A is an ideal in $\tilde{A}$ and that the map $a \mapsto (a, 0)$ is an isometric $*$-homomorphism.

Recall that the spectrum of an element $a \in A$ of a unital C*-algebra, is the set $\sigma(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}$, where we consider the unitization when A is nonunital.

Definition 1.1.13. Let A and B be C*-algebras. An element $a \in A$ is **positive** if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$. We denote $a \geq 0$ and $A_+ := \{a \in A \mid a \geq 0\}$. Moreover, a linear transformation $\varphi : A \to B$ is **positive** if $\varphi(a) \geq 0$ for every $a \geq 0$. We denote $\varphi \geq 0$.

The following theorem is very useful to determine when a given functional is a positive element. A nonunital version of the theorem and its proof can be found in [Murphy, 2014, Theorem 3.3.3].

Theorem 1.1.14. *Let A be a unital C*-algebra and $\tau : A \to \mathbb{C}$ be a continuous linear functional. Then the following are equivalent:*

(i) $\tau \geq 0$ *;*

(ii) $\tau(1) = \|\tau\|$. □

Definition 1.1.15. Let A be a C*-algebra. A positive linear functional $\tau : A \to \mathbb{C}$ is called a **state** if $\|\tau\| = 1$. The set of all states on A is denoted by $S(A)$. Furthermore, a state τ is said to be **faithful** if $\tau(a^*a) = 0$ implies $a = 0$, for all $a \in A$; and it is called **tracial** if $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

As a consequence of Theorem 1.1.14, if one wants to show that τ is a state on a unital C*-algebra, it is enough to show that $\|\tau\| \leq 1$ and $\tau(1) = 1$. There is another tool to deal with nonunital C*-algebras.

Definition 1.1.16. Let A be a C*-algebra and Λ a directed set, this means that for every $\lambda_1, \lambda_2 \in \Lambda$ there is $\lambda \in \Lambda$ such that $\lambda_1, \lambda_2 \leq \lambda$. An **approximate unit** for A is a net $(e_\lambda)_{\lambda\in\Lambda}$ of positive elements of A with $\|e_\lambda\| \leq 1$ for every $\lambda \in \Lambda$, such that $\lambda_1 \leq \lambda_2$ implies $e_{\lambda_1} \leq e_{\lambda_2}$ and, for all $a \in A$,

$$\lim_{\lambda} e_\lambda a = a \,.$$

Definition 1.1.17. A **representation** of a C*-algebra A is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and $\pi : A \to B(\mathcal{H})$ is a $*$-homomorphism. We say that π is **faithful** if π is injective. The same definition applies when A is just a $*$-algebra.

Remark 1.1.18. Given a family of representations $\{(\mathcal{H}_i, \pi_i)\}_{i\in I}$ of a C*-algebra A, we can construct a representation that is the sum of the representations as the map

$$\oplus_i \pi_i : A \longrightarrow B(\oplus_i \mathcal{H}_i)\,,$$

where, for any $a \in A$, $\oplus_i \pi_i(a) : \oplus_i \mathcal{H}_i \to \oplus_i \mathcal{H}_i$ is given by

$$(\xi_i)_{i\in I} \longmapsto (\pi_i(a)(\xi_i))_{i\in I}\,.$$

Construction 1.1.19 (GNS). Now we sketch the Gelfand-Naimark-Segal construction. Let A be a C*-algebra and τ a state. We define a sesquilinear form $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{C}$ by $\langle a, b \rangle := \tau(b^*a)$. Note that $\langle \cdot, \cdot \rangle$ need not be an inner product, but it is always a semi-inner product. In order to get an inner product, we define $N_\tau := \{a \in A \mid \tau(a^*a) = 0\}$, which is a closed left ideal of A.

Let $\mathcal{H}_\tau^0 := A/N_\tau$ be the quotient space, and equip it with the inner product defined by $\langle \hat{a}, \hat{b} \rangle := \tau(b^*a)$, where $\hat{a} = a + N_\tau$ denotes the equivalence class. Hence $\mathcal{H}_\tau^0$ is a pre-Hilbert space, and its completion $\mathcal{H}_\tau := \overline{A/N_\tau}$ is a Hilbert space.

Given $a \in A$, now define $\pi_\tau^0(a) : \mathcal{H}_\tau^0 \to \mathcal{H}_\tau^0$ by $\pi_\tau^0(a)(\hat{b}) = \widehat{ab}$ (left multiplication), this map is well defined, linear and bounded, therefore can be extended to the completion $\pi_\tau(a) : \mathcal{H}_\tau \to \mathcal{H}_\tau$, so that the map $\pi_\tau : A \to B(\mathcal{H}_\tau)$, $a \mapsto \pi_\tau(a)$ yields a representation $(\mathcal{H}_\tau, \pi_\tau)$. The direct sum of all the representations over the states on A, $\pi = \bigoplus_{\tau \in S(A)} \pi_\tau$ and $\mathcal{H} = \bigoplus_{\tau \in S(A)} \mathcal{H}_\tau$, is called the **universal representation** of A.

The following theorem asserts that every C*-algebra is isometrically isomorphic to a C*-subalgebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$ (see [Murphy, 2014, Theorem 3.4.1.]).

Theorem 1.1.20 (Gelfand-Naimark). *Every C*-algebra A has a faithful representation. More specifically, the universal representation $\bigoplus_{\tau \in S(A)} \pi_\tau : A \to B\big(\bigoplus_{\tau \in S(A)} \mathcal{H}_\tau\big)$ is faithful.* □

The representation theory of C*-algebras is very important. In the following, we present some definitions and propositions that will be used in this work.

Definition 1.1.21. Let $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$ be representations of a C*-algebra A. We say that:

(i) π is **nondegenerate** if $\{\pi(a)\xi \mid a \in A,\ \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

(ii) π is **cyclic** if there exists a vector $\xi \in \mathcal{H}$ such that $\pi(A)\xi := \{\pi(a)\xi \mid a \in A\}$ is dense in $\mathcal{H}$. This vector is called a **cyclic vector** of π.

(iii) π is **irreducible** if there is no closed subspace $\mathcal{H}' \subseteq \mathcal{H}$ that is invariant under the action of $\pi(A)$, i.e., $\pi(A)\mathcal{H}' \subseteq \mathcal{H}'$.

(iv) π is **essential** if $\pi(A)$ contains no nonzero compact operators, i.e., $\pi(A) \cap K(\mathcal{H}) = \{0\}$.

(v) π and ρ are **unitarily equivalent** if there is a unitary operator $u : \mathcal{H} \to \mathcal{K}$ such that $\rho(a) = u\pi(a)u^*$, for every $a \in A$.

Remark 1.1.22. If A is unital and τ is a state, then $\xi_\tau := \hat{1}$ is a unit vector in $\mathcal{H}_\tau$ (from the GNS construction) satisfying

$$\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle,$$

for every $a \in A$. If A is nonunital, one can use an approximate unit to have the same result (see [Blackadar, 2006, II.6.4.2]). Furthermore, ξ_τ is a cyclic vector of π_τ, since $\pi(A)\xi_\tau = A/N_\tau$ is dense in $\mathcal{H}_\tau$.

Conversely, if $(\mathcal{H}, \pi)$ is a cyclic representation with cyclic unit vector ξ, then the map $\tau : A \to \mathbb{C}$ given by

$$a \longmapsto \langle \pi(a)\xi, \xi \rangle ,$$

defines a state. Also, $(\mathcal{H}, \pi)$ is unitarily equivalent to $(\mathcal{H}_\tau, \pi_\tau)$.

The proof of the next proposition can be found in [Murphy, 2014, Theorem 5.1.3.].

Proposition 1.1.23. *Every nondegenerate representation $(\mathcal{H}, \pi)$ of a C*-algebra A is a direct sum of cyclic representations.* □

Proposition 1.1.24. *Let A be a C*-algebra and $B \subseteq A$ be a C*-subalgebra. If $\rho : B \to B(\mathcal{H})$ is a representation, then there exists a Hilbert space $\mathcal{H}' \supset \mathcal{H}$ and a representation $\pi : A \to B(\mathcal{H}')$ extending ρ, i.e., if $p : \mathcal{H}' \to \mathcal{H} \subset \mathcal{H}'$ denotes the orthogonal projection, we have $\rho(b) = p\pi(b)p^*$ for all $b \in B$.*

Proof. Since every representation can be decomposed as the sum of a nondegenerate representation with a zero representation, we may assume that ρ is nondegenerate. Moreover, by Proposition 1.1.23, it is enough to treat the case that ρ is cyclic. Let $\tau : B \to \mathbb{C}$ the state associated with ρ as in Remark 1.1.22.

By the Hahn-Banach Theorem, there exists an extension $\tau' : A \to \mathbb{C}$ of τ such that $\|\tau'\| = \|\tau\|$. In particular, note that τ' is a state on A. Finally, consider the GNS representation relative to τ' and take $\mathcal{H}' = \mathcal{H}_{\tau'}$ and $\pi = \pi_{\tau'}$. □

From this point until the end of this section, we will consider only concrete $*$-algebras $M \subseteq B(\mathcal{H})$. Recall that a net $(u_\lambda)_{\lambda\in\Lambda} \subseteq B(\mathcal{H})$ *converges weakly* to $u \in B(\mathcal{H})$ if, for every $\xi, \eta \in \mathcal{H}$,

$$\langle u_\lambda(\xi), \eta \rangle \longrightarrow \langle u(\xi), \eta \rangle .$$

Also, remember that the commutant of a set $S \subseteq B(\mathcal{H})$ is

$$S' := \{u \in B(\mathcal{H}) \mid uv = vu, \text{ for every } v \in S\} .$$

Definition 1.1.25. A **von Neumann algebra** $M \subseteq B(\mathcal{H})$ is a weakly closed unital $*$-subalgebra of $B(\mathcal{H})$.

The proof of the following theorem can be found in [Murphy, 2014, Theorem 4.1.5].

Theorem 1.1.26 (von Neumann bicommutant). *Let $A \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra. Then A is a von Neumann algebra if and only if $A'' = A$.* □

Definition 1.1.27. Let $\tau : M \to \mathbb{C}$ be a linear functional. We say that τ is **normal** if, for every increasing bounded net $(x_\lambda)_{\lambda\in\Lambda} \subseteq M^+$ with $x = \sup_\lambda x_\lambda$, we have $\tau(x) = \lim_\lambda \tau(x_\lambda)$. Furthermore, let M_* denote the set of normal functionals of M. The set M_* is called the **predual** of M.

Example 1.1.28. Consider $\xi, \eta \in \mathcal{H}$ and define the functional $\tau_{\xi,\eta}(x) := \langle x\xi, \eta\rangle$. It is straightforward that $\tau_{\xi,\eta}$ is a normal functional. In particular, if $\xi = \eta$ and $\|\xi\| = 1$, then $\tau_{\xi,\xi}$ is a state called a *vector state*.

Remark 1.1.29. The predual M_* is the unique Banach space (up to isometric isomorphism) such that $(M_*)^* = M$ (see [Brown and Ozawa, 2008, Theorem 1.3.5.]). In the end of Section 1.4, we present a more concrete version of the predual.

Remark 1.1.30. The *ultraweak topology* of M is the weak* topology that comes from the isomorphism $M = (M_*)^*$. This means that a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq M$ converges to $x \in M$ if

$$\tau(x_\lambda) \longrightarrow \tau(x)\,,$$

for every $\tau \in M_*$.

Let X be a Banach space and $B(X, M)$ be the set of bounded linear maps from X to M. The *point-ultraweak topology* is defined in convergence terms as follows: a bounded net $(T_\lambda)_{\lambda\in\Lambda} \subseteq B(X, M)$ converges to $T \in B(X, M)$ if

$$\tau(T_\lambda(x)) \longrightarrow \tau(T(x))\,,$$

for every $x \in X$ and $\tau \in M_*$.

Furthermore, $B(X, M)$ can be viewed as the dual space of

$$B(X, M)_* := \overline{\operatorname{span}}\{x \otimes \tau \in B(X, M)^* \mid x \in X,\, \tau \in M_*\}\,,$$

where $x \otimes \tau(T) := \tau(T(x))$, for every $T \in B(X, M)$.

In this context, the weak* topology of $B(X, M)$ agrees with the point-ultraweak topology on bounded sets (see [Brown and Ozawa, 2008, §1.3]). The following result is a consequence of the Banach-Alaoglu Theorem (see [Brown and Ozawa, 2008, Theorem 1.3.7]).

Proposition 1.1.31. *Every bounded net $(T_\lambda)_{\lambda\in\Lambda} \subseteq B(X, M)$ has a cluster point in the point-ultraweak topology.* □

1.2 Completely positive maps and nuclearity

This section contains an introduction to completely positive maps, which are a key ingredient in the definition of Følner C*-algebras. One of the goals is to present two cornerstone results in the theory: Stinespring's Theorem and Arverson's Extension Theorem. Additionally, the end of the section contains a quick presentation of a net-based definition of nuclearity. The primary references for this section are [Brown and Ozawa, 2008, Chapters 1 & 2] and [Blackadar, 2006, §II.6.9].

Definition 1.2.1. Let A and B be C*-algebras. A linear map $\varphi : A \to B$ is called **completely positive (c.p.)** if $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive for every $n \in \mathbb{N}$, where $\varphi^{(n)}$ is defined by $\varphi^{(n)}([a_{ij}]) = [\varphi(a_{ij})]$.

Moreover, a c.p. map φ is called **contractive completely positive (c.c.p.)** if $\|\varphi\| \leq 1$. When A and B are unital C*-algebras, a c.p. map φ is called **unital completely positive (u.c.p.)** if φ is unital.

Example 1.2.2. Given $\pi : A \to B$ a $*$-homomorphism and $v \in B$, the map φ defined by $\varphi(a) = v^*\pi(a)v$ is a c.p. map. Indeed, suppose that $[a_{ij}] = [x_{ij}]^*[x_{ij}]$ for some $[x_{ij}] \in M_n(A)$, i.e., $[a_{ij}]$ is positive. Hence,

$$\varphi^{(n)}([a_{ij}]) = [\varphi(a_{ij})] = \left[\varphi(\sum_{k=1}^{n}(x_{ki}^* x_{kj}))\right] = \left[\sum_{k=1}^{n} v^*\pi(x_{ki})^*\pi(x_{kj})v\right] =$$

$$= [\pi(x_{ij})v]^*[\pi(x_{ij})v] \geq 0\,,$$

as desired. In particular, if $v \in B$ is a projection, we say that φ is a *compression.*

Theorem 1.2.3 (**Stinespring**). *Let A be a unital C*-algebra and $\varphi : A \to B(\mathcal{H})$ be a c.p. map. Then, there exist a Hilbert space $\hat{\mathcal{H}}$, a representation $\pi : A \to B(\hat{\mathcal{H}})$ and an operator $v : \mathcal{H} \to \hat{\mathcal{H}}$ such that*

$$\varphi(a) = v^*\pi(a)v\,,$$

for all $a \in A$. Moreover, $\|\varphi\| = \|v^*v\| = \|\varphi(1)\| = \left\|\varphi^{(n)}\right\|$, for all $n \in \mathbb{N}$.

Proof. The proof involves a "twisted" version of the GNS construction. We begin with the definition of the Hilbert space $\hat{\mathcal{H}}$. Consider the tensor product of vector spaces $A \odot \mathcal{H}$ and define a sesquilinear form on $A \odot \mathcal{H}$ by

$$\langle a \otimes \xi, b \otimes \eta \rangle := \langle \varphi(b^*a)\xi, \eta \rangle$$

on elementary tensors.

Next, we will show that $\langle \cdot, \cdot \rangle$ is positive semidefinite, i.e., $\langle x, x \rangle \geq 0$ for all $x \in A \odot \mathcal{H}$. First, observe that, given two elements $x, y \in A \odot \mathcal{H}$, we can use a classical tensor trick to rewrite them as finite sums

$$x = \sum_{i=1}^{n} a_i \otimes \xi_i \text{ and } y = \sum_{i=1}^{n} b_i \otimes \xi_i,$$

where $\{\xi_i\}_{i=1}^{n}$ is a linearly independent set of vectors. Therefore

$$\left\langle \sum_{j=1}^{n} a_j \otimes \xi_j, \sum_{i=1}^{n} b_i \otimes \xi_i \right\rangle = \sum_{i,j=1}^{n} \langle \varphi(b_i^* a_j)\xi_j, \xi_i \rangle\,.$$

In particular, considering $x = \sum_{i=1}^{n} a_i \otimes \xi_i \in A \odot \mathcal{H}$, we have

$$\langle x, x \rangle = \sum_{i,j=1}^{n} \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle\,.$$

First define

$$a = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and observe that $a^*a = [a_i^*a_j]$ is positive on $M_n(A)$, and hence its transpose matrix $[a_j^*a_i]$ is also positive.

Since φ is c.p., $[\varphi(a_j^*a_i)]$ is positive on $M_n(B(\mathcal{H}))$. Recall there is a $*$-isomorphism between $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^n)$ given by

$$M_n(B(\mathcal{H})) \ni [m_{ij}] \longmapsto m \in B(\mathcal{H}^n)$$

where the operator m is defined by $m(\eta_1, \dots, \eta_n) := (\sum_{i=1}^n m_{1i}\eta_i, \dots, \sum_{i=1}^n m_{ni}\eta_i)$, for any $\eta_i \in \mathcal{H}$. In particular, denote by $\hat{m}$ the operator on $B(\mathcal{H}^n)$ that is image of $[\varphi(a_j^*a_i)] \in M_n(B(\mathcal{H}))$ under the $*$-isomorphism above, i.e., $\hat{m}_{ij} = \varphi(a_j^*a_i)$. Hence, $\hat{m}$ is positive and

$$0 \leq \langle \hat{m}(\xi_1, \dots, \xi_n), (\xi_1, \dots, \xi_n) \rangle$$

for the vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$ given before.

Therefore,

$$0 \leq \left\langle (\sum_{i=1}^n \hat{m}_{1i}\xi_i, \dots, \sum_{i=1}^n \hat{m}_{ni}\xi_i), (\xi_1, \dots, \xi_n) \right\rangle \stackrel{?}{=} \sum_{i,j=1}^n \langle \varphi(a_j^*a_i)\xi_i, \xi_j \rangle = \langle x, x \rangle \,,$$

as desired.

Finally, define the 'zero subspace' as $N_\varphi := \{x \in A \odot \mathcal{H} \mid \langle x, x \rangle = 0\}$ and $\hat{\mathcal{H}}$ as the completion of $\hat{\mathcal{H}}_0 := A \odot \mathcal{H}/N_\varphi$. We denote by $\overline{\sum_{i=1}^n a_i \otimes \xi_i}$ the element in $\hat{\mathcal{H}}$ corresponding to $\sum_{i=1}^n a_i \otimes \xi_i \in A \odot \mathcal{H}$.

Now, define the map $v : \mathcal{H} \to \hat{\mathcal{H}}$ by

$$\xi \longmapsto \overline{1 \otimes \xi} \,.$$

It is not hard to see that v is a bounded linear map, since

$$\|v\xi\| = \left\langle \, \overline{1 \otimes \xi}, \overline{1 \otimes \xi} \, \right\rangle^{1/2} = \langle \varphi(1)\xi, \xi \rangle^{1/2} \leq \|\varphi(1)\|^{1/2} \|\xi\| \,.$$

For each $a \in A$, define $\pi(a)$ as the completion of the map $\pi_0(a) : \hat{\mathcal{H}}_0 \to \hat{\mathcal{H}}$ defined on elementary tensors by

$$\overline{b \otimes \xi} \longmapsto \overline{ab \otimes \xi}$$

There are a few verifications to do about $\pi_0(a)$. But will only prove that $\pi_0(a)$ is a well-defined and limited on elementary tensors. For $b \in A$ and $\xi \in \mathcal{H}$, note that $\varphi(b^*a^*ab) \leq \|a\|^2 \varphi(b^*b)$, since $b^*a^*ab \leq \|a\|^2 b^*b$ on A.

Hence

$$\left\|\pi_0(a)(\overline{b\otimes\xi})\right\|^2 = \left\langle\ \overline{ab\otimes\xi}, \overline{ab\otimes\xi}\ \right\rangle = \langle\varphi(b^*a^*ab)\xi,\xi\rangle \leq$$
$$\leq \|a\|^2\langle\varphi(b^*b)\xi,\xi\rangle = \|a\|^2\left\|\overline{b\otimes\xi}\right\|^2 .$$

Thus $\pi_0(a)$ is limited. The same equation shows that $\pi_0(a)$ is well-defined, since $\overline{b\otimes\xi}=0$ implies $\pi_0(a)(\overline{b\otimes\xi})=0$.

Pick an element $a\in A$, to show that

$$\varphi(a) = v^*\pi(a)v\,,$$

it is enough to show that

$$\langle\varphi(a)\xi,\eta\rangle = \langle v^*\pi(a)v\xi,\eta\rangle$$

for every $\xi,\eta\in\mathcal{H}$. Indeed, we have

$$\langle v^*\pi(a)v\xi,\eta\rangle = \langle\pi(a)v\xi, v\eta\rangle = \left\langle\ \overline{a\otimes\xi}, \overline{1\otimes\eta}\ \right\rangle = \langle\varphi(a)\xi,\eta\rangle\,,$$

as desired.

Furthermore, notice that $\varphi(1) = v^*\pi(1)v = v^*v$ and, for any $a\in A$,

$$\|\varphi(a)\| = \|v^*\pi(a)v\| \leq \|v\|^2\|\pi(a)\| \leq \|v^*v\|\,\|a\| .$$

Thus $\|\varphi\| = \|v^*v\| = \|\varphi(1)\|$. Similarly, one can show that $\left\|\varphi^{(n)}\right\| = \|\varphi(1)\| = \|\varphi\|$, for all $n\in\mathbb{N}$. □

Definition 1.2.4. The triplet $(\pi,\hat{\mathcal{H}},v)$ that appears in Theorem 1.2.3 is called a **Stinespring dilation**. The projection $vv^*\in B(\hat{\mathcal{H}})$ is called the **Stinespring projection**.

Remark 1.2.5. It is worth mentioning that there are many different triplets that satisfy the conditions of Stinespring's Theorem, and therefore, could be called a Stinespring dilation. Nevertheless, the triplet constructed in the proof above satisfies a minimality condition, in the sense that $\pi(A)v\mathcal{H}$ is dense in $\hat{\mathcal{H}}$. That said, one can prove that a minimal Stinespring dilation is unique up to unitary equivalence.

Proposition 1.2.6. *Let A be a unital C*-algebra, B be a C*-algebra and $\varphi : A\to B$ be a c.c.p. map.*

(i) We have $\varphi(a)^*\varphi(a)\leq\varphi(a^*a)$, for all $a\in A$.

(ii) Given $a\in A$, suppose that $\varphi(a^*a)=\varphi(a)^*\varphi(a)$ and $\varphi(aa^*)=\varphi(a)\varphi(a)^*$. Then $\varphi(ab)=\varphi(a)\varphi(b)$ and $\varphi(ba)=\varphi(b)\varphi(a)$, for all $b\in A$.

(iii) The subspace $A_\varphi = \{a\in A\mid \varphi(a)^*\varphi(a)=\varphi(a^*a) \text{ and } \varphi(a)\varphi(a)^*=\varphi(aa^*)\}$ is a C-subalgebra.*

Proof. **(i)** Suppose that $B \subset B(\mathcal{H})$ is a faithful representation and let $(\pi, \hat{\mathcal{H}}, v)$ be a Stinespring dilation of $\varphi : A \to B \subset B(\mathcal{H})$. note that, by Stinespring's Theorem (1.2.3),

$$\|vv^*\| = \|v\|^2 = \|v^*v\| = \|\varphi\| \leq 1\,,$$

since φ is a contraction. In particular, this implies that $1_{\hat{\mathcal{H}}} - vv^* \geq 0$. Hence, for any $a \in A$,

$$\varphi(a^*a) - \varphi(a)^*\varphi(a) = v^*\pi(a^*)\pi(a)v - v^*\pi(a)^*vv^*\pi(a)v = v^*\pi(a)^*(1_{\hat{\mathcal{H}}} - vv^*)\pi(a)v \geq 0\,,$$

as desired.
(ii) Let $a \in A$ be an element such that $\varphi(a^*a) - \varphi(a)^*\varphi(a) = 0 = \varphi(aa^*) - \varphi(a)\varphi(a)^*$. note that the element $x \coloneqq (1_{\hat{\mathcal{H}}} - vv^*)^{1/2}\pi(a)v$ satisfies

$$0 = \varphi(a^*a) - \varphi(a)^*\varphi(a) = v^*\pi(a)^*(1_{\hat{\mathcal{H}}} - vv^*)\pi(a)v = x^*x\,.$$

Hence, $0 = \|x^*x\| = \|x\|^2$ implies $x = 0$. In particular,

$$\varphi(ba) - \varphi(b)\varphi(a) = v^*\pi(b)(1_{\hat{\mathcal{H}}} - vv^*)\pi(a)v = v^*\pi(b)(1_{\hat{\mathcal{H}}} - vv^*)^{1/2}x = 0\,,$$

as desired. Analogously, using $y = v^*\pi(a)(1_{\hat{\mathcal{H}}} - vv^*)^{1/2}$, it follows that $\varphi(ab) = \varphi(a)\varphi(b)$.
(iii) That fact that A_φ is closed under multiplication follows easily from *(ii)*, and the closedness in the norm follows from the definition. Finally, A_φ is closed under involution, because, given a Stinespring dilation $(\pi, \hat{\mathcal{H}}, v)$, we have

$$\varphi(a^*) = v^*\pi(a^*)v = (v^*\pi(a)v)^* = \varphi(a)^*\,,$$

as desired. □

Definition 1.2.7. Let $\varphi : A \to B$ be a c.c.p. map. The C*-subalgebra

$$A_\varphi \coloneqq \{a \in A \mid \varphi(a)^*\varphi(a) = \varphi(a^*a) \text{ and } \varphi(a)\varphi(a)^* = \varphi(aa^*)\}$$

is called the **multiplicative domain** of φ.

The multiplicative domain is the largest C*-subalgebra of A on which φ restricts to a $*$-homomorphism and will play a key role in various proofs in this work.

Proposition 1.2.8. *Let A be a C*-algebra and let $\{e_{ij}\}_{i,j=1}^n$ be the canonical basis of $M_n(\mathbb{C})$. Then a linear map $\varphi : M_n(\mathbb{C}) \to A$ is c.p. if and only if $[\varphi(e_{ij})]$ is a positive element of $M_n(A)$.*

Proof. **(⇒)** Suppose that φ is a c.p. map. Notice that $[e_{ij}]$ is positive as an element of $M_n(M_n(\mathbb{C}))$, since it is an orthogonal projection. Hence $\varphi^{(n)}$ being positive implies that $\varphi^{(n)}([e_{ij}]) = [\varphi(e_{ij})]$ is positive on $M_n(A)$.

(⇐) Suppose that $[\varphi(e_{ij})] \geq 0$ and denote $[\varphi(e_{ij})]^{1/2} = [b_{ij}]$. Note, in particular, that

$$\varphi(e_{ij}) = \sum_{k=1}^{n} b_{ki}^* b_{kj} \,. \tag{1.2}$$

Now consider $A \subset B(\mathcal{H})$ a faithful representation. The idea of the proof is to construct a "Stinespring dilation" of $\varphi : M_n(\mathbb{C}) \to A \subset B(\mathcal{H})$. Define the operator $v : \mathcal{H} \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H}$ by

$$v\xi \coloneqq \sum_{j,k=1}^{n} \zeta_j \otimes \zeta_k \otimes b_{kj}\xi \,,$$

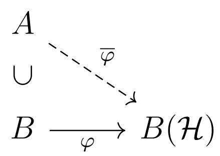
where $\{\zeta_j\}_{j=1}^n$ is the canonical basis of $\mathbb{C}^n$. Hence, for any $m = [m_{ij}] \in M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$ and $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}
\langle v^*(m \otimes 1 \otimes 1)v\xi, \eta\rangle &= \langle (m \otimes 1 \otimes 1)v\xi, v\eta\rangle \\
&= \sum_{j,k,i,l=1}^{n} \langle m\zeta_j \otimes \zeta_k \otimes b_{kj}\xi, \zeta_i \otimes \zeta_l \otimes b_{li}\eta\rangle \\
&= \sum_{i,j,k,l=1}^{n} \langle m\zeta_j, \zeta_i\rangle\langle \zeta_k, \zeta_l\rangle\langle b_{kj}\xi, b_{li}\eta\rangle \\
&\overset{k=l}{=} \sum_{i,j=1}^{n} m_{ij} \Big\langle \sum_{k=1}^{n} b_{ki}^* b_{kj}\xi, \eta\Big\rangle \\
&\overset{(1.2)}{=} \Big\langle \varphi\big(\sum_{i,j=1}^{n} m_{ij}e_{ij}\big)\xi, \eta\Big\rangle \\
&= \langle \varphi(m)\xi, \eta\rangle \,.
\end{aligned}$$

Thus $\varphi(\cdot) = v^*(\,\cdot \otimes 1 \otimes 1)v$ is a c.p. map. □

Definition 1.2.9. Let $B \subset A$ be C*-algebras. A **projection** is a linear map $E : A \to B$ such that $E|_B = \mathrm{id}_B$. Additionally, a c.c.p. projection such that $E(bxb) = bE(x)b'$ for all $x \in A$ and $b, b' \in B$[2], is called a **conditional expectation**.

Theorem 1.2.10 (Arverson's Extension). *Let A be a C^*-algebra and $B \subset A$ be a C^*-subalgebra. Then, every c.p. map $\varphi : B \to B(\mathcal{H})$ extends to a c.p. map $\overline{\varphi} : A \to B(\mathcal{H})$.*



Moreover, if $1_A \in B$ and φ is a u.c.p map, then the extension $\overline{\varphi}$ is u.c.p. as well.

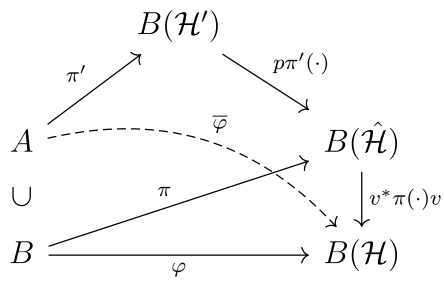
[2] This means that E is a B-bimodule map.

Proof. Suppose that $\varphi : B \to B(\mathcal{H})$ is a c.p. map. By Stinespring's Theorem (1.2.3), there exists a Stinespring dilation $(\pi, \hat{\mathcal{H}}, v)$ such that $\varphi(\cdot) = v^*\pi(\cdot)v$. Hence, by Proposition 1.1.24, there exist a Hilbert space $\mathcal{H}' \supset \hat{\mathcal{H}}$ and a representation $\pi' : A \to B(\mathcal{H}')$ that is an extension of π. Denote by $p : \mathcal{H}' \to \hat{\mathcal{H}}$ the orthogonal projection, thus $\pi(b) = p\pi'(b) = p\pi'(b)p^*$, for all $b \in B$. Note that $p^* : \hat{\mathcal{H}} \to \mathcal{H}'$ is just the inclusion and could be omitted.

Now, define $\overline{\varphi} : A \to B(\mathcal{H})$ by

$$\overline{\varphi}(a) = v^*p\pi'(a)p^*v\,.$$

It is easy to see that $\overline{\varphi}$ is an extension of φ.



With calculations analogous to Example 1.2.2, we conclude that $\overline{\varphi}$ is c.p. Finally, the u.c.p. case is trivial. □

Now we provide a proof of a technical result that will be helpful in the next chapter.

Lemma 1.2.11. *Let A be a unital C^*-algebra and M be a von Neumann algebra. If $(\Phi_i)_{i\in I} \subseteq B(A, M)$ is a net of u.c.p. maps that converges to $\Phi \in B(A, M)$ in the point-ultraweak topology, then Φ is also a u.c.p. map.*

Proof. By definition $\Phi_i \to \Phi$ in the point-ultraweak topology if and only if

$$\tau(\Phi_i(a)) \longrightarrow \tau(\Phi(a))\,,$$

for every $\tau \in M_*$ and $a \in A$.

Note that, if $a \geq 0$, then $\Phi_i(a) \geq 0$, for every $i \in I$. Given $\xi \in \mathcal{H}$, consider the normal functional $\tau_\xi(\cdot) := \langle \cdot\, \xi, \xi\rangle$ and observe that

$$0 \leq \langle \Phi_i(a)\xi, \xi\rangle = \tau_\xi(\Phi_i(a)) \longrightarrow \tau_\xi(\Phi(a)) = \langle \Phi(a)\xi, \xi\rangle\,.$$

Since the limit of a convergent net of non-negative real numbers is a non-negative real number, it follows that $\langle \Phi(a)\xi, \xi\rangle \geq 0$, for every $\xi \in \mathcal{H}$. Thus, $\Phi(a) \geq 0$ and Φ is a positive map. Moreover, if $a = 1$, the above argument leads to $\langle \xi, \xi\rangle \longrightarrow \langle \Phi(1)\xi, \xi\rangle$, for every $\xi \in \mathcal{H}$. Therefore, $\Phi(1) = 1$ and Φ is a unital positive map.

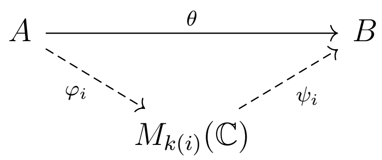
Similarly, for any $n \in \mathbb{N}$, one can prove that $\Phi_i^{(n)} \longrightarrow \Phi^{(n)}$ in the point-ultraweak topology of $B(M_n(A), M_n(M))$. Hence, Φ is a u.c.p. map. □

Now that the fundamental theorems about completely positive maps have been proven, we end this section with a definition of nuclearity in terms of nets of c.c.p. maps. To ease notation, sometimes we omit the directed set of a net, in particular, when we are dealing with nets of maps.

Definition 1.2.12. Let A and B be C*-algebras. A map $\theta : A \to B$ is called **nuclear** if there exists two nets of c.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ and $\psi_i : M_{k(i)}(\mathbb{C}) \to B$ such that, for all $a \in A$,

$$\lim_i \|\psi_i \circ \varphi_i(a) - \theta(a)\| = 0\,.$$

Intuitively, we have the following diagrams, which asymptotically commute pointwise.



Definition 1.2.13. A C*-algebra A is called **nuclear** if the identity map $\mathrm{id}_A : A \to A$ is nuclear.

Remark 1.2.14. The definition of nuclear map and nuclear C*-algebras can also be characterized by a local property. This means, e.g., that a C*-algebra is nuclear if and only if for each finite set $F \subseteq A$ and $\varepsilon > 0$ there exist c.c.p. maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$, for all $a \in F$. Examples of the process of transitioning between a net-based definition and characterization by local properties will be discussed in the next chapter.

In [Brown and Ozawa, 2008, §2.4] we can see several examples of nuclear C*-algebras. For example, using a constant net of a diagonal map, we can prove that finite-dimensional C*-algebras are nuclear. Furthermore, inductive limits of nuclear algebras are nuclear, such as AF-algebras.

Example 1.2.15. Every abelian C*-algebra is nuclear. First, we will prove this when $A = C(X)$ for some compact Hausdorff space X, i.e., when A is unital. Let $F \subseteq A$ be a non-empty finite subset and $\varepsilon > 0$. By compactness there is a finite open cover $U_1, \ldots, U_n$ of X such that

$$|f(x) - f(y)| < \varepsilon\,, \tag{1.3}$$

for all $f \in F$ and $x, y \in U_i$ $(1 \le i \le n)$. Furthermore, let $\sigma_1, \ldots, \sigma_n$ be a partition of the unity subordinate to the cover above, and choose $y_i \in U_i$ arbitrarily. Define $\varphi : A \to M_n(\mathbb{C})$ by

$$f \mapsto \begin{pmatrix} f(y_1) & & 0 \\ & \ddots & \\ 0 & & f(y_n) \end{pmatrix}.$$

Notice that φ is a unital $*$-homomorphism, hence it is a u.c.p. map.

Now define $\psi : M_n(\mathbb{C}) \to A$ by

$$[m_{ij}] \longmapsto \sum_{i=1}^{n} m_{ii}\sigma_i\,.$$

Denote by $\{e_{ij}\}_{i,j=1}^{n}$ the canonical basis of $M_n(\mathbb{C})$ and observe that

$$[\psi(e_{ij})] = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} \geq 0\,.$$

Thus ψ is c.p. by Proposition 1.2.8. More specifically, ψ is u.c.p. since it is unital.

Hence, for any $f \in F$,

$$\begin{aligned} \|f - \psi \circ \varphi(f)\| &= \left\| (\sum_{i=1}^{n} \sigma_i) f - \sum_{i=1}^{n} f(y_i)\sigma_i \right\| \\ &= \left\| \sum_{i=1}^{n} (f - f(y_i)1)\sigma_i \right\| \\ &= \sup_{x \in X} \left| \sum_{i=1}^{n} (f(x) - f(y_i))\sigma_i(x) \right| \leq \varepsilon\,. \end{aligned}$$

Where the last inequality follows from $\sigma_i|_{X \setminus U_i} = 0$ and (1.3). Therefore, $A = C(X)$ is nuclear.

Finally, in the nonunital case, we will prove the general result that the unitization $\tilde{A}$ being nuclear implies that A is nuclear. Let $F \subseteq A$ be a non-empty finite subset and $\varepsilon > 0$. Notice that $F \subseteq \tilde{A}$, hence there exists c.c.p. maps $\varphi : \tilde{A} \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to \tilde{A}$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon/2$, for all $a \in F$. Furthermore, there exist a positive element $u \in A^+$ such that $\|u\| \leq 1$ and $\|uau - a\| < \varepsilon/2$, for all $a \in F$ (this u can be viewed as an element of an approximate unit of A). Therefore, define $\Phi : A \to M_n(\mathbb{C})$ as $\varphi|_A$ and $\Psi : M_n(\mathbb{C}) \to A$ by

$$m \longmapsto u\psi(m)u\,.$$

Note that Φ and Ψ are c.c.p. maps and satisfy, for each $a \in F$,

$$\|\Psi \circ \Phi(a) - a\| = \|u\psi(\varphi(a))u - a\| \leq \underbrace{\|u\psi(\varphi(a))u - uau\|}_{\leq \|\psi(\varphi(a)) - a\|} + \|uau - a\| < \varepsilon\,.$$

Therefore, A is nuclear.

1.3 Crossed products and group C*-algebras

This section contains an introduction to crossed products of C*-dynamical systems. Although we do not use crossed products again until Chapter 3, they are presented here because the construction group C*-algebras is a mere “corollary” of the following presentation. The main references to this section are [Brown and Ozawa, 2008, Chapter 4], [Blackadar, 2006, §II.10.3], and [Davidson, 1996, Chapter VIII].

We denote by G a discrete group. Similarly to the C*-case, we say that $(\rho, \mathcal{H})$ is a **(unitary) representation** of G if the map $\rho : G \to \mathcal{U}B(\mathcal{H})$ is a group homomorphism. We will use the notation $\rho_g = \rho(g)$.

Definition 1.3.1. An **action** of G on A is a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ denote the group of $*$-automorphisms on A. We will denote $\alpha(g)$ by α_g, for each $g \in G$. The triplet (A, G, α) is called a **C*-dynamical system**.

Remark 1.3.2. A topological dynamical system is an action θ of a (discrete) group G on a topological space X. If X is locally compact Hausdorff, this action induces an action α of G on the abelian C*-algebra $C_0(X)$ by

$$\alpha_g(f)(x) := f(\theta_{g^{-1}}(x)) \qquad (g \in G, f \in C_0(X), x \in X)\,.$$

Hence, C*-dynamical systems can be viewed as a noncommutative version of topological dynamical systems.

Consider the vector space $C_c(G, A) = \{f : G \to A \mid f \text{ has compact support}\}$. Since G is a discrete group this means that f has finite support, hence an element $f \in C_c(G, A)$ can be seen as a finite formal sum $f = \sum_{g\in G} a_g\delta_g$, where $a_g \in G$ and δ_g are seen as “place markers”. In other words, $a_g\delta_g$ can be seen as the function that has the value a_g at g and is 0 otherwise.

Definition 1.3.3. Let (A, G, α) be a C*-dynamical system. We define the “α-twisted multiplication” on $C_c(G, A)$ by

$$\left(\sum_{g\in G} a_g\delta_g\right)\left(\sum_{h\in G} b_h\delta_h\right) := \sum_{g,h\in G} a_g\alpha_g(b_h)\delta_{gh}\,, \tag{1.4}$$

and define the involution by

$$\left(\sum_{g\in G} a_g\delta_g\right)^* := \sum_{g\in G} \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}}\,. \tag{1.5}$$

Remark 1.3.4. The definition above may seem unusual at first glance, but the following calculations aim to provide some intuition for the reader. The goal of the crossed product is to encode the action α in such a way that the action becomes inner. Intuitively, this means that

$$\alpha_g(a) = \delta_g a\delta_{g^{-1}}, \qquad (g \in G,\ a \in A). \tag{†}$$

Using this idea, we calculate the product in the crossed product algebra as follows:

$$(a\delta_g)(b\delta_h) = a\delta_g b(\delta_{g^{-1}}\delta_g)\delta_h = a(\delta_g b\delta_{g^{-1}})\delta_g\delta_h \overset{(†)}{=} a\alpha_g(b)\delta_{gh}.$$

For the involution, we interpret it as an inversion (i.e., "$g^* = g^{-1}$"), leading to:

$$(a\delta_g)^* = \delta_g^* a^* = \delta_g^* a^* (\delta_g \delta_{g^{-1}}) = (\delta_{g^{-1}} a^* \delta_g)\delta_{g^{-1}} \overset{(\dagger)}{=} \alpha_{g^{-1}}(a^*)\delta_{g^{-1}}.$$

It is straightforward to verify that $C_c(G, A)$ is a $*$-algebra under the operations defined above. The next question is: *Does there exist a C*-norm on $C_c(G, A)$ that allows us to complete it into a C*-algebra?*

The answer is affirmative, as we shall demonstrate.

Definition 1.3.5. Let (A, G, α) be a C*-dynamical system. A **covariant representation** is a triplet $(\pi, \rho, \mathcal{H})$ where $\pi : A \to B(\mathcal{H})$ is a representation of A and $\rho : G \to B(\mathcal{H})$ is a unitary representation of G such that, for every $a \in A$ and $g \in G$,

$$\rho_g \pi(a) \rho_g^* = \pi(\alpha_g(a))\,. \tag{1.6}$$

Construction 1.3.6 (Integrated form). Given $(\pi, \rho, \mathcal{H})$ a covariant representation we will construct a $*$-representation $\pi \rtimes \rho$ of $C_c(G, A)$, which is sometimes called the *integrated form*. Given $f = \sum_{g\in G} a_g \delta_g$, we define

$$\pi \rtimes \rho(f) := \sum_{g\in G} \pi(a_g)\rho_g\,. \tag{1.7}$$

If $f_1 = \sum_{g\in G} a_g \delta_g$ and $f_2 = \sum_{h\in G} b_h \delta_h$, then

$$\begin{aligned}
\pi \rtimes \rho(f_1) \cdot \pi \rtimes \rho(f_2) &= \left(\sum_{g\in G} \pi(a_g)\rho_g\right) \cdot \left(\sum_{h\in G} \pi(b_h)\rho_h\right) \\
&= \sum_{g,h\in G} \pi(a_g)(\rho_g \pi(b_h)\rho_g^*)\rho_g \rho_h \\
&\overset{(1.6)}{=} \sum_{g,h\in G} \pi(a_g)\pi(\alpha_g(b_h))\rho_{gh} \\
&= \pi \rtimes \rho \left(\sum_{g,h\in G} a_g \alpha_g(b_h)\delta_{gh}\right) \\
&\overset{(1.4)}{=} \pi \rtimes \rho(f_1 \cdot f_2)\,,
\end{aligned}$$

and

$$\begin{aligned}
\pi \rtimes \rho(f)^* &= \left(\sum_{g\in G} \pi(a_g)\rho_g\right)^* \\
&= \sum_{g\in G} \rho_g^* \pi(a_g)^* \\
&= \sum_{g\in G} \rho_g^* \pi(a_g^*)\rho_g\rho_g^* \\
&= \sum_{g\in G} \rho_{g^{-1}} \pi(a_g^*)\rho_{g^{-1}}^* \rho_{g^{-1}} \\
&\overset{(1.6)}{=} \sum_{g\in G} \pi(\alpha_{g^{-1}}(a_g^*))\rho_{g^{-1}} \\
&= \pi \rtimes \rho\left(\sum_{g\in G} \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}}\right) \\
&\overset{(1.5)}{=} \pi \rtimes \rho(f^*)\,.
\end{aligned}$$

Conversely, every nondegenerate $*$-representation $(\sigma, \mathcal{H})$ of $C_c(G, A)$ yields a covariant representation by

$$\pi(a) := \sigma(a\delta_e) \text{ and } \rho_g := \sigma(1_A\delta_g)\,,$$

if A is unital. Indeed, observe that,

$$\pi \rtimes \rho\left(\sum_{g\in G} a_g\delta_g\right) = \sum_{g\in G} \pi(a_g)\rho_g = \sum_{g\in G} \sigma(a_g\delta_e)\sigma(1_A\delta_g) = \sigma\left(\sum_{g\in G} a_g\delta_g\right).$$

Similarly, if A is nonunital take an approximate unity $(e_\lambda)_{\lambda\in\Lambda}$ and define

$$\rho_g = \lim_\lambda \sigma(e_\lambda\delta_g),$$

where the limit is taken with respect to the strong topology. Using that ρ is nondegenerate, one checks that the limit exists and that (π, ρ) forms a covariant pair.

Construction 1.3.7 (**Regular representation**)**.** Consider the complex Hilbert space

$$\ell^2(G) := \left\{ (x_g)_{g\in G} \mid \sum_{g\in G} |x_g|^2 < \infty \right\}.$$

Elements of $\ell^2(G)$ can be viewed as complex-valued functions, and we denote by $\{\delta_g\}_{g\in G}$ the canonical orthonormal basis. (This is a slight abuse of notation, as the same symbol δ_g is used in $C_c(G, A)$; in both cases, the δ's serve as a "place marker.") With this convention, any element of $\ell^2(G)$ can be expressed as a formal sum $\sum_{g\in G} x_g\delta_g$.

Next, let $A \subset \mathcal{B}(\mathcal{H})$ be a faithful representation of A on a Hilbert space $\mathcal{H}$, and define the map $\pi : A \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ on elementary tensors by:

$$\pi(a)(\xi \otimes \delta_g) := \alpha_{g^{-1}}(a)(\xi) \otimes \delta_g\,, \tag{1.8}$$

for all $a \in A$, $g \in G$, and $\xi \in \mathcal{H}$. It is straightforward to check that π defines a faithful nondegenerate representation.

Now consider the *regular representation* $\lambda : G \to \mathcal{B}(\ell^2(G))$, defined on the orthonormal basis by:

$$\lambda_g(\delta_h) = \delta_{gh}, \qquad (g, h \in G) \tag{1.9}$$

Hence the map $1_\mathcal{H} \otimes \lambda : G \to B(\mathcal{H} \otimes \ell^2(G))$ given by

$$1_\mathcal{H} \otimes \lambda_g(\xi \otimes \delta_h) := \xi \otimes \delta_{gh}\,, \quad (g, h \in G,\, \xi \in \mathcal{H})$$

is a unitary representation of G[3].

Finally, observe that $(\pi, 1_\mathcal{H} \otimes \lambda, \mathcal{H} \otimes \ell^2(G))$ forms a *covariant representation*. Indeed, for any $a \in A$, $g \in G$, and $\xi \in \mathcal{H}$:

$$\begin{aligned}
(1_\mathcal{H} \otimes \lambda_g)\pi(a)(1_\mathcal{H} \otimes \lambda_g^*)(\xi \otimes \delta_h) &\overset{(1.9)}{=} (1_\mathcal{H} \otimes \lambda_g)\pi(a)(\xi \otimes \delta_{g^{-1}h}) \\
&\overset{(1.8)}{=} (1_\mathcal{H} \otimes \lambda_g)(\alpha_{h^{-1}g}(a)(\xi) \otimes \delta_{g^{-1}h}) \\
&\overset{(1.9)}{=} \alpha_{h^{-1}}(\alpha_g(a)(\xi)) \otimes \delta_h \\
&\overset{(1.8)}{=} \pi(\alpha_g(a))(\xi \otimes \delta_h).
\end{aligned}$$

Furthermore, $\pi \rtimes (1_\mathcal{H} \otimes \lambda)$ is *faithful*. To see this, suppose:

$$\pi \rtimes (1_\mathcal{H} \otimes \lambda)\left(\sum_{g \in G} a_g \delta_g\right) = 0.$$

This implies:

$$\sum_{g \in G} \pi(a_g)(1_\mathcal{H} \otimes \lambda_g) = 0.$$

Applying this to a generic elementary tensor $\xi \otimes \delta_h \in \mathcal{H} \otimes \ell^2(G)$ yields:

$$0 = \sum_{g \in G} \pi(a_g)(1_\mathcal{H} \otimes \lambda_g)(\xi \otimes \delta_h) = \sum_{g \in G} \alpha_{(gh)^{-1}}(a_g)(\xi) \otimes \delta_{gh}.$$

Since the set $\{\delta_{gh}\}_{g \in G}$ is linearly independent, it follows that:

$$\alpha_{(gh)^{-1}}(a_g)(\xi) = 0, \quad \text{for all } g \in G.$$

As ξ is arbitrary, we conclude that $\alpha_{(gh)^{-1}}(a_g) = 0$. Since $\alpha_{(gh)^{-1}}$ is an automorphism, it follows that $a_g = 0$ for all $g \in G$. Thus, $\sum_{g \in G} a_g \delta_g = 0$, proving that $\pi \rtimes (1_\mathcal{H} \otimes \lambda)$ is faithful.

Definition 1.3.8. Let (A, G, α) be a C*-dynamic system. The **(full) crossed product**, denoted $A \rtimes_\alpha G$, is the completion of $C_c(G, A)$ with respect of the C*-norm:

$$\|f\| := \sup_\sigma \|\sigma(f)\|\ , \tag{1.10}$$

where the supremum is taken over all (nondegenerate) $*$-representations of $C_c(G, A)$.

[3] About the existence of tensor product of operators, see Remark A.2.1.

Remark 1.3.9. The supremum is taken over a non-empty class because of the regular representation (see Construction 1.3.7). Moreover, the supremum exists because each representation σ can be viewed as the integrated form $\pi \rtimes \rho$ of some covariant system $(\pi, \rho, \mathcal{H})$. Consequently, for any σ,

$$\left\|\sigma\big(\sum_{g\in G} a_g\delta_g\big)\right\| = \left\|\sum_{g\in G}\pi(a_g)\rho_g\right\| \leq \sum_{g\in G}\underbrace{\|\pi(a_g)\|}_{\leq\|a_g\|}\underbrace{\|\rho_g\|}_{=1} \leq \sum_{g\in G}\|a_g\|\,. \tag{1.11}$$

This shows that the supremum is well-defined.

Definition 1.3.10. Let (A, G, α) be a C*-dynamical system. The **reduced crossed product**, denoted by $A \rtimes_{\alpha,r} G$, is defined as the completion of $C_c(G, A)$ with respect of the C*-norm:

$$\|f\|_r := \|\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)(f)\|\ , \tag{1.12}$$

where $\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)$ is the regular representation defined in Construction 1.3.7.

One can show that the reduced norm does not depend on the choice of the faithful representation $A \subset B(\mathcal{H})$(see [Brown and Ozawa, 2008, Theorem 4.1.5.]).

Proposition 1.3.11. *The reduced cross product $A \rtimes_{\alpha,r} G$ does not depend on the choice of the faithful representation $A \subset B(\mathcal{H})$.* □

Remark 1.3.12. Thanks to the regular representation, the reduced crossed product can be concretely realized as:

$$A \rtimes_{\alpha,r} G = \overline{\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)(C_c(G, A))} \subseteq B(\mathcal{H} \otimes \ell^2(G))\,.$$

Proposition 1.3.13 (Universal Property of the Crossed Product). *Let (A, G, α) be a C*-dynamical system and $(\pi, \rho, \mathcal{H})$ be a covariant representation. Then, there is a unique $*$-homomorphism $\sigma : A \rtimes_\alpha G \to B(\mathcal{H})$ such that*

$$\sigma\left(\sum_{g\in G} a_g\delta_g\right) = \sum_{g\in G}\pi(a_g)\rho_g\,,$$

for every $\sum_{g\in G} a_g\delta_g \in C_c(G, A)$.

Proof. We just need to check that $\sigma = \pi \rtimes \rho$ defined in (1.7) is continuous in the full norm. But this follows from the definition of the full norm as the supremum over all σ as in (1.10). □

Example 1.3.14 (Group C*-algebras). Although group C*-algebras can be constructed independently of crossed products, their definition emerges naturally as a "corollary" of the crossed product framework.

Consider the C*-algebra $A = \mathbb{C}$ with the trivial action $\nu_g = \mathrm{id}_{\mathbb{C}}$ for all $g \in G$. In this case, $C_c(G, \mathbb{C}) \cong \mathbb{C}[G]$, the group algebra of G, with multiplication given by:

$$(x\delta_g)(y\delta_h) = xy\delta_{gh}\,,$$

and involution given by:

$$(x\delta_g)^* = \overline{x}\delta_{g^{-1}}\,.$$

Then we say that the **(full) group C*-algebra** is $C^*(G) = \mathbb{C} \rtimes_\nu G$ and the **reduced group C*-algebra** is $C^*_r(G) = \mathbb{C} \rtimes_{\nu,r} G$.

The universal property of the crossed product (Proposition 1.3.13) translates directly to the group C*-algebra $C^*(G)$ as follows. Since every unitary representation $\rho : G \to B(\mathcal{H})$ gives rise to a covariant representation of $(\mathbb{C}, G, \nu)$, there exists a unique $*$-homomorphism $\pi : C^*(G) \to B(\mathcal{H})$ such that $\pi(\delta_g) = \rho_g$ for all $g \in G$.

In particular, the trivial representation $G \to \mathbb{C}$, given by $g \mapsto 1$, induces a $*$-homomorphism $\epsilon : C^*(G) \to \mathbb{C}$. Thus, the group C*-algebra $C^*(G)$ always admits a character.

On the other hand, the regular representation $\lambda : G \to B(\ell^2(G))$ can be linearly extended to $\mathbb{C}[G]$. Then, the reduced group C*-algebra can be concretely represented as:

$$C^*_r(G) = \overline{\lambda(\mathbb{C}[G])} \subseteq B(\ell^2(G))\,.$$

More specifically:

$$C^*_r(G) = C^*\big(\{\lambda_g \mid g \in G\}\big) \subseteq B(\ell^2(G))\,.$$

Furthermore, the map $\mathbb{C}[G] \to C^*_r(G)$ defined on generators by $\delta_g \mapsto \lambda_g$, is continuous when $\mathbb{C}[G]$ is equipped with the full norm. Hence, there is a surjective $*$-homomorphism $\pi : C^*(G) \to C^*_r(G)$, called the **canonical quotient map**.

Finally, in this context, the **group von Neumann algebra** is defined as:

$$L(G) \coloneqq C^*_r(G)'' \subseteq B(\ell^2(G))\,,$$

where $C^*_r(G)''$ denotes the bicommutant of $C^*_r(G)$ in $B(\ell^2(G))$.

Remark 1.3.15. Let A be a unital C*-algebra, G be a discrete group and consider the $*$-homomorphism $\iota : \mathbb{C}[G] \to A \rtimes_{\alpha,r} G$ given by

$$\sum_{g\in G} x_g\delta_g \longmapsto \sum_{g\in G} x_g 1_A \delta_g\,,$$

where the domain is equipped with the reduced C*-norm from $C_r^*(G)$. Note that, for any $\sum_{g\in G} x_g\delta_g \in \mathbb{C}[G]$,

$$\begin{aligned}\big\|\iota(\sum_{g\in G} x_g\delta_g)\big\|_{A\rtimes_{\alpha,r}G} &= \big\|\pi\rtimes(1_{\mathcal{H}}\otimes\lambda)(\sum_{g\in G} x_g 1_A\delta_g)\big\|_{B(\mathcal{H}\otimes\ell^2(G))}\\ &= \big\|\sum_{g\in G} x_g \underbrace{\pi(1_A)}_{=1}(1_{\mathcal{H}}\otimes\lambda_g)\big\|_{B(\mathcal{H}\otimes\ell^2(G))}\\ &= \big\|1_{\mathcal{H}}\otimes\big(\sum_{g\in G} x_g\lambda_g\big)\big\|_{B(\mathcal{H}\otimes\ell^2(G))}\\ &= \|1_{\mathcal{H}}\|_{B(\mathcal{H})}\,\big\|\sum_{g\in G} x_g\lambda_g\big\|_{B(\ell^2(G))}\\ &= \big\|\sum_{g\in G} x_g\lambda_g\big\|_{B(\ell^2(G))} = \big\|\sum_{g\in G} x_g\delta_g\big\|_{C_r^*(G)}\,.\end{aligned}$$

Therefore, ι is an isometry an can be extended to a unital injective $*$-homomorphism from $C_r^*(G)$ to $A\rtimes_{\alpha,r} G$.

Example 1.3.16 (Rotational algebras). Denote the unit circle of $\mathbb{C}$ by $\mathbb{T}$ and let $\theta > 0$ be a real number. The rotational algebra is the universal C*-algebra A_θ generated by two unitary operators u and v that satisfies

$$uv = e^{2\pi i\theta} vu\,.$$

Equivalently, A_θ can be described as the crossed product $C(\mathbb{T})\rtimes_{\alpha_\theta}\mathbb{Z}$, where α_θ is given by:

$$\alpha_\theta(f)(z) := f(e^{2\pi i\theta}z)\,,$$

for every $f\in C(\mathbb{T})$ and $z\in\mathbb{T}$ (see [Davidson, 1996, Example VIII.1.1]). Since α_θ is a $*$-automorphism of $C(\mathbb{T})$, it induces an action of $\mathbb{Z}$ on $C(\mathbb{T})$, where $n\cdot f = \alpha_\theta^n(f)$ for all $n\in\mathbb{Z}$.

Example 1.3.17 (Uniform Roe algebras). The uniform Roe algebra of a discrete group G can be defined as the C*-subalgebra $C_u^*(G)$ of $B(\ell^2(G))$ generated by the reduced group C*-algebra $C_r^*(G)$ and $\ell^\infty(G)$. An equivalent characterization is provided on [Brown and Ozawa, 2008, Proposition 5.1.3]:

$$C_u^*(G)\cong \ell^\infty(G)\rtimes_{\alpha,r} G\,,$$

where the action of G on $\ell^\infty(G)$ is given by:

$$\alpha_g(f)(h) := f(g^{-1}h)\,,$$

for every $f\in\ell^\infty(G)$ and $h\in G$.

Notes and Remarks. The universal property of the (full) crossed product can be slightly improved: for every $*$-homomorphism $\varphi : C_c(A,G)\to B$ there is a unique $*$-homomorphism

$\overline{\varphi} : A \rtimes_\alpha G \to B$ extending φ. In particular, the inclusion $C_c(A,G) \hookrightarrow A \rtimes_{\alpha,r} G$ can be extended to a surjective $*$-homomorphism $\pi : A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$. When π is injective, it follows that $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$, which is called the *weak containment* property.

Furthermore, the full group C*-algebra can be viewed as the universal C*-algebra generated by the set of operators $\{\delta_g \mid g \in G\}$ subject to relations

$$\delta_e = 1\,,\ \delta_g\delta_h = \delta_{gh} \quad \text{and} \quad \delta_g^* = \delta_{g^{-1}}\,.$$

Then $C^*(G)$ has the following universal property: for every unitary representation $\rho : G \to B(\mathcal{H})$ there exists a unique $*$-homomorphism $\overline{\rho} : C^*(G) \to B(\mathcal{H})$ such that $\overline{\rho}(\delta_g) = \rho_g$ for every $g \in G$.

1.4 Hilbert-Schmidt and trace-class operators

In this section, we step back from the C*-algebraic prerequisites and visit Hilbert space theory. Although the content of this section is considered standard, it is often omitted from introductory courses in Functional Analysis or Operator Algebras. Therefore, we provide a brief and self-contained presentation of Hilbert-Schmidt and trace-class operators. The results here will prove useful in Section 2.2 and beyond. This presentation is largely based on [Murphy, 2014, §2.4] and, to a lesser extent, on [Blackadar, 2006, §1.8], while the proof of the main technical result, the Powers-Størmer inequality (1.4.18), draws from [Brown, 2006, Proposition 3.1.3] and [Brown and Ozawa, 2008, Proposition 6.2.4].
Notation convention: Unless otherwise stated, $\{\xi_i\}_{i\in I}$ will denote an orthonormal basis of a Hilbert space $\mathcal{H}$. Also, if $u \in B(\mathcal{H})$, then $\operatorname{rank}(u) = \dim(\overline{u(\mathcal{H})})$.

Definition 1.4.1. Let $u \in B(\mathcal{H})$ and $\{\xi_i\}_{i\in I} \subseteq \mathcal{H}$ be an orthonormal basis of $\mathcal{H}$. Define the **Hilbert-Schmidt norm** of u by

$$\|u\|_2 := \left(\sum_{i\in I} \|u(\xi_i)\|^2\right)^{1/2}.$$

If $\|u\|_2 < \infty$, we say that u is a **Hilbert-Schmidt operator**. We denote by $\mathcal{L}^2(\mathcal{H})$ the set of Hilbert-Schmidt operators.

Proposition 1.4.2. *Let $u, v \in B(\mathcal{H})$, and $\lambda \in \mathbb{C}$. Then*

(i) $\|.\|_2$ *is independent of the choice of orthonormal basis;*

(ii) $\|u+v\|_2 \le \|u\|_2 + \|v\|_2$, $\|\lambda u\|_2 = |\lambda|\,\|u\|_2$, *and* $\|u^*\|_2 = \|u\|_2$ *;*

(iii) $\|u\| \le \|u\|_2$ *;*

(iv) $\|uv\|_2 \le \|u\|\,\|v\|_2$ *and* $\|uv\|_2 \le \|u\|_2\,\|v\|$ *.*

Proof. **(i)** Let $\{\xi_i\}_{i\in I}$ and $\{\zeta_i\}_{i\in I}$ be orthonormal bases of $\mathcal{H}$. Hence, for each non-empty finite set $F \subseteq I$,

$$\sum_{i\in F} \|u(\xi_i)\|^2 = \sum_{i\in F}\sum_{j\in I} |\langle u(\xi_i), \zeta_j\rangle|^2 = \sum_{j\in I}\sum_{i\in F} |\langle \xi_i, u^*(\zeta_j)\rangle|^2 \leq \sum_{j\in I} \|u^*(\zeta_j)\|^2 .$$

Therefore, letting the sets F go bigger and bigger we conclude that

$$\sum_{i\in I} \|u(\xi_i)\|^2 \leq \sum_{i\in I} \|u^*(\zeta_i)\|^2 .$$

By symmetry,

$$\sum_{i\in I} \|u^*(\zeta_i)\|^2 \leq \sum_{i\in I} \|u(\xi_i)\|^2 , \qquad (u \rightsquigarrow u^*,\ \xi_i \rightsquigarrow \zeta_i)$$

thus

$$\sum_{i\in I} \|u(\xi_i)\|^2 = \sum_{i\in I} \|u^*(\zeta_i)\|^2 .$$

Again by symmetry, changing ξ_i by ζ_i in the right side of the equality above, we conclude that

$$\sum_{i\in I} \|u(\xi_i)\|^2 = \sum_{i\in I} \|u(\zeta_i)\|^2 \ \Big(= \sum_{i\in I} \|u^*(\zeta_i)\|^2 \Big) .$$

Thus $\|\cdot\|_2$ does not depend of the choice of basis.
(ii) Notice that the equation above implies that $\|u^*\|_2 = \|u\|_2$. For the first inequality, assume that $F \subseteq I$ is a non-empty finite set. Then

$$\sqrt{\sum_{i\in F} \|u(\xi_i) + v(\xi_i)\|^2} \leq \sqrt{\sum_{i\in F} \big(\|u(\xi_i)\| + \|v(\xi_i)\| \big)^2} \leq$$

$$\overset{(\dagger)}{\leq} \sqrt{\sum_{i\in F} \|u(\xi_i)\|^2} + \sqrt{\sum_{i\in F} \|v(\xi_i)\|^2} \leq \sqrt{\sum_{i\in I} \|u(\xi_i)\|^2} + \sqrt{\sum_{i\in I} \|v(\xi_i)\|^2} ,$$

where ($\dagger$) follows from Minkowski's inequality. Letting the set F on the left side growing towards I, we conclude that $\|u+v\|_2 \leq \|u\|_2 + \|v\|_2$, as desired. On the other hand, the equality $\|\lambda u\|_2 = |\lambda|\,\|u\|_2$ is trivial.
(iii) Let $\xi \in \mathcal{H}$ be an arbitrary unit vector. Hence, there is an orthonormal basis $\{\xi_i\}_{i\in I}$ that contains ξ. From $\|u(\xi)\|^2 \leq \sum_{i\in I} \|u(\xi_i)\|^2 = \|u\|_2^2$, it follows that $\|u\| \leq \|u\|_2$.
(iv) The first inequality follows from the following calculation.

$$\|uv\|_2^2 = \sum_{i\in I} \|uv(\xi_i)\|^2 \leq \sum_{i\in I} \|u\|^2 \|v(\xi_i)\|^2 = \|u\|^2 \|v\|_2^2 .$$

In particular, using the first inequality and *(ii)* it follows that

$$\|uv\|_2 = \|v^*u^*\|_2 \leq \|v^*\|\,\|u^*\|_2 = \|v\|\,\|u\|_2 ,$$

as desired. □

Remark 1.4.3. It follows from conditions *(ii) - (iv)* that $(\mathcal{L}^2(\mathcal{H}), \|\cdot\|_2)$ is a Banach $*$-algebra. Additionally, condition *(iv)* implies that $\mathcal{L}^2(\mathcal{H})$ is a self-adjoint ideal of $B(\mathcal{H})$, although not closed in general. Furthermore, every $u \in \mathcal{L}^2(\mathcal{H})$ is a compact operator, since it is a limit of finite-rank operators. Indeed, define $F_n := \{i \in I \mid \|u(\xi_i)\| \leq 1/n\}$, for each $n \in \mathbb{N}$. Also define a finite-rank operator u_n by

$$u_n(\zeta) := \sum_{i \in F_n} \langle \zeta, \xi_i \rangle u(\xi_i) \,,$$

for all $\zeta \in \mathcal{H}$. In particular, note that $u_n(\xi_i) = u(\xi_i)$ for every $i \in F_n$. Hence,

$$\|u - u_n\|_2^2 = \sum_{i \in I} \|(u - u_n)(\xi_i)\|^2 = \sum_{i \in I \setminus F_n} \|u(\xi_i)\|^2 \longrightarrow 0 \,,$$

since the last sum is the tail of a convergent series. From condition *(iii)*, it follows that $u_n \longrightarrow u$ in norm, as desired.

Lemma 1.4.4 (Polarization Identity). *Let $u_1, u_2 \in \mathcal{L}^2(\mathcal{H})$ and $\{\xi_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. If $v = u_1^* u_2$, then*

$$\sum_{i \in I} \langle v(\xi_i), \xi_i \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \left\| u_2 + i^k u_1 \right\|_2^2 \,. \tag{1.13}$$

Proof. First, we will prove that the left side of the equation above is finite. Indeed, let $F \subseteq I$ be a non-empty finite subset. Hence,

$$\begin{aligned}\sum_{i \in F} \langle v(\xi_i), \xi_i \rangle = \sum_{i \in F} \langle u_2(\xi_i), u_1(\xi_i) \rangle \leq \sum_{i \in F} \|u_2(\xi_i)\| \, \|u_1(\xi_i)\| \leq \\ \overset{\text{CBS}}{\leq} \sqrt{\sum_{i \in F} \|u_2(\xi_i)\|^2} \sqrt{\sum_{i \in F} \|u_1(\xi_i)\|^2} \leq \|u_2\|_2 \, \|u_1\|_2 \,,\end{aligned}$$

where (CBS) stands for the Cauchy-Bunyakovsky-Schwarz inequality. Therefore,

$$\sum_{i \in I} \langle v(\xi_i), \xi_i \rangle < \infty \,.$$

Finally, Equation 1.13 follows from the polarization identity below

$$\langle v(\zeta), \zeta \rangle = \langle u_2(\zeta), u_1(\zeta) \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \left\| u_2(\zeta) + i^k u_1(\zeta) \right\|^2 \,,$$

for all $\zeta \in \mathcal{H}$. Thus

$$\sum_{i \in I} \langle v(\xi_i), \xi_i \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \sum_{i \in I} \left\| (u_2 + i^k u_1)(\xi_i) \right\|^2 = \frac{1}{4} \sum_{k=0}^{3} i^k \left\| u_2 + i^k u_1 \right\|_2^2 \,,$$

as desired. ∎

Definition 1.4.5. Let $u \in B(\mathcal{H})$ and $\{\xi_i\}_{i\in I} \subseteq \mathcal{H}$ be an orthonormal basis of $\mathcal{H}$. Define the **trace-class norm** of u to be

$$\|u\|_1 = \sum_{i\in I} \langle |u|(\xi_i), \xi_i\rangle \,,$$

where $|u| = (u^*u)^{1/2} \geq 0$. If $\|u\|_1 < \infty$ we say that u is a **trace-class operator**. We denote by $\mathcal{L}^1(\mathcal{H})$ the set of trace-class operators.

Remark 1.4.6. It os not hard to see that

$$\|u\|_1 = \left\||u|^{\frac{1}{2}}\right\|_2^2 \text{ and } \|u\|_2 = \|u^*u\|_1^{1/2} \,.$$

Hence, we could define trace-class norm in terms of Hilbert-Schmidt norm and vice versa. In particular, the definition of $\|\cdot\|_1$ does not depend on the choice of orthonormal basis.

The trace-class operators have similar properties to the Hilbert-Schmidt operators, but the proofs will be often more difficult. One key ingredient on those proofs will be the polar decomposition.

Proposition 1.4.7 (Polar decomposition). *If $v \in B(\mathcal{H})$, then there is a unique partial isometry*[4] *$u \in B(\mathcal{H})$ satisfying*

$$v = u|v| \text{ and } \ker(v) = \ker(u) \,.$$

Furthermore, $u^*v = |v|$.

Proof. Note that, for every $\xi \in \mathcal{H}$,

$$\||v|(\xi)\|^2 = \langle |v|^2(\xi), \xi\rangle = \langle v^*v(\xi), \xi\rangle = \|v(\xi)\|^2 \,.$$

Thus the map $u_0 : |v|(\mathcal{H}) \to \mathcal{H}$ defined by

$$|v|(\xi) \longmapsto v(\xi)$$

is well-defined, linear and isometric. Hence, the extension $u_1 : \overline{|v|(\mathcal{H})} \to \mathcal{H}$ is isometric as well. Now, define $u \in B(\mathcal{H})$ by

$$u(\xi) := \begin{cases} u_1(\xi) & \text{, if } \xi \in \overline{|v|(\mathcal{H})} \\ 0 & \text{, if } \xi \in \overline{|v|(\mathcal{H})}^{\perp} \,. \end{cases}$$

Note that u is a partial isometry, because it is isometric on $\ker(u)^\perp = \overline{|v|(\mathcal{H})}$ by construction. Furthermore, $v = u|v|$ follows directly from the definition. Applying u^* on both sides we obtain $u^*v = u^*u|v|$, which implies $u^*v = |v|$, since u^*u is the projection onto $\ker(u)^\perp = \overline{|v|(\mathcal{H})}$.

[4] This means that u is isometric on $\ker(u)^\perp$, or, equivalently, that $uu^*u = u$ (see [Murphy, 2014, Theorem 2.3.3]).

Additionally, observe that $\ker(u) = \overline{|v|(\mathcal{H})}^{\perp} = \ker(|v|)$. On the other hand, from $v = u|v|$ and $u^*v = |v|$, it follows that $\ker(v) = \ker(|v|)$. Therefore, $\ker(v) = \ker(u)$, as desired.

Finally, suppose that $w \in B(\mathcal{H})$ is a partial isometry satisfying $v = w|v|$ and $\ker(v) = \ker(w)$. note that $u|_{\overline{|v|(\mathcal{H})}} = w|_{\overline{|v|(\mathcal{H})}}$, and $\ker(u) = \ker(v) = \overline{|v|(\mathcal{H})}^{\perp} = \ker(w)$. Therefore, $u = w$. □

Proposition 1.4.8. *Let $v \in B(\mathcal{H})$. The following are equivalent:*

(i) v is trace-class;

(ii) $|v|$ is trace-class;

(iii) $|v|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator;

(iv) There exist $u_1, u_2 \in \mathcal{L}^2(\mathcal{H})$ such that $v = u_1u_2$.

Proof. The implications **(i)⇒(ii)⇒(iii)** are trivial.
(iii)⇒(iv) Suppose that $|v|^{\frac{1}{2}} \in \mathcal{L}^2(\mathcal{H})$, and consider the polar decomposition of v, with $v = u|v|$. Define $u_1 := u|v|^{\frac{1}{2}}$ and $u_2 := |v|^{\frac{1}{2}}$. Then $u_1, u_2 \in \mathcal{L}^2(\mathcal{H})$ and $v = u_1u_2$.
(iv)⇒(i) Suppose that $v = u_1u_2$ with $u_1, u_2 \in \mathcal{L}^2(\mathcal{H})$. Let $v = w|v|$ be the polar decomposition of v. Then, $|v| = w^*v = w^*u_1u_2 = (u_1^*w)^*u_2$. By Lemma 1.4.4,

$$\|v\|_1 = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|u_2 + i^k u_1^* w\right\|_2^2 < \infty\,.$$

Thus $v \in \mathcal{L}^1(\mathcal{H})$. □

Definition 1.4.9. We define the **trace** of a trace-class operator $u \in \mathcal{L}^1(\mathcal{H})$ to be

$$\operatorname{Tr}(u) := \sum_{i\in I}\langle u(\xi_i), \xi_i\rangle\,.$$

Remark 1.4.10. Observe that $\operatorname{Tr}(u) < \infty$ by Proposition 1.4.8. But the converse is not true, since any operator with a zero diagonal has finite trace, even if it is not compact. Furthermore, when $\mathcal{H} = \mathbb{C}^n$ the trace of a matrix $[x_{ij}] \in M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$ is given by

$$\operatorname{Tr}([x_{ij}]) = \sum_{i=1}^{n}\langle [x_{ij}]e_i, e_i\rangle = \sum_{i=1}^{n} x_{ii}\,,$$

as expected. Here, $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$.

Theorem 1.4.11. *Let $u, v \in B(\mathcal{H})$. If either u and v are Hilbert-Schmidt operators or v is trace-class, then*

$$\operatorname{Tr}(uv) = \operatorname{Tr}(vu)\,.$$

Proof. Suppose that $u, v \in \mathcal{L}^2(\mathcal{H})$. note that, for any $k \in \{0,1,2,3\}$,

$$\left\|(v + i^k u^*)^*\right\|_2 = \underbrace{|i^k|}_{=1} \left\|\overline{i^k} u + v^*\right\|_2 = \left\|i^k(\overline{i^k} u + v^*)\right\|_2 = \left\|u + i^k v^*\right\|_2 . \tag{1.14}$$

Hence, by the polarization identity (Lemma 1.4.4), it follows that

$$\begin{aligned}\operatorname{Tr}(uv) = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|v + i^k u^*\right\|_2^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|(v + i^k u^*)^*\right\|_2^2 = \\ \overset{(1.14)}{=} \frac{1}{4}\sum_{k=0}^{3} i^k \left\|v + i^k u^*\right\|_2^2 = \operatorname{Tr}(vu) .\end{aligned} \tag{1.15}$$

Now suppose $u \in B(\mathcal{H})$ and $v \in \mathcal{L}^1(\mathcal{H})$. By Proposition 1.4.8, there exist u_1, $u_2 \in \mathcal{L}^2(\mathcal{H})$ such that $v = u_1 u_2$. Since $\mathcal{L}^2(\mathcal{H})$ is an ideal of $B(\mathcal{H})$, it follows that

$$\operatorname{Tr}(uv) = \operatorname{Tr}\big((uu_1)u_2\big) \overset{(1.15)}{=} \operatorname{Tr}\big(u_2(uu_1)\big) = \operatorname{Tr}\big((u_2 u)u_1\big) \overset{(1.15)}{=} \operatorname{Tr}\big(u_1(u_2 u)\big) = \operatorname{Tr}(vu) ,$$

as desired. □

Proposition 1.4.12. *Let $u, v \in B(\mathcal{H})$, and $\lambda \in \mathbb{C}$. Then*

(i) $\|u + v\|_1 \leq \|u\|_1 + \|v\|_1$, $\|\lambda u\|_1 = |\lambda| \, \|u\|_1$ *and* $\|u^*\|_1 = \|u\|_1$ *;*

(ii) $\|u\| \leq \|u\|_1$ *;*

(iii) $\|vu\|_1 \leq \|u\| \, \|v\|_1$ *and* $\|vu\|_1 \leq \|u\|_1 \, \|v\|$ *;*

(iv) $\|u\|_2 \leq \|u\|_1$.

Proof. **(i)** The equality $\|\lambda u\|_1 = |\lambda| \, \|u\|_1$ is trivial. Let $u = w|u|$, $v = \tilde{w}|v|$ and $u + v = \hat{w}|u + v|$ be the polar decompositions of u, v and $u + v$, respectively. Hence

$$|u + v| = \hat{w}^*(u + v) = \hat{w}^* w|u| + \hat{w}^* \tilde{w}|v| .$$

Then

$$\begin{aligned}
\|u+v\|_1 &= \sum_{i\in I}\langle |u+v|(\xi_i),\xi_i\rangle \\
&= \Big|\sum_{i\in I}\langle \hat{w}^* w|u|(\xi_i),\xi_i\rangle + \sum_{i\in I}\langle \hat{w}^*\tilde{w}|v|(\xi_i),\xi_i\rangle\Big| \\
&\le \sum_{i\in I}\big|\langle |u|^{1/2}(\xi_i), |u|^{1/2}w^*\hat{w}(\xi_i)\rangle\big| + \sum_{i\in I}\big|\langle |v|^{1/2}(\xi_i), |v|^{1/2}\tilde{w}^*\hat{w}(\xi_i)\rangle\big| \\
&\overset{\text{CBS}}{\le} \sqrt{\sum_{i\in I}\||u|^{1/2}(\xi_i)\|^2}\sqrt{\sum_{i\in I}\||u|^{1/2}w^*\hat{w}(\xi_i)\|^2} \\
&\qquad + \sqrt{\sum_{i\in I}\||v|^{1/2}(\xi_i)\|^2}\sqrt{\sum_{i\in I}\||v|^{1/2}\tilde{w}^*\hat{w}(\xi_i)\|^2} \\
&= \|u\|_1^{1/2}\left\||u|^{1/2}w^*\hat{w}\right\|_2 + \|v\|_1^{1/2}\left\||v|^{1/2}\tilde{w}^*\hat{w}\right\|_2 \\
&\le \|u\|_1^{1/2}\left\||u|^{1/2}\right\|_2 \underbrace{\|w^*\hat{w}\|}_{\le 1} + \|v\|_1^{1/2}\left\||v|^{1/2}\right\|_2 \underbrace{\|\tilde{w}^*\hat{w}\|}_{\le 1} \\
&\le \|u\|_1^{1/2}\|u\|_1^{1/2} + \|v\|_1^{1/2}\|v\|_1^{1/2} \\
&= \|u\|_1 + \|v\|_1 ,
\end{aligned}$$

as desired.

Furthermore, notice that $uu^* = w|u|^2w^*$ and $w^*w|u| = w^*u = |u|$. Hence

$$(w|u|w^*)^2 = w|u|w^*w|u|w^* = w|u|^2w^* = uu^* = |u^*|^2 .$$

Since $w|u|w^* \ge 0$, it follows that $|u^*| = w|u|w^*$. Therefore

$$\|u^*\|_1 = \mathrm{Tr}(|u^*|) = \mathrm{Tr}((w|u|)w^*) = \mathrm{Tr}(w^*(w|u|)) = \mathrm{Tr}(|u|) = \|u\|_1 .$$

(ii) Note that

$$\|u\|_1 = \left\||u|^{1/2}\right\|_2^2 \overset{(\dagger)}{\ge} \left\||u|^{1/2}\right\|^2 \overset{(\ddagger)}{=} \||u|\| = \|u\| ,$$

where (†) follows from *(iii)* in Proposition 1.4.2 and (‡) follows from the C*-identity.

(iii) Recall that, by Theorems 2.2.5 and 2.2.6. in [Murphy, 2014], if $a, b, c \in B(\mathcal{H})$, then

$$0 \le a \le b \text{ implies } c^*ac \le c^*bc \text{ and } a^{1/2} \le b^{1/2} .$$

Now apply these with $a = u^*u$, $b = \|u\|^2 1$ and $c = v$, then

$$|uv|^2 = (uv)^*(uv) = v^*u^*uv \le v^*v\,\|u\|^2 = |v|^2\,\|u\|^2 .$$

Applying the square root, it follows that $|uv| \le |v|\,\|u\|$. Therefore,

$$\|uv\|_1 = \sum_{i\in I}\langle |uv|(\xi_i),\xi_i\rangle \le \sum_{i\in I}\|u\|\,\langle |v|(\xi_i),\xi_i\rangle = \|u\|\,\|v\|_1 .$$

Additionally, $\|uv\|_1 = \|v^*u^*\|_1 \le \|v^*\|\,\|u^*\|_1 = \|u\|_1\,\|v\|$.

(iv) Note that

$$\|u\|_2^2 = \|u^*u\|_1 \le \|u^*\|\,\|u\|_1 \le \|u^*\|_1\,\|u\|_1 = \|u\|_1^2 .$$

Hence, $\|u\|_2 \le \|u\|_1$, as desired. □

Remark 1.4.13. It follows from the above proposition that $(\mathcal{L}^1(\mathcal{H}), \|\cdot\|_1)$ is a Banach $*$-algebra. Additionally, condition *(iii)* implies that $\mathcal{L}^1(\mathcal{H})$ is a self-adjoint ideal of $B(\mathcal{H})$, although not closed in general. Furthermore, it follows from condition *(iv)* that $\mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^2(\mathcal{H})$.

Theorem 1.4.14. *The function*

$$\begin{aligned} \operatorname{Tr}\colon \mathcal{L}^1(\mathcal{H}) &\to \mathbb{C} \\ u &\mapsto \operatorname{Tr}(u) \end{aligned}$$

is linear and, for every $v \in B(\mathcal{H})$ and $u \in \mathcal{L}^1(\mathcal{H})$,

$$|\operatorname{Tr}(vu)| \leq \|v\| \, \|u\|_1 \ .$$

Furthermore, for every $u, v \in \mathcal{L}^2(\mathcal{H})$,

$$|\operatorname{Tr}(vu)| \leq \|v\|_2 \, \|u\|_2 \ .$$

Proof. It is easy to see that the trace is linear. Suppose that $v \in B(\mathcal{H})$, $u \in \mathcal{L}^1(\mathcal{H})$ and consider $u = w|u|$ the polar decomposition of u. Hence

$$\begin{aligned} |\operatorname{Tr}(vu)| &= |\sum_{i\in I} \langle vu(\xi_i), \xi_i\rangle| \\ &\leq \sum_{i\in I} |\langle |u|^{1/2}(\xi_i), |u|^{1/2} w^* v^*(\xi_i)\rangle| \\ &\overset{\text{CBS}}{\leq} \sum_{i\in I} \big\| |u|^{1/2}(\xi_i)\big\| \, \big\| |u|^{1/2} w^* v^*(\xi_i)\big\| \\ &\overset{\text{CBS}}{\leq} \Big(\sum_{i\in I} \big\| |u|^{1/2}(\xi_i)\big\|^2\Big)^{1/2} \Big(\sum_{i\in I} \big\| |u|^{1/2} w^* v^*(\xi_i)\big\|^2\Big)^{1/2} \\ &= \underbrace{\big\| |u|^{1/2}\big\|_2} \big\| |u|^{1/2} w^* v^*\big\|_2 \\ &\leq \|u\|_1^{1/2} \ \big\| |u|^{1/2}\big\|_2 \underbrace{\|w^*\|}_{\leq 1} \|v^*\| \\ &\leq \|u\|_1 \, \|v\| \ . \end{aligned}$$

Similarly, if $u, v \in \mathcal{L}^2(\mathcal{H})$, then

$$\begin{aligned} |\operatorname{Tr}(vu)| &= |\sum_{i\in I} \langle vu(\xi_i), \xi_i\rangle| \\ &\leq \sum_{i\in I} |\langle u(\xi_i), v^*(\xi_i)\rangle| \\ &\overset{\text{CBS}}{\leq} \sum_{i\in I} \|u(\xi_i)\| \, \|v^*(\xi_i)\| \\ &\overset{\text{CBS}}{\leq} \Big(\sum_{i\in I} \|u(\xi_i)\|^2\Big)^{1/2} \Big(\sum_{i\in I} \|v^*(\xi_i)\|^2\Big)^{1/2} \\ &= \|u\|_2 \, \|v\|_2 \ , \end{aligned}$$

as desired. □

Remark 1.4.15. The last inequality of the theorem above is known as Cauchy-Schwarz inequality. This is because $\mathcal{L}^2(\mathcal{H})$ has a Hilbert space structure with inner product given by $\langle u, v\rangle := \mathrm{Tr}(v^*u)$, i.e., the Hilbert-Schmidt norm is induced by an inner product.

Remark 1.4.16. Denote by $\mathcal{F}(\mathcal{H})$ the set of all finite-rank operators of $B(\mathcal{H})$. It is standard knowledge in Functional Analysis that

$$\mathcal{F}(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^2(\mathcal{H}) \subseteq K(\mathcal{H}),$$

where the inclusion is proper if $\mathcal{H}$ is infinite dimensional. To see that $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{L}^1(\mathcal{H})$ holds, consider a finite-rank operator u. Since u can be written as a linear combination of positive finite-rank operators, we can suppose that $u \geq 0$. Consider $\{\xi_i\}_{i\in I}$ an orthonormal basis of $\mathcal{H}$ such that $u(\mathcal{H}) = \mathrm{span}\{\xi_i \mid i \in F\}$, where $F \subseteq I$ is a finite set. Since $\langle u(\xi_i), \xi_i\rangle = 0$ if $i \in I \setminus F$, it follows that

$$\|u\|_1 = \sum_{i\in I}\langle u(\xi_i), \xi_i\rangle = \sum_{i\in F}\langle u(\xi_i), \xi_i\rangle < \infty.$$

Thus, $u \in \mathcal{L}^1(\mathcal{H})$.

In particular, if $u = p \in B(\mathcal{H})$ is a finite-rank orthogonal projection, then

$$\|p\|_1 = \mathrm{Tr}(p) = \mathrm{rank}(p) \text{ and } \|p\|_2 = \sqrt{\mathrm{rank}(p)}.$$

Indeed, observe that the cardinality of F is equal to $\mathrm{rank}(p)$. Hence,

$$\|p\|_1 = \sum_{i\in I}\langle p(\xi_i), \xi_i\rangle = \sum_{i\in F}\langle \xi_i, \xi_i\rangle = \mathrm{rank}(p),$$

and

$$\|p\|_2 = \sqrt{\|p^*p\|_1} = \sqrt{\|p\|_1} = \sqrt{\mathrm{rank}(p)}.$$

Remark 1.4.17. Before the next result, it is necessary to observe that the Hilbert-Schmidt and trace-class norms are unitarily invariant. Let u be a unitary operator on $B(\mathcal{H})$ and $v \in \mathcal{L}^2(\mathcal{H})$. Hence,

$$\|uv\|_2^2 = \sum_{i\in I}\|uv(\xi_i)\|^2 = \sum_{i\in I}\langle \underbrace{u^*u}_{=1} v(\xi_i), v(\xi_i)\rangle = \sum_{i\in I}\|v(\xi_i)\|^2 = \|v\|_2^2.$$

Analogously, $\|vu\|_2 = \|u^*v^*\|_2 = \|v^*\|_2 = \|v\|_2$, since u^* is a unitary operator. On the other hand, if $v \in \mathcal{L}^1(\mathcal{H})$, then

$$\|uv\|_1 = \left\||uv|^{1/2}\right\|_2^2 = \left\|(v^*u^*uv)^{1/2}\right\|_2^2 = \left\|(v^*v)^{1/2}\right\|_2^2 = \|v\|_1.$$

Finally, $\|vu\|_1 = \|u^*v^*\|_1 = \|v^*\|_1 = \|v\|_1$.

Theorem 1.4.18 (Powers-Størmer inequality). *For any $0 \leq h, k \in \mathcal{L}^2(\mathcal{H})$, one has*

$$\|h-k\|_2^2 \leq \left\|h^2 - k^2\right\|_1 \leq \|h+k\|_2\,\|h-k\|_2.$$

In particular, if $u \in \mathcal{U}(B(\mathcal{H}))$ and $h \in \mathcal{F}(\mathcal{H})_+$ then

$$\left\|uh^{1/2} - h^{1/2}u\right\|_2 = \left\|uh^{1/2}u^* - h^{1/2}\right\|_2 \leq \|uhu^* - h\|_1^{1/2}.$$

Proof. The right inequality follows from the fact that

$$h^2 - k^2 = \frac{1}{2}((h+k)(h-k) + (h-k)(h+k)) \tag{1.16}$$

and $\|uv\|_1 \leq \|u\|_2 \|v\|_2$, for every $u, v \in \mathcal{L}^2(\mathcal{H})$.

Since $h-k \in \mathcal{L}^2(\mathcal{H})$ is a self-adjoint compact operator, consider the orthonormal basis of eigenvectors $\{\xi_i\}_{i\in I}$ with the corresponding set of (real) eigenvalues $\{\lambda_i\}_{i\in I}$.

Note that $h-k \leq h+k$ and $-(h-k) \leq h+k$ implies that, for each $i \in I$,

$$|\lambda_i| = |\langle (h-k)\xi_i, \xi_i\rangle| \leq \langle (h+k)\xi_i, \xi_i\rangle. \tag{1.17}$$

Also, note that

$$|\langle v\xi, \xi\rangle| \leq \langle |v|\xi, \xi\rangle\,, \tag{1.18}$$

for any self-adjoint operator $v \in B(\mathcal{H})$ and $\xi \in \mathcal{H}$.

Therefore,

$$\begin{aligned}
\|h-k\|_2^2 &= \sum_{i\in I} \langle (h-k)\xi_i, (h-k)\xi_i\rangle \\
&= \sum_{i\in I} \langle |h-k|^2 \xi_i, \xi_i\rangle \\
&= \sum_{i\in I} |\lambda_i|^2 \\
&\overset{(1.17)}{\leq} \sum_{i\in I} |\lambda_i| \langle (h+k)\xi_i, \xi_i\rangle \\
&= \sum_{i\in I} \left| \left\langle (h+k)(\underbrace{\tfrac{1}{2}\lambda_i\xi_i}_{\frac{1}{2}(h-k)\xi_i}), \xi_i \right\rangle + \left\langle (h+k)\xi_i, \underbrace{\tfrac{1}{2}\lambda_i\xi_i}_{\frac{1}{2}(h-k)\xi_i} \right\rangle \right| \\
&= \sum_{i\in I} \left| \left\langle \frac{1}{2}((h+k)(h-k) + (h-k)(h+k))\xi_i, \xi_i \right\rangle \right| \\
&\overset{(1.16)}{=} \sum_{i\in I} |\langle (h^2-k^2)\xi_i, \xi_i\rangle| \\
&\overset{(1.18)}{\leq} \sum_{i\in I} \langle |h^2-k^2|\xi_i, \xi_i\rangle \\
&= \left\|h^2-k^2\right\|_1
\end{aligned}$$

Finally, in the particular case which u is a unitary and $h \geq 0$ has finite-rank, the inequality follows from the first inequality and the fact that the trace-class and Hilbert-Schmidt norms are unitarily invariant. □

Notes and Remarks. As promised, we give a concrete presentation of the predual of a von Neumann algebra $M \subseteq B(\mathcal{H})$. First, for each $v \in B(\mathcal{H})$ define the linear map $\mathrm{Tr}(\cdot\, v) : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ by

$$u \longmapsto \mathrm{Tr}(uv)\,.$$

Since, $|\operatorname{Tr}(uv)| \leq \|u\|_1 \|v\|$, it follows that $\operatorname{Tr}(\cdot\, v)$ is continuous, i.e., $\operatorname{Tr}(\cdot\, v) \in \mathcal{L}^1(\mathcal{H})^*$.

The map $\Phi : B(\mathcal{H}) \to \mathcal{L}^1(\mathcal{H})^*$ given by

$$v \longmapsto \operatorname{Tr}(\cdot\, v)$$

is called the canonical map from $B(\mathcal{H})$ to $\mathcal{L}^1(\mathcal{H})^*$, and is an isometric linear isomorphism (see [Murphy, 2014, Theorem 4.2.3.]). In particular, $\mathcal{L}^1(\mathcal{H})$ is the predual of $B(\mathcal{H})$.

As a consequence, the weak* topology of $B(\mathcal{H})$ coming from the identification with $\mathcal{L}^1(\mathcal{H})^*$ is the ultraweak topology. In terms of convergence, this means that a net $(v_\lambda)_{\lambda\in\Lambda} \subseteq B(\mathcal{H})$ converges ultraweakly to an operator $v \in B(\mathcal{H})$ if and only if

$$\operatorname{Tr}(uv_\lambda) \longrightarrow \operatorname{Tr}(uv)\,,$$

for every $u \in \mathcal{L}^1(\mathcal{H})$. Since $M \subseteq B(\mathcal{H}) = \mathcal{L}^1(\mathcal{H})^*$, we can define the pre-annihilator of M as

$$M_\perp \coloneqq \{u \in \mathcal{L}^1(\mathcal{H}) \mid \operatorname{Tr}(ux) = 0\,, \text{for every } x \in M\}\,.$$

Thus $M_* \cong \mathcal{L}^1(\mathcal{H})/M_\perp$.

Furthermore, the isomorphism $\mathcal{L}^1(\mathcal{H})^* \cong B(\mathcal{H})$ implies that the set of normal states on $B(\mathcal{H})$, denoted by S, is weak* dense in the set of states on $B(\mathcal{H})$. Observe that the elements of S are of the form $\operatorname{Tr}(h\,\cdot)$, where $0 \leq h \in \mathcal{L}^1(\mathcal{H})$ and $\operatorname{Tr}(h) = 1$. Moreover, note that S is a convex subset.

In order to show that the weak* closure of S is $S(B(\mathcal{H}))$, we will show that if $x \in B(\mathcal{H})$ is self-adjoint and $\varphi(x) \geq 0$ for every $\varphi \in S$, then $x \geq 0$ (see [Murphy, 2014, Theorem 5.1.14]). Indeed, suppose for the sake of contradiction that x is not positive. Hence, there exists a unit vector $\eta \in \mathcal{H}$ such that $\langle x\eta, \eta\rangle < 0$. Now consider the rank-one projection $h_\eta \coloneqq \langle \cdot, \eta\rangle\eta$ and an orthonormal basis $\{\xi_i\}_{i\in I} \subseteq \mathcal{H}$ that contains η. Hence

$$\operatorname{Tr}(h_\eta x) = \operatorname{Tr}(xh_\eta) = \sum_{i\in I}\langle xh_\eta\xi_i, \xi_i\rangle = \langle x\eta, \eta\rangle < 0\,,$$

which is a contradiction. Therefore, the set of normal states on $B(\mathcal{H})$ is weak* dense in the set of states on $B(\mathcal{H})$.

2 Amenability and Følner C*-Algebras

The amenability phenomenon is protean and is not readily 'pigeon-holed'.

Paterson, Amenability (1988)

This chapter contains the main theorems of this work, connecting amenable traces, Følner nets and Følner C*-algebras. We begin in Section 2.1 with the definition and characterization of amenable groups. Next, we work with amenable traces on unital C*-algebras and prove a characterization via net of u.c.p. maps in Section 2.2. In Section 2.3, we define Følner nets and present the relation that exists with amenable traces. Then we speak about quasidiagonality and his connections to Følner nets in Section 2.4. Finally, in Section 2.5 we define and prove the core theorem that characterizes Følner C*-algebras (Theorem 2.5.7), and present some consequences of the developed theory.

2.1 Amenable groups

Amenable groups consist of one of the most significant classes of groups and have a huge number of characterizations[1]. Due to the importance of amenability, many of the definitions provided later in this chapter are inspired by or analogous to the definition in the amenable group context. So, the goal of this section is to provide a background for understanding future concepts and their basic examples. The main references are [Brown and Ozawa, 2008, Chapter 2], [Paterson, 1988], and [Biz, 2017, Capítulo 2].

Notation convention: We use $|\cdot|$ to denote the cardinality of a set, $\sqcup$ to denote disjoint union, and G to denote a discrete group. Additionally, X_1 will denote the (closed) unit ball of a Banach space X.

As briefly mentioned in Example 1.3.17, G has a canonical action on the C*-algebra $\ell^\infty(G)$. Given $g \in G$ and $f \in \ell^\infty(G)$, we define $g \cdot f$ as

$$(g \cdot f)(h) = f(g^{-1}h)\,,$$

for every $h \in G$. A series of easy verifications shows that $g \cdot f \in \ell^\infty(G)$ and $\cdot$ is indeed an action.

This action is spatially implemented by the left regular representation. To view that, let $\{\delta_h\}_{h\in G}$ be the canonical orthonormal basis of $\ell^2(G)$ and $f \in \ell^\infty(G)$. Note that

[1] Approximately $10^{10^{10}}$, according to [Brown and Ozawa, 2008, p. 48].

$f \in B(\ell^2(G))$ if we think about f as the multiplication operator defined by

$$\delta_h \longmapsto f(h)\delta_h .$$

Thus $\ell^\infty(G) \subset B(\ell^2(G))$. Finally, observe that $\lambda_g f \lambda_g^* = g \cdot f$, for all $g \in G$, since

$$(\lambda_g f \lambda_g^*)(\delta_h) = (\lambda_g f)(\delta_{g^{-1}h}) = \lambda_g(f(g^{-1}h)\delta_{g^{-1}h}) = f(g^{-1}h)\delta_h = (g \cdot f)(\delta_h) .$$

Definition 2.1.1. Let G be a discrete group. We say that G is **amenable** if there exists an **invariant mean** on $\ell^\infty(G)$. This means that there exists a state $\mu : \ell^\infty(G) \to \mathbb{C}$ which is invariant under left translation action, i.e.,

$$\mu(g \cdot f) = \mu(f) ,$$

for all $g \in G$ and $f \in \ell^\infty(G)$.

Remark 2.1.2. Although we do not present a proof here, amenability passes to subgroups, extensions, quotients, and inductive limits (see [Biz, 2017] or [Paterson, 1988] for more details).

Example 2.1.3. Every finite group G is amenable. Define a state $\mu : \ell^\infty(G) \to \mathbb{C}$ by

$$f \longmapsto \frac{1}{|G|} \sum_{g \in G} f(g) .$$

It is straightforward to verify that the "mean" is an invariant mean when G is finite.

Example 2.1.4. Not every group is amenable; for example, the free group $\mathbb{F}_2$ is not amenable. In fact, denote the generators of $\mathbb{F}_2$ as x and y, and define five subsets of $\mathbb{F}_2$,

$$X^+ = \{\text{all reduced word starting with } x\} \quad X^- = \{\text{all reduced word starting with } x^{-1}\}$$
$$Y^+ = \{\text{all reduced word starting with } y\} \quad Y^- = \{\text{all reduced word starting with } y^{-1}\}$$
$$\text{and} \quad Z = \{e, y, y^2, \ldots\} .$$

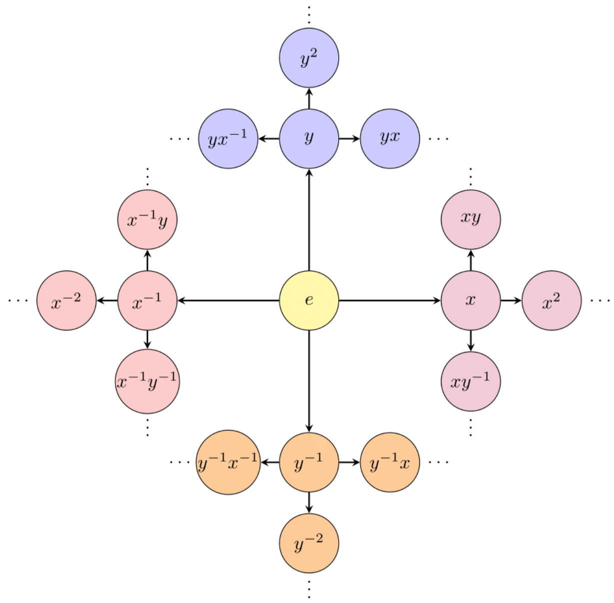
Notice that we have the following partitions (see Figure 1),

$$\begin{aligned}\mathbb{F}_2 &= X^+ \sqcup X^- \sqcup (Y^+ \setminus Z) \sqcup (Y^- \cup Z) \\ &= X^+ \sqcup xX^- \\ &= y^{-1}(Y^+ \setminus Z) \sqcup (Y^- \cup Z) .\end{aligned}$$

Now suppose that $\mathbb{F}_2$ has an invariant mean μ. Denote by $\chi_W \in \ell^\infty(\mathbb{F}_2)$ the characteristic function of a set $W \subseteq \mathbb{F}_2$, and observe that $g \cdot \chi_W = \chi_{gW}$ for every $g \in \mathbb{F}_2$. Hence,

$$\begin{aligned}1 = \mu(\chi_{\mathbb{F}_2}) &= \mu(\chi_{X^+}) + \mu(\chi_{X^-}) + \mu(\chi_{Y^+\setminus Z}) + \mu(\chi_{Y^-\cup Z}) \\ &= \mu(\chi_{X^+}) + \mu(x \cdot \chi_{X^-}) + \mu(y^{-1} \cdot \chi_{Y^+\setminus Z}) + \mu(\chi_{Y^-\cup Z}) \\ &= \mu(\chi_{X^+} + \chi_{xX^-}) + \mu(\chi_{y^{-1}(Y^+\setminus Z)} + \chi_{Y^-\cup Z}) \\ &= \mu(\chi_{\mathbb{F}_2}) + \mu(\chi_{\mathbb{F}_2}) = 2 ,\end{aligned}$$

Figure 1 – Cayley graph of $\mathbb{F}_2$.



Source: Prepared by the author (2025).

which is a contradiction. Analogously, every free group $\mathbb{F}_n$ ($n \geq 2$) is not amenable. Furthermore, since amenability passes to subgroups, any group that has a subgroup isomorphic to a free group is not amenable.

Now, observe that an invariant mean is an element of $\ell^\infty(G)^*$, which is a "very large" space. To contour that, we can use the canonical isomorphism $\ell^1(G)^* \cong \ell^\infty(G)$ that identifies[2] $f \in \ell^\infty(G)$ with a functional $\ell^1(G) \to \mathbb{C}$ given by

$$f(x) := \sum_{g \in G} f(g)x(g) \qquad (x \in \ell^1(G))\,.$$

In particular, there is an isomorphism $\ell^1(G)^{**} \cong \ell^\infty(G)^*$. With this identification in mind, notice that and invariant mean is an element of the unit ball of $\ell^1(G)^{**}$, since it is an state. Now recall a classical theorem of Functional Analysis (see [Botelho et al., 2012, Teorema 6.4.4]):

Theorem 2.1.5 (Goldstine). *Let X be a Banach space. The image of the unit ball of X under the canonical embedding is weak* dense in the unit ball of X^{**}.* □

In our case, the unit ball of $\ell^1(G)$ is weak* dense in the unit ball of $\ell^1(G)^{**}$. In particular, there is a net $(x_i)_{i\in I} \subseteq \ell^1(G)_1$ that converges to μ. Here, an element $x \in \ell^1(G)$ is identified, by the canonical embedding, with a functional $x \in \ell^1(G)^{**} \cong \ell^\infty(G)^*$ such that

$$x(f) := \sum_{g \in G} f(g)x(g) \qquad (f \in \ell^\infty(G))\,.$$

[2] To ease notation, we will not distinguish an object and his image under an isomorphism.

On the other hand, note $\mu \geq 0$ and $\|\mu\| = 1$, since the invariant mean is a state. The next question is: *Can we take a net in $\ell^1(G)_1$ converging to μ such that the net has "good" properties?*

This discussion motivates the following definition.

Definition 2.1.6. We say that

$$\mathrm{Prob}(G) = \{\mu \in \ell^1(G) \mid \mu \geq 0 \text{ and } \sum_{g \in G} \mu(g) = 1\}$$

is **space of probabilities measures on** G.

Observe that we deliberately used the same Greek letter to denote an element of $\mathrm{Prob}(G) \subseteq \ell^1(G)$ and an invariant mean $\mu \in \ell^\infty(G)^*$, since $\ell^1(G) \hookrightarrow \ell^1(G)^{**} \cong \ell^\infty(G)^*$. The next result provides a positive answer to our previous question (the argument is based on [Biz, 2017, Teorema 2.24]).

Lemma 2.1.7. *Let $\varphi : \ell^\infty(G) \to \mathbb{C}$ be a state. Then, there is a net $(\mu_i)_{i \in I} \subseteq \mathrm{Prob}(G)$ that converges to φ in the weak* topology of $\ell^\infty(G)^*$.*

Proof. Let φ be a state on $\ell^\infty(G)$. By Goldstine's Theorem, we can find a net $(x_i)_{i \in I} \subseteq \ell^1(G)_1$ which converges to φ in the weak*-topology of $\ell^\infty(G)^*$. Since $\mathbb{C}[G]$ is norm dense in $\ell^1(G)$, we can assume, without loss of generality, that $x_i \in \mathbb{C}[G]$ and $\|x_i\|_{\ell^1} \leq 1$, for all $i \in I$.

First, define $x_i' \coloneqq \dfrac{x_i + \overline{x_i}}{2}$ and observe that $\|x_i'\|_{\ell^1} \leq 1$. Furthermore, for any $f \in \ell^\infty(G)$,

$$\begin{aligned} x_i'(f) = f(x_i') &= f\left(\frac{x_i + \overline{x_i}}{2}\right) \\ &= \frac{1}{2}\big(f(x_i) + \overline{\overline{f}(x_i)}\big) \\ &= \frac{1}{2}\big(x_i(f) + \overline{x_i(\overline{f})}\big) \longrightarrow \frac{1}{2}\big(\varphi(f) + \overline{\varphi(\overline{f})}\big) = \varphi(f)\,. \end{aligned}$$

Hence, $x_i' \xrightarrow{w^*} \varphi$.

Moreover, note that $\|x_i'\|_{\ell^1} \longrightarrow 1$. Indeed,

$$1 = |\varphi(1_{\ell^\infty})| = \lim_i |x_i'(1_{\ell^\infty})| = \lim_i |\sum_{g \in G} x_i'(g)| \overset{(\dagger)}{\leq} \liminf_i \sum_{g \in G} |x_i'(g)| = \liminf_i \|x_i'\|_{\ell^1} \leq 1\,,$$

where $(\dagger)$ follows from the general fact about bounded nets of real numbers: $0 \leq a_i \leq b_i$ implies $\liminf_i a_i \leq \liminf_i b_i$.

Secondly, define $x_i'' \coloneqq |x_i'|$ and note that $\|x_i''\|_{\ell^1} = \|x_i'\|_{\ell^1} \leq 1$. Moreover, note that, if $\|x_i'' - x_i'\|_{\ell^1} \longrightarrow 0$, then $(x_i')_{i \in I}$ and $(x_i'')_{i \in I}$ have the same weak* limit, i.e., $x_i'' \xrightarrow{w^*} \varphi$.

Indeed,

$$\begin{aligned}\|x_i'' - x_i'\|_{\ell^1} &= \sum_{g\in G} \big||x_i'(g)| - x_i'(g)\big| \\ &= \sum_{g\in G} |x_i'(g)| - \sum_{g\in G} 1_{\ell^\infty}(g) x_i'(g) \\ &= \|x_i'\|_{\ell^1} - x_i'(1_{\ell^\infty}) \longrightarrow 1 - \varphi(1_{\ell^\infty}) = 1 - 1 = 0\,.\end{aligned}$$

Thus $(x_i'')_{i\in I}$ converges to φ in the weak* topology. Furthermore, since $\|x_i'\|_{\ell^1} \longrightarrow 1$ and $\|x_i'' - x_i'\|_{\ell^1} \longrightarrow 0$, it follows that $\|x_i''\|_{\ell^1} \longrightarrow 1$.

Finally, define $\mu_i := \dfrac{x_i''}{\|x_i''\|_{\ell^1}}$. It is straightforward to verify that $\mu_i \in \mathrm{Prob}(G)$, for all $i \in I$, and $(\mu_i)_{i\in I}$ converges to φ in the weak* topology. □

Remark 2.1.8. Note that the action of G on $\ell^\infty(G)$ induces an action on $\mathrm{Prob}(G)$. In fact, if $\mu \in \mathrm{Prob}(G)$, we can view $\mu = \big(\mu(h)\big)_{h\in G}$ as an element of $\ell^\infty(G)$ and calculate $g\cdot\mu$, for every $g \in G$. It is straightforward that $g\cdot\mu \in \mathrm{Prob}(G)$, for every $g \in G$ and $\mu \in \mathrm{Prob}(G)$.

Definition 2.1.9. We say that G has an **approximate invariant mean** if for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists $\mu \in \mathrm{Prob}(G)$ such that

$$\max_{g\in E} \|g\cdot\mu - \mu\|_1 < \varepsilon\,. \tag{2.1}$$

Moreover, this is equivalent to saying that there is a net $(\mu_i)_{i\in I} \subseteq \mathrm{Prob}(G)$ satisfying, for every $g \in G$,

$$\|g\cdot\mu_i - \mu_i\|_1 \longrightarrow 0\,.$$

In the following, we present a concept that intuitively defines "approximate invariance" of subsets.

Definition 2.1.10. We say that G satisfies **Følner condition** if for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists a non-empty finite subset $F \subseteq G$ such that

$$\max_{g\in E} \frac{|gF \triangle F|}{|F|} < \varepsilon\,,$$

where $gF = \{gh \mid h \in F\}$ and $\triangle$ denotes the symmetric difference, i.e., $E\triangle F = (E\backslash F) \sqcup (F\backslash E)$. Moreover, a net $(F_i)_{i\in I}$ of non-empty finite subsets of G is called a **Følner net** if, for every $g \in G$,

$$\frac{|gF_i \triangle F_i|}{|F_i|} \longrightarrow 0\,.$$

Proposition 2.1.11. *A group G satisfies the Følner condition if and only if G has a Følner net.*

Proof. The following argument contains the basic process for transitioning between a net-based definition to a characterization by local property. For instance, an analogous argument shows the equivalence stated in the definition of approximate invariant mean.

First, observe that the proof is trivial if G is a finite group. Therefore, suppose that G is an infinite group. To show the forward direction, consider the directed set $\mathfrak{F}$ of non-empty finite subsets of G. For each $E \in \mathfrak{F}$ there exists a finite subset $F_E \subseteq G$ such that

$$\max_{g\in E} \frac{|gF_E \triangle F_E|}{|F_E|} < \frac{1}{|E|} \,.$$

Then $(F_E)_{E\in\mathfrak{F}}$ is a Følner net for G. In fact, given $\varepsilon > 0$ and $g \in G$, there is $E \in \mathfrak{F}$ such that $g \in E$, $\frac{1}{|E|} \leq \varepsilon$, and

$$\frac{|gF_E \triangle F_E|}{|F_E|} < \varepsilon \,.$$

Hence, for any $\mathfrak{F} \ni E' \supseteq E$, we have,

$$\frac{|gF_{E'} \triangle F_{E'}|}{|F_{E'}|} < \frac{1}{|E'|} \leq \frac{1}{|E|} \leq \varepsilon \,.$$

Therefore, $\dfrac{|gF_E \triangle F_E|}{|F_E|} \longrightarrow 0$, for every $g \in G$.

On the other hand, suppose that G has a Følner net $(F_i)_{i\in I}$. Given a non-empty finite subset $E \subseteq G$ and $\varepsilon > 0$, for each $g \in E$, there exists $i_g \in I$ such that

$$\frac{|gF_{i_g} \triangle F_{i_g}|}{|F_{i_g}|} < \varepsilon \,.$$

Thus, taking and index $i \geq i_g$, for every $g \in E$, it follows that

$$\max_{g\in E} \frac{|gF_i \triangle F_i|}{|F_i|} < \varepsilon \,.$$

Therefore, G satisfy the Følner condition. □

Remark 2.1.12. Note that $gF \triangle F = [gF \backslash (F \cap gF)] \sqcup [F \backslash (F \cap gF)]$. Hence,

$$\begin{aligned} \frac{|gF \triangle F|}{|F|} &= \frac{|gF \backslash (F \cap gF)|}{|F|} + \frac{|F \backslash (F \cap gF)|}{|F|} \\ &= \frac{|gF| - |F \cap gF|}{|F|} + \frac{|F| - |F \cap gF|}{|F|} \\ &= 2 - 2\frac{|F \cap gF|}{|F|}. \end{aligned}$$

Applying this to Følner nets we conclude,

$$\frac{|gF_i \triangle F_i|}{|F_i|} \longrightarrow 0 \quad \text{if and only if} \quad \frac{|F_i \cap gF_i|}{|F_i|} \longrightarrow 1 \,.$$

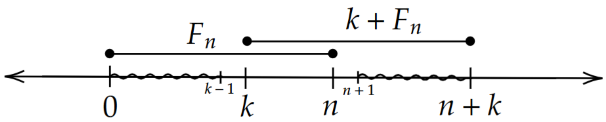
This equivalence will be very helpful in future calculations.

Example 2.1.13. The group $G = \mathbb{Z}$ satisfy the Følner condition. Indeed, for each $n \in \mathbb{N}$, define $F_n := [0, n] = \{0, \ldots, n\}$. Hence, for any $k \in \mathbb{Z}$,

$$\frac{|(k + F_n)\triangle F_n|}{|F_n|} \overset{n \geq k}{=} \frac{|[0, k-1] \sqcup [n+1, n+k]|}{|[0, n]|} = \frac{2k}{n+1} \longrightarrow 0 .$$

Therefore, $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence for $\mathbb{Z}$.

Figure 2 − Symmetric difference for $n > k > 0$.



Source: Prepared by the author (2024).

Remark 2.1.14. Note that the Følner condition implies the existence of an approximate invariant mean given by normalized characteristic functions. Let $(F_i)_{i\in I}$ be a Følner net in G and consider $\mu_i := \frac{1}{|F_i|}\chi_{F_i} \in \mathrm{Prob}(G)$. Then, for any $g \in G$,

$$\|g \cdot \mu_i - \mu_i\|_1 = \left\| g \cdot \left(\frac{1}{|F_i|}\chi_{F_i} \right) - \frac{1}{|F_i|}\chi_{F_i} \right\|_1 = \frac{1}{|F_i|} \sum_{h\in G} |\chi_{F_i}(g^{-1}h) - \chi_{F_i}(h)| =$$

$$= \frac{1}{|F_i|} \sum_{h\in G} |\chi_{gF_i}(h) - \chi_{F_i}(h)| = \frac{|gF_i \triangle F_i|}{|F_i|} \longrightarrow 0 ,$$

where the last equality follows from the fact that $|\chi_{gF_i}(h) - \chi_{F_i}(h)| = 1$ if and only if $h \in gF_i \triangle F_i$. Therefore, $(\mu_i)_{i\in I}$ is an approximate invariant mean.

From the theory developed until now, notice that there is a connection between being amenable, the existence of approximate invariant mean, and the existence of a Følner net. The next result provides a few characterizations of amenable groups (the proof is inspired by [Brown and Ozawa, 2008, Theorem 2.6.8]).

Theorem 2.1.15. *Let G be a discrete group. The following are equivalent:*

(i) G is amenable;

(ii) G has an approximate invariant mean;

(iii) G has a Følner net;

(iv) $C_r^*(G)$ is nuclear;

(v) $C^*(G) = C_r^*(G)$;

(vi) $C_r^*(G)$ has a character.

Proof. We will prove **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)** $\Rightarrow$ **(iv)** $\Rightarrow$ **(i)** and **(iii)** $\Rightarrow$ **(v)** $\Rightarrow$ **(vi)** $\Rightarrow$ **(i)** .
(i) $\Rightarrow$ **(ii)** Let μ be an invariant mean on $\ell^\infty(G)$. By Lemma 2.1.7, there is a net $(\mu_i)_{i\in I} \subseteq \mathrm{Prob}(G)$ that converges to μ in the weak* topology of $\ell^\infty(G)^*$. Moreover, for any $g \in G$, the net $(g.\mu_i - \mu_i)_{i\in I}$ converges to zero weakly in $\ell^1(G)$. Indeed, for every $f \in \ell^\infty(G) \cong \ell^1(G)^*$,

$$f(g\cdot\mu_i) = \sum_{h\in G}\mu_i(g^{-1}h)f(h) = \sum_{h\in G}\mu_i(h)f(gh) = (g^{-1}\cdot f)(\mu_i) \longrightarrow (g^{-1}\cdot f)(\mu) = \mu(g^{-1}\cdot f)$$

and

$$f(\mu_i) \longrightarrow f(\mu) = \mu(f)\,.$$

Since μ is an invariant mean, we have the desired weakly convergence in $\ell^1(G)$.

Let $E \subseteq G$ be a finite subset and $\varepsilon > 0$. Consider the Banach space $\bigoplus_{g\in E}\ell^1(G)$ with the maximum norm, this means that $\|(x_g)_{g\in E}\|_{\max} = \max_{g\in E}\|x_g\|_{\ell^1}$. Recall that there is net in $\mathrm{Prob}(G)$ such that $g\cdot\mu_i - \mu_i$ converges weakly to zero in $\ell^1(G)$, for every $g \in E$. Therefore, zero is an element of the weak closure of the convex subset

$$\bigoplus_{g\in E}\{g.\mu - \mu \mid \mu \in \mathrm{Prob}(G)\} \subseteq \bigoplus_{g\in E}\ell^1(G)\,.$$

Since the weak and norm closures of convex subsets coincide, by Hahn-Banach Theorem, it follows that:

$$0 \in \overline{\bigoplus_{g\in E}\{g.\mu - \mu \mid \mu \in \mathrm{Prob}(G)\}}^{\|\cdot\|_{\max}}\,.$$

Hence, there exists $\mu \in \mathrm{Prob}(G)$ such that

$$\max_{g\in E}\|g\cdot\mu - \mu\|_1 < \varepsilon\,.$$

Thus G has an approximate invariant mean, as desired.
(ii) $\Rightarrow$ **(iii)** Fix a finite subset $E \subseteq G$ and $\varepsilon > 0$. By the approximate invariant mean property, without loss of generality, we can choose $\mu \in \mathrm{Prob}(G)$ such that

$$\sum_{g\in E}\|g.\mu - \mu\|_1 < \varepsilon\,.$$

Note that the maximum is replaced by a sum in the equation above.

Let $r \geq 0$ and let $f \in \ell^1(G)$ be a positive function bounded above by 1. Define the set

$$F(f,r) := \{h \in G \mid f(h) > r\}\,,$$

and consider $\chi_{F(f,r)} \in \ell^\infty(G)$ the characteristic function of this set. In particular, note that $F(f,r) = \emptyset$ if $r \geq 1$.

Let $h \in G$ and $f, \tilde f \in \ell^1(G)$ be positive functions bounded above by 1. Observe that

$$\chi_{F(f,r)}(h) = \chi_{F(\tilde f,r)}(h) \quad \text{if and only if} \quad f(h), \tilde f(h) \leq r \text{ or } \tilde f(h), f(h) > r\,.$$

Negating both sides, it follows that

$$|\chi_{F(f,r)}(h) - \chi_{F(\tilde{f},r)}(h)| = 1 \quad \text{if and only if} \quad f(h) \leq r < \tilde{f}(h) \text{ or } \tilde{f}(h) \leq r < f(h)$$
$$\text{if and only if} \quad h \in F(f,r) \triangle F(\tilde{f},r) \tag{2.2}$$

Furthermore, if both f and $\tilde{f}$ are bounded above by 1, it follows that

$$|f(h) - \tilde{f}(h)| = \int_0^1 |\chi_{F(f,r)}(h) - \chi_{F(\tilde{f},r)}(h)| dr \,. \tag{2.3}$$

This (Lebesgue) integral can be interpreted as follows. Suppose that $f(h) \geq \tilde{f}(h)$ and note that the map $r \mapsto |\chi_{F(f,r)}(h) - \chi_{F(\tilde{f},r)}(h)|$ is just the characteristic function of the set $\{r \in \mathbb{R}_{\geq 0} \mid \tilde{f}(h) \leq r < f(h)\}$. Hence, the integral is just the length of the interval $[\tilde{f}(h), f(h))$ as (2.3) claims. Additionally, if $\tilde{f} = 0$, the same reasoning yields

$$f(h) = \int_0^1 \chi_{F(f,r)}(h) dr \,. \tag{2.4}$$

Now consider $\mu, g.\mu \in \mathrm{Prob}(G)$, which are positive functions bounded above by 1. Moreover, observe that $F(g \cdot \mu, r) = gF(\mu, r)$ for any $r \geq 0$. Hence[3],

$$\begin{aligned} \|g.\mu - \mu\|_1 &= \sum_{h\in G} |g.\mu(h) - \mu(h)| \\ &\overset{(2.3)}{=} \sum_{h\in G} \int_0^1 |\chi_{F(g.\mu,r)}(h) - \chi_{F(\mu,r)}(h)| dr \\ &= \int_0^1 \sum_{h\in G} |\chi_{gF(\mu,r)}(h) - \chi_{F(\mu,r)}(h)| dr \\ &\overset{(2.2}{=} \int_0^1 |gF(\mu,r) \triangle F(\mu,r)| dr \,. \end{aligned}$$

Also observe that

$$1 = \sum_{h\in G} \mu(h) \overset{(2.4)}{=} \sum_{h\in G} \int_0^1 \chi_{F(\mu,r)}(h) dr = \int_0^1 \sum_{h\in G} \chi_{F(\mu,r)}(h) dr = \int_0^1 |F(\mu,r)| dr \,. \tag{2.5}$$

Therefore,

$$\int_0^1 \sum_{g\in E} |gF(\mu,r) \triangle F(\mu,r)| dr = \sum_{g\in G} \|g.\mu - \mu\|_1 < \varepsilon \overset{(2.5)}{=} \int_0^1 \varepsilon |F(\mu,r)| dr \,.$$

Hence, for some $0 \leq r \leq 1$, we must have

$$\sum_{g\in E} |gF(\mu,r) \triangle F(\mu,r)| < \varepsilon |F(\mu,r)| \quad \text{, which implies} \quad \frac{|gF(\mu,r) \triangle F(\mu,r)|}{|F(\mu,r)|} < \varepsilon \,,$$

for every $g \in E$. Therefore, G satisfies the Følner condition.

[3] In the third equality we may switch the integral and the sum by Tonelli's Theorem.

(iii) ⇒ (iv) First of all, given $h, k \in G$, define the operator $e_{h,k} \in B(\ell^2(G))$ to be $e_{h,k}(\delta_k) = \delta_h$ and zero otherwise.

Let $(F_i)_{i\in I}$ be a Følner net on G. For each i denote by p_i the finite-rank projection onto the subspace $\ell^2(F_i) \coloneqq \operatorname{span}\{\delta_g \mid g \in F_i\}$. With the identification $p_i B(\ell^2(G))p_i \cong M_{F_i}(\mathbb{C})$, the operators $\{e_{h,k}\}_{h,k\in F_i}$ are the canonical matrix units of $M_{F_i}(\mathbb{C})$. In particular, note that

$$e_{h,h}\lambda_g e_{k,k} = \begin{cases} e_{h,k} & \text{, if } gk = h\,, \\ 0 & \text{, if } gk \neq h\,. \end{cases} \tag{2.6}$$

Indeed, for example, given $\delta_k \in \ell^2(F_i)$, if $gk = h$ it follows that

$$e_{h,h}\lambda_g e_{k,k}(\delta_k) = e_{h,h}\lambda_g(\delta_k) = e_{h,h}(\delta_{gk}) = \delta_h = e_{h,k}(\delta_k)\,.$$

Additionally, observe that $p_i = \sum_{h\in F_i} e_{h,h}$ implies that

$$p_i\lambda_g p_i = \sum_{h,k\in F_i} e_{h,h}\lambda_g e_{k,k} \overset{(2.6)}{=} \sum_{h\in F_i\cap gF_i} e_{h,g^{-1}h}\,, \tag{2.7}$$

for every $g \in G$.

Now consider the map

$$\begin{aligned} \varphi_i : C^*_r(G) &\longrightarrow M_{F_i}(\mathbb{C}) \\ x &\longmapsto p_i x p_i\,, \end{aligned}$$

that is a compression of a unital representation, hence it is a u.c.p. map. Also consider the map

$$\begin{aligned} \psi_i : M_{F_i}(\mathbb{C}) &\longrightarrow C^*_r(G) \\ e_{h,k} &\longmapsto \frac{1}{|F_i|}\lambda_{hk^{-1}}\,. \end{aligned}$$

Note that ψ_i is unital and $[\psi_i(e_{h,k})] = \frac{1}{|F_i|}[\lambda_h\lambda_k^*] \geq 0$, hence ψ_i is a u.c.p. map by Proposition 1.2.8.

Since $C^*_r(G) = C^*(\{\lambda_g \mid g \in G\})$, by density it suffices to check that

$$\|\lambda_g - \psi_i \circ \varphi_i(\lambda_g)\| \to 0$$

for every $g \in G$. Indeed,

$$\psi_i \circ \varphi_i(\lambda_g) \overset{(2.7)}{=} \psi_i\left(\sum_{h\in F_i\cap gF_i} e_{h,g^{-1}h}\right) = \sum_{h\in F_i\cap gF_i} \frac{1}{|F_i|}\lambda_{h(g^{-1}h)^{-1}} = \frac{|F_i\cap gF_i|}{|F_i|}\lambda_g\,, \tag{2.8}$$

note that the right side converges to λ_g, since $\dfrac{|F_i\cap gF_i|}{|F_i|} \longrightarrow 1$ by Remark 2.1.12. Therefore, $C^*_r(G)$ is nuclear.

(iv) ⇒ (i) Since $C_r^*(G)$ is nuclear, let $\varphi_i : C_r^*(G) \to M_{k(i)}(\mathbb{C})$ and $\psi_i : M_{k(i)}(\mathbb{C}) \to C_r^*(G)$ be u.c.p. maps such that

$$\|\psi_i \circ \varphi_i(x) - x\| \longrightarrow 0\,,$$

for every $x \in C_r^*(G)$. By Arvenson's Extension Theorem (1.2.10), we can extend each φ_i to a u.c.p. map $\tilde{\varphi}_i : B(\ell^2(G)) \to M_{k(i)}(\mathbb{C})$. Now define the u.c.p. maps

$$\begin{aligned}\Phi_i : B(\ell^2(G)) &\longrightarrow C_r^*(G) \subseteq L(G)\\ x &\longmapsto \psi_i \circ \tilde{\varphi}_i(x)\,,\end{aligned}$$

and note that $\Phi_i(x) \to x$ for every $x \in C_r^*(G)$. Since the codomain of each Φ_i can be taken as the von Neumann algebra $L(G) = C_r^*(G)''$, there exists a cluster point of $\{\Phi_i\}_{i\in I}$ on the point ultraweak topology by Proposition 1.1.31. Furthermore, this cluster point is a u.c.p. map by Lemma 1.2.11.

Denote this cluster point by $\Phi : B(\ell^2(G)) \to L(G)$, and note that $\Phi(x) = x$, for every $x \in C_r^*(G)$. Furthermore, observe that $\Phi(x^*x) = x^*x = \Phi(x)^*\Phi(x)$ and $\Phi(xx^*) = xx^* = \Phi(x)\Phi(x)^*$, for every $x \in C_r^*(G)$. Hence, $C_r^*(G)$ is contained in the multiplicative domain of Φ by Proposition 1.2.6.

Define a vector state $\tau : L(G) \to \mathbb{C}$ by

$$x \longmapsto \langle x\delta_e, \delta_e\rangle\,,$$

and observe that τ is a tracial. In fact, to show that τ is tracial, it suffices to calculate on the generators of $L(G)$:

$$\tau(\lambda_g\lambda_h) = \langle \lambda_{gh}\delta_e, \delta_e\rangle = \begin{cases} 1\,, \text{ if } gh = 1\,,\\ 0\,, \text{ otherwise.}\end{cases}$$

for every $g, h \in G$. Hence, $\tau(\lambda_g\lambda_h) = \tau(\lambda_h\lambda_g)$, as desired.

Now consider the state

$$\eta \coloneqq \tau \circ \Phi : B(\ell^2(G)) \to \mathbb{C}\,.$$

To get an invariant mean, we restrict η to $\ell^\infty(G) \subset B(\ell^2(G))$. Indeed, since the action of G on $\ell^\infty(G)$ is spatially implemented,

$$\eta(g.f) = \eta(\lambda_g f \lambda_g^*) = \tau(\Phi(\lambda_g f \lambda_g^*)) \overset{(\dagger)}{=} \tau(\lambda_g \Phi(f)\lambda_g^*) = \tau(\underbrace{\lambda_g^*\lambda_g}_{=1}\Phi(f)) = \tau(\Phi(f)) = \eta(f)\,,$$

for every $g \in G$ and $f \in \ell^\infty(G)$. Where $(\dagger)$ follows from the fact that $\lambda_g, \lambda_g^* \in C_r^*(G)$ are elements of the multiplicative domain of Φ and $\Phi|_{C_r^*(G)} = \mathrm{id}_{C_r^*(G)}$. Therefore, $\eta|_{\ell^\infty(G)}$ is an invariant mean and G is amenable.

(iii) ⇒ (v) Consider $\pi : C^*(G) \to C_r^*(G)$ the canonical quotient map. We want to show that π is injective.

First, we claim that π is injective if there exists a net of u.c.p maps $\Psi_i : C_r^*(G) \to C^*(G)$ such that, for every $x \in \mathbb{C}[G]$,

$$\|x - \Psi_i \circ \pi(x)\| \longrightarrow 0 .$$

Assume that there is a net with the property above. Since π restricted to $\mathbb{C}[G]$ is the identity, consider $x \in C^*(G) \setminus \mathbb{C}[G]$ such that $\pi(x) = 0$. Hence there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}[G]$ converging to x on $C^*(G)$.

The claim follows from a standard $\varepsilon/3$ argument. Consider $\varepsilon > 0$. Since $x_n \longrightarrow x$, there is $n_0 \in \mathbb{N}$ such that $\|x - x_{n_0}\| < \varepsilon/3$. However, observe that $x_{n_0} \in \mathbb{C}[G]$ and $\|x - \Psi_i \circ \pi(x)\| \longrightarrow 0$. Hence, there exists $i_0 \in I$ such that

$$i \geq i_0 \quad \text{implies } \|x_{n_0} - \Psi_i \circ \pi(x_{n_0})\| < \varepsilon/3 .$$

Therefore, $i \geq i_0$ implies

$$\|x - \Psi_i \circ \pi(x)\| \leq \|x - x_{n_0}\| + \|x_{n_0} - \Psi_i \circ \pi(x_{n_0})\| + \|\Psi_i \circ \pi(x_{n_0}) - \Psi_i \circ \pi(x)\| < \varepsilon ,$$

since $\|\Psi_i \circ \pi\| \leq 1$. Therefore,

$$\|x\| = \|x - \Psi_i \circ \underbrace{\pi(x)}_{=0}\| \longrightarrow 0 ,$$

which implies $x = 0$ and π injective, as desired.

Now we claim that, in fact, there exists a net of u.c.p maps $\Psi_i : C_r^*(G) \to C^*(G)$ such that, for every $x \in \mathbb{C}[G]$,

$$\|x - \Psi_i \circ \pi(x)\| \longrightarrow 0 .$$

Let $(F_i)_{i\in I}$ be a Følner net on G and take the same nets of u.c.p maps defined in the proof that **(iii)⇒(iv)**, i.e., $\varphi : C_r^*(G) \to M_{F_i}(\mathbb{C})$ and $\psi_i : M_{F_i}(\mathbb{C}) \to \mathbb{C}[G] \subseteq C^*(G)$. Note that the codomain of each ψ_i can be taken as $C^*(G)$.

Define $\Psi_i : C_r^*(G) \to C^*(G)$ as the composition $\Psi_i \coloneqq \psi_i \circ \varphi_i$, thus each Ψ_i is a u.c.p. map. By equation 2.8, for any $g \in G$,

$$\Psi_i(\delta_g) = \psi_i \circ \varphi_i(\delta_g) = \frac{|F_i \cap gF_i|}{|F_i|}\delta_g .$$

Hence,

$$\|\delta_g - \Psi_i \circ \pi(\delta_g)\| = \|\delta_g - \Psi_i(\delta_g)\| = \left\|\delta_g - \frac{|F_i \cap gF_i|}{|F_i|}\delta_g\right\| \longrightarrow 0 .$$

Since the set $\{\delta_g \mid g \in G\}$ is dense in $C^*(G)$, the claim follows. Therefore, our first claim is fulfilled and $C^*(G) \cong C_r^*(G)$, as desired.

(v) ⇒ (vi) By the universal property of $C^*(G)$, the trivial representation $\epsilon : G \to \mathbb{C}$ representation extends to a character on $C_r^*(G) = C^*(G)$.

(vi) ⇒ (i) Let $\tau : C_r^*(G) \to \mathbb{C}$ be any character, i.e., a nonzero $*$-homomorphism. By Arverson's Extension Theorem (1.2.10), we can extend τ to a u.c.p. map $\overline{\tau} : B(\ell^2(G)) \to$

$\mathbb{C}$. Since $\ell^\infty(G) \subset B(\ell^2(G))$, the restriction $\mu := \overline{\tau}|_{\ell^\infty(G)}$ is a state and a good candidate for an invariant mean. In fact, since the left translation action is spatially implemented,

$$\mu(g.f) = \overline{\tau}(\lambda_g f \lambda_g^*) \overset{(\dagger)}{=} \overline{\tau}(\lambda_g)\overline{\tau}(f)\overline{\tau}(\lambda_g^*) = \underbrace{\tau(\lambda_g)\overline{\tau(\lambda_g)}}_{=1}\tau(f) = \overline{\tau}(f) = \mu(f)\,,$$

for all $g \in G$ and $f \in \ell^\infty(G)$. Where ($\dagger$) follows from the fact that $\lambda_g, \lambda_g^* \in C_r^*(G)$ are elements of the multiplicative domain of $\overline{\tau}$. Therefore, μ is an invariant mean on $\ell^\infty(G)$ and G is amenable. □

Example 2.1.16. As a consequence of the previous theorem, notice that all abelian groups G are amenable. In fact, note that $C_r^*(G)$ is abelian when G is abelian. Since all abelian C*-algebras are nuclear, as shown in Example 1.2.15, it follows that $C_r^*(G)$ is nuclear. Therefore, G is amenable by Theorem 2.1.15.

Example 2.1.17. Since $\mathbb{F}_2$ is not amenable, Theorem 2.1.15 implies that $C_r^*(\mathbb{F}_2)$ is not nuclear.

Remark 2.1.18. Let (A, G, α) be a C*-dynamical system. Although the proof of the following is beyond the scope of this work (see [Brown and Ozawa, 2008, §4.2]), when G is amenable we have the following:

(i) $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$;

(ii) A is nuclear if and only if $A \rtimes_{\alpha,r} G$ is nuclear.

In particular, the proof of *(i)* is similar to the proof that $C^*(G) = C_r^*(G)$.

Notes and Remarks. The definition of amenable groups goes back to the work of von Neumann to understand why the Banach-Tarski Theorem occurs only for dimensions ≥ 3. The definition of von Neumann says that a (discrete) group is amenable if there exists a (left) invariant finitely additive measure $m : \mathcal{P}(G) \to [0, +\infty]$ such that $m(G) = 1$. It was M.M. Day who changed the perspective from measures to functionals. The key ideal is that, given a finitely additive measure m, we can define a functional $\mu : \ell^\infty(G) \to \mathbb{C}$ by

$$\mu(\chi_E) = m(E) \qquad (E \subseteq G)\,.$$

One can prove that this induces a correspondence between (left) invariant finitely additive measures and (left) invariant means (for more historical background and more details in this discussion, see [Paterson, 1988, Chapter 0]).

Another approach to amenability of groups is towards paradoxical decompositions (see [Tomkowicz and Wagon, 2016]). A group G is *paradoxical* is there exists pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq G$ and elements $g_1, \ldots, g_n, h_1 \ldots, h_m \in G$ such that

$$G = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j\,.$$

One can prove that G is amenable if and only if G is not paradoxical. In particular, observe that the proof that $\mathbb{F}_2$ is not amenable involves a paradoxical decomposition. Moreover, the dichotomy amenable/paradoxical is also present in semigroups, see [Ara et al., 2020].

Even when G is not amenable, the action α of G on A can behave "nicely". This actions are called *amenable actions*, and they are a sufficient condition to ensure weak containment; i.e., the canonical surjective $*$-homomorphism $\tilde{\pi} : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is also injective (see [Brown and Ozawa, 2008, §4.3]).

Unlike the regular case where $C^*_r(G) \subset A \rtimes_{\alpha,r} G$, sometimes we have $C^*(G) \not\subset A \rtimes_\alpha G$. In fact, suppose that G is non-amenable and we have an amenable action on a unital C*-algebra A. Recall that there is an injective unital $*$-homomorphism $\iota : C^*_r(G) \to A \rtimes_{\alpha,r} G$. In addition, by universality, there exists a unital $*$-homomorphism $\phi : C^*(G) \to A \rtimes_\alpha G$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow[\cong]{\tilde{\pi}} & A \rtimes_{\alpha,r} G \\ \phi \uparrow & & \uparrow \iota \\ C^*(G) & \xrightarrow[\pi]{} & C^*_r(G) \end{array}$$

Since G is non-amenable, $C^*(G) \ncong C^*_r(G)$, i.e., π is not injective. Therefore, ϕ is not injective.

2.2 Amenable traces

In this section, we introduce amenable traces, a fundamental ingredient in the study of Følner C*-algebras. While the definition of amenable trace can be extended to the nonunital case, we will focus on unital C*-algebras, since the main theorem of this section, a characterization of amenable traces, is presented in the unital case. Additionally, we apply the developed theory to nuclear C*-algebras and the reduced group C*-algebra. This section is largely based on [Brown and Ozawa, 2008, Chapter 6], along with previous and related works [Brown, 2004] and [Brown, 2006].

Notation convention: In this section, we will work with unital concrete C*-algebras $A \subset B(\mathcal{H})$, unless stated otherwise. As usual, we may assume $1_\mathcal{H} = 1_A$. Moreover, $\mathcal{U}(A)$ denote the set of unitaries, $\mathrm{tr} = \mathrm{tr}_k$ denote the canonical tracial state on $M_k(\mathbb{C})$, $A \odot B$ denote the algebraic tensor product, and $A \otimes B$ denote the minimal tensor product.

There is a canonical action of the (discrete) group $\mathcal{U}(A)$ on $B(\mathcal{H})$ given by adjunction; i.e., $\mathrm{Ad}_u(x) := uxu^*$ for every $u \in \mathcal{U}(A)$ and $x \in B(\mathcal{H})$.

Definition 2.2.1. Let A be a unital C*-algebra. A state τ on A is called an **amenable trace** if there is a state φ on $B(\mathcal{H})$ that extends τ and is invariant under the action of $\mathcal{U}(A)$ on $B(\mathcal{H})$. This means that $\varphi|_A = \tau$ and $\varphi(uxu^*) = \varphi(x)$ for every unitary $u \in A$ and $x \in B(\mathcal{H})$. The state φ is called a **hypertrace** on A.

Remark 2.2.2. Let φ denote the hypertrace on A associated with the amenable trace τ. Then, for any $u \in \mathcal{U}(A)$ and $x \in B(\mathcal{H})$,

$$\varphi(ux) = \varphi((ux)uu^*) = \varphi(u(xu)u^*) = \varphi(xu)\,.$$

Since A is linearly generated by the elements of $\mathcal{U}(A)$, we conclude that the hypertrace φ satisfies

$$\varphi(ax) = \varphi(xa)\,,$$

for every $a \in A$ and $x \in B(\mathcal{H})$. In particular, this implies that an amenable trace is indeed a tracial state, as the definition suggests. Conversely, if φ satisfies the equation above, then φ is a hypertrace. Furthermore, the expression above define amenable traces on nonunital C*-algebras.

Remark 2.2.3. Notice that there is certain similarity between the definition of amenable traces/hypertraces and invariant means. This similarity is not accidental, since the definition of hypertraces as an analogue of an invariant mean can be traced back to the work of Alain Connes on [Connes, 1976a] and [Connes, 1976b].

Example 2.2.4. We claim that $C_r^*(G)$ admits an amenable trace when G is amenable. Recall that $\ell^\infty(G) \subset B(\ell^2(G))$, since each function $f \in \ell^\infty(G)$ can be viewed as an operator defined by $f(\delta_h) = f(h)\delta_h$ in the canonical orthonormal basis of $\ell^2(G)$. Additionally, remember that the action of G on $\ell^\infty(G)$ is spatially implemented by the left regular representation, i.e., $\lambda_g f \lambda_g^* = g \cdot f$, for every $g \in G$ and $f \in \ell^\infty(G)$.

Now define the "diagonal map" $\Phi : B(\ell^2(G)) \to \ell^\infty(G) \subset B(\ell^2(G))$ by

$$x \longmapsto \sum_{g \in G} e_g x e_g\,,$$

where the sum is taken in the strong (operator) topology, and $e_g \in B(\ell^2(G))$ denote the rank-one projection onto span$\{\delta_g\}$; i.e., $e_g(\cdot) = \langle \cdot\,, \delta_g\rangle \delta_g$. Observe that, at first sight, the codomain of Φ should be $B(\ell^2(G))$. However, the following calculations show that the range of Φ is contained in $\ell^\infty(G)$:

$$\Phi(x)(\delta_h) = \sum_{g \in G} e_g x e_g(\delta_h) \overset{(\dagger)}{=} e_h x e_h(\delta_h) = \langle x(\delta_h), \delta_h\rangle \delta_h\,, \tag{2.9}$$

for each $x \in B(\ell^2(G))$ and $h \in G$. Note that the sum drop to a single term in (†) because $e_g(\delta_h) \neq 0$ if and only if $g = h$.

An essential step to our construction is to show that Φ is a unital and contractive[4]. In fact, $\Phi(1) = \sum_{g \in G} e_g = 1$, hence Φ is unital. On the other hand, for any $x \in B(\ell^2(G))$

[4] Furthermore, one can prove the stronger statement that Φ is a conditional expectation.

and $\sum_{h\in G}\alpha_h\delta_h \in \ell^2(G)$,

$$\begin{aligned}\Big\|\Phi(x)\big(\sum_{h\in G}\alpha_h\delta_h\big)\Big\|^2 &\overset{(2.9)}{=} \Big\langle \sum_{h\in G}\alpha_h\langle x(\delta_h),\delta_h\rangle\delta_h, \sum_{g\in G}\alpha_g\langle x(\delta_g),\delta_g\rangle\delta_g\Big\rangle \\ &= \sum_{h\in G}|\alpha_h|^2|\langle x(\delta_h),\delta_h\rangle|^2 \\ &\overset{\text{CBS}}{\leq} \sum_{h\in G}|\alpha_h|^2\,\|x\|^2 \\ &= \|x\|^2\,\Big\|\sum_{h\in G}\alpha_h\delta_h\Big\|^2_{\ell^2(G)}.\end{aligned}$$

Hence, $\|\Phi(x)\|\leq\|x\|$, as desired.

Moreover, for any $g\in G$ and $x\in B(\ell^2(G))$, we aim to prove that

$$\Phi(\lambda_g x\lambda_g^*) = \lambda_g\Phi(x)\lambda_g^* = g\cdot\Phi(x)\,. \tag{2.10}$$

Indeed, for any $h\in G$,

$$\Phi(\lambda_g x\lambda_g^*)(\delta_h) = \sum_{s\in G} e_s\lambda_g x\lambda_g^* e_s(\delta_h) = e_h\lambda_g x\lambda_g^*(\delta_h) = e_h\lambda_g x(\delta_{g^{-1}h})$$

and

$$(\lambda_g\Phi(x)\lambda_g^*)(\delta_h) = \sum_{s\in G}\lambda_g e_s x e_s\lambda_g^*(\delta_h) = \sum_{s\in G}\lambda_g e_s x e_s(\delta_{g^{-1}h}) = \lambda_g e_{g^{-1}h}x(\delta_{g^{-1}h}).$$

Therefore (2.10) holds if $e_h\lambda_g = \lambda_g e_{g^{-1}h}$. But this is easy to see, since

$$e_h\lambda_g(\delta_s) = \begin{cases}\delta_h & ,\text{if } h = gs,\\ 0 & ,\text{if } h\neq gs.\end{cases} \quad\text{and}\quad \lambda_g e_{g^{-1}h}(\delta_s) = \begin{cases}\delta_h & ,\text{if } s = g^{-1}h,\\ 0 & ,\text{if } s\neq g^{-1}h.\end{cases}$$

Let μ be an invariant mean on $\ell^\infty(G)$. Define a functional φ on $B(\ell^2(G))$ by $\varphi := \mu\circ\Phi$. Since

$$\varphi(1) = \mu\circ\Phi(1) = \mu(1) = 1 \quad\text{and}\quad \|\varphi\|\leq\|\mu\|\,\|\Phi\|\leq 1\,,$$

we conclude that φ is a state. Moreover, for each $g\in G$ and $x\in B(\ell^2(G))$,

$$\varphi(\lambda_g x\lambda_g^*) = \mu\circ\Phi(\lambda_g x\lambda_g^*) \overset{(2.10)}{=} \mu(g\cdot\Phi(x)) = \mu(\Phi(x)) = \varphi(x)\,.$$

As a consequence $\varphi(ax) = \varphi(xa)$, for every $a\in\mathbb{C}[G]$ and $x\in B(\ell^2(G))$. By a density argument the same is true for every $a\in C_r^*(G)$. Therefore, φ is a hypertrace and the restriction $\tau := \varphi|_{C_r^*(G)}$ is an amenable trace on $C_r^*(G)$.

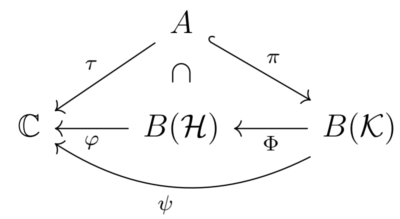
As everyone should expect, the definition of amenable trace does not depends on the choice of the embedding $A\subset B(\mathcal{H})$.

Proposition 2.2.5. *Let τ be an amenable trace on $A \subset B(\mathcal{H})$. For every faithful representation $\pi : A \to B(\mathcal{K})$, there exists a state ψ on $B(\mathcal{K})$ such that $\psi \circ \pi = \tau$ and*

$$\psi(\pi(u)\ x\ \pi(u^*)) = \psi(x)\,,$$

for every $u \in \mathcal{U}(A)$ and $x \in B(\mathcal{K})$.

Proof. First, consider the inverse $*$-homomorphism $\pi^{-1} : \pi(A) \to A \subset B(\mathcal{H})$. By Arverson's Extension Theorem (1.2.10), there exists a u.c.p. map $\Phi : B(\mathcal{K}) \to B(\mathcal{H})$ that extends π^{-1}. Let φ be a hypertrace on A extending τ, and define $\psi \coloneqq \varphi \circ \Phi$. Intuitively, we have the following commutative diagram.



Since $\psi(1_{\mathcal{K}}) = \varphi \circ \Phi(1_{\mathcal{K}}) = \varphi(1_{\mathcal{H}}) = 1$ and $\|\psi\| \leq \|\varphi\|\,\|\Phi\| \leq 1$, we conclude that ψ is a state on $B(\mathcal{K})$. Furthermore, $\psi \circ \pi = \tau$ by definition.

Next, note that $\Phi|_{\pi(A)} = \pi^{-1}$ is a $*$-homomorphism. Hence $\pi(A)$ is contained in the multiplicative domain of Φ. Therefore, for every $u \in \mathcal{U}(A)$ and $x \in B(\mathcal{K})$,

$$\begin{aligned}\psi(\pi(u)\ x\ \pi(u^*)) = \varphi \circ \Phi(\pi(u)\ x\ \pi(u^*)) &= \varphi(\Phi(\pi(u))\Phi(x)\Phi(\pi(u^*))) \\ &= \varphi(u\Phi(x)u^*) = \varphi(\Phi(x)) = \psi(x)\,,\end{aligned}$$

as desired. □

Although the next result will be crucial to the proof of the theorem that characterizes amenable traces, the proof is technical and can be found in [Brown and Ozawa, 2008, Lemma 6.2.5]. However, we provide a sketch of the proof in Appendix A.3.

Lemma 2.2.6. *Let $h \in B(\mathcal{H})$ be a positive finite-rank operator with rational eigenvalues such that* $\mathrm{Tr}(h) = 1$. *Then, there is $k \in \mathbb{N}$ and a u.c.p. map $\varphi : B(\mathcal{H}) \to M_k(\mathbb{C})$ satisfying*

$$\mathrm{tr}(\varphi(x)) = \mathrm{Tr}(hx)\,,$$

for every $x \in B(\mathcal{H})$, and

$$|\mathrm{tr}(\varphi(uu^*) - \varphi(u)\varphi(u^*))| \leq 2\,\|uhu^* - h\|_1^{1/2}\,,$$

for every unitary operator $u \in \mathcal{U}(A)$. □

Definition 2.2.7. Let ω be a state on a C*-algebra A. Define the seminorm $\|\cdot\|_{2,\omega}$ on A by

$$\|a\|_{2,\omega} \coloneqq \omega(a^*a)^{1/2},\ a \in A.$$

Remark 2.2.8. In particular, observe that $\|\cdot\|_{2,\omega}$ is a norm if and only if ω is faithful. Moreover, if ω is faithful, for any $a \in A$,

$$\|a\|_{2,\omega} = \omega(a^*a)^{1/2} \leq \|a^*a\|^{1/2} = \|a\|\,,$$

hence the topology on A generated by $\|\cdot\|_{2,\omega}$ is weaker than the usual topology on A (generated by the C*-norm).

Lemma 2.2.9. *Let A and B be unital C*-algebras. If $\varphi : A \to B$ is a u.c.p. map and ω is a state on B, then*

$$\|\varphi(ab) - \varphi(a)\varphi(b)\|_{2,\omega} \leq \|a\|\,\omega(\varphi(b^*b) - \varphi(b^*)\varphi(b))^{1/2},$$

for all $a, b \in A$.

Proof. By the GNS construction, we can assume that $B \subset B(\mathcal{H})$ and that there exists $\xi \in \mathcal{H}$ such that $\omega(\cdot) = \langle \cdot\, \xi, \xi\rangle$ (see Remark 1.1.22). In particular, for any $b \in B$,

$$\|b\|_{2,\omega} = \omega(b^*b)^{1/2} = \langle b^*b\xi, \xi\rangle^{1/2} = \|b\xi\|\,. \tag{2.11}$$

By Stinespring's Theorem (1.2.3), there exists a representation $\pi : A \to B(\hat{\mathcal{H}})$ and an isometry $v : \mathcal{H} \to \hat{\mathcal{H}}$ such that $\varphi(\cdot) = v^*\pi(\cdot)v$. Therefore, for any $a, b \in A$,

$$\begin{aligned}
\|\varphi(ab) - \varphi(a)\varphi(b)\|_{2,\omega} &\overset{(2.11)}{=} \|(\varphi(ab) - \varphi(a)\varphi(b))\xi\| \\
&= \|v^*\pi(a)(1 - vv^*)\pi(b)v\xi\| \\
&\leq \left\|v^*\pi(a)(1 - vv^*)^{1/2}\right\| \left\|(1 - vv^*)^{1/2}\pi(b)v\xi\right\| \\
&\leq \underbrace{\|v^*\|}_{\leq 1} \|\pi(a)\| \underbrace{\left\|(1 - vv^*)^{1/2}\right\|}_{\leq 1} \Big\langle \underbrace{v^*\pi(b^*)(1 - vv^*)\pi(b)v}_{=\varphi(b^*b) - \varphi(b^*)\varphi(b)}\,\xi, \xi\Big\rangle^{1/2} \\
&\leq \|a\|\,\omega(\varphi(b^*b) - \varphi(b^*)\varphi(b))^{1/2},
\end{aligned}$$

as desired. □

Remark 2.2.10. Let tr be the canonical faithful tracial state on $M_k(\mathbb{C})$. This means that, for any matrix $x = [x_{ij}] \in M_k(\mathbb{C})$,

$$\operatorname{tr}(x) = \frac{1}{k}\sum_{i=1}^{k} x_{ii}\,.$$

In particular, note that $\operatorname{tr}(x) = \frac{1}{k}\operatorname{Tr}(x)$. Then,

$$\|x\|_{2,\mathrm{tr}} = \operatorname{tr}(x^*x)^{1/2} = \frac{1}{\sqrt{k}}\operatorname{Tr}(x^*x)^{1/2} = \frac{1}{\sqrt{k}}\|x\|_2\,.$$

Furthermore, since $M_k(\mathbb{C})$ is finite-dimensional, the identity is a Hilbert-Schmidt operator,

$$|\operatorname{tr}(x)| = \frac{1}{k}|\operatorname{Tr}(x1)| \overset{\text{CBS}}{\leq} \frac{1}{k}\|x\|_2 \underbrace{\|1\|_2}_{=\sqrt{k}} = \frac{1}{\sqrt{k}}\|x\|_2 = \|x\|_{2,\mathrm{tr}}\,.$$

Therefore, $|\operatorname{tr}(x)| \leq \|x\|_{2,\mathrm{tr}}$, for every $x \in M_k(\mathbb{C})$.

In order to prove the next theorem, we need to define some technicalities with regard to A^{op}, the **opposite C*-algebra** of A. Recall that A^{op} is just A with reversed multiplication: $a \cdot b = ba$. In particular, A^{op} is a C*-algebra. Some of the claims made here are justified in the Appendix A.3, and we refer to [Brown and Ozawa, 2008, §6.1] for a complete presentation.

Fix τ a tracial state and consider the GNS triplet $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau = \hat{1})$. Then, there is a representation $\pi_\tau^{\text{op}} : A^{\text{op}} \to B(\mathcal{H}_\tau)$ given by

$$\pi_\tau^{\text{op}}(a)\hat{b} = \widehat{ba} \qquad (a \in A,\, \hat{b} \in A/N_\tau)\,.$$

One can verify that π_τ^{op} is well-defined because τ is tracial. Furthermore, the representations π_τ and π_τ^{op} have interesting properties: they have commuting ranges, i.e., $\pi_\tau(A)' \supset \pi_\tau^{\text{op}}(A^{\text{op}})$; and they satisfy $\pi_\tau(A)'' = \pi_\tau^{\text{op}}(A^{\text{op}})'$.

Since π_τ and π_τ^{op} have commuting ranges, the product $*$-homomorphism $\pi_\tau \times \pi_\tau^{\text{op}} : A \odot A^{\text{op}} \to B(\mathcal{H}_\tau)$, given on elementary tensors by

$$a \otimes b \longmapsto \pi_\tau(a)\pi_\tau^{\text{op}}(b)\,,$$

is well-defined. Composing with the canonical vector state $x \mapsto \langle x\xi_\tau, \xi_\tau\rangle_\tau$, we get a positive linear functional $\mu_\tau : A \odot A^{\text{op}} \to \mathbb{C}$ that satisfies, on elementary tensors,

$$\mu_\tau(a \otimes b) = \langle \pi_\tau(a)\pi_\tau^{\text{op}}(b)\xi_\tau, \xi_\tau\rangle_\tau = \langle \pi_\tau(ab)\xi_\tau, \xi_\tau\rangle_\tau = \tau(ab)\,,$$

since $\pi_\tau^{\text{op}}(b)\xi_\tau = \widehat{1b} = \hat{b} = \widehat{b1} = \pi_\tau(b)\xi_\tau$.

In particular, consider $A = M_k(\mathbb{C})$ and $\tau = \text{tr}$ the tracial state. Then, the linear functional $\mu_{\text{tr}} : M_k(\mathbb{C}) \odot M_k(\mathbb{C})^{\text{op}} \to \mathbb{C}$ satisfies

$$\mu_{\text{tr}}(m \otimes \tilde{m}) = \text{tr}(m\tilde{m})\,,$$

on elementary tensors.

Finally, we are ready to proof the main theorem of the section (argument based on [Brown and Ozawa, 2008, Theorem 6.2.7]).

Theorem 2.2.11. *Let A be a unital C*-algebra and τ be a tracial state on A. Then, the following are equivalent:*

(i) τ is amenable;

(ii) there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that, for every $a, b \in A$,

$$\text{tr}(\varphi_i(a)) \longrightarrow \tau(a) \quad \textit{and} \quad \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\text{tr}} \longrightarrow 0\,. \tag{2.12}$$

(iii) the positive linear functional $\mu_\tau : A \odot A^{op} \to \mathbb{C}$ is continuous with respect to the minimal tensor product norm;

(iv) the product $*$-homomorphism $\pi_\tau \times \pi_\tau^{op} : A \odot A^{op} \to B(L^2(A,\tau))$ is continuous with respect to the minimal tensor product norm;

(v) for any faithful representation $A \subset B(\mathcal{H})$ there exists a u.c.p. map $\Phi : B(\mathcal{H}) \to \pi_\tau(A)''$ such that $\Phi(a) = \pi_\tau(a)$ for every $a \in A$.

Proof. **(i) ⇒ (ii)** Let $A \subset B(\mathcal{H})$ a faithful representation and ψ a hypertrace extending the amenable trace τ.

Step 1: Given a unitary $u \in \mathcal{U}(A)$ and $\varepsilon > 0$, find $h \in \mathcal{L}^1(\mathcal{H})_+$ such that

$$\operatorname{Tr}(h) = 1\,,\ \ |\operatorname{Tr}(uh) - \tau(u)| < \varepsilon \text{ and } \|h - uhu^*\|_1 < \varepsilon\,. \tag{2.13}$$

Recall that the set of normal states on $B(\mathcal{H})$ is weak* dense in the set of states on $B(\mathcal{H})$. Hence, there is a net of positive operators $(h_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}^1(\mathcal{H})_+$ such that $\operatorname{Tr}(h_\lambda x) \longrightarrow \psi(x)$, for every $x \in B(\mathcal{H})$. Since ψ is a hypertrace, it follows that

$$\operatorname{Tr}((h_\lambda - uh_\lambda u^*)x) = \operatorname{Tr}(h_\lambda x) - \operatorname{Tr}(h_\lambda(u^*xu)) \longrightarrow \psi(x) - \psi(u^*xu) = 0\,, \tag{2.14}$$

for every $x \in B(\mathcal{H})$ and $u \in \mathcal{U}(A)$.

Fix a unitary $u \in \mathcal{U}(A)$ and $\varepsilon > 0$. By (2.14), $\operatorname{Tr}(uh) \longrightarrow \tau(u)$ and $h_\lambda - uh_\lambda u^* \longrightarrow 0$ in the weak topology of $\mathcal{L}^1(\mathcal{H})$. In particular, there exist $\lambda_0 \in \Lambda$ such that

$$\lambda \geq \lambda_0 \quad \text{implies} \quad |\operatorname{Tr}(uh_\lambda) - \tau(u)| < \varepsilon\,.$$

Define the subnet $\Lambda_0 = \{\lambda \in \Lambda \mid \lambda \geq \lambda_0\}$ and consider the convex hull

$$B \coloneqq \operatorname{co}\{h_\lambda - uh_\lambda u^* \mid \lambda \in \Lambda_0\} \subseteq \mathcal{L}^1(\mathcal{H})\,.$$

Note that 0 is contained in the weak closure of B, since $h_\lambda - uh_\lambda u^* \longrightarrow 0$ in the weak topology of $\mathcal{L}^1(\mathcal{H})$. Furthermore, by the Hahn-Banach Theorem, the weak and norm closures of B coincide. Thus, 0 is an element of the norm closure of B; i.e, some convex combination of elements in $\{h_\lambda - uh_\lambda u^*\}_{\lambda\in\Lambda}$ will tend to zero in the trace-class norm.

Therefore, there exists $\lambda_1, \ldots, \lambda_n \in \Lambda_0$ and positive real numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$, such that the positive trace-class operator $h \coloneqq \sum_{i=1}^n \alpha_i h_{\lambda_i}$ satisfies

$$\|h - uhu^*\|_1 < \varepsilon\,,$$

and

$$|\operatorname{Tr}(uh) - \tau(u)| = \Big|\sum_{i=1}^n \alpha_i(\operatorname{Tr}(uh_{\lambda_i}) - \tau(u))\Big| \leq \sum_{i=1}^n \alpha_i\Big|\operatorname{Tr}(uh_{\lambda_i}) - \tau(u)\Big| < \varepsilon\,.$$

Moreover, normalizing, if necessary, we may assume that $\operatorname{Tr}(h) = \|h\|_1 = 1$.

Step 2: Extend the previous step to a finite set of unitaries of A with a "nicer" h.

A direct sum trick will suffice. In fact, given $F = \{u_1, \ldots, u_m\} \subseteq A$ a finite set and $\varepsilon > 0$, observe that the n-tuple $(h_\lambda - u_1h_\lambda u_1^*, \ldots, h_\lambda - u_mh_\lambda u_m^*)$ converges weakly to zero in $\bigoplus_{j=1}^m \mathcal{L}^1(\mathcal{H})$. Similarly to step 1, we can find a positive trace class operator $h \in \mathcal{L}^1(\mathcal{H})$ satisfying (2.13) for each $u_i \in F$. Furthermore, since positive finite-rank operators with rational eigenvalues are norm dense $\mathcal{L}^1(\mathcal{H})_+$, we may assume that h has finite-rank and rational eigenvalues.

Step 3: Define the net of u.c.p. maps satisfying (2.12).

Consider the directed set $\mathfrak{F}$ of finite subsets of $\mathcal{U}(A)$, and given $F \in \mathfrak{F}$ define $\varepsilon_F := 1/|F|$. By step 2, there is a positive trace-class operator $h_F \in \mathcal{L}^1(\mathcal{H})$ with finite-rank and rational eigenvalues such that (2.13) holds. This means that, for each $u \in F \subseteq \mathcal{U}(A)$,

$$\operatorname{Tr}(h_F) = 1\,,\ \ |\operatorname{Tr}(uh_F) - \tau(u)| < \varepsilon_F \tag{2.15}$$

$$\text{and} \quad \|h_F - uh_F u^*\|_1 < \varepsilon_F\,. \tag{2.16}$$

By Lemma 2.2.6, there is a u.c.p. map $\varphi_F : B(\mathcal{H}) \to M_{k(F)}(\mathbb{C})$ satisfying

$$\operatorname{tr}(\varphi_F(x)) = \operatorname{Tr}(h_F x) \quad \text{and} \tag{2.17}$$

$$\left|\operatorname{tr}\left(\varphi_F(uu^*) - \varphi_F(u)\varphi_F(u^*)\right)\right| \leq 2\,\|uh_F u^* - h_F\|_1^{1/2}\,, \tag{2.18}$$

for every $x \in B(\mathcal{H})$ and $u \in \mathcal{U}(A)$.

Hence, for every $u \in \mathcal{U}(A)$,

$$\operatorname{tr}(\varphi_F(u)) \overset{(2.17)}{=} \operatorname{Tr}(h_F u) \overset{(2.15)}{\longrightarrow} \tau(u)\,,$$

and, for every $u \in \mathcal{U}(A)$ and $a \in A$,

$$\begin{aligned}\|\varphi_F(au) - \varphi_F(a)\varphi_F(u)\|_{2,\mathrm{tr}} &\leq \|a\|\left|\operatorname{tr}\left(\varphi_F(uu^*) - \varphi_F(u)\varphi_F(u^*)\right)\right|^{1/2} \quad \text{(by Lemma 2.2.9)}\\ &\overset{(2.18)}{\leq} \sqrt{2}\,\|a\|\,\|h_F - uh_F u^*\|_1^{1/4} \overset{(2.16)}{\longrightarrow} 0\,.\end{aligned}$$

Since the unitaries linearly generate A, we conclude that the net of u.c.p. maps $(\varphi_F)_{F\in\mathfrak{F}}$ satisfies (2.12), as desired.

(ii) ⇒ (iii) Since we aim to prove that $|\mu_\tau(x)| \leq \|x\|_{\min}$, for every $x \in A \odot A^{\mathrm{op}}$; it will suffice to prove that there exists a net of states $\phi_i : A \otimes A^{\mathrm{op}} \to \mathbb{C}$ converging pointwise to μ_τ. Indeed, in this case, for any $x \in A \odot A^{\mathrm{op}}$,

$$|\mu_\tau(x)| = \lim_i |\phi_i(x)| \leq \|x\|_{\min}\,.$$

Consider a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ satisfying condition *(ii)*: for all $a, b \in A$,

$$\operatorname{tr}(\varphi_i(a)) \longrightarrow \tau(a) \quad \text{and} \quad \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} \longrightarrow 0\,.$$

Note that, since opposite C*-algebras do not change the order structure, each $\varphi_i^{\mathrm{op}} : A^{\mathrm{op}} \to M_{k(i)}(\mathbb{C})^{\mathrm{op}}$ is also a u.c.p. map. Here, φ_i^{op} stands for the same map as φ_i, but with a "different name".

By Proposition A.2.5, u.c.p. maps induce u.c.p. maps on the minimal tensor product. Hence, there is a u.c.p. map

$$\begin{aligned}\varphi_i \otimes \varphi_i^{\mathrm{op}} : A \otimes A^{\mathrm{op}} &\longrightarrow M_{k(i)}(\mathbb{C}) \otimes M_{k(i)}(\mathbb{C})^{\mathrm{op}} \\ a \otimes b &\longmapsto \varphi_i(a) \otimes \varphi_i(b)\,.\end{aligned}$$

Additionally, denote by $\mu_i : M_{k(i)}(\mathbb{C}) \otimes M_{k(i)}(\mathbb{C})^{\mathrm{op}} \to \mathbb{C}$ the state $\mu_{\mathrm{tr}_{k(i)}}$ that satisfies

$$m \otimes \tilde{m} \longmapsto \mathrm{tr}_{k(i)}(m\tilde{m})\,,$$

on elementary tensors.

We claim that $\mu_i \circ (\varphi_i \otimes \varphi_i^{\mathrm{op}}) \longrightarrow \mu_\tau$ pointwise. Indeed, take an elementar tensor $a \otimes b \in A \odot A^{\mathrm{op}}$, and note that

$$\mu_i(\varphi_i \otimes \varphi_i^{\mathrm{op}}(a \otimes b)) = \mathrm{tr}(\varphi_i(a)\varphi_i(b))$$

Furthermore, the inequality $|\operatorname{tr}(x)| \leq \|x\|_{2,\mathrm{tr}}$, for every $x \in M_{k(i)}(\mathbb{C})$, implies that

$$|\operatorname{tr}(\varphi_i(a)\varphi_i(b)) - \operatorname{tr}(\varphi_i(ab))| \leq \|\varphi_i(a)\varphi_i(b) - \varphi_i(ab)\|_{2,\mathrm{tr}} \longrightarrow 0$$

Finally, from $\operatorname{tr}(\varphi_i(ab)) \longrightarrow \tau(ab)$, it follows that

$$|\mu_i(\varphi_i \otimes \varphi_i^{\mathrm{op}}(a \otimes b)) - \tau(ab)| \leq |\operatorname{tr}(\varphi_i(a)\varphi_i(b)) - \operatorname{tr}(\varphi_i(ab))| + |\operatorname{tr}(\varphi_i(ab)) - \tau(ab)| \longrightarrow 0\,.$$

In other words, $\mu_i(\varphi_i \otimes \varphi_i^{\mathrm{op}}(a \otimes b)) \longrightarrow \tau(ab) = \mu_\tau(a \otimes b)$. Since the span of elementary tensors is dense in $A \otimes A^{\mathrm{op}}$, we have the desired result.

(iii) ⇒ (iv) Recall that, by definition,

$$\mu_\tau(a \otimes b) = \langle \pi_\tau \times \pi_\tau^{\mathrm{op}}(a \otimes b)\xi_\tau, \xi_\tau\rangle_\tau\,,$$

on elementary tensor $a \otimes b \in A \odot A^{\mathrm{op}}$. To simplify notation, we aim to prove a more general result: suppose that B $(= A \odot A^{\mathrm{op}})$ is a unital $*$-subalgebra dense in a unital C*-algebra C $(= A \otimes A^{\mathrm{op}})$. Let $\varphi : B \to \mathbb{C}$ be a positive linear functional, and $\rho : B \to B(\mathcal{H})$ be a nondegenerate representation with cyclic vector $\xi \in \mathcal{H}$ such that

$$\varphi(b) = \langle \rho(b)\xi, \xi\rangle\,.$$

If φ is continuous; i.e., $|\varphi(b)| \leq \|b\|$; then ρ is also continuous. Indeed, for every $b, c \in B$,

$$\begin{aligned}\|\rho(b)\rho(c)\xi\|^2 = \langle \rho(bc)\xi, \rho(bc)\xi\rangle = \langle \rho(c^*b^*bc)\xi, \xi\rangle = \\ = \varphi(c^*b^*bc) \leq \varphi(c^*c)\,\|b\|^2 = \|\rho(c)\xi\|^2\,\|b\|^2\,,\end{aligned}$$

where the inequality follows from $b^*b \leq \|b\|^2 1$. Since vectors of the form $\rho(c)\xi$ linearly generate B, we conclude that ρ is continuous. Identifying φ with μ_τ and ρ with $\pi_\tau \times \pi_\tau^{\mathrm{op}}$, we conclude that the product $*$-homomorphism is continuous in the minimal tensor norm.

(iv) ⇒ (v) Suppose that condition *(iv)* is satisfied. First, we aim to prove that there exists a u.c.p. map $\Phi : B(\mathcal{H}) \to \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})'$ that extends π_τ. Consider the extension of the product $*$-homomorphism to $A \otimes A^{\mathrm{op}}$,

$$\pi_\tau \times \pi_\tau^{\mathrm{op}} : A \otimes A^{\mathrm{op}} \to B(\mathcal{H}_\tau)\,,$$

and notice that $A \otimes A^{\mathrm{op}} \subset B(\mathcal{H}) \otimes A^{\mathrm{op}}$. By Arverson's Extension Theorem (1.2.10), there exists a u.c.p. map

$$\Psi : B(\mathcal{H}) \otimes A^{\mathrm{op}} \to B(\mathcal{H}_\tau)$$

extending $\pi_\tau \otimes \pi_\tau^{\mathrm{op}}$.

Define $\Phi : B(\mathcal{H}) \to \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})' \subseteq B(\mathcal{H}_\tau)$ by

$$x \longmapsto \Psi(x \otimes 1_A)\,.$$

Note that Φ is just the composition of the unital $*$-homomorphism $x \mapsto x \otimes 1_A$ with the u.c.p. map Ψ, thus Φ is a u.c.p. map. Additionally, observe that Φ extends π_τ, since

$$\Phi(a) = \Psi(a \otimes 1_A) = \pi_\tau \otimes \pi_\tau^{\mathrm{op}}(a \otimes 1_A) = \pi_\tau(a)\pi_\tau(1_A) = \pi_\tau(a)\,,$$

for all $a \in A$. Furthermore, in order to show that the range of Φ is contained in $\pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})'$, we use a multiplicative domain argument. We aim to prove that, for any $x \in B(\mathcal{H})$ and $a \in A^{\mathrm{op}}$,

$$\Phi(x)\pi_\tau^{\mathrm{op}}(a) = \pi_\tau^{\mathrm{op}}(a)\Phi(x)\,.$$

However, note that $\pi_\tau^{\mathrm{op}}(a) = \Psi(1_A \otimes a)$, and $\mathbb{C}1_A \otimes A^{\mathrm{op}}$ is contained in the multiplicative domain of Ψ. Hence,

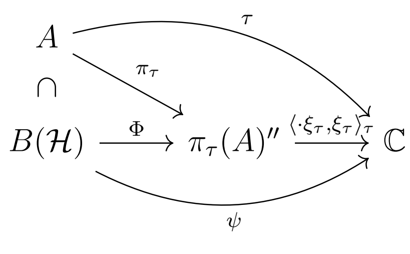
$$\Phi(x)\pi_\tau^{\mathrm{op}}(a) = \Psi(x \otimes 1_A)\Psi(1_A \otimes a) = \Psi((x \otimes 1_A)(1_A \otimes a)) = \\ = \Psi((1_A \otimes a)(x \otimes 1_A)) = \pi_\tau^{\mathrm{op}}(a)\Phi(x)\,.$$

Finally, recall that $\pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})' = \pi_\tau(A)''$. Therefore, $\Phi : B(\mathcal{H}) \to \pi_\tau(A)''$ is a u.c.p. map that extends π_τ, as desired.

(v) ⇒ (i) Suppose that $A \subset B(\mathcal{H})$ is a faithful representation and $\Phi : B(\mathcal{H}) \to \pi_\tau(A)''$ is a u.c.p. map such that $\Phi(a) = \pi_\tau(a)$, for every $a \in A$. Recall that, by the GNS construction, for every $a \in A$,

$$\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle\,.$$

Hence, we can extend to a map $\langle \cdot \xi_\tau, \xi_\tau \rangle_\tau$ that is a tracial state on $\pi_\tau(A)''$. Furthermore, observe that A is contained in the multiplicative domain of Φ, since $\Phi|_A = \pi_\tau$ is a $*$-homomorphism.



Define $\psi : B(\mathcal{H}) \to \mathbb{C}$ by

$$\psi(x) = \langle \Phi(x)\xi_\tau, \xi_\tau \rangle_\tau .$$

Then, for every $a \in A$ and $x \in B(\mathcal{H})$,

$$\begin{aligned}\psi(ax) &= \langle \Phi(ax)\xi_\tau, \xi_\tau \rangle_\tau \\ &= \langle \Phi(a)\Phi(x)\xi_\tau, \xi_\tau \rangle_\tau \\ &= \langle \pi_\tau(a)\Phi(x)\xi_\tau, \xi_\tau \rangle_\tau \\ &= \langle \Phi(x)\pi_\tau(a)\xi_\tau, \xi_\tau \rangle_\tau \\ &= \langle \Phi(xa)\xi_\tau, \xi_\tau \rangle_\tau \\ &= \psi(xa) ,\end{aligned}$$

and

$$\psi(a) = \langle \Phi(a)\xi_\tau, \xi_\tau \rangle_\tau = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle_\tau = \tau(a) .$$

Therefore, ψ is a hypertrace and τ is an amenable trace. □

As consequences of the characterization of amenable traces, we present the following results.

Proposition 2.2.12. *Every tracial state on a unital nuclear C*-algebra is amenable.*

Proof. Let $A \subset B(\mathcal{H})$ be a unital nuclear C*-algebra and τ be any tracial state on A. We aim to prove that τ is amenable using consition *(iv)*. Recall that, by the universality of the maximal tensor product, the product $*$-homomorphism $\pi_\tau \times \pi_\tau^{\mathrm{op}}$ is continuous with respect to the maximal tensor product norm. However, since A is nuclear, the maximal and minimal norm coincide (see [Brown and Ozawa, 2008, Theorem 3.8.7.]). Therefore, the product $*$-homomorphism $\pi_\tau \times \pi_\tau^{\mathrm{op}}$ is continuous with respect to the minimal tensor product norm; i.e., τ is an amenable trace. □

Proposition 2.2.13. *Let G be a discrete group. the following are equivalent:*

(i) G is amenable;

(ii) $C_r^*(G)$ has an amenable trace;

(iii) every trace on $C_r^*(G)$ is amenable.

Proof. **(i) ⇒ (ii)** Follows from the previous calculations in Example 2.2.4.
(ii) ⇒ (i) If $C_r^*(G)$ has an amenable trace, let φ be a hypertrace on $B(\ell^2(G))$ such that $\varphi(uxu^*) = \varphi(x)$, for every $u \in \mathcal{U}(C_r^*(G))$ and $x \in B(\ell^2(G))$. In particular, if $f \in \ell^\infty(G)$ and $g \in G$,

$$\varphi(g \cdot f) = \varphi(\lambda_g f \lambda_g^*) = \varphi(f) ,$$

since $\lambda_g \in \mathcal{U}(C_r^*(G))$ and the action is spatially implemented. Therefore, the restriction $\mu := \varphi|_{\ell^\infty(G)}$ is an invariant mean.

(i) ⇒ (iii) If G is amenable, then $C_r^*(G)$ is nuclear by Theorem 2.1.15. Therefore, Proposition 2.2.12 implies that every trace on $C_r^*(G)$ is amenable.
(iii) ⇒ (i) It is obvious, since *(iii)* implies *(ii)*. □

Remark 2.2.14 (C*-algebras without amenable traces). We say that a unital C*-algebra A is *properly infinite* if there exist two isometries $x, y \in A$ such that $xx^*+yy^* \leq 1$. Suppose that A has a tracial state τ. Then,

$$2 = \tau(x^*x) + \tau(y^*y) = \tau(xx^*) + \tau(yy^*) = \tau(xx^* + yy^*) \leq \tau(1) = 1\,,$$

which is a contradiction. Therefore, properly infinite C*-algebras does not admit tracial states, and, in particular, does not admit amenable traces.

As an example of properly infinite C*-algebras, we have the Cuntz algebra $\mathcal{O}_n$ ($n \geq 2$), the universal C*-algebra generated by operators $s_1, \ldots, s_n$ such that, for every $i = 1, \ldots, n$,

$$s_i^*s_i = 1\,, \quad s_i^*s_j = 0 \quad (i \neq j) \quad \text{and} \quad s_1s_1^* + \ldots + s_ns_n^* = 1.$$

Hence, $\mathcal{O}_n$ is properly infinite and does not admit tracial states.

Another classical example of a C*-algebra without tracial states is $K(\mathcal{H})$, when $\mathcal{H}$ is infinite-dimensional. Suppose, for the sake of contradiction, that $K(\mathcal{H})$ admits a tracial state τ. If $\xi \in \mathcal{H}$ is a unit vector, denote by p_ξ the rank-one projection given by $\eta \mapsto \langle \eta, \xi \rangle \xi$. In particular, note that, if $\xi, \xi' \in \mathcal{H}$ are orthogonal unit vectors, then there is a unitary $u \in B(\mathcal{H})$ such that $u(\xi) = \xi'$. Furthermore, observe that $up_\xi = p_{\xi'}u$, since

$$up_\xi(\eta) = \langle \eta, \xi \rangle u(\xi) = \langle \eta, u^*(\xi') \rangle \xi' = \langle u(\eta), \xi' \rangle \xi' = p_{\xi'}u(\eta)\,,$$

for every $\eta \in \mathcal{H}$. Hence,

$$\tau(p_\xi) = \tau(u^*p_{\xi'}u) = \tau(\underbrace{u^*p_{\xi'}}_{\in K(\mathcal{H})}\;\underbrace{p_{\xi'}u}_{\in K(\mathcal{H})}) = \tau(p_{\xi'}uu^*p_{\xi'}) = \tau(p_{\xi'})\,.$$

Finally, let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. Observe that $\tau(p_{\xi_i})$ has the same value for each $i \in I$, let say $c > 0$. For any finite subset $F \subseteq I$, consider the projection $p_F = \sum_{i \in F} p_{\xi_i}$, then

$$\tau(p_F) = \sum_{i \in F} \tau(p_{\xi_i}) = \sum_{i \in F} c = |F|c\,.$$

On the other hand, $p_F \leq 1$ implies $\tau(p_F) \leq 1$. Hence $|F| \leq 1/c$ for every finite subset, which is an absurdity. Therefore, $K(\mathcal{H})$ does not admits tracial states. Furthermore, the same argument shows that $B(\mathcal{H})$ does not admit tracial states.

Notes and Remarks. The first version of Theorem 2.2.11 appeared in [Kirchberg, 1994]. Also, in that work, the maps equivalent to amenable traces were called *liftable traces.*

In [Bekka, 1990], the author defines, given a unitary representation $\rho : G \to B(\mathcal{H})$, a G-invariant mean as an state φ on $B(\mathcal{H})$ such that, for any $g \in G$ and $x \in B(\mathcal{H})$,

$$\varphi(\rho(g)x\rho(g^{-1})) = \varphi(x)\,.$$

Therefore, if ρ is faithful we can extend to a representation of $C^*(G)$. In this setting, the formula above essentially defines a hypertrace.

The proof that *(iv)* implies *(v)* in Theorem 2.2.11 is essentially the proof of The Trick (for more details, see [Brown and Ozawa, 2008, Proposition 3.6.5]).

In [Brown, 2004, Theorem 2.3], the author gives an elegant (and shorter) proof of the equivalence between conditions *(i)*, *(ii)* and *(v)* in Theorem 2.2.11 when A is separable. The author uses $\mathcal{R}$, the hyperfinite II_1 factor, and ultraproducts in the argument.

2.3 Følner-type conditions for C*-algebras

In the same sense that amenable traces are analogues to invariant means, there is an analogue to Følner nets on C*-algebras. This section presents Følner type conditions for C*-algebras and its connections with the existence of amenable traces. The main references are [Ara and Lledó, 2014], [Bédos, 1995] and [Bédos, 1996].

Definition 2.3.1. Let $A \subset B(\mathcal{H})$ be a C*-algebra. A net of nonzero finite rank orthogonal projections $(p_i)_{i\in I} \subseteq B(\mathcal{H})$ is a called a **Følner net** for A if

$$\frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} \longrightarrow 0\,, \tag{2.19}$$

for every $a \in A$. Moreover, $(p_i)_{i\in I}$ is said to be a **proper Følner net** if it converges to $1_{\mathcal{H}}$ in the strong operator topology.

Before we present examples, we present a useful lemma, since it shows that we only need to verify the convergence of the Følner net on a set of generators.

Lemma 2.3.2. *Let $A \subset B(\mathcal{H})$ be a C*-algebra generated by $S \subseteq A$ and $(p_i)_{i\in I} \subset B(\mathcal{H})$ be a net of nonzero finite rank orthogonal projections. Then, $(p_i)_{i\in I}$ is a Følner net for A if and only if, for every $a \in S$,*

$$\frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} \longrightarrow 0\,. \tag{2.20}$$

Proof. The forward direction is trivial. Now suppose that (2.20) holds for every $a \in S$. Given an element $b \in A$, we need to consider four cases. First, consider $b = a^* \in S^*$, since the involution is isometric in $\|\cdot\|_2$ (see Proposition 1.4.2), we conclude that (2.20) holds for b.

Second, suppose that $b = a_1 a_2$ with $a_1, a_2 \in S \cup S^*$. Note that,

$$\begin{aligned}\frac{\|bp_i - p_i b\|_2}{\|p_i\|_2} &= \frac{\|a_1(a_2 p_i - p_i a_2) + (a_1 p_i - p_i a_1)a_2\|_2}{\|p_i\|_2}\\ &\leq \|a_1\|\,\frac{\|a_2 p_i - p_i a_2\|_2}{\|p_i\|_2} + \frac{\|a_1 p_i - p_i a_1\|_2}{\|p_i\|_2}\,\|a_2\| \longrightarrow 0\,,\end{aligned}$$

which implies that (2.20) holds for b. The proof when b is an arbitrary finite product of elements of $S \cup S^*$ follows from induction. The set formed by these products will be called $\hat{S}$.

Third, consider the case where $b = \alpha a_1 + a_2$ is a linear combination of elements of $\hat{S}$. Then

$$\frac{\|bp_i - p_i b\|_2}{\|p_i\|_2} \leq |\alpha| \frac{\|a_1 p_i - p_i a_1\|_2}{\|p_i\|_2} + \frac{\|a_2 p_i - p_i a_2\|_2}{\|p_i\|_2} \longrightarrow 0\,.$$

Hence (2.20) holds for any $b \in \operatorname{span}(\hat{S})$.

Finally, consider $b \in A$ a generic element and $\varepsilon > 0$. Then, there exists an element $a \in \operatorname{span}(\hat{S})$ such that $\|b - a\| < \varepsilon/4$. Since (2.20) holds for a, there is $i_0 \in I$, such that

$$i \geq i_0 \quad \text{implies} \quad \frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} < \frac{\varepsilon}{2}\,.$$

Therefore, for any $i \geq i_0$,

$$\begin{aligned} \frac{\|bp_i - p_i b\|_2}{\|p_i\|_2} &\leq \frac{\|(b-a)p_i - p_i(b-a)\|_2}{\|p_i\|_2} + \frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} \\ &\leq 2\,\|b-a\|\,\frac{\|p_i\|_2}{\|p_i\|_2} + \frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} < \varepsilon\,, \end{aligned}$$

which implies that (2.20) holds for any $b \in A$; i.e., $(p_i)_{i\in I}$ is a Følner net for A. □

Example 2.3.3. Let G be an amenable group and $(F_i)_{i\in I}$ be a Følner net for the group G. For each $i \in I$, consider $p_i \in B(\ell^2(G))$ the finite-rank projection of $\ell^2(G)$ onto $\ell^2(F_i) = \operatorname{span}\{\lambda_g \mid g \in F_i\}$. The following calculations show that $(p_i)_{i\in I}$ is a Følner net for $C_r^*(G) \subset B(\ell^2(G))$.

Since $C_r^*(G) = C^*(\{\lambda_g \mid g \in G\})$, by Lemma 2.3.2, it is enough to show that

$$\frac{\|\lambda_g p_i - p_i \lambda_g\|_2}{\|p_i\|_2} \longrightarrow 0\,,$$

for every $g \in G$. Denote by $\{\delta_h\}_{h\in G}$ the canonical orthonormal basis of $\ell^2(G)$ and recall that $\lambda_g(\delta_h) = \delta_{gh}$.

By the definition of the Hilbert-Schmidt norm,

$$\|p_i\|_2^2 = \sum_{h\in G} \|p_i(\delta_h)\|^2 = \sum_{h\in F_i} \|\delta_h\|^2 = \sum_{h\in F_i} 1 = |F_i|\,. \tag{2.21}$$

Now, note that

$$\lambda_g p_i(\delta_h) = \begin{cases} \delta_{gh} & \text{, if } h \in F_i\,; \\ 0 & \text{, if } h \notin F_i\,; \end{cases} \tag{2.22}$$

and

$$p_i \lambda_g(\delta_h) = \begin{cases} \delta_{gh} & \text{, if } gh \in F_i\,; \\ 0 & \text{, if } gh \notin F_i\,. \end{cases} \tag{2.23}$$

In particular, note that $gh \in F_i$ if and only if $h \in g^{-1}F_i$. Hence, using (2.22) and (2.23), we conclude that

$$(\lambda_g p_i - p_i \lambda_g)(\delta_h) = \begin{cases} \delta_{gh} & \text{, if } h \in F_i \backslash g^{-1}F_i\,; \\ -\delta_{gh} & \text{, if } h \in g^{-1}F_i \backslash F_i\,; \\ 0 & \text{, if } h \notin g^{-1}F_i \triangle F_i\,. \end{cases} \tag{2.24}$$

Therefore,

$$\|\lambda_g p_i - p_i\lambda_g\|_2^2 = \sum_{h\in G} \|(\lambda_g p_i - p_i \lambda_g)(\delta_h)\|^2 = \sum_{h \in g^{-1}F_i \triangle F_i} \|\pm \delta_{gh}\|^2 = |g^{-1}F_i \triangle F_i|$$

Finally, using the equation above and (2.21) we conclude that

$$\frac{\|\lambda_g p_i - p_i \lambda_g\|_2}{\|p_i\|_2} = \sqrt{\frac{|g^{-1}F_i \triangle F_i|}{|F_i|}} \longrightarrow 0\,,$$

since $(F_i)_{i\in I}$ is a Følner net for the group G.

Moreover, observe that $(p_i)_{i\in I}$ does not need to be a proper Følner net. For instance, consider the Følner sequence of $\mathbb{Z}$ in Example 2.1.13, then p_n is the projection onto $\ell^2([0,n])$ and converge strongly to the projection onto $\ell^2(\mathbb{N}\cup\{0\})$, which is different from $1_{\ell^2(\mathbb{Z})}$.

Example 2.3.4. Consider $\mathcal{T}$ the Toeplitz algebra, which is the the C*-algebra generated by the unilateral shift $s \in B(\ell^2(\mathbb{N}))$. Denoting by $\{\delta_n\}_{n\in\mathbb{N}}$ the canonical basis of $\ell^2(\mathbb{N})$, we have $s(\delta_n) = \delta_{n+1}$. Now consider, for each $n \in \mathbb{N}$, the orthogonal projections $p_n \in B(\ell^2(\mathbb{N}))$ onto the subspace $\text{span}\{\delta_1, \dots, \delta_n\}$. Hence, for any $n \in \mathbb{N}$,

$$\|sp_n - p_n s\|_2^2 = \sum_{m\in\mathbb{N}} \|(sp_n - p_n s)(\delta_m)\|^2 = \sum_{m=1}^{n} \|s(\delta_m) - p_n(\delta_{m+1})\|^2 = \|\delta_{n+1}\|^2 = 1$$

and

$$\frac{\|sp_n - p_n s\|_2}{\|p_n\|_2} = \frac{1}{\sqrt{n}} \longrightarrow 0.$$

Therefore, $(p_n)_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ by Lemma 2.3.2.

As before, we can always "localize" our definition.

Definition 2.3.5. We say that A satisfies the **Følner condition** if for any finite subset $F \subseteq A$ and any $\varepsilon > 0$ there exists a finite rank orthogonal projection $p \in B(\mathcal{H})$ such that

$$\frac{\|ap - pa\|_2}{\|p\|_2} < \varepsilon\,, \tag{2.25}$$

for every $a \in F$.

Proposition 2.3.6. *Let A be a C*-algebra, then A has a Følner net if and only if A satisfies the Følner condition.*

Proof. (⇒) Denote by $(p_i)_{i\in I}$ a proper Følner net for A. Consider a finite subset $F \subseteq A$, $\varepsilon > 0$ and a finite-rank orthogonal projection q. For each $a \in F$ there exists $i_a \in I$ such that

$$i \geq i_a \quad \text{implies} \quad \frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} < \varepsilon\,.$$

Taking $i_0 \geq i_a$, for every $a \in F$, it follows that

$$\frac{\|ap_{i_0} - p_{i_0} a\|_2}{\|p_{i_0}\|_2} < \varepsilon\,,$$

for every $a \in F$.
(⇐) Suppose that A satisfies the Følner condition and consider the directed set $\mathfrak{F} = \{F \mid F \subseteq A \text{ is a non-empty finite subset}\}$, where $F \leq F'$ if $F \subseteq F'$.

Then, for each $F \in \mathfrak{F}$, there exists a finite-rank orthogonal projection $p_F \in B(\mathcal{H})$ such that

$$\frac{\|ap_F - p_F a\|_2}{\|p_F\|_2} < \frac{1}{|F|}\,,$$

for every $a \in F$.

We claim that $(p_F)_{F\in\mathfrak{F}}$ is a Følner net for A. Indeed, fix an element $a \in A$ and $\varepsilon > 0$, and consider $E \in \mathfrak{F}$ such that $a \in E$ and $\frac{1}{|E|} < \varepsilon$. Then

$$F \geq E \quad \text{implies} \quad \frac{\|ap_F - p_F a\|_2}{\|p_F\|_2} < \frac{1}{|F|} \leq \frac{1}{|E|} < \varepsilon\,.$$

Therefore, $(p_F)_{F\in\mathfrak{F}}$ is a Følner net for A, as desired. □

Remark 2.3.7. Furthermore, if A is separable and satisfies the Følner condition, then we may take a Følner sequence instead of a Følner net. In fact, if $D \subseteq A$ is a countable dense set, when defining the directed set $\mathfrak{F}$ we consider only finite subsets $F \subseteq D$. This argument is valid in general, i.e., whenever we have an equivalence between a local property and a net-based definition, if the underlying space is separable (or countable), we may consider sequences instead of nets.

The following proposition will show that we could use slightly different expressions to define Følner nets or Følner conditions.

Proposition 2.3.8. *Let $(p_i)_{n\in I} \subseteq B(\mathcal{H})$ be a net of nonzero finite-rank orthogonal projections. Then $(p_i)_{i\in I}$ is a Følner net for A if and only if, for all $a \in A$,*

$$\frac{\|(1-p_i)ap_i\|_2}{\|p_i\|_2} \longrightarrow 0\,. \tag{2.26}$$

Proof. (⇒) Suppose that, for any $a \in A$,

$$\frac{\|ap_i - p_i a\|_2}{\|p_i\|_2} \longrightarrow 0\,.$$

Note that, for any $i \in I$,

$$(1-p_i)(ap_i - p_i a) = ap_i - p_i a - p_i a p_i + p_i a = ap_i - p_i a p_i = (1-p_i)ap_i \,.$$

Hence,

$$\frac{\|(1-p_i)ap_i\|_2}{\|p_i\|_2} = \frac{\|(1-p_i)(ap_i - p_i a)\|_2}{\|p_i\|_2} \leq \underbrace{\|1-p_i\|}_{=1} \frac{\|(ap_i - p_i a)\|_2}{\|p_i\|_2} \longrightarrow 0 \,.$$

(⇐) Conversely, if (2.26) holds for every $a \in A$. Observe that, for any $a \in A$ and $i \in I$,

$$\frac{\|p_i a(1-p_i)\|_2}{\|p_i\|_2} = \frac{\|(p_i a(1-p_i))^*\|_2}{\|p_i\|_2} = \frac{\|p_i a^*(1-p_i)\|_2}{\|p_i\|_2} \longrightarrow 0 \,.$$

Since

$$\|ap_i - p_i a\|_2 = \|ap_i - p_i a p_i + p_i a p_i - p_i a\|_2 \leq \|(1-p_i)ap_i\|_2 + \|p_i a(1-p_i)\|_2 \ ,$$

it follows that $\dfrac{\|ap_i - p_i a\|_2}{\|p_i\|_2} \longrightarrow 0$, as desired. □

Before we state the main theorem of this section, recall that $\mathrm{tr}(\cdot)$ denotes the canonical tracial state on $M_k(\mathbb{C})$ and $\mathrm{Tr}(\cdot)$ denotes the canonical trace on $\mathcal{L}^1(\mathcal{H})$. If we take an isometry $v : \mathbb{C}^k \to \mathcal{H}$ with a finite-rank orthogonal projection $p = vv^*$, $\mathrm{rank}(p) = k$, one can verify that, for any $a \in B(\mathcal{H})$,

$$\mathrm{tr}(v^* a v) = \frac{\mathrm{Tr}(pa)}{\mathrm{Tr}(p)} \,. \tag{2.27}$$

Indeed, let $\{e_j\}_{j=1}^k$ denote the canonical basis of $\mathbb{C}^k$ and let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$ such that $\xi_i = v(e_i)$, for $i \in J \subseteq I$. Then,

$$\mathrm{tr}(v^* a v) = \frac{1}{k}\sum_{j=1}^k \langle v^* a v e_j, e_j\rangle = \frac{1}{k}\sum_{j=1}^k \langle a v e_j, v e_j\rangle = \frac{1}{k}\sum_{i \in J}\langle a\xi_i, \xi_i\rangle \,.$$

On the other hand,

$$\frac{\mathrm{Tr}(ap)}{\mathrm{Tr}(p)} = \frac{1}{\mathrm{Tr}(p)}\sum_{i \in I}\langle a v v^* \xi_i, \xi_i\rangle = \frac{1}{k}\sum_{i \in J}\langle a\xi_i, \xi_i\rangle \,.$$

Thus (2.27) holds. This equation will be crucial in the calculations of the next theorem.

The next result is a synthesis between [Ara and Lledó, 2014, Proposition 3.1] and [Bédos, 1995, Theorem 1.1].

Theorem 2.3.9. *Let $A \subset B(\mathcal{H})$ be a unital C^*-algebra.*

(i) Then, A has a Følner net $(p_i)_{i \in I}$ if and only if A has an amenable trace.

(ii) If A and $\mathcal{H}$ are separable, $A \cap K(\mathcal{H}) = \{0\}$, and τ is an amenable trace on A. Then A has a Følner sequence $(p_n)_{n\in\mathbb{N}}$ satisfying

$$\tau(a) = \lim_n \frac{\operatorname{Tr}(ap_n)}{\operatorname{Tr}(p_n)} . \tag{2.28}$$

Proof. **(i)** Suppose that $(p_i)_{i\in I}$ is a Følnet net for A. Consider the net of linear functionals $\varphi_i : B(\mathcal{H}) \to \mathbb{C}$ given by

$$x \mapsto \frac{\operatorname{Tr}(xp_i)}{\operatorname{Tr}(p_i)} .$$

Observe that each φ_i is a state, since

$$\varphi_i(1) = \frac{\operatorname{Tr}(p_i)}{\operatorname{Tr}(p_i)} = 1$$

and

$$\|\varphi_i(x)\| = \frac{|\operatorname{Tr}(xp_i)|}{|\operatorname{Tr}(p_i)|} \leq \frac{\|x\| \, \|p_i\|_1}{\|p_i\|_1} = \|x\|$$

for every $x \in B(\mathcal{H})$. Where the inequality follows from $\operatorname{Tr}(p_i) = \|p_i\|_1$ and Theorem 1.4.14.

Since $(\varphi_i)_{i\in I}$ is contained in the unit ball of $B(\mathcal{H})^*$, by weak*-compactness there is a cluster point φ for the net. We claim that this cluster point is a hypertrace on A. In fact, taking a subnet $K \subseteq I$, if necessary, we can say that

$$\varphi = \lim_{k\in K} \varphi_k .$$

For any $u \in \mathcal{U}(A)$ and $x \in B(\mathcal{H})$, note that

$$\begin{aligned} \varphi(uxu^*) = \varphi(x) \quad & \text{if and only if} \quad \varphi(uxu^* - x) = 0 \\ & \text{if and only if} \quad \lim_{k\in K} \varphi_k(uxu^* - x) = 0 . \end{aligned}$$

Now, observe that, for each $k \in K$,

$$\begin{aligned} |\varphi_k(uxu^* - x)| &= \frac{|\operatorname{Tr}((uxu^* - x)p_k)|}{|\operatorname{Tr}(p_k)|} \\ &\overset{(\dagger)}{=} \frac{|\operatorname{Tr}((u^*p_ku - p_k)x)|}{|\operatorname{Tr}(p_k)|} \\ &\leq \frac{\|u^*p_ku - p_k\|_1}{\|p_k\|_1} \|x\| , \end{aligned} \tag{2.29}$$

where $(\dagger)$ comes from

$$\operatorname{Tr}((uxu^* - x)p_k) = \operatorname{Tr}(uxu^*p_k - xp_k) = \operatorname{Tr}(u^*p_kux - p_kx) = \operatorname{Tr}((u^*p_ku - p_k)x) .$$

Observe carefully that the right side of (2.29) is expressed in terms of the trace-class norm. To change to the Hilbert-Schmidt norm, we use the Powers-Størmer inequality (see 1.4.18). If $0 \leq a, b \in \mathcal{L}^2(\mathcal{H})$, then $\|a^2 - b^2\|_1 \leq \|a+b\|_2 \|a-b\|_2$. Applying this inequality to $a = u^* p_k u \, (= a^2)$ and $b = p_k \, (= b^2)$, we have

$$\begin{aligned} \|u^* p_k u - p_k\|_1 &\leq \|u^* p_k u + p_k\|_2 \|u^* p_k u - p_k\|_2 \\ &= \|p_k u + u p_k\|_2 \|p_k u - u p_k\|_2 \\ &\leq (\|p_k u\|_2 + \|u p_k\|_2) \|p_k u - u p_k\|_2 \\ &\leq 2 \|u\| \|p_k\|_2 \|p_k u - u p_k\|_2 \,, \end{aligned}$$

where the second equality follows because the Hilbert-Schmidt norm is unitarily invariance (see Remark 1.4.17). Recall that $\|p_k\|_1 = \|p_k\|_2^2$ and apply the estimation above to (2.29), then

$$|\varphi_k(uxu^* - x)| \leq \frac{\|u^* p_k u - p_k\|_1}{\|p_k\|_1} \|x\| \leq 2 \|u\| \frac{\|p_k\|_2 \|p_k u - u p_k\|_2}{\|p_k\|_2^2} \|x\| \longrightarrow 0$$

Therefore, φ is a hypertrace and $\tau := \varphi|_A$ is an amenable trace on A.

The proof of the converse can be found in [Bédos, 1995, Theorem 1.1]. It will not be presented here because it relies in an argument on [Bekka, 1990, Theorem 6.2], which in turn, relies on [Connes, 1976a] and [Greenleaf, 1969].

(ii) Suppose that A and $\mathcal{H}$ are separable, $A \cap K(\mathcal{H}) = \{0\}$, and τ is an amenable trace on A. We aim to prove that, for any finite subset $F \subseteq A$ and $1 > \varepsilon > 0$, there exists a finite-rank orthogonal projection $p \in B(\mathcal{H})$ such that,

$$\frac{\|(1-p)ap\|_2}{\|p\|_2} < \varepsilon \text{ and } \left| \tau(a) - \frac{\mathrm{Tr}(ap)}{\mathrm{Tr}(p)} \right| < \varepsilon \,, \tag{2.30}$$

for every $a \in F$. Observe that (2.30) describes exactly the local property of a Følner sequence $(p_n)_{n \in \mathbb{N}}$ that satisfies $\dfrac{\mathrm{Tr}(ap_n)}{\mathrm{Tr}(p_n)} \longrightarrow \tau(a)$.

First, remember that, by Theorem 2.2.11 there exist $k \in \mathbb{N}$ and a u.c.p. map $\varphi : A \to M_k(\mathbb{C})$ such that

$$|\tau(a) - \mathrm{tr}(\varphi(a))| < \varepsilon \tag{2.31}$$

and

$$|\,\mathrm{tr}(\varphi(a^* a) - \varphi(a^*)\varphi(a))| \leq \|\varphi(a^* a) - \varphi(a^*)\varphi(a)\|_{2,\mathrm{tr}} < \varepsilon \tag{2.32}$$

for every $a \in F$.

Furthermore, by Stinespring's Theorem (1.2.3), there exist an isometry $v : \mathbb{C}^k \to \hat{\mathcal{H}}$ and a representation $\pi : A \to B(\hat{\mathcal{H}})$ such that

$$\varphi(b) = v^* \pi(b) v \tag{2.33}$$

for every $b \in A$. Note that the domain of the isometry v is $\mathbb{C}^k$ because $M_k(\mathbb{C}) \cong B(\mathbb{C}^k)$, hence the codomain of φ is $B(\mathbb{C}^k)$. Denote by $q := vv^* \in B(\hat{\mathcal{H}})$ the Stinespring (orthogonal) projection and note that q has finite-rank.

We claim that, for every $a \in F$,

$$\frac{\|(1-q)\pi(a)q\|_2}{\|q\|_2} < \sqrt{\varepsilon}. \tag{2.34}$$

Fix an element $a \in F$. Since q is an orthogonal projection, take a orthogonal basis $\{\xi_i\}_{i\in I}$ of $\hat{\mathcal{H}}$ such that there is a finite subset $J \subseteq I$ with $q(\xi_i) = \xi_i$ if $i \in J$ and zero otherwise. In particular, observe that $\|q\|_2^2 = \operatorname{rank}(q) = |J|$. Therefore,

$$\begin{aligned}\|(1-q)\pi(a)q\|_2^2 &= \sum_{i\in I} \|((1-q)\pi(a)q)(\xi_i)\|_2^2 &(2.35)\\ &= \sum_{i\in I} \langle q^*\,\pi(a^*)(1-q)^*(1-q)\pi(a)q\,\xi_i, \xi_i\rangle\\ &= \sum_{i\in J} \langle q\,\pi(a^*)(1-q)\pi(a)q\,\xi_i, \xi_i\rangle &(2.36)\end{aligned}$$

On the other hand, by (2.33),

$$\begin{aligned}\varphi(a^*a) - \varphi(a^*)\varphi(a) &= v^*\pi(a^*)\pi(a)v - v^*\pi(a^*)vv^*\pi(a)v\\ &= v^*\pi(a^*)(1-vv^*)\pi(a)v\,, &(2.37)\end{aligned}$$

Thus,

$$\begin{aligned}|\operatorname{tr}(\varphi(a^*a) - \varphi(a^*)\varphi(a))| &= |\operatorname{tr}(v^*\pi(a^*)(1-q)\pi(a)v)|\\ &= |\operatorname{tr}(\underbrace{v^*v}_{=1}\,v^*\pi(a^*)(1-q)\pi(a)v)|\\ &\overset{(2.27)}{=} \frac{1}{\operatorname{Tr}(q)}|\operatorname{Tr}(vv^*\pi(a^*)(1-q)\pi(a)vv^*)|\\ &= \frac{1}{\|q\|_2^2}\Big|\sum_{i\in J}\langle q\,\pi(a^*)(1-q)\pi(a)q\,\xi_i, \xi_i\rangle\Big| &(2.38)\end{aligned}$$

Using (2.32), (2.36) and (2.38), we conclude that

$$\frac{\|(1-q)\pi(a)q\|_2}{\|q\|_2} = \sqrt{|\operatorname{tr}(\varphi(a^*a) - \varphi(a^*)\varphi(a))|} < \sqrt{\varepsilon}\,, \tag{2.39}$$

for every $a \in F$, as desired.

Unfortunately, q is a projection in $B(\hat{\mathcal{H}})$ and not in $B(\mathcal{H})$, as we expected. To "correct the space" we will use Voiculescu's Theorem (A.1.4). Consider the faithful representation $\iota : A \hookrightarrow B(\mathcal{H})$ and observe that $\pi|_{A\cap K(\mathcal{H})} = \pi|_{\{0\}} = 0$. By Voiculescu's Theorem, ι and $\iota \oplus \pi$ are approximately unitarily equivalent. Therefore, there is a unitary $u : \mathcal{H} \to \mathcal{H} \oplus \hat{\mathcal{H}}$ such that

$$\|\iota(a) - u^*(\iota \oplus \pi)(a)u\| = \|a - u^*(a \oplus \pi(a))u\| < \varepsilon \tag{2.40}$$

for every $a \in F$. Hence, define a orthogonal projection on $\mathcal{H}$ by

$$p := u^*(0 \oplus q)u.$$

Observe that, by definition, $\|p\|_2^2 = \operatorname{rank}(p) = \operatorname{rank}(q) = \|q\|_2^2$. Denoting $p^\perp = 1 - p$ and $q^\perp = 1 - q$, observe that

$$\left\|p^\perp ap\right\|_2 \leq \left\|p^\perp(a - u^*(a \oplus \pi(a))u)p\right\|_2 + \left\|p^\perp u^*(a \oplus \pi(a))up\right\|_2 \tag{2.41}$$

The first term can be estimated has follows

$$\left\|p^\perp(a - u^*(a \oplus \pi(a))u)p\right\|_2 \leq \underbrace{\left\|p^\perp\right\|}_{\leq 1} \|a - u^*(a \oplus \pi(a))u\| \, \|p\|_2 \overset{(2.40)}{<} \varepsilon \, \|p\|_2 \tag{2.42}$$

For the other term, first observe that $p^\perp = u^*(1 \oplus q^\perp)u$ implies

$$\begin{aligned}
p^\perp u^*(a \oplus \pi(a))up &= u^*(1 \oplus q^\perp) \underbrace{u\, u^*}_{=1}(a \oplus \pi(a)) \underbrace{u\, u^*}_{=1}(0 \oplus q)u \\
&= u^*(1 \oplus q^\perp)(a \oplus \pi(a))(0 \oplus q)u \\
&= u^*(0 \oplus q^\perp \pi(a)q)u\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|p^\perp u^*(a \oplus \pi(a))up\right\|_2 &= \left\|u^*(0 \oplus q^\perp \pi(a)q)u\right\|_2 \\
&\leq \underbrace{\|u^*\|}_{\leq 1} \left\|0 \oplus q^\perp \pi(a)q\right\|_2 \underbrace{\|u\|}_{\leq 1} \\
&\leq \left\|q^\perp \pi(a)q\right\|_2 \overset{(2.34)}{<} \sqrt{\varepsilon}\, \|q\|_2
\end{aligned} \tag{2.43}$$

Using (2.41), (2.42) and (2.43), we conclude that

$$\frac{\|(1-p)ap\|_2}{\|p\|_2} = \frac{\left\|p^\perp ap\right\|_2}{\|p\|_2} < \varepsilon + \sqrt{\varepsilon} \leq 2\sqrt{\varepsilon}\,.$$

Therefore, we obtained the left side of (2.30) with $2\sqrt{\varepsilon}$, instead of ε.

For the second condition of (2.30) observe that, for any $a \in F$,

$$\begin{aligned}
\operatorname{tr}(\varphi(a)) &= \operatorname{tr}(v^*\pi(a)v) \\
&\overset{(2.27)}{=} \frac{\operatorname{Tr}(q\pi(a))}{\operatorname{Tr}(q)} \\
&= \frac{\operatorname{Tr}((0 \oplus q)(a \oplus \pi(a))}{\operatorname{Tr}(q)} \\
&= \frac{\operatorname{Tr}(p(u^*(a \oplus \pi(a))u))}{\operatorname{Tr}(p)}
\end{aligned}$$

where the last equality came from $0 \oplus q = upu^*$ and $\operatorname{Tr}(q) = \operatorname{rank}(q) = \operatorname{Tr}(p)$. Therefore,

$$\begin{aligned}
\left|\tau(a) - \frac{\operatorname{Tr}(pa)}{\operatorname{Tr}(p)}\right| &\leq |\tau(a) - \operatorname{tr}(\varphi(a))| + \left|\operatorname{tr}(\varphi(a)) - \frac{\operatorname{Tr}(ap)}{\operatorname{Tr}(p)}\right| \\
&\overset{(2.31)}{<} \varepsilon + \left|\frac{\operatorname{Tr}\left(p(u^*(a \oplus \pi(a))u - a)\right)}{\operatorname{Tr}(p)}\right| \\
&\leq \varepsilon + \frac{\|p\|_1}{\|p\|_1} \|u^*(a \oplus \pi(a))u - a\| \overset{(2.40)}{<} 2\varepsilon
\end{aligned}$$

for every $a \in F$. □

Notes and Remarks. Følner sequences were introduced by Connes on [Connes, 1976a].

By [Bédos, 1996, Lemma 1], we can replace the Hilbert-Schmidt norm in the definition of Følner nets by the trace-class norm.

Observe that in the proof of Theorem 2.3.9, part *(ii)*, we use separability only to invoke Voiculescu's Theorem. Now, the question is: *How can we get rid of separability?*

By the non-separable version of Voiculescu's Theorem (A.1.6), to ensure that ι and $\iota \oplus \pi$ are approximately unitarily equivalent we need to prove that $\operatorname{rank}(\iota(a)) = \operatorname{rank}(\iota \oplus \pi(a))$, for every $a \in A$. In fact, observe that we basically suppose $A \cap K(\mathcal{H}) = \{0\}$ to guarantee the equality of the ranks. Hence, using the notation of [Hadwin, 1981], define $K_\kappa(\mathcal{H}) \coloneqq \{x \in B(\mathcal{H}) \mid \operatorname{rank}(x) < \kappa\}$, where $\kappa = \dim(\mathcal{H})$. Hence, if $A \cap K_\kappa(\mathcal{H}) = \{0\}$, a similar proof should ensure the existence of a Følner net (p_i) such that $\dfrac{\operatorname{Tr}(ap_i)}{\operatorname{Tr}(p_i)} \longrightarrow \tau(a)$.

2.4 Quasidiagonality

The concept of quasidiagonality came from operator theory and has a wide range. Historically, quasidiagonality precedes the introduction of Følner-type conditions. Although we will not be able to present all the nuances of quasidiagonality, the goal of this section is to present the relation between quasidiagonality and Følner nets and, later, with Følner C*-algebras. The main references in this section are [Brown and Ozawa, 2008, Chapter 7] and [Blackadar, 2006, §V.4].

Definition 2.4.1. Let A be a C*-algebra. We say that A is **quasidiagonal** if there exists a net of c.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that

$$\lim_i \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\| = 0\,, \tag{2.44}$$

for every $a, b \in A$, and

$$\|a\| = \lim_i \|\varphi_i(a)\|\ , \tag{2.45}$$

for every $a \in A$. As usual, if A is separable we can replace a net by a sequence.

Example 2.4.2. If we consider $A = C_0(X)$ an abelian C*-algebra, where X is a locally compact Hausdorff space, then A is a quasidiagonal C*-algebra. Indeed, denote by $\mathfrak{F}$ the directed set of non-empty finite subsets of X. Given $F = \{x_1, \ldots, x_n\} \subseteq X$ define a map $\varphi_F : A \to M_{|F|}(\mathbb{C})$ given by

$$f \mapsto \begin{pmatrix} f(x_1) & & 0 \\ & \ddots & \\ 0 & & f(x_n) \end{pmatrix}.$$

Since A is abelian, each φ_F is a $*$-homomorphism and thus a c.c.p. map. Moreover, for every $f, g \in C(X)$,

$$\lim_F \|\varphi_F(fg) - \varphi_F(f)\varphi_F(g)\| = \|0\| = 0$$

and

$$\lim_F \|\varphi_F(f)\| = \lim_F \max_{x\in F} |f(x)| = \|f\|_\infty \, .$$

Since quasidiagonality is very "sensitive" to separability conditions, from now until the end of this section, let A be a separable C*-algebra and let $\mathcal{H}$ be a separable Hilbert space.

Definition 2.4.3. Let $S \subseteq B(\mathcal{H})$ be a separable set of operators. We say that S is a **quasidiagonal set** if there exists an increasing sequence $(p_n)_{n\in\mathbb{N}} \subseteq B(\mathcal{H})$ of finite-rank projections converging strongly to $1_\mathcal{H}$ such that

$$\|ap_n - p_n a\| \longrightarrow 0\,, \tag{2.46}$$

for every $a \in S$.

Definition 2.4.4. A representation $\pi : A \to B(\mathcal{H})$ is called a **quasidiagonal representation** if $\pi(A)$ is a quasidiagonal set.

The proof of the following theorem, which connects both approaches to quasidiagonality, can be found in [Brown and Ozawa, 2008, Theorem 7.2.5].

Theorem 2.4.5 (Voiculescu). *Let A be a unital separable C*-algebra. The following are equivalent:*

(i) A is quasidiagonal;

(ii) A has a faithful quasidiagonal representation. □

Remark 2.4.6. One tricky observation about the previous results is that not every faithful representation of a quasidiagonal C*-algebra is a quasidiagonal representation (see [Brown and Ozawa, 2008, Remark 7.5.4]).

Finally, since equation 2.46 resembles the definition of Følner nets, one could ask if there is any relation between them. The next proposition shows that the answer is affirmative.

Proposition 2.4.7. *If A is a unital separable C*-algebra and $A \subset B(\mathcal{H})$ is a faithful quasidiagonal representation, then A has a proper Følner sequence.*

Proof. Let $(p_n)_{n\in\mathbb{N}} \subseteq B(\mathcal{H})$ be an increasing sequence converging strongly to $1_\mathcal{H}$ such that $\|ap_n - p_n a\| \longrightarrow 0$, for every $a \in A$.

Hence, for any $n \in \mathbb{N}$ and $a \in A$,

$$\begin{aligned}\|ap_n - p_n a\|_2 &= \|(ap_n - p_n a)p_n + p_n(ap_n - p_n a)\|_2 \\ &\le \|ap_n - p_n a\| \, \|p_n\|_2 + \|p_n\|_2 \, \|ap_n - p_n a\| \\ &= 2\,\|ap_n - p_n a\| \, \|p_n\|_2 \, .\end{aligned}$$

Therefore $\dfrac{\|ap_n - p_n a\|_2}{\|p_n\|_2} \longrightarrow 0$, and A has a proper Følner sequence. □

Remark 2.4.8. Let $A \subset B(\mathcal{H})$ be quasidiagonal representation of a unital separable C*-algebra A. Hence, there is a sequence $(p_n)_{n\in\mathbb{N}}$ of finite-rank orthogonal projections such that

$$\lim_n \frac{\|ap_n - p_n a\|}{\|p_n\|} = \lim_n \|ap_n - p_n a\| = 0 \quad \text{and} \quad \lim_n \frac{\|ap_n - p_n a\|_2}{\|p_n\|_2} = 0\,,$$

where the second limit follows from the previous proposition. The question is: *What is the difference between using the operator norm and the Hilbert-Schmidt norm?* Intuitively, we have the following answer: note that $\|p_n\| = 1$ and $\|p_n\|_2 = \operatorname{rank}(p_n)^{1/2}$. Hence, when we consider the Hilbert-Schmidt norm, we are considering the growth of the dimension of the underlying spaces [Ara and Lledó, 2014, p. 166].

Notes and Remarks. As we shall see in the next section, if $C_r^*(G)$ is quasidiagonal, then G is amenable. The converse is known as Rosenberg's Conjecture, it was stated on [Hadwin and Rosenberg, 1987] and proved true thirty years later on [Tikuisis et al., 2017].

2.5 Følner C*-algebras

As a culmination of all theory developed until now, in this section we finally define Følner C*-algebras. By Theorem 2.2.11 and Theorem 2.3.9, we can see the intimate relation that exists between amenable traces, nets of u.c.p. maps, and Følner nets. Inspired by the the approach of quasidiagonality via c.c.p. maps, Ara and Llédo defined (unital) Følner C*-algebras via nets of u.c.p. maps (see [Ara and Lledó, 2014, p.164]). This approach has the advantage of being abstract, i.e., it does not need a concrete representation. The main references for this section are [Ara and Lledó, 2014] and [Ara et al., 2018].

Notation convention: We assume that all representations of a unital C*-algebra are nondegenerate, i.e., they are unital $*$-homomorphism.

Definition 2.5.1. Let A be a unital C*-algebra. We say that A is a **Følner C*-algebra** if there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that

$$\lim_i \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} = 0\,, \tag{2.47}$$

for every $a, b \in A$. Moreover, if the net satisfies

$$\|a\| = \lim_i \|\varphi_i(a)\|\,, \tag{2.48}$$

for every $a \in A$, then A is called a **proper Følner C*-algebra**. As usual, when A is separable, we may replace a net with a sequence.

Remark 2.5.2. We say that a net $(\varphi_i)_{i\in I}$ that satisfies (2.47) is *asymptotically multiplicative*. Intuitively, this means that "A almost has a finite-dimensional representation when we take dimension big enough". Furthermore, when (2.48) holds, we say that the net is *asymptotically isometric*.

Example 2.5.3. It is not hard to see that any (nonzero) finite dimensional C*-algebra A is a proper Følner C*-algebra. Indeed, assume that $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ (see [Murphy, 2014, Theorem 6.3.8]), then there is a faithful representation φ of A into $M_{n_1+\ldots+n_k}(\mathbb{C}) \cong B(\mathbb{C}^{n_1+\ldots+n_k})$ given by a block diagonal map

$$(a_1, \ldots, a_k) \mapsto \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{pmatrix}.$$

Taking the constant sequence $(\varphi)_{n\in\mathbb{N}}$, we can see that

$$\|\varphi(ab) - \varphi(a)\varphi(b)\|_{2,\mathrm{tr}} = \|0\|_{2,\mathrm{tr}} = 0 \text{ and } \|a\| = \|\varphi(a)\|,$$

for every $a, b \in A$. Thus, A is a proper Følner C*-algebra. Furthermore, observe that the same argument shows that every unital C*-algebra with a nonzero finite-dimensional representation is a Følner C*-algebra, in particular, $C^*(G)$ is always a Følner C*-algebra.

Example 2.5.4. Note that every unital quasidiagonal C*-algebra is a proper Følner C*-algebra. In fact, let $(\varphi_i)_{i\in I}$ be a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that (2.44) and (2.45) hold. Since $\|\cdot\|_{2,\mathrm{tr}} \leq \|\cdot\|$ (see Remark 2.2.8), it follows that

$$\|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} \leq \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\| \xrightarrow{(2.44)} 0.$$

In particular, by Example 2.4.2, every unital abelian C*-algebra is a Følner C*-algebra.

Proposition 2.5.5. *Let A be a unital C*-algebra and $B \subseteq A$ be a C*-subalgebra with $1_A \in B$. If A is a (proper) Følner C*-algebra, then B is a (proper) Følner C*-algebra.*

Proof. Suppose that there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ which is asymptotically multiplicative and asymptotically isometric; i.e., A is a proper Følner C*-algebra. Since $1_A \in B \subseteq A$, it follows that each positive element on B (or $M_n(B)$) is also a positive element on A (or $M_n(A)$)[5]. Hence, the restrictions $\varphi_i|_B : B \to M_{k(i)}(\mathbb{C})$ form a net of u.c.p. maps that are asymptotically multiplicative and asymptotically isometric; i.e., B is a proper Følner C*-algebra. □

Remark 2.5.6. Recall that a representation $\pi : A \to B(\mathcal{H})$ is essential if $\pi(A) \cap K(\mathcal{H}) = \{0\}$. Consider a faithful representation $\rho : A \to B(\mathcal{H})$ and define the *infinite inflation* $\hat{\rho} : A \to B(\oplus_n \mathcal{H})$ by

$$\hat{\rho}(a) := \oplus_n \rho(a) \quad (a \in A).$$

[5] In fact, $\sigma_A(b) = \sigma_B(b)$, for every $b \in B$ (see [Murphy, 2014, Theorem 2.1.11]).

We aim to prove that $\hat{\rho}(A) \cap K(\oplus_n \mathcal{H}) = \{0\}$. In fact, if $\rho(a) \neq 0$ consider $\xi \in \mathcal{H}$ such that $\|\rho(a)(\xi)\| \geq \|\rho(a)\|/2$. Then, define a bounded sequence $(\xi_k)_{k\in\mathbb{N}} \subseteq \oplus_n \mathcal{H}$ such that the k-th entry of ξ_k is ξ and it is zero otherwise, i.e., $\xi_k = (0, \ldots, 0, \xi, 0, \ldots)$. Hence, if $k < m$,

$$\begin{aligned}\|\oplus_n \rho(a)(\xi_k) - \oplus_n \rho(a)(\xi_m)\| &= \|(0, \ldots, 0, \rho(a)(\xi), 0, \ldots, 0, -\rho(a)(\xi), 0, \ldots)\| \\ &\geq \sqrt{2}\,\|\rho(a)(\xi)\| \geq \frac{\|\rho(a)(\xi)\|}{\sqrt{2}}\,.\end{aligned}$$

Hence $(\oplus_n \rho(a)(\xi_k))_{k\in\mathbb{N}}$ has no convergent subsequence, i.e., $\oplus_n \rho(a)$ is not compact. Moreover, $\hat{\rho}$ is clearly faithful.

Now, we present the main result of this chapter, the theorem of characterization of Følner C*-algebras. The result and the proof is inspired by [Ara and Lledó, 2014, Theorem 4.3] and [Ara et al., 2018, Theorem 3.8].

Theorem 2.5.7. *Let A be a unital C*-algebra. Then, the following are equivalent:*

(i) There exists a nonzero representation $\pi : A \to B(\mathcal{H})$ such that $\pi(A)$ has a Følner net;

(ii) There exists a faithful representation $\pi : A \to B(\mathcal{H})$ such that $\pi(A)$ has a Følner net;

(iii) There exists a faithful essential representation $\pi : A \to B(\mathcal{H})$ such that $\pi(A)$ has a Følner net;

(iv) There exists a nonzero representation $\pi : A \to B(\mathcal{H})$ such that $\pi(A)$ has an amenable trace;

(v) Every faithful representation $\pi : A \to B(\mathcal{H})$ satisfies that $\pi(A)$ has an amenable trace;

(vi) There exists a net $\{(\mathcal{H}_i, \pi_i, p_i)\}_{i\in I}$, where each $\pi_i : A \to B(\mathcal{H}_i)$ is a nonzero representation, and each $p_i \in B(\mathcal{H}_i)$ is a nonzero finite-rank orthogonal projection, such that

$$\lim_i \frac{\|\pi_i(a)p_i - p_i\pi_i(a)\|_2}{\|p_i\|_2} = 0\,, \tag{2.49}$$

for every $a \in A$;

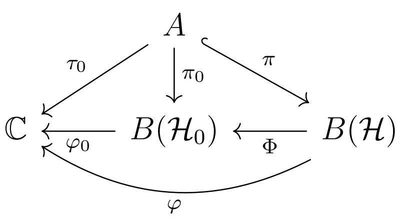
(vii) A is a Følner C-algebra.*

Proof. We aim to prove that **(ii)** $\Rightarrow$ **(iii)** $\Rightarrow$ **(iv)** $\Rightarrow$ **(v)** $\Rightarrow$ **(vii)** $\Rightarrow$ **(vi)** $\Rightarrow$ **(i)** $\Rightarrow$ **(iv)** and **(v)** $\Rightarrow$ **(ii)**.

(ii) $\Rightarrow$ (iii) Suppose that $\pi : A \to B(\mathcal{H})$ is a faithful representation and $(p_i)_{i\in I} \in B(\mathcal{H})$ is a Følner net for $\pi(A)$. By Remark 2.5.6, the infinite inflation $\hat{\pi} : A \to B(\oplus_n \mathcal{H})$ is essential and faithful. Furthermore, define the finite-rank projections $\hat{p_i} := p_i \oplus 0 \oplus 0 \ldots \in B(\oplus_n \mathcal{H})\,, i \in I$. Then

$$\lim_i \frac{\|\hat{\pi}(a)\hat{p_i} - \hat{p_i}\hat{\pi}(a)\|_2}{\|\hat{p_i}\|_2} = \lim_i \frac{\|\pi(a)p_i - p_i\pi(a)\|_2}{\|p_i\|_2} = 0\,,$$

for every $a \in A$. Hence $(\hat{p_i})_{i\in I}$ is a Følner net for $\hat{\pi}(A)$.
(iii) $\Rightarrow$ **(iv)** Since π is faithful, we can identify A with $\pi(A)$. If A has a Følner net, then A as an amenable trace by part *(i)* of Theorem 2.3.9.
(iv) $\Rightarrow$ **(v)** The proof is similar to the prove that the definition of amenable trace does not depend of $B(\mathcal{H})$ (Proposition 2.2.5). Let $\pi_0 : A \to B(\mathcal{H}_0)$ be a nonzero representation such that $\pi_0(A)$ has an amenable trace τ_0 which extends to a hypertrace φ_0, and let $\pi : A \to B(\mathcal{H})$ be a faithful representation. By Arverson's Extension Theorem 1.2.10, there exists a u.c.p. map $\Phi : B(\mathcal{H}) \to B(\mathcal{H}_0)$ extending $\pi_0 \circ \pi^{-1} : \pi(A) \to B(\mathcal{H}_0)$. Define $\varphi := \varphi_0 \circ \Phi$ and $\tau := \varphi|_{\pi(A)}$.



Note that, by construction, φ and τ are states. Furthermore, observe that and $\pi(A)$ is contained in the multiplicative domain of Φ, since $\Phi|_{\pi(A)} = \pi_0 \circ \pi^{-1}$ is a $*$-homomorphism. Therefore, φ is a hypertrace on $\pi(A)$ and τ is an amenable trace, since for any $a \in A$ and $x \in B(\mathcal{H})$,

$$\varphi(\pi(a)x) = \varphi_0(\Phi\big(\pi(a)x\big)) = \varphi_0(\underbrace{\Phi(\pi(a))}_{\in\pi_0(A)}\ \underbrace{\Phi(x)}_{\in B(\mathcal{H}_0)}\ ) = \varphi_0(\Phi(x)\Phi(\pi(a))) = \varphi(x\pi(a)).$$

(v) $\Rightarrow$ **(vii)** Let $\pi : A \to B(\mathcal{H})$ be a faithful representation. Since π is faithful we can identify A with $\pi(A)$, hence A has an amenable trace τ. By Theorem 2.2.11, there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that

$$\lim_i \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} = 0\,,$$

for every $a, b \in A$. Therefore A is a Følner C*-algebra.
(vii) $\Rightarrow$ **(vi)** Suppose that A is a Følner C*-algebra and denote by $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ the net of u.c.p. maps such that

$$\lim_i \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|_{2,\mathrm{tr}} = 0\,,$$

for every $a, b \in A$. By Stinespring's Theorem 1.2.3, for each $i \in I$ there exists a Hilbert space $\mathcal{H}_i$, a representation $\pi_i : A \to B(\mathcal{H}_i)$ and an isometry $v_i : \mathbb{C}^{k(i)} \to \mathcal{H}_i$ such that

$$\varphi(a) = v_i^* \pi_i(a) v_i$$

for every $a \in A$. Defining $p_i := v_i v_i^* \in B(\mathcal{H}_i)$, observe that p_i is a nonzero orthogonal projection with rank $k(i)$.

Observe that, for any $x \in B(\mathcal{H}_i)$,

$$\|v_i^* x v_i\|_{2,\mathrm{tr}} = \sqrt{\mathrm{tr}(v_i^* x^* p_i x v_i)} \overset{(2.27)}{=} \sqrt{\frac{\mathrm{Tr}(p_i x^* p_i x)}{\mathrm{Tr}(p_i)}} = \frac{\|p_i x p_i\|_2}{\|p_i\|_2}\,, \tag{2.50}$$

since $\|p_i x p_i\|_2 = \sqrt{\mathrm{Tr}((p_i x p_i)^*\, p_i x p_i)} = \sqrt{\mathrm{Tr}(p_i x^* p_i x)}$.

Thus, for any $a \in A$

$$\begin{aligned}\|\varphi_i(a^* a) - \varphi_i(a^*)\varphi_i(a)\|_{2,\mathrm{tr}} &= \|v_i^* \pi_i(a^*)(1-p_i)\pi_i(a) v_i\|_{2,\mathrm{tr}} \\ &\overset{(2.50)}{=} \frac{\|p_i \pi_i(a^*)(1-p_i)\pi_i(a) p_i\|_2}{\|p_i\|_2}\end{aligned} \tag{2.51}$$

Since p_i is a projection, for any $x \in B(\mathcal{H}_i)$,

$$\|x p_i\|_2 = \mathrm{Tr}(p_i x^* x p_i)^{1/2} = \mathrm{Tr}(p_i\, p_i x^* x p_i)^{1/2} \overset{\mathrm{CBS}}{\leq} \|p_i\|_2^{1/2}\, \|p_i x^* x p_i\|_2^{1/2}\,, \text{ and} \tag{2.52}$$

$$\begin{aligned}\|p_i x\|_2 &= \mathrm{Tr}(x^* p_i x)^{1/2} = \mathrm{Tr}(p_i x x^*)^{1/2} = \mathrm{Tr}(p_i\, p_i x x^* p_i)^{1/2} \\ &\overset{\mathrm{CBS}}{\leq} \|p_i\|_2^{1/2}\, \|p_i x x^* p_i\|_2^{1/2}\,.\end{aligned} \tag{2.53}$$

Therefore

$$\begin{aligned}\frac{\|\pi_i(a)p_i - p_i\pi_i(a)\|_2}{\|p_i\|_2} &= \frac{\|(1-p_i)\pi_i(a)p_i - p_i\pi_i(a)(1-p_i)\|_2}{\|p_i\|_2} \\ &\leq \frac{\|(1-p_i)\pi_i(a)p_i\|_2}{\|p_i\|_2} + \frac{\|p_i\pi_i(a)(1-p_i)\|_2}{\|p_i\|_2} \\ &\overset{(2.52)+(2.53)}{\leq} \left(\frac{\|p_i\pi_i(a^*)(1-p_i)\pi_i(a)p_i\|_2}{\|p_i\|_2}\right)^{1/2} + \left(\frac{\|p_i\pi_i(a)(1-p_i)\pi_i(a^*)p_i\|_2}{\|p_i\|_2}\right)^{1/2} \\ &\overset{(2.51)}{=} \|\varphi_i(a^*a) - \varphi_i(a^*)\varphi_i(a)\|_{2,\mathrm{tr}}^{1/2} + \|\varphi_i(aa^*) - \varphi_i(a)\varphi_i(a^*)\|_{2,\mathrm{tr}}^{1/2} \longrightarrow 0\,,\end{aligned}$$

as desired.

(vi) ⇒ (i) Let $\{(\mathcal{H}_i, \pi_i, p_i)\}_{i\in I}$ be a net satisfying condition *(vi)*, i.e., each π_i is a nonzero representation and each $p_i \in B(\mathcal{H}_i)$ is a nonzero finite-rank orthogonal projection, such that, for every $a \in A$,

$$\lim_i \frac{\|\pi_i(a)p_i - p_i\pi_i(a)\|_2}{\|p_i\|_2} = 0\,.$$

Define the nonzero representation $\oplus_i \pi_i : A \to B(\oplus_i \mathcal{H}_i)$ and a nonzero finite-rank orthogonal projection $q_i \in B(\oplus_i \mathcal{H}_i)$ with p_i in the i-th position and 0 otherwise. Note that $(q_i)_{i\in I}$ is a Følner net for $\oplus_i \pi_i(A)$, since

$$\lim_i \frac{\|(\oplus_i\pi_i)(a)q_i - q_i(\oplus_i\pi_i)(a)\|_2}{\|q_i\|_2} = \lim_i \frac{\|\pi_i(a)p_i - p_i\pi_i(a)\|_2}{\|p_i\|_2} = 0\,.$$

(i) ⇒ (iv) Follows from Theorem 2.3.9 part *(i)*, since the existence of a Følner net implies the existence of an amenable trace.

(v) ⇒ (ii) Also follows from Theorem 2.3.9 part *(i)*, since the existence of an amenable trace implies the existence of a Følner net. □

Remark 2.5.8. This class of Følner C*-algebras was previously considered by Bédos under the name *weakly hypertracial* [Bédos, 1995, §2]. According to Bédos, a representation $\pi : A \to B(\mathcal{H})$ is called *hypertracial* if $\pi(A)$ has a hypertrace; i.e., an amenable trace. Additionally, A is called weakly hypertracial if every faithful representation is hypertracial, i.e., if A satisfies condition *(v)* of the theorem above.

Remark 2.5.9. In [Ara and Lledó, 2014] and [Ara et al., 2018], the authors give another equivalence in Theorem 2.5.7:

(iii)' Every faithful *essential* representation $\pi : A \to B(\mathcal{H})$ satisfies that $\pi(A)$ has a *proper* Følner net.

The main technical difficulty is to construct a proper Følner net from a general Følner net. Since the proof uses techniques that are beyond the scope of this work, we do not present a proof here.

As a consequence of the characterization of Følner C*-algebras, we have an alternative proof for Proposition 2.5.5, since $1_A \in B \subseteq A$ implies that any nondegenerate (i.e., unital) faithful representation $\pi : A \to B(\mathcal{H})$ restricts to nondegenerate (i.e., unital) faithful representation $\pi|_B : B \to B(\mathcal{H})$. Hence, if $\pi(A)$ has an amenable trace, then $\pi(B)$ also has an amenable trace. Another consequence of Theorem 2.5.7 is the following:

Corollary 2.5.10. *Let A be a unital C*-algebra. If a nonzero quotient of A is a Følner C*-algebra, then A is a Følner C*-algebra.*

Proof. Suppose that $J \subset A$ is a proper ideal and the quotient A/J is a Følner C*-algebra. By condition *(v)* of Theorem 2.5.7, there exists a faithful representation $\pi : A/J \to B(\mathcal{H})$ such that $\pi(A/J)$ has an amenable trace. Composing π with the surjective $*$-homomorphism $\rho : A \to A/J$, note that $\pi \circ \rho : A \to B(\mathcal{H})$ is a nonzero representation. Since $\pi \circ \rho(A) = \pi(A/J)$ has an amenable trace, it follows that A is amenable by condition *(iv)* of Theorem 2.5.7. □

Remark 2.5.11. The converse of the previous corollary is not true. Let A be a Følner C*-algebra and $\mathcal{O}_2$ be the Cuntz algebra. Observe that $\mathcal{O}_2$ is not a Følner C*-algebra, because it has no tracial states. However, $A \oplus \mathcal{O}_2$ is a Følner C*-subalgebra, since the nonzero quotient $A \oplus \mathcal{O}_2/(0 \oplus \mathcal{O}_2) \cong A$ is a Følner C*-algebra. On the other hand, $A \oplus \mathcal{O}_2/(A \oplus 0) \cong \mathcal{O}_2$ is not a Følner C*-algebra.

On the other hand, note that $0 \oplus \mathcal{O}_2 \cong \mathcal{O}_2$ is not a Følner C*-algebra and is a C*-subalgebra of $A \oplus \mathcal{O}_2$ that does not have the same unit, i.e., $1_{A \oplus \mathcal{O}_2} \notin 0 \oplus \mathcal{O}_2$. Hence, the hypothesis that $1_A \in B$ is really necessary in Proposition 2.5.5.

Since one of the motivations to study Følner C*-algebras is the relation with amenability, it is natural to ask which relations exist between nuclear C*-algebras and Følner C*-algebras.

Example 2.5.12. Recall that $C^*(\mathbb{F}_2)$ admits a character $\epsilon : C^*(\mathbb{F}_2) \to \mathbb{C}$ induced by the trivial representation of $\mathbb{F}_2$ (see Example 1.3.14). Therefore, $C^*(\mathbb{F}_2)$ satisfies condition *(iv)* of Theorem 2.5.7, since $\epsilon(C^*(\mathbb{F}_2)) = \mathbb{C}$ has an amenable trace. Thus $C^*(\mathbb{F}_2)$ is Følner C*-algebra. On the other hand, $C^*(\mathbb{F}_2)$ is not nuclear, since $\mathbb{F}_2$ is not amenable.

Nevertheless, the Cuntz algebra $\mathcal{O}_n$ $(n \geq 2)$ is nuclear[6], but it is not a Følner C*-algebra, since it does not admit amenable traces (see Example 2.2.14).

Therefore, the class of Følner C*-algebras is different from the class of nuclear C*-algebras. However, the next result shows that they have a close relation.

Corollary 2.5.13. *Let A be a unital nuclear C*-algebra. Then A is a Følner C*-algebra if and only if A has at least one tracial state.*

Proof. ($\Leftarrow$) Suppose that A is faithfully represented on $B(\mathcal{H})$ and has a tracial state τ. Then τ is an amenable trace by Proposition 2.2.12. Therefore, A is a Følner C*-algebra by condition *(v)* of Theorem 2.5.7.
($\Rightarrow$) Now suppose that A is a Følner C*-algebra faithfully represented on $B(\mathcal{H})$. By condition *(v)* of Theorem 2.5.7, it follows that $A \subset B(\mathcal{H})$ admits at least one tracial state. □

Example 2.5.14. We say that A is a unital AF-algebra if there exists a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ of finite-dimensional C*-subalgebras such that $1_A \in A_i$ and $\overline{\cup_n A_n} = A$. Denote by tr_n the canonical tracial state of A_n. Then, for each $a \in \cup_n A_n$, define $\mathrm{tr}(a) := \mathrm{tr}_m(a)$, where m is the smaller n such that $a \in A_n$. One can prove that tr extends to a tracial state on A. Therefore, A is a Følner C*-algebra, since it is nuclear and has a tracial state.

Remark 2.5.15. It is worth mentioning that the class of Følner C*-algebras is bigger than the class of unital quasidiagonal C*-algebras. In fact, recall that the Toeplitz algebra $\mathcal{T} \subset B(\ell^2(\mathbb{N}))$ has a Følner sequence (see Example 2.3.4), hence it is a Følner C*-algebra by condition *(i)* of Theorem 2.5.7. However, the Toeplitz algebra is not quasidiagonal, since every quasidiagonal algebra does not admit proper isometries, i.e., an isometry s with $ss^* \neq 1$ (see [Brown and Ozawa, 2008, Proposition 7.1.15]).

The next result summarizes the relations between amenability for groups and the concepts studied in this chapter. Furthermore, it may be viewed as a justification for the study of Følner C*-algebras as an amenability-like property.

[6] One way of viewing this is the following: Cuntz algebras are graph algebras, and all graph algebras are nuclear.

Theorem 2.5.16. *Let G be a discrete group. Then, the following are equivalent:*

(i) G is amenable;

(ii) $C_r^*(G)$ is nuclear;

(iii) $C_r^*(G)$ is a Følner C-algebra;*

(iv) $C_r^*(G)$ is quasidiagonal.

Proof. **(i) $\iff$ (ii)** Already proved in Theorem 2.1.15.
(i) $\iff$ (iii) By Proposition 2.2.13, G is amenable if and only if $C_r^*(G) \subset B(\ell^2(G))$ has an amenable trace. Since the existence of amenable traces is equivalent to saying that $C_r^*(G)$ is a Følner C*-algebra (Theorem 2.5.7), the assertion follows.
(iv)$\Longrightarrow$(i) It follows from the fact that $C_r^*(G)$ quasidiagonal implies $C_r^*(G)$ Følner (see Example 2.5.4), and *(iii)$\Rightarrow$(i)*.
(i)$\Longrightarrow$(iv) This implication is known as Rosenberg's Conjecture, which was proved in [Tikuisis et al., 2017, Corollary 6.6]. □

Example 2.5.17 (Non-Følner C*-algebras). Observe that a C*-algebra without any tracial state is not a Følner C*-algebra. For example, Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$, $K(\mathcal{H})$ and $B(\mathcal{H})$, for $\mathcal{H}$ infinite dimensional (see Example 2.2.14).

The last result of this section may seem surprising, as it states that assuming that the net is asymptotically isometric (2.48) does not provide additional information in the separable case.

Proposition 2.5.18. *Let A be a unital separable C*-algebra. Then A is a Følner C*-algebra if and only if A is a proper Følner C*-algebra.*

Proof. The backward direction is obvious. Suppose that A is a separable Følner C*-algebra, hence there is a sequence of u.c.p. maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$, such that (2.47) holds. We always can assume that the sequence $k(n)$ is growing in such way that

$$\lim_n \frac{n}{k(n)} = 0\,. \tag{2.54}$$

For example, if we define $\varphi_n' : A \to M_{n^2\,k(n)}(\mathbb{C})$ by taking n^2 direct sums of each φ_n, then $(\varphi_n')_{n\in N}$ still satisfies (2.47) and also (2.54). We will keep the original sequence to ease notation.

Let $\pi : A \to B(\mathcal{H})$ be a faithful representation of A on a separable Hilbert space $\mathcal{H}$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$, and $p_n \in B(\mathcal{H})$ the orthogonal projection onto $\operatorname{span}\{\xi_1, \ldots, \xi_n\}$. In particular, $(p_n)_{n\in\mathbb{N}}$ is an increasing sequence converging strongly to $1_{\mathcal{H}}$, and $\operatorname{rank}(p_n) = n$, for each $n \in \mathbb{N}$.

Also recall that we may view $p_n\pi(a)p_n$ as an element of $M_n(\mathbb{C})$ defining

$$p_n\pi(a)p_n(e_i) = \langle \pi(a)\xi_i, \xi_i \rangle \,, \tag{2.55}$$

where $\{e_i\}_{i=1}^n$ is the orthonormal basis of $\mathbb{C}^n$. Furthermore, define $\sigma_n : A \to M_n(\mathbb{C})$ using the identification in (2.55), i.e., $\sigma_n(a) = p_n\pi(a)p_n$. Note that σ_n is a u.c.p. map, since it is the compression by an isometry $p_n : \mathbb{C}^n \to \mathcal{H}$, $p_n(e_i) = \xi_i$.

Define $\psi_n : A \to M_{k(n)+n}(\mathbb{C})$ by:

$$a \mapsto \begin{pmatrix} \varphi_n(a) & 0 \\ 0 & \sigma_n(a) \end{pmatrix} = \varphi_n(a) \oplus \sigma_n(a) \,,$$

and note that ψ_n is a u.c.p. map, since it is a direct sum of u.c.p. maps.

Observe that

$$\begin{aligned} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\|^2_{2,\mathrm{tr}_{k(n)+n}} &= \underbrace{\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|^2_{2,\mathrm{tr}_{k(n)+n}}}_{\longrightarrow 0} + \\ &\qquad + \|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\|^2_{2,\mathrm{tr}_{k(n)+n}} \,, \end{aligned}$$

where the first term goes to zero because $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative. For the estimations on the second term observe that:

$$\sigma_n(ab) - \sigma_n(a)\sigma_n(b) = p_n\pi(ab)p_n - p_n\pi(a)p_n\pi(b)p_n = p_n\pi(a)(1-p_n)\pi(b)p_n =: x_n \,,$$

thus,

$$\|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\|^2_{2,\mathrm{tr}_{k(n)+n}} = \mathrm{tr}_{k(n)+n}(x_n^* x_n) = \frac{\mathrm{Tr}(x_n^* x_n)}{k(n)+n}$$

but,

$$\begin{aligned} \mathrm{Tr}(x_n^* x_n) &= \mathrm{Tr}(p_n\pi(b)^*(1-p_n)\pi(a)^*p_n\pi(a)(1-p_n)\pi(b)p_n) \\ &\leq \|p_n\|_2 \, \|\pi(b)^*(1-p_n)\pi(a)^*p_n\pi(a)(1-p_n)\pi(b)p_n\|_2 \\ &\leq \|p_n\|_2 \, \|\pi(b)^*(1-p_n)\pi(a)^*p_n\pi(a)(1-p_n)\pi(b)\| \, \|p_n\|_2 \\ &\leq \|p_n\|_2^2 \underbrace{\|1-p_n\|^2 \, \|p_n\|}_{=1} \|\pi(a)\|^2 \, \|\pi(b)\|^2 \\ &= n \, \|a\|^2 \, \|b\|^2 \,. \end{aligned}$$

Hence,

$$\|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\|^2_{2,\mathrm{tr}} \leq \frac{n\,\|a\|^2\,\|b\|^2}{k(n)+n} \longrightarrow 0 \,,$$

since (2.54) holds. Therefore, $\|\psi_n(ab) - \psi_n(a)\psi_n(b)\|^2_{2,\mathrm{tr}_{k(n)+n}} \longrightarrow 0$.

In particular, observe that $s - \lim_n p_n\pi(a)p_n = \pi(a)$, for every $a \in A$. Hence $\|a\| = \|\pi(a)\| = \lim_n \|p_n\pi(a)p_n\|$ by the Banach-Steinhaus Theorem. Furthermore, note that $\|\sigma_n(a)\| = \|p_n\pi(a)p_n\| \leq \|\psi_n(a)\|$. Then,

$$\|a\| - \|\psi_n(a)\| \leq \|a\| - \|p_n\pi(a)p_n\| \longrightarrow 0 \,.$$

Therefore, $(\psi_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative and isometric, i.e., A is a proper Følner C*-algebra. □

Notes and Remarks. According to Lledó in [Lledó, 2013], the name "Følner algebra" came from the work of [Hagen et al., 2000, §7.2.1], given a sequence $(p_n)_{n\in\mathbb{N}} \subseteq B(\mathcal{H})$ of finite-rank orthogonal projections converging strongly to $1_{\mathcal{H}}$, the authors define the Følner algebra as

$$\mathfrak{F}(\mathcal{H}) = \left\{ x \in B(\mathcal{H}) \;\middle|\; \lim_n \frac{\mathrm{Tr}(|ap_n - p_n a|)}{\mathrm{Tr}(p_n)} = 0 \right\}.$$

In [Lledo and Yakubovich, 2013] and [Ara et al., 2014], the authors show that Følner sequences have connections with operator theory and operator algebras. For example, when $\mathcal{H}$ is separable, they mention that an operator $x \in B(\mathcal{H})$ is finite if and only if $C^*(x, 1_{\mathcal{H}})$ has an amenable trace (Williams' Theorem), hence $C^*(x, 1_{\mathcal{H}})$ is a Følner C*-algebra.

In [Lledó and Martínez, 2022] the authors define Følner C*-algebras in the nonunital case. They changed the net of u.c.p maps by a net of c.c.p. maps and maintained the asymptotically isometric condition (2.48). They proved that this new definition is equivalent to usual definition when the C*-algebra is unital. With this new approach, $K(\mathcal{H})$ is a nonunital Følner C*-algebra.

Moreover, Følner C*-algebras are well-behaved when dealing with tensor products. In fact, if γ C*-norm on $A \odot B$, then $A \otimes_\gamma B$ is a Følner C*-algebra if and only if A and B are Følner C*-algebras.

3 Følner Crossed Products

I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring up ideas; things emerge and I follow them up.

Michael Atiyah

After developing the theory of Følner C*-algebra, this chapter focuses on applications to crossed products. To be more specific, in Section 3.1 we characterize when the reduced crossed product $A \rtimes_{\alpha,r} G$ is a Følner C*-algebra, and in Section 3.3, as an original contribution, we aim to do the same when α is a partial action. Since we have not presented partial actions yet, Section 3.2 gives an introduction to partial actions and partial crossed products.

3.1 Følner C*-algebras and crossed products

In this section, we present a characterization of Følner reduced crossed products and its consequences to rotational algebras and the uniform Roe algebra. The results presented in this section are inspired in [Bédos, 1995].

Observe that the proof of the next result is similar to the calculations of Example 2.2.4, where we show that $C_r^*(G)$ has an amenable trace when G is amenable.

Lemma 3.1.1. *Let A be a unital C^*-algebra, G be a discrete group and $(\pi, \rho, \mathcal{H})$ a covariant representation. If $\pi(A) \subseteq B(\mathcal{H})$ has an amenable trace and G is amenable, then $C^*(\pi(A), \rho(G)) \subseteq B(\mathcal{H})$ has an amenable trace.*

Proof. Let τ be an amenable trace on $\pi(A) \subseteq B(\mathcal{H})$ and μ be an invariant mean on $\ell^\infty(G)$. Also, denote by φ the hypertrace associated with τ. Define a linear map $\Phi : B(\mathcal{H}) \to \ell^\infty(G)$ by $x \mapsto \Phi_x$, where

$$\Phi_x(g) := \varphi(\rho_g^* x \rho_g)\,,$$

for each $g \in G$. Note that Φ is well-defined, since Φ_x is a bounded map:

$$|\Phi_x(g)| = |\varphi(\rho_g^* x \rho_g)| \leq \|\varphi\| \left\|\rho_g^*\right\| \|x\| \|\rho_g\| \leq \|x\|\ ,$$

for all $g \in G$. Furthermore, this implies that Φ is a contractive map, $\|\Phi\| \leq 1$. Observe that, for any $g \in G$,

$$\Phi_{1_\mathcal{H}}(g) = \varphi(\rho_g^* \rho_g) = \varphi(1_\mathcal{H}) = 1\,.$$

Thus Φ is also unital.

Now, define the linear map $\psi : B(\mathcal{H}) \to \mathbb{C}$ by

$$x \longmapsto \mu(\Phi_x)\,. \tag{3.1}$$

Note that, for each $x \in B(\mathcal{H})$,

$$|\psi(x)| = |\mu(\Phi_x)| \leq \|\Phi_x\| \leq \|x\|\ ,$$

and

$$\psi(1_{\mathcal{H}}) = \mu(\Phi_{1_{\mathcal{H}}}) = \mu(1_{\ell^\infty(G)}) = 1\,.$$

Hence ψ is a state on $B(\mathcal{H})$.

To show that ψ is a hypertrace on $C^*(\pi(A), \rho(G))$ it is enough to show on the set of generators $\pi(A)$ and $\rho(G)$. First observe that, for each $x \in B(\mathcal{H})$ and $g, h \in G$,

$$\begin{aligned} \Phi_{\rho_h x \rho_h^*}(g) &= \varphi(\rho_g^* \rho_h x \rho_h^* \rho_g) \\ &= \varphi(\rho_{h^{-1}g}^* x \rho_{h^{-1}g}) \\ &= \Phi_x(h^{-1}g) \\ &= h \cdot \Phi_x(g)\,. \end{aligned} \tag{3.2}$$

Since μ is invariant,

$$\psi(\rho_h x \rho_h^*) = \mu(\Phi_{\rho_h x \rho_h^*}) \overset{(3.2)}{=} \mu(h \cdot \Phi_x) = \mu(\Phi_x) = \psi(x)\,.$$

On the other hand, for every $x \in B(\mathcal{H})$, $a \in A$ and $g \in G$,

$$\begin{aligned} \Phi_{\pi(a)x}(g) &= \varphi(\rho_g^* \pi(a) x \rho_g) \\ &= \varphi(\underbrace{\rho_g^* \pi(a) \rho_g}_{\in A}\, \underbrace{\rho_g^* x \rho_g}_{\in B(\mathcal{H})}) \\ &= \varphi(\rho_g^* x \rho_g \rho_g^* \pi(a) \rho_g) \\ &= \varphi(\rho_g^* x \pi(a) \rho_g) \\ &= \Phi_{x\pi(a)}(g)\,. \end{aligned}$$

Hence, $\Phi_{\pi(a)x} = \Phi_{x\pi(a)}$, which implies $\psi(ax) = \psi(xa)$. Therefore, ψ is a hypertrace and $\psi|_{C^*(\pi(A),\rho(G))}$ is an amenable trace. □

Before the characterization of Følner reduced crossed products, we need to remember a few thing about the regular representation (see Section 1.3 for more details). Consider a faithful nondegenerate representation $A \subset \mathcal{B}(\mathcal{H})$. Then we have a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ and a representation $1_{\mathcal{H}} \otimes \lambda : G \to (\mathcal{H} \otimes \ell^2(G))$ such that $(\pi, 1_{\mathcal{H}} \otimes \lambda, \mathcal{H} \otimes \ell^2(G))$ is a covariant representation. Furthermore, $\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)$ is faithful and $A \rtimes_{\alpha,r} G$ can be concretely represented as:

$$A \rtimes_{\alpha,r} G = \overline{\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)(C_c(G, A))} \subseteq B(\mathcal{H} \otimes \ell^2(G))\,.$$

Equivalently, $A \rtimes_{\alpha,r} G$ can be described as the C*-algebra generated by $\pi(A)$ and $1 \otimes \lambda(G)$, i.e.,

$$A \rtimes_{\alpha,r} G = C^*\big(\pi(A), 1_{\mathcal{H}} \otimes \lambda(G)\big)\,.$$

Moreover, the map $\iota : A \to A \rtimes_{\alpha,r} G$ given by

$$a \longmapsto a\delta_e$$

is a unital injective $*$-homomorphism. Hence, we may identify 1_A with $1_{A \rtimes_{\alpha,r} G} = 1_A \delta_e$, and A with a C*-subalgebra of $A \rtimes_{\alpha,r} G$. Similarly, recall that there exists a unital injective $*$-homomorphism from $C_r^*(G)$ to $A \rtimes_{\alpha,r} G$ (see Remark 1.3.15). Hence, we may also identify $1_{C_r^*(G)}$ with $1_{A \rtimes_{\alpha,r} G}$ and $C_r^*(G)$ with a C*-subalgebra of $A \rtimes_{\alpha,r} G$.

Using these identifications and the previous lemma, the characterization of Følner reduced crossed products is straightforward.

Theorem 3.1.2. *Let G be a discrete group, and let α be an action of G on a unital C*-algebra A.*

(i) If $A \rtimes_{\alpha,r} G$ is a Følner C-algebra, then A is a Følner C*-algebra;*

(ii) If $A \rtimes_{\alpha,r} G$ is a Følner C-algebra, then G is amenable;*

(iii) If G is amenable and A is a Følner C-algebra, then $A \rtimes_{\alpha,r} G$ is a Følner C*-algebra.*

Proof. **(i)** Note that A is a C*-subalgebra of $A \rtimes_{\alpha,r} G$ with $1_{A \rtimes_{\alpha,r} G} \in A$. Therefore A is a Følner C*-algebra by Proposition 2.5.5.
(ii) Analogously, $C_r^*(G)$ is a unital C*-algebra of $A \rtimes_{\alpha,r} G$, whence $C_r^*(G)$ is a Følner C*-algebra. Furthermore, we conclude that G amenable by Theorem 2.5.16.
(iii) The construction of the regular representation provides a covariant representation $(\pi, 1_{\mathcal{H}} \otimes \lambda, \mathcal{H} \otimes \ell^2(G))$. Since A is Følner, $\pi(A) \subset B(\mathcal{H} \otimes \ell^2(G))$ has amenable trace by Theorem 2.5.7. Therefore, $A \rtimes_{\alpha,r} G = C^*(\pi(A), 1_{\mathcal{H}} \otimes \lambda(G))$ has an amenable trace by Lemma 3.1.1; i.e., $A \rtimes_{\alpha,r} G$ is a Følner C*-algebra. □

Remark 3.1.3. Assuming G amenable, it is important to observe that we have the weak containment property, $A \rtimes_{\alpha,r} G \cong A \rtimes_\alpha G$. Hence the statements *(i)* and *(iii)* still are true when we consider the full crossed product. Nevertheless, it is more natural to consider the reduced crossed product over the full crossed product, since $C_r^*(G)$ is always a unital C*-subalgebra of $A \rtimes_{\alpha,r} G$. On the other hand, it is possible that $C^*(G)$ is not a unital C*-subalgebra of $A \rtimes_\alpha G$, e.g., when the action α is amenable but G is not amenable (see *Notes and Remarks* at the end of Section 2.1).

We say that an amenable trace τ on A is *G-invariant* if $\tau(\alpha_g(a)) = \tau(a)$ for every $a \in A$ and $g \in G$. As pointed out in [Ara and Lledó, 2014, Proposition 4.9], the above proposition can be rewritten as follows:

Corollary 3.1.4. *Let G be a discrete group, and let α be an action of G on a unital C^*-algebra A. Then, the following are equivalent:*

(i) $A \rtimes_{\alpha,r} G$ is a Følner C^*-algebra;

(ii) G is amenable and A is a Følner C^*-algebra;

(iii) G is amenable and A has a G-invariant amenable trace.

Proof. **(i)** $\Longleftrightarrow$ **(ii)** Follows from Theorem 3.1.2.
(iii)$\Longrightarrow$**(ii)** Follows from Theorem 2.5.7, since a unital C*-algebra A with an amenable trace is a Følner C*-algebra.
(i)$\Longrightarrow$**(iii)** Consider the regular representation $A \rtimes_{\alpha,r} G \subset B(\mathcal{H} \otimes \ell^2(G))$ and an amenable trace τ on $A \rtimes_{\alpha,r} G$. To simplify notation, we identify A with his image on $A \rtimes_{\alpha,r} G$ and, in particular, $A \subset B(\mathcal{H} \otimes \ell^2(G))$. With these identifications, note that $\alpha_g(a) = \delta_g a \delta_g^*$ for every $a \in A$ and $g \in G$. Finally, it is straightforward to verify that $\tau|_A$ is a G-invariant amenable trace, since

$$\tau|_A(\alpha_g(a)) = \tau(\delta_g a \delta_g^*) = \tau(a \delta_g^* \delta_g) = \tau|_A(a)\,,$$

for every $a \in A$ and $g \in G$. □

Example 3.1.5. In Example 1.3.16, we describe the rotational algebra A_θ as the crossed product $C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$. Since $\mathbb{Z}$ is amenable, it follows that the reduced and full crossed products coincide. Furthermore, $C(\mathbb{T})$ is a Følner C*-algebra, since it is abelian. Therefore, the rotational algebra A_θ is a Følner C*-algebra by Theorem 3.1.2.

Example 3.1.6. The uniform Roe algebra of a discrete group G can be described as the reduced crossed product $C_u^*(G) = \ell^\infty(G) \rtimes_{\alpha,r} G$ (see Example 1.3.17). Note that $\ell^\infty(G)$ is a unital abelian C*-algebra, hence it is a Følner C*-algebra. Therefore, if G is amenable, it follows that the uniform Roe algebra $C_u^*(G)$ is a Følner C*-algebra by Theorem 3.1.2. Conversely, if $C_u^*(G)$ is a Følner C*-algebra, then G is amenable.

Notes and Remarks. One may use Følner nets to analyze when the reduced crossed product is a Følner C*-algebra. For instance, let $(q_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ be Følner sequences for a unital separable C*-algebra A and a countable amenable group G, respectively. In [Lledó, 2013, Thm 3.4], the author construct an explicit Følner sequence $(r_n)_{n\in\mathbb{N}}$ for $A \rtimes_{\alpha,r} G$ assuming that the action α satisfies the following technical condition: for all $a \in A$,

$$\lim_n \left(\max_{g\in F_n} \frac{\left\| q_n \alpha_g^{-1}(a) - \alpha_g^{-1}(a) q_n \right\|_2}{\left\| q_n \right\|_2} \right) = 0\,.$$

To be more specific, suppose that $A \subset B(\mathcal{H})$ is a faithful representation and denote by $p_n \in B(\ell^2(G))$ the projection onto $\ell^2(F_n)$. Then, $r_n := q_n \otimes p_n$ is a Følner sequence for $A \rtimes_{\alpha,r} G \subset B(\mathcal{H} \otimes \ell^2(G))$.

3.2 Partial crossed products

The goal of this section is to give a short introduction to partial actions and to define the partial crossed products associated with a "partial covariant system". The main reference for partial actions is [Exel, 2017], but to define the reduced partial crossed product we will mix the approaches of [McClanahan, 1995] and [Exel, 2017, Chapter 9]. Since we already presented crossed products of (global) covariant systems, we will not provide the same level of detail as in previous sections.

Definition 3.2.1. A **(topological) partial action** of G on the topological space X is a pair $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ where each $D_g \subseteq X$ is open and each $\theta_g : D_{g^{-1}} \to D_g$ is a homeomorphism, satisfying

(i) $D_e = X$ and $\theta_e = \mathrm{id}_X$;

(ii) $\theta_h^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}$, for all $g, h \in G$;

(iii) $\theta_g \circ \theta_h(x) = \theta_{gh}(x)$, for all $x \in \theta_h^{-1}(D_h \cap D_{g^{-1}})$ and $g, h \in G$.

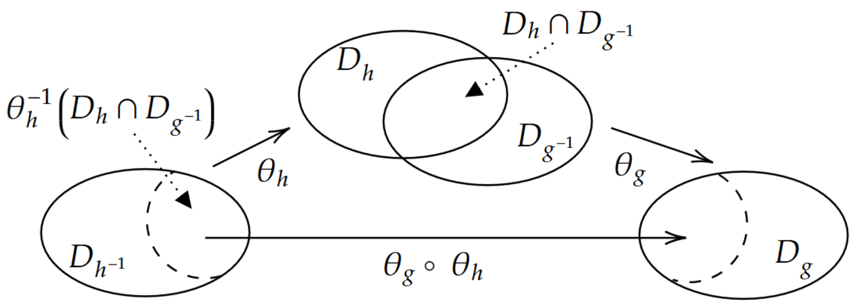
The quadruple $(X, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is called a **(topological) partial dynamical system**.

Remark 3.2.2. Conditions *(ii)* and *(iii)* can be condensed into:

(ii)' $\theta_g \circ \theta_h \subseteq \theta_{gh}$.

Furthermore, this means that the composition of functions is only partially defined. This can be better understood with the help of the following diagram:

Figure 3 – Composing partially defined functions.



Source: Prepared by the author (2024).

It is also worth to mention that conditions *(i)*-*(iii)* lead to $\theta_g^{-1} = \theta_{g^{-1}}$, for every $g \in G$. For more details, we refer the reader to [Exel, 2017, Chapter 2].

Construction 3.2.3 (Restriction of global actions)**.** One way of producing examples of partial actions is via the process of restriction. Let η be a global action of G on a topological space Y, and let $X \subseteq Y$ be an open subset. For each $g \in G$, define

$$D_g := X \cap \eta_g(X),$$

and also $\theta_g : D_{g^{-1}} \to D_g$ by restriction of η, this means that

$$\theta_g(x) \coloneqq \eta_g(x)\,,$$

for every $x \in X \cap \eta_{g^{-1}}(X) = D_{g^{-1}}$, and $g \in G$. Notice that the codomain of θ_g is indeed D_g, since $\eta_g(D_{g^{-1}}) = \eta_g(X) \cap X = D_g$.

Example 3.2.4. In this work, the most important example of topological partial action is a restriction of the Bernoulli action. Let G be a group and consider $\{0,1\}^G$ with the product topology. Note that $\{0,1\}^G$ is compact by the Tychonoff's Theorem. Identify $\{0,1\}^G$ with $\mathcal{P}(G)$ via

$$\begin{aligned} \{0,1\}^G &\to \mathcal{P}(G) \\ f &\mapsto \omega_f = \{g \in G \mid f(g) = 1\} \\ f_\omega &\hookleftarrow \omega\,, \end{aligned}$$

where $f_\omega(g) = 1$ if and only if $g \in \omega$; i.e., f_ω is the characteristic function of ω. With this identification in hand, we define the global Bernoulli action, for each $g \in G$, by

$$\begin{aligned} \eta_g : \{0,1\}^G &\to \{0,1\}^G \\ \omega &\mapsto g\omega \coloneqq \{gh \mid h \in \omega\}\,. \end{aligned}$$

Consider now the set $\Omega_1 \coloneqq \{\omega \in \{0,1\}^G \mid e \in \omega\}$, which is open and compact on $\{0,1\}^G$. The partial Bernoulli action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is the restriction of the global Bernoulli action η to Ω_1. In particular, note that

$$D_g = \Omega_1 \cap \eta_g(\Omega_1) = \{\omega \in \{0,1\}^G \mid e, g \in \omega\}\,,$$

since an element of $\eta_g(\Omega_1)$ is characterized by the fact that $g \in \omega$. Furthermore, each D_g is compact.

Definition 3.2.5. A **(C*-algebraic) partial action** of G on a C*-algebra A is a pair $(\{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ where each $I_g \subseteq A$ is a (closed two-sided) ideal, and each $\alpha_g : I_{g^{-1}} \to I_g$ is a $*$-isomorphism, satisfying

(i) $I_e = A$ and $\alpha_e = \mathrm{id}_A$;

(ii) $\alpha_h^{-1}(I_h \cap I_{g^{-1}}) \subseteq I_{(gh)^{-1}}$, for all $g, h \in G$;

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in \alpha_h^{-1}(I_h \cap I_{g^{-1}})$ and $g, h \in G$.

The quadruple $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is called a **(C*-algebraic) partial dynamical system**.

Remark 3.2.6. Note that a (topological) partial action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ on a locally compact Hausdorff space X induces a (C*-algebraic) partial action $(\{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on the C*-algebra $C_0(X)$. Specifically, $I_g = C_0(D_g)$ and $\alpha_g(f)(x) = f(\theta_{g^{-1}}(x))$ for every $x \in D_g$ and $f \in C_0(D_{g^{-1}})$. In both topological and C*-algebraic cases, we will omit the adjective in parentheses if the context is clear enough.

Similarly to the topological case, we have $\alpha_g^{-1} = \alpha_{g^{-1}}$. Moreover, we can improve the condition *(ii)* as follows:

Lemma 3.2.7. *Let $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial dynamical system. For every $g, h \in G$, we have the following*

$$\alpha_g(I_{g^{-1}} \cap I_h) = I_g \cap I_{gh}\,. \tag{3.3}$$

Proof. Replacing h by g^{-1} and g by h^{-1} in condition *(ii)* of Definition 3.2.5 and observing that the range of α_g is contained in I_g, it follows that

$$\alpha_g(I_{g^{-1}} \cap I_h) \subseteq I_g \cap I_{gh}\,.$$

Composing with $\alpha_{g^{-1}}$ on both sides, we have

$$I_{g^{-1}} \cap I_h \subseteq \alpha_{g^{-1}}(I_g \cap I_{gh})\,.$$

Replacing g^{-1} by g and h by gh, we conclude that

$$I_g \cap I_{gh} \subseteq \alpha_g(I_{g^{-1}} \cap I_h)\,. \qquad \square$$

Henceforth, fix a partial dynamical system $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Consider the vector subspace $A \rtimes^p_{\alpha,\mathrm{alg}} G \subseteq C_c(G, A)$ generated by elements of the form $a\delta_g$ with $a \in I_g$ and $g \in G$. In other words,

$$A \rtimes^p_{\alpha,\mathrm{alg}} G = \left\{ \sum_{g\in G} a_g\delta_g \;\middle|\; a_g \in I_g, \text{ for every } g \in G \right\}.$$

Before defining the product, let us build some intuition and motivation. Recall that, in the case that α is a global action, we already defined the multiplication as

$$(a\delta_g)(b\delta_h) = a\alpha_g(b)\delta_{gh}\,,$$

where $a, b \in A$ and $g, h \in G$. But in the partial case, we need $a\alpha_g(b) \in I_{gh}$, which is not true in general. To contour this problem, we define the α-twisted partial multiplication in $A \rtimes^p_{\alpha,\mathrm{alg}} G$ as

$$(a\delta_g)(b\delta_h) := \alpha_g(\alpha_{g^{-1}}(a)b)\delta_{gh}\,,$$

for every $a \in I_g$, $b \in I_h$, and $g, h \in G$. Notice that everything is in the "proper domain" in the definition above. In fact, since $\alpha_{g^{-1}}(a)$ belongs to the ideal $I_{g^{-1}}$ and b belongs to the ideal I_h, we conclude that $\alpha_{g^{-1}}(a)b \in I_{g^{-1}} \cap I_h$. Hence, by (3.3), it follows that $\alpha_g(\alpha_{g^{-1}}(a)b)$ lies in I_{gh}, as should be. Finally, notice that, when α is a global action, the multiplications defined above are formally the same.

Similarly to the involution on $C_c(G, A)$, we define the involution on $A \rtimes^p_{\alpha,\mathrm{alg}} G$ as

$$(a\delta_g)^* := \alpha_{g^{-1}}(a^*)\delta_{g^{-1}} ,$$

for every $a \in I_g$ and $g \in G$. In particular, note that $a \in I_g$ implies that $a^* \in I_g$ and $\alpha_{g^{-1}}(a^*) \in I_{g^{-1}}$.

The proof that $A \rtimes^p_{\alpha,\mathrm{alg}} G$ is a $*$-algebra is almost trivial, besides the associativity of the α-twisted partial multiplication, which requires an argument with multipliers (see [Exel, 2017]).

Definition 3.2.8. Let $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial dynamical system. The **(full) partial crossed product**, denoted $A \rtimes^p_\alpha G$, is the completion of $A \rtimes^p_{\alpha,\mathrm{alg}} G$ with respect of the C*-norm

$$\|f\| := \sup_\sigma \|\sigma(f)\| ,$$

where the supremum is taken over all (nondegenerate) $*$-representations of $A \rtimes^p_{\alpha,\mathrm{alg}} G$.

Note that, with the definition above, $\|\cdot\|$ is just a seminorm, but with the construction of the regular representation (3.2.14), it follows that $\|\cdot\|$ is indeed a norm.

As usual, the full partial crossed product satisfies a universal property (see [Exel, 2017, Proposition 11.14]).

Proposition 3.2.9. *Let B be a C*-algebra and $\varphi : A \rtimes^p_{\alpha,\mathrm{alg}} G \to B$ a $*$-homomorphism. Then φ extends to a unique $*$-homomorphism $\overline{\varphi} : A \rtimes^p_\alpha G \to B$ such that the following diagram commutes:*

$$\begin{array}{ccc} & & A \rtimes^p_\alpha G \\ & \nearrow & \big\downarrow \overline{\varphi} \\ A \rtimes^p_{\alpha,\mathrm{alg}} G & \xrightarrow[\varphi]{} & B \end{array}$$

□

Definition 3.2.10. We say that a map $u : G \to B(\mathcal{H})$ is a **partial representation** of G in $B(\mathcal{H})$ if, for any $g, h \in G$, we have

(i) $u_e = 1_{\mathcal{H}}$;

(ii) $u_g u_h u_{h^{-1}} = u_{gh} u_{h^{-1}}$;

(iii) $u_g^* = u_{g^{-1}}$.

Remark 3.2.11. Given a partial representation, when we apply the involution to the equation on *(ii)*, we get

$$u_h u_{h^{-1}} u_{g^{-1}} = u_{h^{-1}}^* u_h^* u_g^* = (u_g u_h u_{h^{-1}})^* = (u_{gh} u_{h^{-1}})^* = u_{h^{-1}}^* u_{gh}^* = u_h u_{h^{-1}g^{-1}} .$$

Changing variables, we conclude

(ii)' $u_{g^{-1}} u_g u_h = u_{g^{-1}} u_{gh}$;

which is an equivalent condition to *(ii)*. Furthermore, note that

$$u_g u_g^* u_g \overset{(iii)}{=} u_g u_{g^{-1}} u_g \overset{(ii)}{=} u_{gg^{-1}} u_g = u_e u_g \overset{(i)}{=} u_g ,$$

for every $g \in G$. Hence, each u_g is a partial isometry in $B(\mathcal{H})$. Moreover, if $u : G \to B(\mathcal{H})$ is a representation, then u is also a partial representation, which satisfies $u_g u_h = u_{gh}$.

Definition 3.2.12. Let $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial dynamical system. A **partial covariant representation** is a triplet $(\pi, u, \mathcal{H})$ where $\pi : A \to B(\mathcal{H})$ is a representation of A and $u : G \to B(\mathcal{H})$ is a partial representation of G such that, for every $g \in G$ and $a \in I_{g^{-1}}$,

$$u_g \pi(a) u_g^* = \pi(\alpha_g(a)) .$$

Similarly to the global setting, a partial covariant representation is exactly what we need to get a representation of the algebraic partial crossed product.

Proposition 3.2.13. *Let $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial dynamical system and $(\pi, u, \mathcal{H})$ be a partial covariant representation. Denote by $e_g \coloneqq u_g u_{g^{-1}}$ the range projection. Then,*

(i) $\pi(a) = \pi(a)e_g = e_g \pi(a)$, *for every $g \in G$ and $a \in I_g$;*

(ii) $u_g \pi(a) = \pi(\alpha_g(a)) u_g$, *for every $g \in G$ and $a \in I_{g^{-1}}$;*

(iii) the linear map $\pi \rtimes u : A \rtimes^p_{\alpha,\mathrm{alg}} G \to B(\mathcal{H})$ given by

$$a\delta_g \longmapsto \pi(a) u_g ,$$

is a $*$-homomorphism. In particular, $\pi \rtimes u$ can be extended to $A \rtimes^p_\alpha G$.

Proof. **(i)** Note that, for any $a \in I_g$,

$$e_g \pi(a) e_g = u_g u_{g^{-1}} \pi(a) u_g u_{g^{-1}} = u_g \pi(\alpha_{g^{-1}}(a)) u_{g^{-1}} = \pi(\alpha_g(\alpha_{g^{-1}}(a))) = \pi(a) .$$

Hence,

$$\pi(a) e_g = (e_g \pi(a) e_g) e_g = \underbrace{e_g \pi(a) e_g}_{=\pi(a)} = e_g (e_g \pi(a) e_g) = e_g \pi(a) .$$

(ii) Observe that, for any $a \in I_{g^{-1}}$,

$$u_g \pi(a) \overset{(i)}{=} u_g \pi(a) e_{g^{-1}} = u_g \pi(a) u_{g^{-1}} u_g = \pi(\alpha_g(a)) u_g .$$

(iii) We will need the following fact: $u_g u_h = e_g u_{gh}$ (see [Exel, 2017, Proposition 9.8]). Then, for any $a \in I_g$ and $b \in I_h$,

$$
\begin{aligned}
(\pi \rtimes u)(a\delta_g) \cdot (\pi \rtimes u)(b\delta_h) &= \pi(a)u_g\pi(b)u_h \\
&\overset{(ii)}{=} u_g\pi(\alpha_{g^{-1}}(a))\pi(b)u_h \\
&= u_g\pi(\alpha_{g^{-1}}(a)b)u_h \\
&\overset{(ii)}{=} \pi(\alpha_g(\alpha_{g^{-1}}(a)b))u_g u_h \\
&= \pi(\alpha_g(\alpha_{g^{-1}}(a)b))e_g u_{gh} \\
&\overset{(i)}{=} \pi(\alpha_g(\alpha_{g^{-1}}(a)b))u_{gh} \\
&= (\pi \rtimes u)(\alpha_g(\alpha_{g^{-1}}(a)b)\delta_{gh}) \\
&= (\pi \rtimes u)((a\delta_g)(b\delta_h)) ,
\end{aligned}
$$

and

$$
\begin{aligned}
((\pi \rtimes u)(a\delta_g))^* &= (\pi(a)u_g)^* \\
&= u_g^*\pi(a)^* \\
&\overset{(i)}{=} u_{g^{-1}}\pi(a^*)u_g u_{g^{-1}} \\
&= \pi(\alpha_{g^{-1}}(a^*))u_{g^{-1}} \\
&= (\pi \rtimes u)(\alpha_{g^{-1}}(a^*)\delta_{g^{-1}}) \\
&= (\pi \rtimes u)((a\delta_g)^*) .
\end{aligned}
$$

Therefore, $\pi \rtimes u$ is a $*$-homomorphism that extends to $A \rtimes_\alpha^p G$ by universality. □

Construction 3.2.14 (Regular representation). Let $A \subset B(\mathcal{H})$ be a faithful nondegenerate representation of A. For each $g \in G$, define the representation $\iota_g : I_g \to B(\mathcal{H})$ by

$$a \longmapsto \alpha_{g^{-1}}(a) .$$

Consider an approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ of I_g (see [Murphy, 2014, Theorem 3.1.2]), and define the extension $\pi_g : A \to B(\mathcal{H})$ by

$$a \longmapsto \pi_g(a) = s - \lim_\lambda \iota_g(e_\lambda a) = s - \lim_\lambda \alpha_{g^{-1}}(e_\lambda a) . \tag{3.4}$$

Here $s - \lim$ stands for the strong limit of the operators. It follows from [Dixmier, 1977, Proposition 2.10.4] that π_g is the unique extension of ι_g such that $\pi_g(a)\xi = 0$ for every $\xi \in (\iota_g(I_g)\mathcal{H})^\perp$ and $a \in I_g$. In particular, the definition above does not depend on the choice of the approximate unit.

In order to define the regular partial representation we need to observe two things:

Remark 3.2.15. Consider $I \subseteq A$ a (closed two-sided) ideal and $(f_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $I_g \cap I$. Note that, when $a \in A$, we could change the approximate unit in (3.4) by $(f_\lambda)_{\lambda\in\Lambda}$. In fact, define the map $\pi'_g : I \to B(\mathcal{H})$ by

$$\pi'_g(a) = s - \lim_\lambda \alpha_{g^{-1}}(f_\lambda a)\,.$$

By the uniqueness of the extension we must have $\pi_g|_I(a) = \pi_g(a)$ for every $a \in I$ (see [McClanahan, 1995, §3] for more details)

Remark 3.2.16. Note that, if $(e_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $I_{g^{-1}h} \cap I_{g^{-1}}$, then the net $(\alpha_g(e_\lambda))_{\lambda\in\Lambda}$ is an approximate unit for $\alpha_g(I_{g^{-1}h} \cap I_{g^{-1}}) \overset{(3.2.7)}{=} I_g \cap I_{h^{-1}}$. In fact, for any $a \in I_g \cap I_{h^{-1}}$,

$$\lim_\lambda \alpha_g(e_\lambda)a = \lim_\lambda \alpha_g(e_\lambda \alpha_{g^{-1}}(a)) = \alpha_g\big(\lim_\lambda e_\lambda \alpha_{g^{-1}}(a)\big) = \alpha_g(\alpha_{g^{-1}}(a)) = a\,.$$

Back to our construction, define the linear map $\pi : A \to B(\mathcal{H} \otimes \ell^2(G))$ by

$$\pi(a)(\xi \otimes \delta_g) \coloneqq \pi_g(a)(\xi) \otimes \delta_g\,,$$

for every $a \in A$, $\xi \in \mathcal{H}$ and $g \in G$. Recall that, using the left regular representation $\lambda : G \to \mathcal{B}(\ell^2(G))$, the map $1_\mathcal{H} \otimes \lambda : G \to B(\mathcal{H} \otimes \ell^2(G))$ given by

$$1_\mathcal{H} \otimes \lambda_h(\xi \otimes \delta_g) \coloneqq \xi \otimes \delta_{hg}\,,$$

is a unitary representation of G.

Observe that $(\pi, 1_\mathcal{H} \otimes \lambda, \mathcal{H} \otimes \ell^2(G))$ is a partial covariant representation. Indeed, for any $g, h \in G$, consider $(e_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $I_{g^{-1}h} \cap I_{g^{-1}}$. Then, for any $a \in I_{g^{-1}}$,

$$\begin{aligned}
(1_\mathcal{H} \otimes \lambda_g)\pi(a)(1_\mathcal{H} \otimes \lambda_g^*)(\xi \otimes \delta_h) &= (1_\mathcal{H} \otimes \lambda_g)\pi(a)(\xi \otimes \delta_{g^{-1}h}) \\
&= (1_\mathcal{H} \otimes \lambda_g)(\pi_{g^{-1}h}(a)(\xi) \otimes \delta_{g^{-1}h}) \\
&= \pi_{g^{-1}h}(a)(\xi) \otimes \delta_h \\
&\overset{(3.2.15)}{=} \lim_\lambda \alpha_{h^{-1}g}(e_\lambda a)(\xi) \otimes \delta_h \\
&= \lim_\lambda \alpha_{h^{-1}}(\alpha_g(e_\lambda a))(\xi) \otimes \delta_h \\
&= \lim_\lambda \alpha_{h^{-1}}\big(\alpha_g(e_\lambda)\ \alpha_g(a)\big)(\xi) \otimes \delta_h \\
&\overset{(3.2.16)}{=} \pi_h(\alpha_g(a))(\xi) \otimes \delta_h \\
&= \pi(\alpha_g(a))(\xi \otimes \delta_h)\,.
\end{aligned}$$

Therefore, we can define a $*$-homomorphism $\pi \rtimes (1_\mathcal{H} \otimes \lambda) : A \rtimes^p_{\alpha,\mathrm{alg}} G \to B(\mathcal{H} \otimes \ell^2(G))$, called the *regular representation*, that satisfies

$$\pi \rtimes (1_\mathcal{H} \otimes \lambda)(a\delta_g) = \pi(a)(1_\mathcal{H} \otimes \lambda_g)\,.$$

Definition 3.2.17. Let $(A, G, \{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial dynamical system. The **reduced partial crossed product**, denoted by $A \rtimes^p_{\alpha,r} G$, is defined as the completion of $A \rtimes^p_{\alpha,\mathrm{alg}} G$ with respect to the C*-norm:

$$\|f\|_r := \|\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)(f)\| \ ,$$

where $\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)$ is the regular representation defined in Construction 3.2.14. Equivalently, $A \rtimes^p_{\alpha,r} G$ can be concretely represented as:

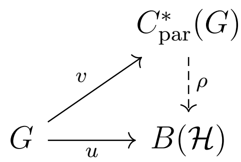
$$A \rtimes^p_{\alpha,r} G = \overline{\pi \rtimes (1_{\mathcal{H}} \otimes \lambda)(A \rtimes^p_{\alpha,\mathrm{alg}} G)} \subseteq B(\mathcal{H} \otimes \ell^2(G)) \, .$$

Example 3.2.18. Recall that $C^*(G)$ has the following universal property: for every unitary representation $u : G \to B(\mathcal{H})$ there exists a unique $*$-homomorphism $\rho : C^*(G) \to B(\mathcal{H})$ such that $\rho(\delta_g) = u_g$ for every $g \in G$.

Analogously, define the partial group C*-algebra as the universal C*-algebra generated by the set of operators $\{v_g \mid g \in G\}$ subject to relations

$$v_e = 1 \, , \ v_g v_h v_{h^{-1}} = v_{gh} v_{h^{-1}} \quad \text{and} \quad v_g^* = v_{g^{-1}} \, .$$

The correspondent universal property of $C^*_{\mathrm{par}}(G)$ is the following: for every partial representation $u : G \to B(\mathcal{H})$ there exists a unique $*$-homomorphism $\rho : C^*(G) \to B(\mathcal{H})$ such that $\rho(\delta_g) = u_g$ for every $g \in G$. In other words, the following diagram commutes.



Where $v : G \to C^*_{\mathrm{par}}(G)$ denote the partial representation $g \mapsto v_g$.

Furthermore, note that the partial Bernoulli action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ induces a partial action $(\{C(D_g)\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on $C(\Omega_1)$. One can prove that

$$C^*_{\mathrm{par}}(G) \cong C(\Omega_1) \rtimes^p_{\alpha} G \, .$$

Remembering that $C^*(G) \cong \mathbb{C} \rtimes_\nu G$, where ν is the trivial action, we conclude that $C^*_{\mathrm{par}}(G)$ is “extremely bigger” than $C^*(G)$. For details about this example, we refer to [Exel, 2017, Chapter 14].

With the above isomorphism in mind, we define $C^*_{\mathrm{par},r}(G) := C(\Omega_1) \rtimes^p_{\alpha,r} G$, in order to avoid technicalities with the definition of $C^*_{\mathrm{par},r}(G)$.

Remark 3.2.19. Recall that the partial action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ on a locally compact Hausdorff space X induces a partial action $(\{C_0(D_g)\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on the abelian C*-algebra $C_0(X)$.

Suppose that x_0 is a fixed point, i.e., $x_0 \in D_g$ and $\theta_g(x_0) = x_0$ for every $g \in G$. Note that the trivial global action of G on $\{x_0\}$ induces a trivial global action ν of G on $C(\{x_0\}) \cong \mathbb{C}$, i.e., $\nu_g = \mathrm{id}_{\{x_0\}}$.

By universality, we have a $*$-homomorphism $\pi : C_0(X) \rtimes^p_\alpha G \to \mathbb{C} \rtimes_\nu G \cong C^*(G)$ given on generators by

$$f\delta g \longmapsto f(x_0)\delta_g \, .$$

Note that π is surjective because the set $\{f(x_0)\delta_g \mid g \in G, f \in C_0(D_g)\}$ is dense in $C^*(G)$.

Analogously, there is a surjective $*$-homomorphism $\pi_r : C_0(X) \rtimes^p_{\alpha,r} G \to C^*_r(G)$[1].

In particular, note that the set $G \in \Omega_1$ is a fixed point for the partial Bernoulli action. Therefore, there is a surjective $*$-homomorphism $\pi_{(r)}$ from the (reduced) partial group C*-algebra $C^*_{\text{par}}(G) = C(\Omega_1) \rtimes^p_{\alpha,(r)} G$ to the (reduced) group C*-algebra $C^*_{(r)}(G)$. Where the parentheses "(r)" mean that the statement is valid on both full and reduced C*-algebras.

Notes and Remarks. Notice that we can define C*-algebraic partial actions similarly to Definition 1.3.1 of global actions. In this case, we need to consider the semigroup of partial automorphism of A, which is the set $\text{pAut}(A)$ of all $*$-isomorphisms between ideals of A. A partial action in this setting is a map $\alpha : G \to \text{pAut}(A)$ satisfying certain properties (see [Exel, 2017, Proposition 4.5]).

3.3 Følner C*-algebras and partial crossed products

The purpose of this section is to show which statements of Theorem 3.1.2 hold in the case of partial actions. These statements are present in the following question:

Question. *Let G be a discrete group and let α be a **partial** action of G on a unital C*-algebra A. Which of the following is true?*

(i) If $A \rtimes^p_{\alpha,r} G$ is a Følner C-algebra, then A is a Følner C*-algebra?*

(ii) If $A \rtimes^p_{\alpha,r} G$ is a Følner C-algebra, then G is amenable?*

(iii) If G is amenable and A is a Følner C-algebra, then $A \rtimes^p_{\alpha,r} G$ is a Følner C*-algebra?*

As we shall see, only question *(i)* has an affirmative answer. The other two can be proved false by some counterexamples. We aim to provide some explanations of why these results do not generalize to the partial setting, and present stronger hypotheses that guarantee a version of *(iii)* to be true. Furthermore, we provide a generalization of Theorem 2.5.16 considering the partial group C*-algebra.

The original results of this section are Proposition 3.3.1, Proposition 3.3.11 and Theorem 3.3.13.

Notation convention: As before, all C*-algebras are assumed to be unital.

Similarly to the global case, note that the map $\iota : A \to A \rtimes^p_{\alpha,r} G$ given by

$$a \longmapsto a\delta_e$$

[1] In fact, the existence of these $*$-homomorphisms follows from the general fact that $C_0(X) \ni f \mapsto f(x_0) \in \mathbb{C}$ is G-equivariant, see [Exel, 2017, Proposition 22.2].

is a unital injective $*$-homomorphism. In particular, $\iota(1_A) = 1_A\delta_e$ is the unit of $A \rtimes^p_{\alpha,r} G$. Hence, we may identify A with its image $A\delta_e \subseteq A \rtimes^p_{\alpha,r} G$, and 1_A with $1_{A\rtimes^p_{\alpha,r}G}$. As a consequence:

Proposition 3.3.1. *Let G be a discrete group and let α be a partial action of G on a unital C*-algebra A. If $A \rtimes^p_{\alpha,r} G$ is a Følner C*-algebra, then A is a Følner C*-algebra.*

Proof. With the identification above, A is a C*-subalgebra of $A \rtimes^p_{\alpha,r} G$ with the same unit. Therefore A is a Følner C*-algebra by Proposition 2.5.5. □

Remark 3.3.2. The answer to question *(ii)* is negative; i.e., $A \rtimes^p_{\alpha,r} G$ being a Følner C*-algebra does not imply that G is amenable. Indeed, let A be a Følner C*-algebra and G a non-amenable group. Consider the partial action $(\{I_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on A with

$$I_g = \{0\}, \text{ if } g \neq e, \text{ and } I_e = A\,,$$

and

$$\alpha_g = \mathrm{id}_{\{0\}}, \text{ if } g \neq e, \text{ and } \alpha_e = \mathrm{id}_A\ .$$

Note that $A \rtimes^p_{\alpha,\mathrm{alg}} G = \{a\delta_e \mid a \in A\} = A\delta_e \cong A$. Therefore the partial reduced crossed product $A \rtimes^p_{\alpha,r} G \cong A$ is a Følner C*-algebra and G is non-amenable.

Moreover, recall that, in the global setting, we proved that G is amenable using the fact that $C^*_r(G)$ is a C*-subalgebra of $A \rtimes_{\alpha,r} G$ with the same unit. Specifically, when α is a global action, we have a unital injective $*$-homomorphism $C^*_r(G) \hookrightarrow A \rtimes_{\alpha,r} G$ given by

$$x\delta_g \longmapsto x1_A\delta_g \qquad (x \in \mathbb{C},\, g \in G)\,.$$

Hence, if α is a partial action, in order to define the same map we should have $1_A \in I_g$ for every $g \in G$. However, this implies that $I_g = A$ and α is a global action. Therefore, the approach used in the global setting to prove that "$A \rtimes_{\alpha,r} G$ Følner implies G amenable" does not work in the partial setting.

The counterexample showing why the answer to question *(iii)* is negative in the partial setting is more technical. It relies on the work of [Scarparo, 2017], which presents some results connecting supramenable groups and partial actions. To understand why "A Følner and G amenable do not imply $A \rtimes^p_{\alpha,r} G$ Følner", we need introduce a few key concepts.

From now until the end of the section, G will denote a discrete amenable group, unless otherwise stated. In particular, reduced and full (partial) crossed products coincide[2].

Definition 3.3.3. A group G is called **supramenable** if, for any nonempty subset $E \subseteq G$, there exists a (left) invariant finitely additive measure $m : \mathcal{P}(G) \to [0, +\infty]$ such that $m(E) = 1$. By *(left) invariant*, we mean $m(gE) = m(E)$ for every $g \in G$ and $E \subseteq G$.

[2] Follows from [Exel, 2017, Theorem 20.7].

Remark 3.3.4. Finite and abelian groups are supramenable. Furthermore, supramenability passes to subgroups and quotients (see [Tomkowicz and Wagon, 2016, Theorem 14.4]). For example, if G is finite and $E \subseteq G$ is nonempty, then $m(F) := |F|/|E|$, $(F \subseteq G)$ defines the desired invariant finitely additive measure such that $m(E) = 1$.

Remark 3.3.5. Recall that the existence of a free subgroup is an obstruction to a group being amenable (see Example 2.1.17). Similarly, the existence of a free semigroup is an obstruction to supramenability. In fact, suppose that $S \subseteq G$ is a free semigroup with two generators a and b and note that $aS \sqcup bS \subseteq S$. If m is an invariant finitely additive measure such that $m(S) = 1$, then

$$2 = m(S) + m(S) = m(aS) + m(bS) = m(aS \sqcup bS) \leq m(S) = 1\,,$$

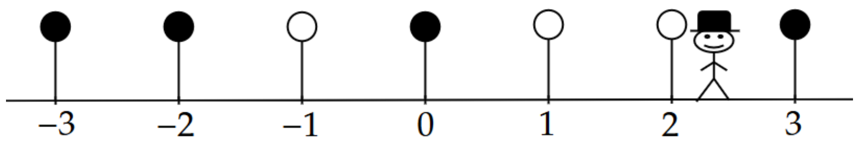
which is a contradiction.

Example 3.3.6. The lamplighter group is amenable but not supramenable. Recall that the lamplighter group is the semidirect product

$$\bigoplus_{k\in\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z} \cong \langle a, t \mid a^2 = e\,,\ [t^m a t^{-m}\,,\ t^n a t^{-n}] = e\,,\ m, n \in \mathbb{Z}\rangle\,,$$

where the right side stands for the group presentation and $[.\,,.]$ denotes the commutator, and the action in the left side is given by reindexing: $m \cdot (a_n)_{n\in\mathbb{Z}} = (a_{n-m})_{n\in\mathbb{Z}}$. Intuitively, the group represents the action of a lamplighter in $\mathbb{Z}$ lamps. The generator a is the action of changing the state of a lamp, and the generator t is the action of moving one lamp to the right. For example, Figure 4 represents the lamps after an action of the element $atat^2at^{-1}$, considering that initially all the lamps were off and the lamplighter was at 0.

Figure 4 – The lamps after an action of $atat^2at^{-1}$.



Source: Prepared by the author (2025).

Note that the lamplighter group is amenable and countable, since it is a semidirect product of amenable countable groups. However, it contains a free semigroup generated by a and t, hence the lamplighter group is not supramenable.

Definition 3.3.7. Let X be a Hausdorff space and $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of G on X. Then,

(i) we say that θ is **free** if $\theta_g(x) = x$ implies $g = e$, for every $g \in G$ and $x \in D_{g^{-1}}$.

(ii) we say that θ is **minimal** if there is no non-trivial invariant closed subsets; i.e., if $F \subseteq X$ is closed and $\theta_g(F \cap D_{g^{-1}}) \subseteq F$ for every $g \in G$, then $F = \emptyset$ or $F = X$.

(iii) if X is totally disconnected, we say that θ is **purely infinite** if for every compact-open subset $K \subseteq X$ there exists $g_1, \ldots, g_{n+m}$ and parwise disjoint compact-open subsets $K_1, \ldots, K_{n+m}$ such that $K_j \subseteq K \cap D_{g_j^{-1}}$ for every $j = 1, \ldots, n+m$, and

$$K = \bigcup_{j=1}^{n} \theta_{g_j}(K_j) = \bigcup_{j=n+1}^{n+m} \theta_{g_j}(K_j)\,.$$

The proof of the next result can be found in [Scarparo, 2017, Proposition 2.10].

Proposition 3.3.8. *Let G be an amenable, non-supramenable, countable group. Then G admits a free, minimal, purely infinite and non-global partial action θ on the Cantor set K. Furthermore, $C(K) \rtimes_\alpha^p G$ is a Kirchberg algebra, where α is the partial action on $C(K)$ induced by θ.* □

Remark 3.3.9. A Kirchberg algebra is a purely infinite, simple, nuclear, separable C*-algebras. A simple C*-algebra is called *purely infinite* if every nonzero hereditary[3] C*-subalgebra contain an infinite projection.

Suppose that a Kirchberg A has a tracial state τ and consider an infinite projection $p \in A$. This means that there exists a projection $q \in A$ such that $q < p$ and $p \sim q$; i.e., there exists $x \in A$ such that $p = x^*x$ and $q = xx^*$. Since τ is a trace, we have

$$\tau(p) = \tau(x^*x) = \tau(xx^*) = \tau(q)\,.$$

Since A is simple, it follows that τ is faithful and $p = q$, which is a contradiction. Therefore, Kirchberg algebras do not admit tracial states and amenable traces; i.e., they are not Følner C*-algebras.

As a consequence of Proposition 3.3.8 and the previous remark, we conclude that $C(K) \rtimes_\alpha^p G$ is not a Følner C*-algebra, although $C(K)$ is a Følner C*-algebra and G is amenable.

On the other hand, since we suppose that G is amenable and non-supramenable in the hypotheses of Proposition 3.3.10, it is natural to ask:

Question. *If G is **supramenable** and A is a Følner C*-algebra, then $A \rtimes_{\alpha,r}^p G$ is a Følner C*-algebra?*

Unfortunately, we only found a proof when A is nuclear. Furthermore, we need the next result, which can be found in [Scarparo, 2017, Theorem 3.6].

[3] This means that, if $a \in A_+$, $b \in B_+$ and $a \leq b$, then $a \in B$.

Theorem 3.3.10. *A group is supramenable if and only if whenever it partially acts on a unital C*-algebra which has a tracial state, then the associated partial crossed product also has a tracial state.* □

As a consequence:

Proposition 3.3.11. *Let G be a discrete group, and let α be a partial action of G on a unital C*-algebra A. If G is supramenable and A is nuclear and Følner, then $A \rtimes^{p}_{\alpha,r} G$ is a Følner C*-algebra.*

Proof. Recall that every Følner C*-algebra has a tracial state. Thus, since G is supramenable and A has a tracial state, it follows that $A \rtimes^{p}_{\alpha,r} G$ has a tracial state by Theorem 3.3.10. Furthermore, it is well known that, if A is nuclear and G is amenable, then $A \rtimes^{p}_{\alpha,r} G$ is nuclear[4]. In conclusion, $A \rtimes^{p}_{\alpha,r} G$ is a unital nuclear C*-algebra with at least one tracial state; i.e., it is a Følner C*-algebra by Corollary 2.5.13. □

Remark 3.3.12. Note that we can replace the hypothesis "A is nuclear and Følner" by "A is nuclear and has at least one tracial state", since they are equivalent (see Corollary 2.5.13).

In Theorem 2.5.16 we saw that G is amenable if and only if $C^*_r(G)$ is nuclear, if and only if $C^*_r(G)$ is a Følner C*-algebra. With the theory developed so far, we can conclude the following.

Theorem 3.3.13. *Let G be a discrete group. Then,*

(i) If G is amenable, then $C^*_{\mathrm{par},(r)}(G)$ is nuclear and Følner;

(ii) If $C^*_{\mathrm{par},r}(G)$ is nuclear, then G is amenable.

Proof. **(i)** Suppose G is amenable and note that $C(\Omega_1)$ is nuclear, since it is abelian. Therefore, the crossed product $C(\Omega_1) \rtimes^{p}_{\alpha,(r)} G = C^*_{\mathrm{par},(r)}(G)$ is nuclear.

Furthermore, there exists a surjective $*$-homomorphism $\pi : C^*_{\mathrm{par},(r)}(G) \to C^*_{(r)}(G)$ (see Remark 3.2.19). In particular, observe that the nonzero quotient

$$\frac{C^*_{\mathrm{par},(r)}(G)}{\ker(\pi)} \cong C^*_{(r)}(G)$$

is a Følner C*-algebra. By Corollary 2.5.10, we conclude that $C^*_{\mathrm{par},(r)}(G)$ is a Følner C*-algebra.

(ii) It follows from the fact that every quotient of a nuclear C*-algebra is nuclear. In fact, considering the quotient

$$\frac{C^*_{\mathrm{par},r}(G)}{\ker(\pi)} \cong C^*_r(G)\,,$$

[4] Although we have not developed the necessary tools to prove this statement, it follows by considering partial crossed products as cross-sectional algebras of Fell bundles [Exel, 2017, Chapters 16 & 17], and applying Theorem 20.7 and Proposition 25.10 of the same book.

we conclude that $C^*_r(G)$ is nuclear. Therefore, G is amenable by Theorem 2.5.16. □

Remark 3.3.14. In the above proof, we have the ingredients to prove that, for every group G, the partial group C*-algebra $C^*_{\text{par}}(G)$ is a Følner C*-algebra, likely the group C*-algebra $C^*(G)$.

Notes and Remarks. After the last result, the natural question is:

Question. *If $C^*_{\text{par},r}(G)$ is a Følner C*-algebra, then G is amenable?*

Unfortunately, since $C^*_{\text{par},r}(G)$ can be viewed as a partial crossed product, the answer of the above question is related to whether or not $A \rtimes^p_{\alpha,r} G$ being Følner implies G amenable, which is a tough question in the partial setting. An intuitive guess for a counterexample will be $C^*_{\text{par},r}(\mathbb{F}_2) = C(\Omega_1) \rtimes^p_{\alpha,r} \mathbb{F}_2$, since α may have some "nice" approximation property that guarantees weak containment $C^*_{\text{par},r}(\mathbb{F}_2) = C^*_{\text{par}}(\mathbb{F}_2)$, which implies that $C^*_{\text{par}}(\mathbb{F}_2)$ is a Følner C*-algebra.

Furthermore, it would be desirable to eliminate the nuclearity hypotheses in Proposition 3.3.11. Note that, if we have

$$A \rtimes^p_{\alpha,r} G \overset{?}{=} C^*\big(\pi(A), 1_{\mathcal{H}} \otimes \lambda(G)\big) \,,$$

similarly to the global setting, then we may use Lemma 3.1.1 to prove that $A \rtimes^p_{\alpha,r} G$ is Følner when G is supramenable and A is Følner.

Conclusion

The study of Følner C*-algebras is very interesting and has deep connections with other important concepts within operator algebras. In particular, this work shows the intricate relation between amenable traces, Følner nets of operators and nets of u.c.p. maps.

However, as pointed out by some members of the Operator Algebra Seminar at UFSC, Følner C*-algebras have a few results that appear unusual when compared to nuclear C*-algebras. Moreover, the definition seems “too weak” to guarantee “nicer” results. One alternative is to modify the definition, allowing only faithful amenable traces or requiring the net to be asymptotically isometric.

Also, in [Bédos, 1995, §3] the author defines a unital C*-algebra A to be *hypertracial* if, for every nondegenerate representation $\pi : A \to B(\mathcal{H})$, $\pi(A)$ has an amenable trace. One may ask which connection hypertracial C*-algebras have with Følner nets and nets of u.c.p. maps.

With regard to our initial goal of studying the relationship between Følner C*-algebras and partial reduced crossed products, the results may be considered unsatisfactory. We conclude that Følner C*-algebras are not as well-behaved as nuclear C*-algebras, when considering partial crossed products.

A Appendix

A.1 Voiculescu theorem

Unitary equivalence between representations is a key concept in operator algebra theory. However, sometimes one can find only an approximate equivalence:

Definition A.1.1. Two maps $\pi : A \to B(\mathcal{H})$ and $\sigma : A \to B(\mathcal{K})$ are called **approximately unitarily equivalent** if there is a net $(U_i)_{i\in I}$ of unitary operators $U_i : \mathcal{H} \to \mathcal{K}$ such that

$$\lim_i \|\sigma(a) - U_i\pi(a)U_i^*\| = 0$$

for every $a \in A$. If $\sigma(a) - U_i\pi(a)U_i^* \in K(\mathcal{K})$ for every $a \in A$ and $i \in I$, then π and σ are called **approximately unitarily equivalent relative to the compacts**. We write $\pi \sim_a \sigma$ ($\pi \sim_K \sigma$) when the representations π and σ are approximately unitarily equivalent (relative to the compacts).

Remark A.1.2. It straightforward that $\sim_a$ and $\sim_K$ are equivalence relation. Furthermore, we can localize the definition, e.g., $\pi \sim_a \sigma$ if and only if for every finite subset $F \subseteq A$ and $\varepsilon > 0$ there exists an unitary $U : \mathcal{H} \to \mathcal{K}$ such that, for each $a \in F$,

$$\|\sigma(a) - U\pi(a)U^*\| < \varepsilon\,.$$

In particular, when A is separable, this means that we can replace a net by a sequence.

The proofs of the next three results can be respectively found in Theorem II.5.3, Corollary II.5.5 and Theorem II.5.8 of [Davidson, 1996].

Theorem A.1.3 (**Noncommutative Weyl-von Neumann**). *Let $A \subset B(\mathcal{H})$ be a separable C*-algebra and $\varphi : A \to B(\mathcal{K})$ be a c.p. map. If $\varphi|_{A\cap K(\mathcal{H})} = 0$, then $\iota \sim_K \varphi$, where $\iota : A \hookrightarrow B(\mathcal{H})$ is the canonical inclusion.* □

Theorem A.1.4 (**Voiculescu's Theorem**). *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $A \subset B(\mathcal{H})$ be a separable C*-algebra with $1_\mathcal{H} \in A$. Let $\iota : A \hookrightarrow B(\mathcal{H})$ be the canonical inclusion and $\rho : A \to B(\mathcal{K})$ be any representation such that $\rho|_{A\cap K(\mathcal{H})} = \{0\}$. Then $\iota \sim_K \iota \oplus \rho$.* □

Theorem A.1.5. *Let A be a separable C*-algebra, and let $\pi : A \to B(\mathcal{H})$ and $\sigma : A \to B(\mathcal{K})$ be nondegenerate representations with $\mathcal{H}$ and $\mathcal{K}$ separable Hilbert spaces. Then, the following are equivalent:*

(i) $\pi \sim_a \sigma$;

(ii) $\pi \sim_K \sigma$ *;*

(iii) $\operatorname{rank}(\pi(a)) = \operatorname{rank}(\sigma(a))$*, for every* $a \in A$*.* □

Although not everything of the previous results holds in the non-separable setting, there is a generalization of the previous theorem. The proof can be found in [Hadwin, 1981, Theorem 3.14].

Theorem A.1.6 (Non-separable Voiculescu's Theorem). *Let* A *be a C*-algebra and* $\pi, \sigma : A \to B(\mathcal{H})$ *be representations. Then* $\pi \sim_a \sigma$ *if and only if* $\operatorname{rank}(\pi(a)) = \operatorname{rank}(\sigma(a))$*, for every* $a \in A$*.* □

A.2 Tensor products of C*-algebras

The purpose of this section is to give a quick presentation of tensor products and a few technical results that will be crucial in Chapter 2. For the reader who is not familiar with tensor products we refer to [Murphy, 2014, Chapter 6] and [Brown and Ozawa, 2008, Chapter 3].

Remark A.2.1. Given $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces there is a unique inner product on the algebraic tensor product $\mathcal{H} \odot \mathcal{K}$ such that

$$\langle \xi \otimes \eta, \zeta \otimes \vartheta \rangle = \langle \xi, \zeta \rangle \langle \eta, \vartheta \rangle \qquad (\xi, \zeta \in \mathcal{H}, \eta, \vartheta \in \mathcal{K}),$$

on elementary tensors. The completion is a Hilbert space denoted by $\mathcal{H} \otimes \mathcal{K}$.

Furthermore, if $u \in B(\mathcal{H})$ and $v \in B(\mathcal{K})$, there exists a unique $u \otimes v \in B(\mathcal{H} \otimes \mathcal{K})$ such that

$$\|u \otimes v\| = \|u\| \, \|v\| \quad \text{and} \quad (u \otimes v)(\xi \otimes \eta) = u(\xi) \otimes v(\eta),$$

for every $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ (see [Murphy, 2014, Lemma 6.3.2]).

Let A and B be C*-algebras and denote by $A \odot B$ the algebraic tensor products as vector spaces. One can check that $A \odot B$ admits a $*$-algebra structure with multiplication given by

$$(a \otimes b)(c \otimes d) := ac \otimes bd$$

and involution given by

$$(a \otimes b) := a^* \otimes b^* .$$

In the algebraic level, we have the following (see [Murphy, 2014, Remark 6.3.2]):

Proposition A.2.2. *Let* A, B *and* C *be C*-algebras. If* $\varphi : A \to C$ *and* $\psi : B \to C$ *are* $*$*-homomorphism with commuting ranges; i.e.,* $\varphi(A) \subseteq \psi(B)'$*; then there is a unique (product)* $*$*-homomorphism* $\varphi \times \psi : A \odot B \to C$ *such that*

$$(\varphi \times \psi)(a \otimes b) = \varphi(a)\psi(b) \qquad (a \in A,\, b \in B).$$ □

However, now the theory becomes tricky. In general, $A \odot B$ admits more than one C*-norm that leads to different completions and different C*-algebras. Suppose that $\pi : A \to B(\mathcal{H})$ and $\rho : B \to B(\mathcal{K})$ are faithful representations, then there exists a faithful representation $\pi \odot \rho : A \odot B \to B(\mathcal{H} \otimes \mathcal{K})$ (see [Murphy, 2014, Theorem 6.3.3.]). Thus we can define the *minimal* or *spatial norm* on $A \odot B$ as

$$\|c\|_{\min} := \|\pi \odot \rho(c)\| ,$$

for all $c \in A \odot B$. The completion of $A \odot B$ with respect to $\|\cdot\|_{\min}$ is denoted by $A \otimes B$ (or $A \otimes_{\min} B$). It is necessary to mention that the above definition does not depend on the choice of the faithful representations (see [Brown and Ozawa, 2008, Proposition 3.3.11]).

Furthermore, we can define the *maximal norm* on $A \odot B$ as

$$\|c\|_{\max} := \sup_{\pi} \|\pi(c)\| ,$$

for all $c \in A \odot B$, where the supremum is taken over all representations. The completion of $A \odot B$ with respect to $\|\cdot\|_{\max}$ is denoted by $A \otimes_{\max} B$ (see [Brown and Ozawa, 2008, §3.3]).

Proposition A.2.3 (Universal property of the maximal tensor product). *Let A, B and C be C*-algebras. If $\varphi : A \odot B \to C$ is a $*$-homomorphism, then there is a unique $*$-homomorphism $\overline{\varphi} : A \otimes_{\max} B \to C$ that extends φ. In particular, there exists a surjective $*$-homomorphism $\pi : A \otimes_{\max} B \to A \otimes B$ induced by the inclusion $A \odot B \hookrightarrow A \otimes B$.* □

Remark A.2.4. In general, it is false that $A \otimes B = A \otimes_{\max} B$. However, it is part of the folklore of C*-algebras that A is nuclear if and only if $A \otimes B = A \otimes_{\max} B$ for every C*-algebra B (see [Brown and Ozawa, 2008, Theorem 3.8.7.]). Indeed, the definition of nuclear C*-algebras is very often given in the tensor product framework.

Tensor products are well-behaved with relation to c.p. maps. The proof of the next result can be found in [Brown and Ozawa, 2008, Theorem 3.5.3].

Proposition A.2.5. *Let A, B and C be C*-algebras. If $\varphi : A \to C$ and $\psi : B \to D$ are c.p. maps, then the algebraic tensor product map $\varphi \odot \psi : A \odot B \to C \odot D$ extends to a c.p. map on both the minimal and maximal tensor products. Furthermore, denoting the extensions by $\varphi \otimes_{(\max)} \psi : A \otimes_{(\max)} B \to C \otimes_{(\max)} D$, we have*

$$\|\varphi \otimes_{\max} \psi\| = \|\varphi \otimes \psi\| = \|\varphi\| \, \|\psi\| .$$ □

A.3 Amenable trace technicalities

This section is based on [Brown and Ozawa, 2008, Chapter 6]. As promised, we provide a sketch to the technical lemma of Section 2.2 (see [Brown and Ozawa, 2008, Lemma 6.2.5], for a complete proof), and present some details about the opposite C*-algebra.

Lemma A.3.1. *Let $h \in B(\mathcal{H})$ be a positive finite-rank operator with rational eigenvalues and such that* $\mathrm{Tr}(h) = 1$. *Then, there is $k \in \mathbb{N}$ and a u.c.p. map $\varphi : B(\mathcal{H}) \to M_k(\mathbb{C})$ satisfying*

$$\mathrm{tr}(\varphi(x)) = \mathrm{Tr}(hx)$$

for every $x \in B(\mathcal{H})$, and

$$|\,\mathrm{tr}(\varphi(uu^*) - \varphi(u)\varphi(u^*))| \leq 2\,\|uhu^* - h\|_1^{1/2}$$

for every unitary operator $u \in A$.

Sketch of Proof. Suppose that $\frac{n_1}{k} < \ldots < \frac{n_m}{k}$ are the the nonzero eigenvalues of h. Then write

$$h = \frac{n_1}{k} q_1 \oplus \ldots \oplus \frac{n_m}{k} q_m\,,$$

where each q_i is the projection onto the eigenspace of $\frac{n_i}{k}$.

Now consider a Hilbert space $\mathcal{K}$ and projections $p_1 \leq \ldots \leq p_m$ on $B(\mathcal{K})$ such that $\mathrm{rank}(p_i) = n_i$. Since the q_i's are pairwise orthogonal, we can define $p \in B(\mathcal{H} \otimes \mathcal{K})$ by

$$p = q_1 \otimes p_i + \ldots + q_m \otimes p_m\,.$$

Then,

$$\mathrm{Tr}(p) = \sum_{i=1}^{m} \mathrm{Tr}(q_i)\,\mathrm{Tr}(p_i) = \sum_{i=1}^{m} \mathrm{Tr}(q_i) n_i = k\,,$$

since $\mathrm{Tr}(h) = 1$.

Define $\varphi : B(\mathcal{H}) \to M_k(\mathbb{C})$ as the compression by p, this means that

$$x \longmapsto p(x \otimes 1_{\mathcal{K}})p\,.$$

In particular

$$\mathrm{tr}(\varphi(x)) = \frac{\mathrm{Tr}(p(x \otimes 1_{\mathcal{K}})p)}{\mathrm{Tr}(p)} = \frac{\sum_{i=1}^{m} \mathrm{Tr}(q_i x q_i)\,\mathrm{Tr}(p_i)}{k} = \sum_{i=1}^{m} \frac{n_i}{k}\,\mathrm{Tr}(q_i x) = \mathrm{Tr}(hx)\,.$$

The challenging part of the proof is to estimate $|\,\mathrm{tr}(\varphi(uu^*) - \varphi(u)\varphi(u^*))|$. □

Definition A.3.2. The **opposite C*-algebra** A^{op} of a C*-algebra A is just A with reversed multiplication, i.e., $a \cdot b = ba$.

Remark A.3.3. Fix τ a tracial state and consider the GNS triplet $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau = \hat{1})$. Define a representation $\pi_\tau^{\mathrm{op}} : A^{\mathrm{op}} \to B(\mathcal{H}_\tau)$ given by

$$\pi_\tau^{\mathrm{op}}(a)\hat{b} = \widehat{ba} \qquad (a \in A,\ \hat{b} \in A/N_\tau)\,.$$

Note that π_τ^{op} is well-defined because τ is tracial. Indeed,

$$\|\pi_\tau^{\mathrm{op}}(a)\hat{b}\|_\tau^2 = \tau((ba)^*ba) = \tau(a^*b^*ba) = \tau(baa^*b^*) \leq \|a\|^2\,\tau(bb^*) = \|a\|^2\,\|\hat{b}\|_\tau\,.$$

Proposition A.3.4. *Let A be a C^*-algebra and τ be a tracial state on A. Then,*

(i) the representations π_τ and π_τ^{op} have commuting ranges, i.e., $\pi_\tau(A)' \supset \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})$;

(ii) we have $\pi_\tau(A)'' = \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})'$

Sketch of Proof. Condition *(i)* is straightforward, since for any $a, b, c \in A$,

$$\pi_\tau^{\mathrm{op}}(a)\pi_\tau(b)\hat{c} = \widehat{bca} = \pi_\tau(b)\pi_\tau^{\mathrm{op}}(a)\hat{c}\,.$$

Furthermore, $\pi_\tau(A)' \supset \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})$ implies $\pi_\tau(A)'' \subset \pi_\tau^{\mathrm{op}}(A^{\mathrm{op}})'$. To prove the equality in *(ii)*, we need a trick (see [Brown and Ozawa, 2008, Theorem 6.1.4]). □

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