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# Generalized $\varphi$-pullback attractors for evolution processes and application to a nonautonomous wave equation

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Orientador: Prof. Matheus Cheque Bortolan, Dr.

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Carlos Pecorari Neto[1]

**Generalized $\varphi$-pullback attractors for evolution processes and application to a nonautonomous wave equation**

O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

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Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de doutor em Matemática.

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To my dear parents.

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*"An equation means nothing to me unless it expresses a thought of God."*
*(Ramanujan)*

# RESUMO

Neste trabalho, definimos os *$\varphi$-atratores pullback generalizados* para processos de evolução em espaços métricos completos, que são famílias compactas e positivamente invariantes, com uma *taxa de atração pullback* determinada pelo comportamento de uma função decrescente $\varphi$ que se anula no infinito. Encontramos condições sob as quais um dado processo de evolução possui um $\varphi$-atrator pullback generalizado, tanto nos casos discreto quanto no contínuo. Apresentamos resultados para os casos especiais para obter um atrator pullback polinomial ou exponencial generalizado e aplicamos esses resultados para obter tais objetos para uma classe de equações da onda não-autônomas.



**Palavras-chave:** $\varphi$-atratores pullback generalizados. $\varphi$-pullback $\kappa$-dissipatividade. Equação da onda não-autônoma. Processos de evolução. Atratores pullback.

**MSC2020:** 35B41,35L20,37L25.

## RESUMO EXPANDIDO

### Introdução

Um importante campo de estudo pertencente à área de Sistemas Dinâmicos é aquele dedicado às equações e sistemas não-autônomos, que são de fundamental importância na compreensão de problemas reais nas áreas de Biologia, Física, Engenharias, entre outras, já que naturalmente os termos independentes que aparecem na modelagem são forças que dependem do tempo. Nas últimas décadas muitos pesquisadores têm dedicado esforços ao estudo de problemas dessa área e obtido avanços significativos. Como exemplo, destacamos os trabalhos de (SELL, 1967a, 1967b; KLOEDEN; RASMUSSEN, 2011; CARABALLO; ŁUKASZEWICZ; REAL, 2006; CARVALHO; LANGA; ROBINSON, 2013; BORTOLAN; CARVALHO; LANGA, 2014).

Uma abordagem natural sobre esse tipo de problema passa pelo entendimento de seu comportamento assintótico, cujos objetos principais são os atratores. Mais especificamente, quando nos referimos aos processos de evolução associados aos problemas não-autônomos, uma dessas principais estruturas é o atrator pullback. O problema que aqui enfrentamos é o seguinte: supondo que o atrator pullback exista, não há informação qualitativa sobre sua respectiva taxa de atração. Neste sentido, vários autores têm trabalhado recentemente com a noção de atrator exponencial pullback, que é uma família compacta, positivamente invariante, que atrai exponencialmente limitados do espaço de fase (no sentido pullback), e que possui dimensão fractal uniformemente limitada. Aqui ressaltamos o importante trabalho de (CARVALHO; SONNER, 2013, 2014).

Por outro lado, trabalhos dedicados a problemas autônomos (através da estrutura de semigrupos) têm sido realizados no sentido de explorar os conjuntos $\varphi$-atraentes, que são conjuntos compactos, positivamente invariantes com relação ao semigrupo e que contém o atrator global, porém com taxa de atração controlada por uma função $\varphi$ que não necessariamente é a função exponencial, mas que satisfaz algumas condições adequadas. Essa ideia foi inicialmente introduzida em 2022 por Zhao, Zhong e Zhu no trabalho (ZHAO; ZHONG; ZHU, 2022).

Em outro trabalho recente, a saber (YAN et al., 2023), os autores estudaram uma versão dessa teoria de $\varphi$-atração para problemas não-autônomos por meio das soluções de Shatah-Struwe e com foco no atrator uniforme. Esta tese também busca ampliar a teoria de $\varphi$-atração ao estudo de problemas não-autônomos (via processos de evolução), porém aqui nosso o foco reside na teoria *pullback*, conforme detalhado a seguir.

### Objetivos

Inspirados pelos trabalhos há pouco citados, e combinando seus respectivos resultados com aqueles presentes em (ZHANG et al., 2017), nosso objetivo é expandir essa teoria para o estudo do comportamento assintótico de problemas não-autônomos. O ponto de partida reside na definição dos $\varphi$-atratores pullback generalizados, nosso principal objeto de estudo deste trabalho.

Buscamos estudar quais condições estes processos devem satisfazer para que possamos garantir a existência de tais atratores generalizados e também queremos entender a relação entre esse novo conceito de atração e o tão conhecido e estudado atrator *pullback*. Por fim, a ideia é aplicar essa nova teoria a uma equação da onda não-autônoma específica.

## Metodologia

A metodologia empregada nesta pesquisa é a usual para a área de Matemática: leitura e estudo intensivos de trabalhos e artigos científicos como estratégia na identificação de lacunas e problemas em aberto. Tais trabalhos serviram como inspiração para essa adaptação/generalização do conhecimento científico já existente às novas abordagens mais amplas ou complexas presentes nesta tese.

Parte do trabalho foi desenvolvida durante intercâmbio na Universidade de Sevilha, Espanha, sob financiamento do programa CAPES-Print, o que permitiu o contato com vários pesquisadores de excelência na área e com excelente infraestrutura de pesquisa.

## Resultados e Discussão

Como já comentado, o objeto central desta tese é o $\varphi$-atrator pullback generalizado, que é uma família $\hat{M} = \{M_t\}_{t\in\mathbb{T}}$ de subconjuntos do espaço de fase $X$ do processo de evolução $S$ associado ao problema não-autônomo ($\mathbb{T}$ denota $\mathbb{R}$ ou $\mathbb{Z}$) satisfazendo:

- $M_t$ é compacto para todo $t \in \mathbb{T}$;
- $S(t,s)M_s \subset M_t$ para todos $t,s \in \mathbb{T}$ com $t \geqslant s$ (invariância positiva);
- $\hat{M}$ é $\varphi$-pullback atraente, isto é, existe uma constante $\omega > 0$ tal que para todo conjunto limitado $D \subset X$ e $t \in \mathbb{T}$, existem $C = C(D,t) \geqslant 0$ e $\tau_0 = \tau_0(D,t) \geqslant 0$ tais que

  $$d_H(S(t,t-\tau)D,M_t) \leqslant C\varphi(\omega\tau) \quad \text{para todo } \tau \geqslant \tau_0,$$

  onde $\varphi$ é uma função decrescente, que se anula no infinito (e sujeita à uma condição adicional que será explicada em mais detalhes posteriormente).

Quando existe uma constante $c > 0$ tal que a dimensão fractal de $M_t$ é menor ou igual a $c$ para todo $t \in \mathbb{R}$, dizemos que $\hat{M}$ é um $\varphi$-atrator pullback, mas este objeto específico não será foco de estudo deste trabalho.

Como resposta a um dos objetivos estipulados para este trabalho, obtivemos resultados de existência de tais atratores generalizados. Mais especificamente, inspirados por (CARVALHO; SONNER, 2013, 2014), dividimos nosso estudo em duas partes: primeiramente estabelecemos um resultado focado no *caso discreto* (veja Teorema 2.6) e, em seguida, e como consequência do primeiro, enunciamos e provamos o resultado para o *caso contínuo* (veja Teorema 2.8).

Também inspirados nos resultados teóricos de (ZHAO; ZHONG; YAN, 2022), apresentamos resultados especificamente desenhados para assegurar a existência de um *atrator polinomial pullback generalizado* ou de um *atrator exponencial pullback generalizado* para um processo de evolução contínuo (veja Teoremas 2.10 e 2.11).

Provamos ainda que quando existe um $\varphi$-atrator pullback generalizado $\hat{M} = \{M_t\}_{t\in\mathbb{T}}$ para um processo de evolução $S$ tal que $\bigcup_{s\leqslant t} M_s$ é limitado para todo $t \in \mathbb{T}$, então $S$ também admite um atrator pullback $\hat{A}$, com $\hat{A} \subset \hat{M}$ (veja Teorema 2.12).

Finalmente, como nosso principal resultado (veja Teorema 3.1), também inspirados por (ZHAO; ZHONG; YAN, 2022), aplicamos essa teoria para provar a existência de um *$\varphi$-atrator pullback generalizado* para uma classe de equações da onda não-autônomas dada por:

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + k(t)\|u_t(t,\cdot)\|^p_{L^2(\Omega)} u_t(t,x) + f(t,u(t,x)) \\ \qquad\qquad = \displaystyle\int_\Omega K(x,y)u_t(t,y)dy + h(x), (t,x) \in [s,\infty) \times \Omega, \\ u(t,x) = 0, (t,x) \in [s,\infty) \times \partial\Omega, \\ u(s,x) = u_0(x),\ u_t(s,x) = u_1(x), x \in \Omega, \end{cases}$$

onde $\Omega \subset \mathbb{R}^3$ é um domínio limitado com fronteira suave $\partial\Omega$, em que $\varphi$ é uma função com decaimento polinomial quando $p > 0$ e com decaimento exponencial quando $p = 0$.

## Considerações Finais

Os resultados esperados foram obtidos nesta pesquisa e, juntamente com aqueles obtidos em (YAN et al., 2023), representam um ponto de partida para uma vasta teoria que pode surgir no estudo de comportamento assintótico de problemas não-autônomos. Um sequência natural ao trabalho já realizado seria estudar os *$\varphi$-atratores pullback*, o que exigiria um trabalho dedicado à dimensão fractal das famílias atraentes. Outra possível linha de pesquisa seria a generalização dessa teoria e o desenvolvimento de aplicações para problemas em espaços de fase tempo-dependentes.

Gostaríamos ainda de ressaltar que dois artigos surgiram em decorrência deste trabalho (ainda em vias de publicação até o momento da finalização desta tese), cujos *preprints* podem ser obtidos em

- M.C. Bortolan, T. Caraballo, and C. Pecorari Neto. Generalized $\varphi$-pullback attractors for evolution processes and application to a nonautonomous wave equation. arXiv, 2023. `https://doi.org/10.48550/arXiv.2311.15630`.
- M.C. Bortolan, T. Caraballo, and C. Pecorari Neto. Generalized exponential pullback attractor for a nonautonomous wave equation. arXiv, 2024. `https://doi.org/10.48550/arXiv.2401.06631`.

**Palavras-chave:** $\varphi$-atratores pullback generalizados. $\varphi$-pullback $\kappa$-dissipatividade. Equação da onda não-autônoma. Processos de evolução. Atratores pullback.

**MSC2020:** 35B41,35L20,37L25.

**ABSTRACT**

In this work we define the *generalized $\varphi$-pullback attractors* for evolution processes in complete metric spaces, which are compact and positively invariant families, with *rate of pullback attraction* determined by the behavior of a decreasing function $\varphi$ that vanishes at infinity. We find conditions under which a given evolution process has a generalized $\varphi$-pullback attractor, both in the discrete and in the continuous cases. We present results for the special cases to obtain a generalized polynomial or exponential pullback attractors, and apply these results to obtain such objects for a class of nonautonomous wave equations.

**Keywords**: Generalized $\varphi$-pullback attractors. $\varphi$-pullback $\kappa$-dissipativity. Nonautonomous wave equation. Evolution processes. Pullback attractors.

**MSC2020:** 35B41,35L20,37L25.

# LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{N}$ | the set of positive integers $\{1,2,3,\ldots\}$ |
| $\mathbb{R}$ | the field of real numbers |
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{T}$ | either $\mathbb{R}$ or $\mathbb{Z}$ |
| $\mathbb{T}^+$ | the set $\{t \in \mathbb{T} \colon t \geqslant 0\}$ |
| $\mathcal{P}$ | the set $\{(t,s) \in \mathbb{T}^2 \colon t \geqslant s\}$ |
| $\mathfrak{F}$ | the class of all families $\hat{D} = \{D_t\}_{t\in\mathbb{T}}$ of $X$ |
| $d_H(\cdot,\cdot)$ | Hausdorff semidistance |
| $\mathrm{diam}(\cdot)$ | diameter of a nonempty set |
| $B_r(x_0)$ or $B_r^X(x_0)$ | open ball of radius $r$ centered in $x_0$ in $X$ |
| $\overline{B}_r(x_0)$ or $\overline{B}_r^X(x_0)$ | closed ball of radius $r$ centered in $x_0$ in $X$ |
| $\kappa(\cdot)$ | Kuratowski measure of non-compactness |
| $\beta(\cdot)$ | ball measure of non-compactness |
| $\mathcal{O}_r(\cdot)$ | $r$-neighborhood of a nonempty set |
| $\omega(\hat{D},t)$ | pullback $\omega$-limit set of a family $\hat{D}$ at time $t$ |
| $\omega(\hat{D})$ | pullback $\omega$-limit of the family $\hat{D}$ |
| $\overline{B}_r^Z$ | closed ball in $Z$ of radius $r$ centered at 0 |
| $\hookrightarrow$ | continuous inclusion |
| $\hookrightarrow\hookrightarrow$ | compact inclusion |
| $\overset{*}{\rightharpoonup}$ | weak$^*$ convergence |

# CONTENTS

# 1 INTRODUCTION

The study of nonautonomous problems has been the focus of many researchers in the last decades, such as (SELL, 1967a, 1967b; KLOEDEN; RASMUSSEN, 2011; CARABALLO; ŁUKASZEWICZ; REAL, 2006; CARVALHO; LANGA; ROBINSON, 2013; BORTOLAN; CARVALHO; LANGA, 2014), for example. Nonautonomous equations and systems are of fundamental importance to model and understand real problems in Chemistry, Biology, Physics, Economics and many other areas since, naturally, the independent terms that usually appear in the models are time-dependent forces.

More specifically, we deal with the asymptotic behavior of such problems and, for that, the key objects are the attracting sets and families. The well recognized existing literature deals with attractors that, in general terms, represent the sets of limit states of the solutions of the nonautonomous equations, and also contain all the bounded solutions defined for all time, in other words, the attractors are the objects that contain the relevant solutions, bearing in mind real world problems.

To be more precise about our goals in this work, we begin by presenting an overview of the theory of evolution processes and their pullback attractors in metric spaces. In what follows we write $\mathbb{T}$ to denote either the set of the real numbers ($\mathbb{R}$) or the integers ($\mathbb{Z}$). We will also denote $\mathbb{T}^+ := \{t \in \mathbb{T} \colon t \geqslant 0\}$.

**Definition 1.1** (Evolution process)**.** By setting $\mathcal{P} = \{(t,s) \in \mathbb{T}^2 \colon t \geqslant s\}$ and considering $(X,d)$ a metric space, we say that a two-parameter family $S = \{S(t,s) \colon (t,s) \in \mathcal{P}\}$ of continuous maps from $X$ into itself is an **evolution process** if

- $S(t,t)x = x$ for all $x \in X$ and $t \in \mathbb{T}$;
- $S(t,r)S(r,s) = S(t,s)$ for all $(t,r), (r,s) \in \mathcal{P}$, that is, $t,r,s \in \mathbb{T}$ and $t \geqslant r \geqslant s$;
- the map $\mathcal{P} \times X \ni (t,s,x) \mapsto S(t,s)x \in X$ is continuous.

The evolution process $S$ is said to be a **continuous**[1] **evolution process** if $\mathbb{T} = \mathbb{R}$ and a **discrete evolution process** if $\mathbb{T} = \mathbb{Z}$.

The framework of evolution processes is what we call the *nonautonomous framework*, due to the explicit dependence of both initial and final times $s$ and $t$, respectively. When there exists a family $T = \{T(t) \colon t \in \mathbb{T}^+\}$ such that $S(t,s) = T(t-s)$ for all $(t,s) \in \mathcal{P}$, we say that $T$ is a **semigroup** and that $S$ is an *autonomous evolution process*. This indicates that $S$ depends on the elapsed time $t-s$ and not explicitly on $t$ and $s$ separately.

To mathematically introduce the framework of the *pullback attraction theory*, we let $\mathfrak{F}$ be the class of all families $\hat{D} = \{D_t\}_{t\in\mathbb{T}}$, where $D_t$ is a nonempty subset of $X$ for each $t \in \mathbb{T}$. A family $\hat{A} \in \mathfrak{F}$ is said to be **closed/compact** if $A_t$ is a closed/compact

[1] We point out that all evolution processes have continuity properties, by definition. The name *continuous evolution process* refers only to the time parameter being taken in $\mathbb{R}$, to separate them from their discrete counterpart.

subset of $X$ for each $t \in \mathbb{T}$. Moreover, given $\hat{A},\hat{B} \in \mathfrak{F}$ we say that $\hat{A} \subset \hat{B}$ if $A_t \subset B_t$ for each $t \in \mathbb{T}$.

**Definition 1.2** (Pullback attraction)**.** For an evolution process $S$ in $X$, we say that a family $\hat{B} \in \mathfrak{F}$ is **pullback attracting**[1] if for each bounded set $D \subset X$ and $t \in \mathbb{T}$ we have

$$\lim_{s\to-\infty} d_H(S(t,s)D, B_t) = 0,$$

where

$$d_H(U,V) = \sup_{u\in U} \inf_{v\in V} d(u,v)$$

denotes the **Hausdorff semidistance** between two nonempty subsets $U$ and $V$ of $X$.

Consider a given evolution process $S$ in $X$. We say that a family $\hat{B}$ is **invariant/positively invariant** for $S$ if for all $(t,s) \in \mathcal{P}$ we have $S(t,s)B_s = B_t$ / $S(t,s)B_s \subset B_t$, respectively. With these definitions, we can present the notion of a *pullback attractor*.

**Definition 1.3** (Pullback attractor)**.** For an evolution process $S$ in a metric space $X$, we say that $\hat{A} \in \mathfrak{F}$ is a **pullback attractor** for $S$ if:

(i) $\hat{A}$ is compact;

(ii) $\hat{A}$ is invariant for $S$;

(iii) $\hat{A}$ is a pullback attracting family;

(iv) $\hat{A}$ is the minimal closed family satisfying (iii), that is, if $\hat{C} \in \mathfrak{F}$ is a pullback attracting closed family, then $\hat{A} \subset \hat{C}$.

The minimality condition (iv) is there to ensure that when a pullback attractor for $S$ exists, it is unique. This condition could be replaced by a number of other properties such as, for instance, that for each $t \in \mathbb{T}$ the set $\cup_{s\leqslant t} A_s$ is bounded in $X$. When a family satisfies this property, we say that it is a **backwards bounded family**.

The framework of pullback attractors is perhaps the most usual form of generalizing the theory of global attractors for semigroups, to describe the asymptotic behavior of nonautonomous problems. For a detailed study of this theory, we refer to (CARVALHO; LANGA; ROBINSON, 2013).

The problem that we face is the following: assuming that a pullback attractor $\hat{A}$ exists, there is no qualitative information regarding the *rate of attraction* of $\hat{A}$. To that end, many authors (for instance, (CARVALHO; SONNER, 2013, 2014)) have worked with the notion of a *pullback exponential attractor*, that is, a compact family $\hat{M} \in \mathfrak{F}$ which is positively invariant for $S$ and *exponentially pullback attracts* bounded subsets of $X$, that is, there exists a positive constant $\omega > 0$ for which for all $D \subset X$ bounded we have

$$\lim_{s\to-\infty} e^{\omega s} d_H(S(t,s)D,M_t) = 0,$$

[1] The term *pullback attraction* emphasizes that we are fixing a final time $t$ and taking the initial time $s \to -\infty$. Hence, we are *pulling back* to the present time the solutions which are starting further in the past.

and, furthermore, the *fractal dimension* of $M_t$ is uniformly bounded for $t \in \mathbb{T}$, that is, there exists $c_0 \geqslant 0$ such that for all $t \in \mathbb{T}$ we have

$$\dim_F(M_t) := \lim_{r \to 0^+} \frac{\ln N(r,M_t)}{\ln \frac{1}{r}} \leqslant c_0,$$

where $N(r,M_t)$ denotes the minimum number of balls of radius $r > 0$ in $X$ that cover $M_t$. The fractal dimension is an upper bound to both the *Hausdorff dimension* and the *topological dimension* of $K$. One of its main use is the following: if $\dim_F(K)$ is finite, then $K$ can be projected injectively in a finite dimensional space, with dimension larger than $2\dim_F(K) + 1$ (see (MAÑÉ, 1981)).

In the work done by Zhao, Zhong and Zhu in (ZHAO; ZHONG; ZHU, 2022), the authors introduce the concept of *compact $\varphi$-attracting sets* in the autonomous framework (that is, for semigroups), which is a fixed set $M$ that contains the global attractor of the system and attracts bounded sets with rate of attraction given by a function $\varphi$, which is not necessarily the exponential function. More recently, in (YAN et al., 2023), the authors studied a nonautonomous version of this theory, using the Shatah–Struwe solutions, but focusing their efforts in the *uniform attractor*, which is a different framework than the one of pullback attractors.

Inspired by those works, and combining the results with the ones of (ZHANG et al., 2017), our goal is to expand this theory for the nonautonomous case and understand the asymptotic behavior of nonautonomous problems (by means of evolution processes) defining the *generalized $\varphi$-pullback attractors*, which we present in what follows. To begin, we will define the functions that will determine the *decay rate* of the pullback attraction, and we chose to name them *decay functions*.

**Definition 1.4** (Decay function)**.** We say that a function $\varphi\colon [k,\infty) \to [0,\infty)$, where $k \geqslant 0$ is an appropriate constant, is a **decay function** if $\varphi$ is decreasing, $\lim\limits_{t\to\infty} \varphi(t) = 0$ and

$$\limsup_{t\to\infty} \frac{\varphi(\omega t + \eta)}{\varphi(\omega t)} < \infty \quad \text{for every } \omega > 0 \text{ and } \eta \in \mathbb{R}.$$

With that, given an evolution process $S$ in a metric space $X$, we can define our main object of study.

**Definition 1.5** (Generalized $\varphi$-pullback attractor)**.** We say that a family $\hat{M} \in \mathfrak{F}$ is a **generalized $\varphi$-pullback attractor** for $S$ if $\hat{M}$ is compact, positively invariant and **$\varphi$-pullback attracting**, that is, there exists a constant $\omega > 0$ such that for *every* bounded set $D \subset X$ and $t \in \mathbb{T}$ there exist $C = C(D,t) \geqslant 0$ and $\tau_0 = \tau_0(D,t) \geqslant 0$ such that

$$d_H(S(t,t-\tau)D,M_t) \leqslant C\varphi(\omega\tau) \quad \text{for all } \tau \geqslant \tau_0.$$

When there exists a constant $c > 0$ such that

$$\dim_F(M_t) \leqslant c \quad \text{for all } t \in \mathbb{R},$$

we say that $\hat{M}$ is a **$\varphi$-pullback attractor**.

Note that the parameter $\omega > 0$ is fixed, and works for all bounded subsets $D$ of $X$ and $t \in \mathbb{T}$, thus, the function $\varphi(\omega\cdot)$ is the one that controls the rate of pullback attraction of the family $\hat{M}$. Although we introduce the notions of generalized $\varphi$-pullback attractors and $\varphi$-pullback attractors, in this work we will focus only on the first concept. Whether their fractal dimension are uniformly bounded or not is an open question, and a future line of work.

In Chapter 2, we study what are the conditions these processes must satisfy to ensure the existence of such generalized attractors. Inspired by (CARVALHO; SONNER, 2013, 2014), we divide our results in the *discrete case* (see Theorem 2.6) and the *continuous case* (see Theorem 2.8). Also, inspired by the theoretical work presented in (ZHAO; ZHONG; YAN, 2022), we present results specifically designed to prove the existence of a generalized *polynomial* or *exponential* pullback attractor for a continuous evolution process (see Theorems 2.10 and 2.11). As expected from the usual theory of exponential pullback attractors, we prove that when a backwards bounded generalized $\varphi$-pullback attractor $\hat{M}$ for an evolution process $S$ exists, then $S$ also has a pullback attractor $\hat{A}$, with $\hat{A} \subset \hat{M}$ (see Theorem 2.12).

As our main result (see Theorem 3.1 in Chapter 3), also inspired by (ZHAO; ZHONG; YAN, 2022), we apply the abstract theory of Chapter 2 to prove the existence of a generalized $\varphi$-pullback attractor for a class of nonautonomous wave equations given by:

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + k(t)\|u_t(t,\cdot)\|^p_{L^2(\Omega)} u_t(t,x) + f(t,u(t,x)) \\ \qquad\qquad = \displaystyle\int_\Omega K(x,y)u_t(t,y)dy + h(x), (t,x) \in [s,\infty) \times \Omega, \\ u(t,x) = 0, (t,x) \in [s,\infty) \times \partial\Omega, \\ u(s,x) = u_0(x),\ u_t(s,x) = u_1(x), x \in \Omega, \end{cases} \tag{NWE}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, where $\varphi$ is a function with polynomial decay, namely $\varphi(t) = t^{-\frac{1}{p}}$, when $p > 0$, and with exponential decay when $p = 0$.

In order to maintain clarity and not impact the main scope of this work, certain technical results have been placed in Appendix A and will be referenced as needed throughout the text.

## 2 GENERALIZED $\varphi$-PULLBACK ATTRACTORS

Now we generalize the theory presented in (ZHAO; ZHONG; ZHU, 2022) to the nonautonomous pullback framework, in a useful manner. At first sight, one might want to work with a map $\varphi(t,s)$ depending on **both** variables $t$ and $s$. This would imply that for each final time $t$ the map $\varphi(t,s)$ describes the decay of the solution as $s \to -\infty$. Since the behavior of $\varphi(t,s)$ can be different for distinct values of $t$ (for instance, it could be polynomial for one parameter $t$ and exponential for another), there would be no correct answer to the question: what is the rate of attraction? The answer would depend heavily on the final time $t$, and even if we could do that, it would not be practical for the applications. Thus, as defined in the introduction, we will work with single-variable functions, which we call *decay functions*. To recall the reader (see Definition 1.4), a function $\varphi\colon [k,\infty) \to [0,\infty)$, where $k \geqslant 0$ is an appropriate constant, is called a decay function if $\varphi$ is decreasing, $\lim_{t\to\infty} \varphi(t) = 0$ and

$$\limsup_{t\to\infty} \frac{\varphi(\omega t + \eta)}{\varphi(\omega t)} < \infty \quad \text{for every } \omega > 0 \text{ and } \eta \in \mathbb{R}. \tag{2.1}$$

Examples of decay functions $\varphi$ are

$$\varphi(t) = ce^{-\beta t}, \quad \varphi(t) = ct^{-\beta}, \quad \text{and} \quad \varphi(t) = c\ln^{-\beta}(t),$$

with $c, \beta$ positive constants. Indeed, they are all decreasing and tend to zero at infinity. For (2.1) we note that

$$\lim_{t\to\infty} \frac{ce^{-\beta(\omega t+\eta)}}{ce^{-\beta(\omega t)}} = e^{-\beta\eta}, \quad \lim_{t\to\infty} \frac{c(\omega t+\eta)^{-\beta}}{c(\omega t)^{-\beta}} = 1 \text{ and } \lim_{t\to\infty} \frac{c\ln^{-\beta}(\omega t+\eta)}{c\ln^{-\beta}(\omega t)} = 1.$$

Condition (2.1) always occurs for a decreasing function $\varphi$ when $\eta \geqslant 0$, but that is not always the case when $\eta < 0$. Indeed, the function $\varphi(t) = t^{-t}$, although decreasing and 0 at infinity, does not satisfy (2.1) for $\eta < 0$, since

$$\lim_{t\to\infty} \frac{(\omega t+\eta)^{-\omega t-\eta}}{(\omega t)^{-\omega t}} = \lim_{t\to\infty} \left(\frac{\omega t}{\omega t+\eta}\right)^{\omega t} (\omega t+\eta)^{-\eta} = \infty.$$

We added (2.1) to avoid having to deal with translations of the function $\varphi$, and thus the rate of attraction will depend only on $\varphi$ and $\omega$, but not on $\eta$. We could have chosen to work without this assumption, but since it is satisfied by the usual decay functions (exponential, polynomial and logarithmic) we have decided to use it. It makes notation a little easier on the eyes.

This definition of a generalized $\varphi$-pullback attractor (see Definition 1.5), that is, a family $\hat{M} \in \mathfrak{F}$ which is compact, positively invariant and $\varphi$-pullback attracts all bounded subsets of $X$, is inspired by the one of an *exponential attractor*, and the word *generalized* is there to emphasize that we are *not* asking hypotheses on the finitude of the fractal dimension. The question of what are the conditions required to obtain the bound for the fractal dimension of such object is still an open problem.

Firstly, we show that the theory of *generalized* $\varphi$-pullback attractors is meaningful only when dealing with problems without properties of compactness on the evolution processes. Recall that we say that a family $\hat{B} \in \mathfrak{F}$ is **backwards bounded** if $\cup_{s\leqslant t} B_s$ is bounded for each $t \in \mathbb{T}$. For a subset $A$ of $X$, we denote by $\overline{A}$ the closure of $A$ in $X$.

**Proposition 2.1.** *Let $X$ be a metric space and $S$ an evolution process in $X$. Assume the following:*

- *$S$ is an **eventually compact** evolution process, that is, there exists $\tau > 0$ such that $S(t,t-\tau)$ is a compact map for each $t \in \mathbb{T}$;*
- *there exists a backwards bounded **pullback absorbing** family $\hat{B}$, that is, given $D \subset X$ bounded and $t \in \mathbb{T}$ there exists $s_0 \leqslant t$ such that $S(t,s)D \subset B_t$ for all $s \leqslant s_0$.*

*Then, given any decay function $\varphi$, $S$ has a generalized $\varphi$-pullback attractor.*

*Proof.* Fix $t \in \mathbb{T}$ and set $B_* := \cup_{s\leqslant t} B_s$, which is bounded by hypotheses. Since $\hat{B}$ is pullback absorbing, there exists $s_0 := s_0(t) \leqslant t$ such that $S(t,s)B_* \subset B_t$ for all $s \leqslant s_0$, which implies, in particular, that

$$S(t,s)B_s \subset B_t \quad \text{for all } s \leqslant s_0.$$

It is clear that if $t_1 \leqslant t_2$ we can choose $s_0(t_1) \leqslant s_0(t_2)$. We define the family $\hat{C}$ by

$$C_t := \bigcup_{s\leqslant s_0(t)} S(t,s)B_s.$$

Clearly $C_t \subset B_t$ and, hence, it is backwards bounded. Also, since for $s \leqslant t$ we have $s_0(s) \leqslant s_0(t)$, we have

$$S(t,s)C_s = \bigcup_{r\leqslant s_0(s)} S(t,r)B_r \subset \bigcup_{r\leqslant s_0(t)} S(t,r)B_r = C_t,$$

which proves that the family $\hat{C}$ is positively invariant. Furthermore, if $D \subset X$ is bounded then for each $s \in \mathbb{T}$ there exists $s_1 = s_1(s) \leqslant s$ such that $S(s,r)D \subset B_s$ for all $r \leqslant s_1$. Thus if $s \leqslant s_1(s_0(t))$ we verify that

$$S(t,s)D = S(t,s_0(t))S(s_0(t),s)D \overset{s\leqslant s_1(s_0(t))}{\subset} S(t,s_0(t))B_{s_0(t)} \subset C_t,$$

which proves that $\hat{C}$ is pullback absorbing.

Now we define the family $\hat{M}$ by

$$M_t = \overline{S(t,t-\tau)C_{t-\tau}} \quad \text{for each } t \in \mathbb{T}.$$

We claim that for any given decay function $\varphi$, $\hat{M}$ is a generalized $\varphi$-pullback attractor for $S$. Firstly we note that, since $C_{t-\tau}$ is bounded and $S(t,t-\tau)$ is compact, $M_t$ is compact. Now for $s \leqslant t$, since $S(t,s)$ is continuous from $X$ into $X$, we have

$$S(t,s)M_s = S(t,s)\overline{S(s,s-\tau)C_{s-\tau}} \subset \overline{S(t,s-\tau)C_{s-\tau}}$$

$$= \overline{S(t,t-\tau)S(t-\tau,s-\tau)C_{s-\tau}} \subset \overline{S(t,t-\tau)C_{t-\tau}} = M_t,$$

hence $\hat{M}$ is positively invariant. Lastly, if $D$ is bounded and $t \in \mathbb{R}$, since $\hat{C}$ is pullback absorbing there exists $s_0 = s_0(t-\tau)$ such that $S(t-\tau,s)D \subset C_{t-\tau}$ for all $s \leqslant s_0$. Thus for $s \leqslant s_0$ we have

$$S(t,s)D = S(t,t-\tau)S(t-\tau,s)D \subset S(t,t-\tau)C_{t-\tau} \subset M_t,$$

which proves that $\hat{M}$ is pullback absorbing and, therefore, it is a generalized $\varphi$-pullback attractor for any given decay function $\varphi$. □

This result shows that for either finite-dimensional problems or problems in infinite-dimensional spaces with compactness properties, the theory of generalized $\varphi$-pullback attractors, as is, is not that difficult, and the existence of a generalized $\varphi$-pullback attractor, for any given decay function $\varphi$, can be achieved by simply showing that $S$ has a backwards bounded pullback absorbing family $\hat{B}^0$. Of course, if one add the hypothesis of finitude of the fractal dimension of the family $\hat{M}$, the situation changes and turns the problem into a more difficult one. For now, as we already mentioned, we will not focus on the issue of the finitude of the fractal dimension.

In the literature, see (ZHANG et al., 2017) for instance, although not always explicit, there is a relationship between the existence of global attractors for semigroups and the decay of the Kuratowski measure of non-compactness for its $\omega$-limits. Hence, the work we present in what follows, inspired by (ZHANG et al., 2017), attempts to unify the results of both (ZHANG et al., 2017) and (ZHAO; ZHONG; ZHU, 2022) for the nonautonomous pullback setting. For what follows, unless clearly stated otherwise:

$(X,d)$ denotes a *complete* metric space.

Recall that for a nonempty bounded subset $C \subset X$, its **diameter** is defined as

$$\operatorname{diam}(C) := \sup_{x,y \in C} d(x,y),$$

and we have $\operatorname{diam}(C) = \operatorname{diam}(\overline{C})$. For $x_0 \in X$ and $r > 0$, the **open ball of radius** $r$ **centered in** $x_0$ will be denoted by

$$B_r(x_0) := \{x \in X : d(x,x_0) < r\},$$

and the **closed ball of radius** $r$ **centered in** $x_0$ will be denoted by

$$\overline{B}_r(x_0) := \{x \in X : d(x,x_0) \leqslant r\}.$$

When there is a need to highlight the space $X$ in which the balls are being considered, we will use the notation $B_r^X(x_0)$ for open balls in $X$ and $\overline{B}_r^X(x_0)$ for closed balls in $X$.

**Definition 2.2** (Kuratowski and ball measures of non-compactness)**.** For a bounded set $B \subset X$ we define its **Kuratowski measure of non-compactness** by

$$\kappa(B) = \inf\{\delta > 0 \colon B \text{ admits a finite cover by sets of diameter} \\ \text{less than or equal to } \delta\}.$$

In relation to the Kuratowski measure, we have the **ball measure of non-compactness**, defined by

$$\beta(B) = \inf\{r > 0 \colon B \text{ admits a finite cover by open balls of radius } r\}.$$

In order to keep the objectivity of our work, the main properties and results regarding the Kuratowski and the ball measures of non-compactness are presented in the Appendix A.

**Remark 2.3.** We could replace the "less than or equal to" in the definition of $\kappa$ with "less than" without changing its values, and for $\beta$ we could choose the covers consisting by either open or closed balls (see Proposition A.1).

**Definition 2.4** ($\varphi$-pullback $\kappa$-dissipativity)**.** We say that an evolution process $S$ in $X$ is **$\varphi$-pullback $\kappa$-dissipative** if there exists $\omega > 0$ such that for every bounded $D \subset X$ and $t \in \mathbb{T}$ there exists $C \geqslant 0$ and $\tau_0 \geqslant 0$ such that

$$\kappa\Big(\bigcup_{\sigma \geqslant \tau} S(t,t-\sigma)D\Big) \leqslant C\varphi(\omega\tau) \quad \text{for all } \tau \geqslant \tau_0.$$

As mentioned in the introduction, inspired by (CARVALHO; SONNER, 2013, 2014), we first construct generalized $\varphi$-pullback attractors in the *discrete case*, that is, when $S = \{S(n,m) \colon n \geqslant m \in \mathbb{Z}\}$ is a discrete evolution process in a complete metric space $X$. After that, we use the results of the discrete case to prove the existence of generalized $\varphi$-pullback attractors for the continuous case.

## 2.1 THE DISCRETE CASE

When working in the construction of a generalized $\varphi$-pullback attractor, it is easy to realize that we need the existence of a pullback absorbing family. A deeper study, when proving several results, shows the pullback absorption might not be enough, and we require a stronger absorption property, which we define below.

**Definition 2.5** (Uniform pullback absorption)**.** Let $S$ be an evolution process in a complete metric space $X$. We say that $\hat{B} \in \mathfrak{F}$ is **uniformly pullback absorbing** if given $D \subset X$ bounded and $t \in \mathbb{T}$, there exists $T > 0$ such that $S(s,s-r)D \subset B_s$ for all $s \leqslant t$ and $r \geqslant T$.

**Theorem 2.6** (Existence of generalized $\varphi$-pullback attractors for discrete evolution processes)**.** *Assume that there exists a closed and backwards bounded family $\hat{B} = \{B_k\}_{k\in\mathbb{Z}}$, which is uniformly pullback absorbing and positively invariant for the process $S$, and that, for a given decay function $\varphi$, $S$ is $\varphi$-pullback $\kappa$-dissipative. Then there exists a generalized $\varphi$-pullback attractor $\hat{M}$ for $S$, with $\hat{M} \subset \hat{B}$.*

*Proof.* Let $\hat{B}$ as in the hypothesis. Since $S$ is $\varphi$-pullback $\kappa$-dissipative, there exists $\omega > 0$ such that given $k \in \mathbb{Z}$ there exist $C \geqslant 0$ and an integer $m_0 \geqslant 1$ where

$$\kappa\Big(\bigcup_{n\geqslant m} S(k,k-n)B_{k-n}\Big) < C\varphi(\omega m) \text{ for all } m \geqslant m_0.$$

For $k \in \mathbb{Z}$ fixed

$$\begin{aligned}\beta\left(S(k,k-m_0)B_{k-m_0}\right) &\leqslant \kappa\left(S(k,k-m_0)B_{k-m_0}\right)\\ &\leqslant \kappa\Big(\bigcup_{n\geqslant m_0} S(k,k-n)B_{k-n}\Big) < C\varphi(\omega m_0),\end{aligned}$$

which means $S(k,k-m_0)B_{k-m_0}$ can be covered by a finite number of balls of radius $C\varphi(\omega m_0)$. Thus, there exist points $x_i^{(m_0)} \in X$ for $i = 1,\dots,r(m_0)$ such that

$$S(k,k-m_0)B_{k-m_0} \subset \bigcup_{i=1}^{r(m_0)} B_{C\varphi(\omega m_0)}(x_i^{(m_0)})$$

and, consequently, there exists $y_i^{(m_0)} \in B_{k-m_0}$ for $i = 1,\dots,r(m_0)$ such that

$$S(k,k-m_0)B_{k-m_0} \subset \bigcup_{i=1}^{r(m_0)} B_{2C\varphi(\omega m_0)}(z_i^{(m_0)}),$$

where $z_i^{(m_0)} = S(k,k-m_0)y_i^{(m_0)}$. Analogously, there exists $y_i^{(m_0+1)} \in B_{k-(m_0+1)}$, for $i = 1,\dots,r(m_0+1)$, in such a manner that

$$S(k,k-(m_0+1))B_{k-(m_0+1)} \subset \bigcup_{i=1}^{r(m_0+1)} B_{2C\varphi(\omega(m_0+1))}(z_i^{(m_0+1)}),$$

where $z_i^{(m_0+1)} = S(k,k-(m_0+1))y_i^{(m_0+1)}$, and so on.

Now, for each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define

$$\mathcal{J}_k^n = \Big\{S(k,k-(m_0+n))y_i^{(m_0+n)} \colon i = 1,\dots,r(m_0+n)\Big\},$$

recalling that the number $m_0$ and the points $y_j$ depend on the value of $k$. We also define $\mathcal{K}_k^0 = \mathcal{J}_k^0$ and $\mathcal{K}_k^n = \mathcal{J}_k^n \cup S(k,k-1)\mathcal{K}_{k-1}^{n-1}$ for each $k \in \mathbb{Z}, n \in \mathbb{N}, n \geqslant 1$. These sets satisfy, for each $k \in \mathbb{Z}$ and $n,p \in \mathbb{N}$, the following properties (which we will later prove - see Proposition 2.7):

(i) $\mathcal{J}_k^n \subset S(k,k-(m_0+n))B_{k-(m_0+n)} \subset B_k$,

(ii) $S(k,k-(m_0+n))B_{k-(m_0+n)} \subset \bigcup_{z\in\mathcal{J}_k^n} B_{2C\varphi(\omega(m_0+n))}(z)$,

(iii) $S(k,k-1)\mathcal{K}_{k-1}^{n} \subset \mathcal{K}_{k}^{n+1}$,

(iv) $S(k,k-(m_0+n))B_{k-(m_0+n)} \subset \bigcup_{z\in\mathcal{K}_k^n} B_{2C\varphi(\omega(m_0+n))}(z)$,

(v) $\mathcal{K}_k^n \subset S(k,k-n)B_{k-n} \subset B_k$,

(vi) $S(k+p,k)\mathcal{K}_k^n \subset \mathcal{K}_{k+p}^{n+p}$,

(vii) $\mathcal{K}_k^n \subset S(k,k-m)B_{k-m}$ for each $m \in \mathbb{N}$ e $n \geqslant m$.

Lastly, we define the family $\hat{E} = \{E_k\}_{k\in\mathbb{Z}}$ by

$$E_k = \bigcup_{n=0}^{\infty} \mathcal{K}_k^n \quad \text{for each integer } k \in \mathbb{Z}.$$

Now we prove that $\hat{E}$ is precompact, positively invariant, and $\varphi$-pullback attracting. For $m \in \mathbb{N}$, using (vii) it follows that $\bigcup_{n=m+1}^{\infty} \mathcal{K}_k^n \subset S(k,k-m)B_{k-m}$. Thus,

$$E_k = \bigcup_{n=0}^{\infty} \mathcal{K}_k^n = \bigcup_{n=0}^{m} \mathcal{K}_k^n \ \cup \bigcup_{n=m+1}^{\infty} \mathcal{K}_k^n \subset \left(\bigcup_{n=0}^{m} \mathcal{K}_k^n\right) \cup S(k,k-m)B_{k-m}.$$

If $m \geqslant m_0$ we obtain

$$\begin{aligned}\kappa(E_k) &\leqslant \kappa\left(\left(\bigcup_{n=0}^{m} \mathcal{K}_k^n\right) \cup S(k,k-m)B_{k-m}\right) \leqslant \max\left\{\kappa\left(\bigcup_{n=0}^{m} \mathcal{K}_k^n\right),\ \kappa(S(k,k-m)B_{k-m})\right\} \\ &= \kappa(S(k,k-m)B_{k-m}) \leqslant \kappa\left(\bigcup_{n\geqslant m} S(k,k-n)B_{k-n}\right) < C\varphi(\omega m),\end{aligned}$$

and making $m \to \infty$ we obtain $\kappa(E_k) = 0$, which means that $E_k$ is precompact.

The positive invariance follows directly from (vi), since for $k \in \mathbb{Z}$ and $p \in \mathbb{N}$, we have

$$S(k+p,k)E_k = S(k+p,k)\bigcup_{n=0}^{\infty} \mathcal{K}_k^n = \bigcup_{n=0}^{\infty} S(k+p,k)\mathcal{K}_k^n \subset \bigcup_{n=0}^{\infty} \mathcal{K}_{k+p}^{n+p} \subset \bigcup_{n=0}^{\infty} \mathcal{K}_{k+p}^{n} = E_{k+p}.$$

For the $\varphi$-pullback attraction, fix $D \subset X$ bounded and $k \in \mathbb{Z}$. From the hypothesis there exists an integer $q \geqslant 1$ such that $S(r,r-n)D \subset B_r$ for all $r \leqslant k$ and $n \geqslant q$. If $n > q+m_0$ there exist $a,b \in \mathbb{N}$ such that $n = a+q$, $n = b+m_0$, $a > m_0$, $b > q$ and $a = m_0+b-q$. Now we have

$$\begin{aligned}d_H(S(k,k-n)D, E_k) &= d_H(S(k,k-a-q)D,E_k) \\ &= d_H\left(S(k,k-a)S(k-a,k-a-q)D, \bigcup_{j=0}^{\infty} \mathcal{K}_k^j\right) \\ &\leqslant d_H\left(S(k,k-a)B_{k-a}, \bigcup_{j=0}^{\infty} \mathcal{K}_k^j\right) \leqslant d_H\left(S(k,k-a)B_{k-a},\ \mathcal{K}_k^{b-q}\right) \\ &\leqslant 2C\varphi(\omega(m_0+b-q)) = 2C\varphi(\omega(n-q)) = 2C\varphi(\omega n - \omega q),\end{aligned}$$

where the last inequality follows from (iv), since

$$S(k,k-a)B_{k-a} = S(k,k-(m_0+(b-q)))B_{k-(m_0+(b-q))}$$

$$\subset \bigcup_{z\in\mathcal{K}_k^{b-q}} B_{2C\varphi(\omega(m_0+b-q))}(z).$$

By the conditions imposed on $\varphi$ , there exists $C_1 > 0$ such that

$$d_H(S(k,k-n)D, E_k) \leqslant 2C\varphi(\omega n - \omega q) \leqslant C_1\varphi(\omega n)$$

for $n$ bigger than a sufficiently large number $n_0$. This proves the $\varphi$-pullback attraction property of the family $\hat{E}$.

Finally, we define the family $\hat{M} = \{M_k\}_{k\in\mathbb{Z}}$ by $M_k = \overline{E_k}$ for each $k \in \mathbb{Z}$. Since $M_k$ is the closure of a precompact set $E_k$, the compactness of each $M_k$ is obvious. Also, since $E_k \subset B_k$ and $\hat{B}$ is closed, we have $M_k \subset B_k$ for each $k \in \mathbb{Z}$, that is, $\hat{M} \subset \hat{B}$. For the positive invariance let $k \in \mathbb{Z}$ and $p \in \mathbb{N}$. Then, by the continuity properties of $S$ and the positive invariance of $\hat{E}$,

$$S(k+p,k)M_k = S(k+p,k)\overline{E_k} \subset \overline{S(k+p,k)E_k} \subset \overline{E_{k+p}} = M_{k+p}.$$

Lastly, notice that if $n > n_0$,

$$\begin{aligned} d_H(S(k,k-n)D,M_k) &= d_H(S(k,k-n)D,\overline{E_k}) \\ &\leqslant d_H(S(k,k-n)D, E_k) \leqslant C_1\varphi(\omega n). \end{aligned}$$

This concludes the proof that $\hat{M}$ is a generalized $\varphi$-pullback attractor for $S$. □

To conclude the proof, we just have to show properties (i)-(vii).

**Proposition 2.7.** *The sets $\mathcal{J}_k^n$ and $\mathcal{K}_k^n$ defined in the proof of the last theorem satisfy the properties (i)-(vii) therein.*

*Proof.* (i). The result follows immediately from the definition of $\mathcal{J}_k^n$ and the positive invariance of $\hat{B}$.

(ii) By the construction we did previously, there exist $y_i^{(m_0+n)} \in B_{k-(m_0+n)}$, for $i = 1,\ldots,r(m_0+n)$, such that

$$S(k,k-(m_0+n))B_{k-(m_0+n)} \subset \bigcup_{i=1}^{r(m_0+n)} B_{2C\varphi(\omega(m_0+n))}(z_i^{(m_0+n)})$$

where $z_i^{(m_0+n)} = S(k,k-(m_0+n))y_i^{(m_0+n)}$, for $i = 1,...,r(m_0+n)$. Noting that $\mathcal{J}_k^n = \{z_i^{(m_0+n)} \colon i = 1,\ldots,r(m_0+n)\}$, this item is also obvious.

(iii) For all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, it follows from the definition of sets $\mathcal{K}_k^n$ that

$$S(k,k-1)\mathcal{K}_{k-1}^n \subset \mathcal{J}_k^{n+1} \cup S(k,k-1)\mathcal{K}_{k-1}^n = \mathcal{K}_k^{n+1}.$$

(iv) It follows from (ii) and the fact that $\mathcal{J}_k^n \subset \mathcal{K}_k^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(v) For each integer $k$, it follows from (i) that $\mathcal{K}_k^0 = \mathcal{J}_k^0 \subset B_k = S(k,k-0)B_{k-0}$. Using the positive invariance of $\hat{B}$ and item (i) again we also have

$$\begin{aligned}\mathcal{K}_k^1 &= \mathcal{J}_k^1 \cup S(k,k-1)\mathcal{K}_{k-1}^0 = \mathcal{J}_k^1 \cup S(k,k-1)\mathcal{J}_{k-1}^0 \\ &\subset S(k,k-(m_0+1))B_{k-(m_0+1)} \cup S(k,k-1)B_{k-1} \\ &= S(k,k-1)S(k-1,k-m_0-1)B_{k-m_0-1} \cup S(k,k-1)B_{k-1} \\ &\subset S(k,k-1)B_{k-1}.\end{aligned}$$

Now let $k \in \mathbb{Z}$ fixed and suppose $n \in \mathbb{N}, n \geqslant 1$. By property (i), positive invariance of $\hat{B}$ and the Principle of Induction on $n$,

$$\begin{aligned}\mathcal{K}_k^n &= \mathcal{J}_k^n \cup S(k,k-1)\mathcal{K}_{k-1}^{n-1} \\ &\subset S(k,k-m_0-n)B_{k-m_0-n} \cup S(k,k-1)S(k-1,k-1-(n-1))B_{k-1-(n-1)} \\ &= S(k,k-m_0-n)B_{k-m_0-n} \cup S(k,k-n)B_{k-n} \\ &= S(k,k-n)S(k-n,k-n-m_0)B_{k-n-m_0} \cup S(k,k-n)B_{k-n} \\ &\subset S(k,k-n)B_{k-n}.\end{aligned}$$

(vi) For $p = 0$ it is clear that $S(k+0,k)\mathcal{K}_k^n = \mathcal{K}_{k+0}^{n+0}$ and for $p = 1$ it is a immediate consequence of property (iii), indeed $S(k+1,k)\mathcal{K}_k^n \subset \mathcal{K}_{k+1}^{n+1}$. Now, by the Principle of Induction on $p$, and using (iii) again, we have

$$\begin{aligned}S(k+p+1,k)\mathcal{K}_k^n &= S(k+p+1,k+p)S(k+p,k)\mathcal{K}_k^n \\ &\subset S(k+p+1,k+p)\mathcal{K}_{k+p}^{n+p} \\ &\subset \mathcal{K}_{k+p+1}^{n+p+1}.\end{aligned}$$

(vii) Let $m \in \mathbb{N}$ fixed. If $n \geqslant m$,

$$\mathcal{K}_k^n \subset S(k,k-n)B_{k-n} = S(k,k-m)S(k-m,k-n)B_{k-n} \subset S(k,k-m)B_{k-m},$$

where we have used property (v) and the positive invariance of $\hat{B}$. □

## 2.2 THE CONTINUOUS CASE

Using the proof of the discrete case we are able to the obtain a result for the continuous case.

**Theorem 2.8** (Existence of generalized $\varphi$-pullback attractors for continuous evolution processes)**.** *Let $S$ be a $\varphi$-pullback $\kappa$-dissipative continuous evolution process in $X$ and assume that there exists a closed and backwards bounded family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$, which is uniformly pullback absorbing and positively invariant family. Suppose also that there exists $\gamma > 0$ such that for $s \in \mathbb{R}$ and $0 \leqslant \tau \leqslant \gamma$ there exists a constant $L_{\gamma,s} > 0$ for which*

$$d(S(s+\tau,s)x,S(s+\tau,s)y) \leqslant L_{\gamma,s}d(x,y) \qquad \text{for all } x,y \in B_s.$$

*Then there exists a generalized $\varphi$-pullback attractor $\hat{M}$ for $S$, with $\hat{M} \subset \hat{B}$.*

*Proof.* Consider the discrete evolution process $S_d$ defined by

$$S_d(m,n) = S(\gamma m,\gamma n) \quad \text{for } m,n \in \mathbb{Z} \text{ with } m \geqslant n.$$

Then $S_d$ is a $\varphi$-pullback $\kappa$-dissipative discrete evolution process and there exists a uniformly pullback absorbing, backwards bounded and positively invariant family $\hat{H} = \{H_k\}_{k\in\mathbb{Z}}$ where $H_k = B_{k\gamma}$ for each $k$. In the proof of Theorem 2.6, we verified that there exists a precompact, positively invariant and $\varphi$-pullback attracting family $\hat{E} = \{E_k\}_{k\in\mathbb{Z}}$ for the discrete evolution process $S_d$, with $\hat{E} \subset \hat{H}$.

Now, define the family $\hat{G} = \{G_t\}_{t\in\mathbb{R}}$ by $G_t = S(t,k\gamma)E_k$ for $t \in [k\gamma, (k+1)\gamma)$. Note that $G_{k\gamma} = E_k$ for each $k \in \mathbb{N}$ and $\hat{G} \subset \hat{B}$. We will prove that $\hat{G}$ is precompact, positively invariant and $\varphi$-pullback attracting for the process $S$.

The precompactness of each $G_t$ follows immediately from its definition, since $E_k$ is precompact and $S(t,k\gamma)$ is continuous. Now, let $t,s \in \mathbb{R}$, $t \geqslant s$. We have $s = p\gamma + p_1$, $t = q\gamma + q_1$ where $p,q \in \mathbb{Z}$, $q \geqslant p$ and $p_1,q_1 \in [0,\gamma)$. Then,

$$\begin{aligned} S(t,s)G_s &= S(q\gamma + q_1,p\gamma + p_1)G_{p\gamma+p_1} = S(q\gamma + q_1,p\gamma + p_1)S(p\gamma + p_1,p\gamma)E_p \\ &= S(q\gamma + q_1,p\gamma)E_p = S(q\gamma + q_1,q\gamma)S(q\gamma,p\gamma)E_p \subset S(q\gamma + q_1,q\gamma)E_q \\ &= G_{q\gamma+q_1} = G_t, \end{aligned}$$

since $S(q\gamma,p\gamma)E_p = S_d(q,p)E_p \subset E_q$ by the positive invariance of family $\{E_k\}_{k\in\mathbb{Z}}$ related to the discrete process $S_d$.

It remains to prove that $\hat{G}$ is $\varphi$-pullback attracting for $S$. Let $D \subset X$ bounded and $t \in \mathbb{R}$ (we can write $t = q\gamma + t_0$ where $q \in \mathbb{Z}$ and $t_0 \in [0,\gamma)$). Taking the time $q \in \mathbb{Z}$, since $S_d$ is $\varphi$-pullback $\kappa$-dissipative, there exist $C \geqslant 0$ and $m_0 \in \mathbb{N}_*$ such that $\kappa\left(\bigcup_{n\geqslant m} S_d(q,q-n)H_{q-n}\right) < C\varphi(\omega m)$ for all $m \geqslant m_0$, where $H_k = B_{k\gamma}$. Since $\hat{B}$ is uniformly pullback absorbing, there exists $T > 0$ such that $S(\varsigma,\varsigma - r)D \subset B_\varsigma$ for all $\varsigma \leqslant t$ and $r \geqslant T$. Let $s \geqslant (m_0 + 2)\gamma + T$. It implies that $s \geqslant \gamma + T + t_0$ and, then, there exist $p \in \mathbb{N}$, $t_1 \in [0,\gamma[$ such that $s = p\gamma + T + t_0 + t_1$ and $p \geqslant m_0$. Now,

$$\begin{aligned} S(t,t-s)D &= S(t, (q-p)\gamma - T - t_1)D \\ &= S(t, q\gamma)S(q\gamma,(q-p)\gamma)S((q-p)\gamma,(q-p)\gamma - (T + t_1))D \\ &\subset S(t,q\gamma)S(q\gamma,(q-p)\gamma)B_{(q-p)\gamma}, \end{aligned}$$

since $(q-p)\gamma \leqslant t$ and $T + t_1 \geqslant T$, and thus

$$\begin{aligned} d_H(S(t,t-s)D,G_t) &= d_H(S(t,t-s)D, S(t, q\gamma)E_q) \\ &\leqslant d_H(S(t,q\gamma)S(q\gamma,(q-p)\gamma)B_{(q-p)\gamma},S(t,q\gamma)E_q) \\ &\overset{(*)}{\leqslant} Ld_H(S(q\gamma,(q-p)\gamma)B_{(q-p)\gamma}, E_q) = Ld_H(S_d(q,q-p)H_{q-p}, E_q), \end{aligned} \tag{2.2}$$

where in $(*)$ we used the facts that $S(q\gamma,(q-p)\gamma)B_{(q-p)\gamma} \subset B_{q\gamma}$ and $E_q = \bigcup_{n=0}^{\infty}\mathcal{K}_q^n \subset H_q = B_{q\gamma}$, which are consequences of the positive invariance of $\hat{B}$ and item (v) of the proof of Theorem 2.6. Now, using item (iv) from the proof of Theorem 2.6, since $p - m_0 \geqslant 0$, we have

$$\begin{aligned} S_d(q,q-p)H_{q-p} &= S_d(q,q-(m_0+(p-m_0)))H_{q-(m_0+(p-m_0))} \\ &\subset \bigcup_{z\in\mathcal{K}_q^{p-m_0}} B^X_{2C\varphi(\omega p)}(z), \end{aligned}$$

where $C$ is an appropriate constant obtained from Theorem 2.6. This ensures that

$$d_H(S_d(q,q-p)H_{q-p}, \mathcal{K}_q^{p-m_0}) \leqslant 2C\varphi(\omega p). \tag{2.3}$$

It follows from (2.2) and (2.3) that for $s \geqslant (m_0+2)\gamma + T$,

$$\begin{aligned} d_H(S(t,t-s)D,G_t) &\leqslant Ld_H(S_d(q,q-p)H_{q-p}, E_q) \\ &= Ld_H(S_d(q,q-p)H_{q-p}, \bigcup_{n=0}^{\infty}\mathcal{K}_q^n) \leqslant Ld_H(S_d(q,q-p)H_{q-p},\mathcal{K}_q^{p-m_0}) \\ &\leqslant 2LC\varphi(\omega p) = 2LC\varphi\left(\omega\left(\tfrac{s-T-t_0-t_1}{\gamma}\right)\right) \\ &= 2CL\varphi\left(\tfrac{\omega}{\gamma}s - \tfrac{\omega}{\gamma}T - \tfrac{\omega}{\gamma}(t_0+t_1)\right) \leqslant 2CL\varphi\left(\tfrac{\omega}{\gamma}s - \left[\tfrac{\omega}{\gamma}T + 2\omega\right]\right). \end{aligned}$$

By the conditions imposed on $\varphi$, there exists $C_1 > 0$ such that $d_H(S(t,t-s)D,G_t) \leqslant C_1\varphi\left(\frac{\omega}{\gamma}s\right)$ for $s$ bigger than a sufficiently large number $s_0$.

Finally, define the family $\hat{M} = \{M_t\}_{t\in\mathbb{R}}$ by $M_t = \overline{G_t}$ for each $t \in \mathbb{R}$. Since $\hat{B}$ is closed and $\hat{G} \subset \hat{B}$, we have $\hat{M} \subset \hat{B}$. Furthermore, we have:

- $\hat{M}$ is a compact, since $\hat{G}$ is precompact,
- $\hat{M}$ is positively invariant, since for all $t \geqslant s$ we know that

$$S(t,s)M_s = S(t,s)\overline{G_s} \subset \overline{S(t,s)G_s} \subset \overline{G_t} = M_t,$$

- $\hat{M}$ is $\varphi$-pullback attracting for $S$, since for $s \geqslant s_0$,

$$d_H(S(t,t-s)D,M_t) = d_H(S(t,t-s)D,\overline{G_t}) \leqslant d_H(S(t,t-s)D,G_t) \leqslant C_1\varphi\left(\tfrac{\omega}{\gamma}s\right).$$

□

## 2.3 EXISTENCE OF GENERALIZED EXPONENTIAL AND POLYNOMIAL PULLBACK ATTRACTORS

We first present a proposition that will be paramount for the proof of the existence of a generalized polinomial (or exponential) pullback attractor for a process $S$ satisfying suitable conditions.

**Proposition 2.9.** *Let $\hat{B} \in \mathfrak{F}$ be a uniformly pullback absorbing family. Suppose that there exist a decay function $\varphi$ and $\omega > 0$ such that for each $t \in \mathbb{R}$ there exist $C \geqslant 0$, $\tau_0 > 0$ such that*

$$\kappa(S(t,t-\tau)B_{t-\tau}) \leqslant C\varphi(\omega\tau) \quad \textit{for all } \tau \geqslant \tau_0.$$

*Then $S$ is $\varphi$-pullback $\kappa$-dissipative.*

*Proof.* Let $D \subset X$ bounded and $t \in \mathbb{R}$. Since $\hat{B}$ is uniformly pullback absorbing, there exists $T > 0$ such that $S(s,s-r)D \subset B_s$ for all $s \leqslant t$ and $r \geqslant T$. Take $\sigma \geqslant \tau_0 + T > 0$ and note that if $s \geqslant 2\sigma$, since $t-\sigma \leqslant t$ and $s - \sigma \geqslant T$, we have

$$\begin{aligned} S(t,t-s)D &= S(t,t-\sigma)S(t-\sigma,t-s)D \\ &= S(t,t-\sigma)S(t-\sigma,(t-\sigma)-(s-\sigma))D \subset S(t,t-\sigma)B_{t-\sigma}, \end{aligned}$$

which implies that $\bigcup_{s\geqslant 2\sigma} S(t,t-s)D \subset S(t,t-\sigma)B_{t-\sigma}$. Thus, since $\sigma \geqslant \tau_0$, it follows that

$$\kappa\Big(\bigcup_{s\geqslant 2\sigma} S(t,t-s)D\Big) \leqslant \kappa(S(t,t-\sigma)B_{t-\sigma}) \leqslant C\varphi(\omega\sigma)$$

for all $\sigma \geqslant T + \tau_0$. This is equivalent to

$$\kappa\Big(\bigcup_{s\geqslant \tau} S(t,t-s)D\Big) \leqslant C\varphi\Big(\tfrac{\omega}{2}\tau\Big)$$

for all $\tau \geqslant 2\tau_0 + 2T$, and the proof is complete. □

We now present two definitions that will serve us for what follows. Let $X$ be a complete metric space and $B \subset X$. A function $\psi\colon X \times X \to \mathbb{R}^+$ is called **contractive** on $B$ if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ we have

$$\liminf_{m,n\to\infty} \psi(x_n,x_m) = 0.$$

We denote the set of such functions by $\mathrm{contr}(B)$. Equivalently, a function $\psi$ is contractive on $B$ if for each sequence $\{x_n\} \subset B$ there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k,\ell\to\infty} \psi(x_{n_k},x_{n_\ell}) = 0.$$

Recall that a **pseudometric** in a set $X$ is a function $\rho\colon X \times X \to [0,\infty)$ that satisfies:

- $\rho(x,x) = 0$ for all $x \in X$;
- $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;
- $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$.

Let $\rho$ be a pseudometric on $X$ and consider a subset $\varnothing \neq B \subset X$. We say that $\rho$ is **precompact on** $B$ if given $\delta > 0$, there exists a finite set of points $\{x_1,\ldots,x_r\} \subset B$ such that

$$B \subset \bigcup_{j=1}^{r} B^{\rho}_{\delta}(x_j)$$

where $B^{\rho}_{\delta}(x_j) = \{y \in X : \rho(y,x_j) < \delta\}$. It follows from Proposition A.13 that $\rho$ is precompact on $B$ if and only if any sequence $\{x_n\} \subset B$ has a Cauchy subsequence $\left\{x_{n_j}\right\}$ with respect to $\rho$.

### 2.3.1 Existence of a generalized polynomial pullback attractor

The purpose of this subsection is to prove Theorem 2.10, which will be used to ensure the existence of a generalized polynomial pullback attractor for the nonautonomous wave equation (NWE) in Chapter 3. To state this result, we make the following definition: we say that a family $\hat{B}$ is **uniformly bounded** if $\cup_{t\in\mathbb{T}} B_t$ is a bounded subset of $X$.

This result for polynomial pullback $\kappa$-dissipativity is somehow technical and, in order to simplify the notation, for an evolution process $S$ and a fixed $t \in \mathbb{R}$ and $T > 0$ as in the statement of Theorem 2.10 we write

$$S_n = S(t-(n-1)T, t-nT) \quad \text{for each } n \in \mathbb{N}. \tag{2.4}$$

We point out that $S_n$ depends on both $t$ and $T$, but since we consider them fixed, we trust there is no confusion with the notation.

**Theorem 2.10** (Existence of generalized polynomial pullback attractors for continuous evolution processes)**.** *Let $X$ be a complete metric space and $S$ be a continuous evolution process in $X$ such that there exists a closed, uniformly bounded, and positively invariant uniformly pullback absorbing family $\hat{B}$ for $S$. Suppose that there exists $\gamma > 0$ such that for each $s \in \mathbb{R}$ and $0 \leqslant \tau \leqslant \gamma$ there exists a constant $L_{\gamma,s} > 0$ such that*

$$d(S(s+\tau,s)x, S(s+\tau,s)y) \leqslant L_{\gamma,s} d(x,y) \text{ for all } x,y \in B_s.$$

*Assume also that there exist $\beta \in (0,1)$, $r > 0$, $T > 0$, $C > 0$ satisfying: given $t \in \mathbb{R}$, there exist functions $g_1, g_2 \colon (\mathbb{R}^+)^m \to \mathbb{R}^+$, $\psi_1, \psi_2 \colon X \times X \to \mathbb{R}^+$ and pseudometrics $\rho_1, \ldots, \rho_m$ on $X$ such that:*

*(i) $g_i$ is non-decreasing with respect to each variable, $g_i(0,\ldots,0) = 0$ and it is continuous at $(0,\ldots,0)$ for $i = 1,2$;*

*(ii) For each $n \in \mathbb{N}$, $\rho_1, \ldots, \rho_m$ are precompact on $B_{t-nT}$;*

*(iii) $\psi_1, \psi_2 \in \operatorname{contr}(B_{t-nT})$ for all $n \in \mathbb{N}$;*

*(iv) for each $n \in \mathbb{N}$ and all $x,y \in B_{t-nT}$ we have*

$$d(S_n x, S_n y)^r \leqslant d(x,y)^r + g_1(\rho_1(x,y),\ldots,\rho_m(x,y)) + \psi_1(x,y);$$

*and*

$$\begin{aligned} d(S_n x, S_n y)^r \leqslant C\Big[& d(x,y)^r - d(S_n x, S_n y)^r \\ &+ g_1(\rho_1(x,y),\ldots,\rho_m(x,y)) + \psi_1(x,y)\Big]^\beta + g_2(\rho_1(x,y),\ldots,\rho_m(x,y)) + \psi_2(x,y), \end{aligned}$$

*where $S_n := S(t-(n-1)T, t-nT)$ for each $n \in \mathbb{N}$.*

*Then $S$ is $\varphi$-pullback $\kappa$-dissipative, with the decay function $\varphi$ given by $\varphi(s) = s^{\frac{\beta}{r(\beta-1)}}$. Also, $S$ has a uniformly bounded generalized $\varphi$-pullback attractor $\hat{M}$.*

*Proof.* Consider the function $u\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $u(s) = (3C)^{-1/\beta}s^{1/\beta} + s$. Since $u$ is an increasing bijective function, it has an inverse function that we will denote $v\colon \mathbb{R}^+ \to \mathbb{R}^+$, which also is increasing. The composite functions $u^n$ and $v^n$ are also increasing for $n \geqslant 2$ and satisfy $u \leqslant u^2 \leqslant u^3 \leqslant \cdots$ and $v \geqslant v^2 \geqslant v^3 \geqslant \cdots$ for any $s \geqslant 0$.

To simplify the notation even further, we set $D_n(x,y) = d(S_n x, S_n y)$, $G_1(x,y) = g_1(\rho_1(x,y), \cdots, \rho_m(x,y))$ and $G_2(x,y) = g_2(\rho_1(x,y), \cdots, \rho_m(x,y))$. Then,

$$D_n(x,y)^r \leqslant d(x,y)^r + G_1(x,y) + \psi_1(x,y)$$

and

$$D_n(x,y)^r \leqslant C\left[d(x,y)^r - D_n(x,y)^r + G_1(x,y) + \psi_1(x,y)\right]^\beta + G_2(x,y) + \psi_2(x,y).$$

Observe that

$$\begin{aligned}
u(D_n(x,y)^r) &= (3C)^{-\frac{1}{\beta}} D_n(x,y)^{\frac{r}{\beta}} + D_n(x,y)^r \\
&\leqslant (3C)^{-\frac{1}{\beta}} \left\{C\left[d(x,y)^r - D_n(x,y)^r + G_1(x,y) + \psi_1(x,y)\right]^\beta + G_2(x,y) + \psi_2(x,y)\right\}^{\frac{1}{\beta}} \\
&\quad + D_n(x,y)^r \\
&\leqslant 3^{-\frac{1}{\beta}} C^{-\frac{1}{\beta}} 2^{\frac{1}{\beta}} \left\{C^{\frac{1}{\beta}}\left[d(x,y)^r - D_n(x,y)^r + G_1(x,y) + \psi_1(x,y)\right] + \left[G_2(x,y) + \psi_2(x,y)\right]^{\frac{1}{\beta}}\right\} \\
&\quad + D_n(x,y)^r \\
&= \left(\frac{2}{3}\right)^{\frac{1}{\beta}} \left[d(x,y)^r - D_n(x,y)^r + G_1(x,y) + \psi_1(x,y)\right] + \left(\frac{2}{3}\right)^{\frac{1}{\beta}} C^{-\frac{1}{\beta}} \left[G_2(x,y) + \psi_2(x,y)\right]^{\frac{1}{\beta}} \\
&\quad + D_n(x,y)^r \\
&\leqslant d(x,y)^r - D_n(x,y)^r + G_1(x,y) + \psi_1(x,y) + \left(\frac{2}{3}\right)^{\frac{1}{\beta}} C^{-\frac{1}{\beta}} 3^{\frac{1}{\beta}} \left[G_2(x,y)^{\frac{1}{\beta}} + \psi_2(x,y)^{\frac{1}{\beta}}\right] \\
&\quad + D_n(x,y)^r \\
&= d(x,y)^r + G_1(x,y) + \psi_1(x,y) + \left(\frac{C}{2}\right)^{-\frac{1}{\beta}} G_2(x,y)^{\frac{1}{\beta}} + \left(\frac{C}{2}\right)^{-\frac{1}{\beta}} \psi_2(x,y)^{\frac{1}{\beta}},
\end{aligned}$$

where we have used the inequality $(a+b)^{\frac{1}{\beta}} \leqslant 2^{\frac{1}{\beta}-1}(a^{\frac{1}{\beta}} + b^{\frac{1}{\beta}})$ for $a,b \geqslant 0$, proven in Proposition A.6.

Define $g(\alpha_1, \cdots, \alpha_m) = g_1(\alpha_1, \ldots, \alpha_m) + \left(\frac{C}{2}\right)^{-\frac{1}{\beta}} g_2(\alpha_1, \ldots, \alpha_m)^{\frac{1}{\beta}}$, where $g$ is a function from $(\mathbb{R}^+)^m$ to $\mathbb{R}^+$, and $\psi\colon X \times X \to \mathbb{R}^+$ by $\psi(x,y) = \psi_1(x,y) + \left(\frac{C}{2}\right)^{-\frac{1}{\beta}} \psi_2(x,y)^{\frac{1}{\beta}}$. Note that $g(0,\ldots,0) = 0$, $g$ is continuous and non decreasing with respect to each variable. Furthermore, $\psi \in \operatorname{contr}(B_{t-nT})$ for all $n \in \mathbb{N}$. Since $v$ is the inverse function of $u$ and it is increasing, for $G(x,y) = g(\rho_1(x,y), \ldots, \rho_m(x,y))$ it follows that

$$D_n(x,y)^r \leqslant v\Big(d(x,y)^r + G(x,y) + \psi(x,y)\Big). \tag{2.5}$$

For $A \subset B_{t-T}$ and $\varepsilon > 0$, there exist sets $E_1, \cdots, E_p$ such that

$$A \subset \bigcup_{j=1}^{p} E_j \quad \text{and} \quad \operatorname{diam}(E_j) < \kappa(A) + \varepsilon \text{ for } j = 1, \cdots, p.$$

If $\{x_i\} \subset A$, there exists $j \in \{1, \cdots, p\}$ and a subsequence $\{x_{i_k}\} \subset E_j$, and thus

$$d(x_{i_k}, x_{i_\ell}) \leqslant \operatorname{diam}(E_j) < \kappa(A) + \varepsilon \text{ for all } k, \ell \in \mathbb{N}. \tag{2.6}$$

Since $\rho_1, \cdots, \rho_m$ are precompact on $B_{t-T}$ and $\psi \in \operatorname{contr}(B_{t-T})$ we have

$$\liminf_{k,\ell\to\infty} G(x_{i_k}, x_{i_\ell}) = 0 \quad \text{and} \quad \liminf_{k,\ell\to\infty} \psi(x_{i_k}, x_{i_\ell}) = 0. \tag{2.7}$$

Joining (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned} \liminf_{k,\ell\to\infty} D_1(x_{i_k}, x_{i_\ell})^r &\leqslant \liminf_{k,\ell\to\infty} v\Big(d(x_{i_k}, x_{i_\ell})^r + G(x_{i_k}, x_{i_\ell}) + \psi(x_{i_k}, x_{i_\ell})\Big) \\ &\leqslant v\left((\kappa(A) + \varepsilon)^r\right). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that for any sequence $\{x_i\} \subset A$ we have

$$\begin{aligned} \liminf_{m,p\to\infty} d(S(t,t-T)x_m, S(t,t-T)x_p)^r &= \liminf_{m,p\to\infty} D_1(x_m, x_p)^r \\ &\leqslant \liminf_{k,\ell\to\infty} D_1(x_{i_k}, x_{i_\ell})^r \leqslant v\left(\kappa(A)^r\right). \end{aligned}$$

Now, let $A \subset B_{t-2T}$, $\varepsilon > 0$ and $\{x_i\} \subset A$. As before, there exists a subsequence $\{x_{i_k}\}$ for which $d(x_{i_k}, x_{i_\ell}) < \kappa(A) + \varepsilon$ for all $k, \ell \in \mathbb{N}$. Since $\rho_1, \cdots, \rho_m$ are precompact on $B_{t-T}$ and $B_{t-2T}$, $\psi \in \operatorname{contr}(B_{t-T}) \cap \operatorname{contr}(B_{t-2T})$ and $S_2 B_{t-2T} \subset B_{t-T}$, we obtain

$$\begin{aligned} &\liminf_{k,\ell\to\infty} G(S_2 x_{i_k}, S_2 x_{i_\ell}) = 0, \quad \liminf_{k,\ell\to\infty} G(x_{i_k}, x_{i_\ell}) = 0, \\ &\liminf_{k,\ell\to\infty} \psi(S_2 x_{i_k}, S_2 x_{i_\ell}) = 0 \quad \text{and} \quad \liminf_{k,\ell\to\infty} \psi(x_{i_k}, x_{i_\ell}) = 0. \end{aligned}$$

Since for any $x, y \in B_{t-2T}$,

$$\begin{aligned} d(S(t,t-2T)x, S(t,t-2T)y)^r &= d(S_1 S_2 x, S_1 S_2 y)^r = D_1(S_2 x, S_2 y)^r \\ &\leqslant v\Big(d(S_2 x, S_2 y)^r + G(S_2 x, S_2 y) + \psi(S_2 x, S_2 y)\Big) \\ &= v\Big(D_2(x,y)^r + G(S_2 x, S_2 y) + \psi(S_2 x, S_2 y)\Big) \\ &\leqslant v\Big(v\Big(d(x,y)^r + G(x,y) + \psi(x,y)\Big) + G(S_2 x, S_2 y) + \psi(S_2 x, S_2 y)\Big), \end{aligned}$$

we obtain

$$\begin{aligned} &\liminf_{m,p\to\infty} d(S(t,t-2T)x_m, S(t,t-2T)x_p)^r \\ &\qquad \leqslant \liminf_{k,\ell\to\infty} d(S(t,t-2T)x_{i_k}, S(t,t-2T)x_{i_\ell})^r \leqslant v^2\left((\kappa(A) + \varepsilon)^r\right). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we conclude, for any sequence $\{x_n\} \subset A$, that

$$\liminf_{m,p\to\infty} d(S(t,t-2T)x_m, S(t,t-2T)x_p)^r \leqslant v^2\left(\kappa(A)^r\right).$$

Inductively, for any $n \in \mathbb{N}$, $A \subset B_{t-nT}$ and $\{x_i\} \subset A$ we obtain

$$\liminf_{m,p\to\infty} d(S(t,t-nT)x_m, S(t,t-nT)x_p)^r \leqslant v^n\left(\kappa(A)^r\right). \tag{2.8}$$

Denote by $M$ a positive constant such that $\kappa(B_t) \leqslant M$ for every $t \in \mathbb{R}$. We claim that for $n \in \mathbb{N}$ and $A \subset B_{t-nT}$ we have

$$\kappa\left(S(t,t-nT)A\right)^r \leqslant 2^r v^n\left(\kappa(A)^r\right) \leqslant 2^r v^n(M^r). \tag{2.9}$$

Assume that the first inequality in (2.9) fails. Then we can choose $a > 0$ such that

$$2^r v^n(\kappa(A)^r) < a < \kappa(S(t,t-nT)A)^r.$$

Thus it implies that

$$\beta(S(t,t-nT)A) \geqslant \frac{1}{2}\kappa(S(t,t-nT)A) > \frac{a^{1/r}}{2},$$

that is, $S(t,t-nT)A$ has no finite cover of balls of radius less than or equal to $\frac{a^{1/r}}{2}$. Take an arbitrary $x_1 \in A$. Then, there exists $x_2 \in A$ such that $d(S(t,t-nT)x_1, S(t,t-nT)x_2) > \frac{a^{1/r}}{2}$, for otherwise $S(t,t-nT)A \subset \overline{B}_{\frac{a^{1/r}}{2}}(S(t,t-nT)x_1)$. Following this idea, there exists $x_3 \in A$ such that $d(S(t,t-nT)x_3,S(t,t-nT)x_i) > \frac{a^{1/r}}{2}$ for $i = 1,2$, for otherwise $S(t,t-nT)A$ would be contained in the union of two balls of radius $\frac{a^{1/r}}{2}$. This process gives us a sequence $\{x_i\}_{i\in\mathbb{N}} \subset A$ such that $d(S(t,t-nT)x_i,S(t,t-nT)x_j) > \frac{a^{1/r}}{2}$ for all $i \neq j$. Therefore

$$d(S(t,t-nT)x_i,S(t,t-nT)x_j)^r > \frac{a}{2^r} > v^n(\kappa(A)^r),$$

which contradicts (2.8). The second inequality of (2.9) follows immediately from the fact that $v^n$ is non-decreasing.

From Proposition A.8, there exists $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$ we have

$$v^n(M^r) \leqslant \left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + M^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{\beta-1}}.$$

Hence, if $n \geqslant n_0$ and $A \subset B_{t-nT}$ we have

$$\begin{aligned}\kappa(S(t,t-nT)A) &\leqslant 2v^n\left(M^r\right)^{1/r}\\ &\leqslant 2\left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + M^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}.\end{aligned}$$

In particular, for $n \geqslant n_0$ we obtain

$$\kappa(S(t,t-nT)B_{t-nT}) \leqslant 2\left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + M^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}.$$

If $s \geqslant (n_0+1)T$ and $n \in \mathbb{N}$ is such that $\frac{s}{T} - 1 < n \leqslant \frac{s}{T}$, we have $n > n_0$ and since $S(t-nT,t-s)B_{t-s} \subset B_{t-nT}$,

$$\begin{aligned}\kappa(S(t,t-s)B_{t-s}) &= \kappa\left(S\left(t,t-nT\right)S\left(t-nT,t-s\right)B_{t-s}\right)\\ &\leqslant \kappa\left(S\left(t,t-nT\right)B_{t-nT}\right)\\ &\leqslant 2\left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + M^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}\end{aligned}$$

$$\begin{aligned}
&\overset{(*)}{\leqslant} 2\left[\left(\tfrac{s}{T}-1-n_0\right)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}\\
&= 2\left[\tfrac{1}{T}\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}s-(1+n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}\\
&= 2(\omega s-\eta)^{\frac{\beta}{r(\beta-1)}},
\end{aligned}$$

where

$$\omega=\frac{1}{T}\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} \text{ and } \eta=(1+n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}},$$

and in $(*)$ we used the fact that the exponent $\frac{\beta}{r(\beta-1)}$ is negative. Now, there exist $C_1>0$ and $s_0>0$ such that $2(\omega s-\eta)^{\frac{\beta}{r(\beta-1)}} \leqslant C_1(\omega s)^{\frac{\beta}{r(\beta-1)}}$ for $s\geqslant s_0$. Then, taking $s\geqslant s_1 := \max\{s_0,(n_0+1)T\}$, we have

$$\kappa(S(t,t-s)B_{t-s}) \leqslant C_1(\omega s)^{\frac{\beta}{r(\beta-1)}}$$

for $s\geqslant s_1$.

It follows from Proposition 2.9 that $S$ is $\varphi$-pullback $\kappa$-dissipative, with the decay function $\varphi$ given by $\varphi(s)=s^{\frac{\beta}{r(\beta-1)}}$, and therefore, from Theorem 2.8 there exists a generalized $\varphi$-pullback attractor $\hat{M}$ for $S$ with $\hat{M}\subset\hat{B}$. Since $\hat{B}$ is uniformly bounded, so is $\hat{M}$. □

### 2.3.2 Existence of a generalized exponential pullback attractor

The result for exponential attraction is far less complicated than its polynomial counterpart, as we can see in what follows.

**Theorem 2.11** (Existence of generalized exponential pullback attractors for continuous evolution processes)**.** *Let $X$ be a complete metric space and $S$ be a continuous evolution process in $X$ such that there exists a closed, uniformly bounded, and positively invariant uniformly pullback absorbing family $\hat{B}$ for $S$. Suppose that there exists $\gamma>0$ such that for each $s\in\mathbb{R}$ and $0\leqslant\tau\leqslant\gamma$ there exists a constant $L_{\gamma,s}>0$ such that*

$$d(S(s+\tau,s)x,S(s+\tau,s)y)\leqslant L_{\gamma,s}d(x,y) \text{ for all } x,y\in B_s.$$

*Assume also that there exist $\mu\in(0,1)$, $T>0$, $r>0$ satisfying: given $t\in\mathbb{R}$, there exist functions $g\colon(\mathbb{R}^+)^m\to\mathbb{R}^+$, $\psi: X\times X\to\mathbb{R}^+$ and pseudometrics $\rho_1,\dots,\rho_m$ on $X$ such that*

*(i) $g$ is non-decreasing with respect to each variable, $g(0,\dots,0)=0$ and $g$ is continuous at $(0,\dots,0)$;*

*(ii) for each $n\in\mathbb{N}$, the pseudometrics $\rho_1,\dots,\rho_m$ are precompact on $B_{t-nT}$;*

*(iii) $\psi\in\operatorname{contr}(B_{t-nT})$ for all $n\in\mathbb{N}$;*

*(iv) for each $n\in\mathbb{N}$ and $x,y\in B_{t-nT}$ we have*

$$d(S_nx,S_ny)^r\leqslant\mu d(x,y)^r+g(\rho_1(x,y),\dots,\rho_m(x,y))+\psi(x,y),$$

*where* $S_n := S(t-(n-1)T,t-nT)$ *for each* $n \in \mathbb{N}$.

*Then* $S$ *is* $\varphi$*-pullback* $\kappa$*-dissipative, with the decay function* $\varphi$ *given by* $\varphi(s) = \mu^s$*. Also,* $S$ *has a uniformly bounded generalized* $\varphi$*-pullback attractor* $\hat{M}$*.*

*Proof.* For $A \subset B_{t-T}$ and $\varepsilon > 0$, there exist sets $E_1,...,E_p$ such that

$$A \subset \bigcup_{j=1}^{p} E_j \quad \text{and} \quad \operatorname{diam}(E_j) < \kappa(A) + \varepsilon \text{ for } j = 1,..,p. \tag{2.10}$$

If $\{x_i\} \subset A$, there exists $j \in \{1,\cdots,p\}$ and a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ such that $\{x_{i_k}\} \subset E_j$. Thus,

$$d(x_{i_k},x_{i_l}) \leqslant \operatorname{diam}(E_j) < \kappa(A) + \varepsilon \text{ for all } k,l \in \mathbb{N}. \tag{2.11}$$

Since $\rho_1,\cdots,\rho_m$ are precompact on $B_{t-T}$ and $\psi \in \operatorname{contr}(B_{t-T})$, we have

$$\liminf_{k,l\to\infty} g(\rho_1(x_{i_k},x_{i_l}),\cdots,\rho_m(x_{i_k},x_{i_l})) = 0 \text{ and } \liminf_{k,l\to\infty} \psi(x_{i_k},x_{i_l}) = 0. \tag{2.12}$$

Joining (2.11), (2.12) and hypothesis (iv), we obtain

$$\begin{aligned}
\liminf_{k,l\to\infty} d(S_1x_{i_k},S_1x_{i_l})^r &= \liminf_{k,l\to\infty} d(S(t,t-T)x_{i_k},S(t,t-T)x_{i_l})^r \\
&\leqslant \liminf_{k,l\to\infty} [\mu d(x_{i_k},x_{i_l})^r + g(\rho_1(x_{i_k},x_{i_l}),\cdots,\rho_m(x_{i_k},x_{i_l})) + \psi(x_{i_k},x_{i_l})] \\
&\leqslant \mu(\kappa(A)+\varepsilon)^r,
\end{aligned}$$

and, since $\varepsilon > 0$ is arbitrary, we conclude that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ we have

$$\liminf_{k,\ell\to\infty} d(S_1x_k,S_1x_\ell)^r \leqslant \mu\kappa(A)^r.$$

Now, let $A \subset B_{t-2T}$, $\varepsilon > 0$ and $\{x_i\} \subset A$. As before, there exists a subsequence $\{x_{i_k}\}$ for which $d(x_{i_k},x_{i_l}) < \kappa(A) + \varepsilon$ for all $k,l \in \mathbb{N}$. Since $\rho_1,\cdots,\rho_m$ are precompact on $B_{t-T}$ and $B_{t-2T}$, $\psi \in \operatorname{contr}(B_{t-T}) \cap \operatorname{contr}(B_{t-2T})$ and $S_2B_{t-2T} \subset B_{t-T}$, we obtain

$$\liminf_{k,l\to\infty} g(\rho_1(x_{i_k},x_{i_l}),\cdots,\rho_m(x_{i_k},x_{i_l})) = 0\ ,\ \liminf_{k,l\to\infty} \psi(x_{i_k},x_{i_l}) = 0,$$

$$\liminf_{k,l\to\infty} g(\rho_1(S_2x_{i_k},S_2x_{i_l}),\cdots,\rho_m(S_2x_{i_k},S_2x_{i_l})) = 0\ ,\ \liminf_{k,l\to\infty} \psi(S_2x_{i_k},S_2x_{i_l}) = 0.$$

Since for any $x,y \in B_{t-2T}$,

$$\begin{aligned}
d(S(t,t-2T)x,&S(t,t-2T)y)^r = d(S_1S_2x,S_1S_2y)^r \\
&\leqslant \mu d(S_2x,S_2y)^r + g(\rho_1(S_2x,S_2y),\ldots,\rho_m(S_2x,S_2y)) + \psi(S_2x,S_2y) \\
&\leqslant \mu[\mu d(x,y)^r + g(\rho_1(x,y),\ldots,\rho_m(x,y)) + \psi(x,y)] \\
&\quad + g(\rho_1(S_2x,S_2y),\ldots,\rho_m(S_2x,S_2y)) + \psi(S_2x,S_2y),
\end{aligned}$$

we obtain

$$\liminf_{k,l\to\infty} d(S(t,t-2T)x_{i_k},S(t,t-2T)x_{i_l})^r \leqslant \mu^2(\kappa(A)+\varepsilon)^r,$$

and, again, since $\varepsilon > 0$ is arbitrary, we conclude that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$,

$$\liminf_{m,p\to\infty} d(S(t,t-2T)x_m, S(t,t-2T)x_p)^r \leqslant \mu^2 \kappa(A)^r.$$

Inductively, for any $n \in \mathbb{N}$, $A \subset B_{t-nT}$ and $\{x_n\}_{n\in\mathbb{N}} \subset A$ we obtain

$$\liminf_{m,p\to\infty} d(S(t,t-nT)x_m, S(t,t-nT)x_p)^r \leqslant \mu^n \kappa(A)^r. \tag{2.13}$$

Denote by $M$ a positive constant such that $\kappa(B_t) \leqslant M$ for all $t \in \mathbb{R}$. We claim that for $n \in \mathbb{N}$ and $A \subset B_{t-nT}$ we have

$$\kappa(S(t,t-nT)A)^r \leqslant 2^r \mu^n . \kappa(A)^r \leqslant 2^r \mu^n M^r, \tag{2.14}$$

for which the proof is completely analogous to that one presented for (2.9). In particular, we conclude that

$$\kappa(S(t,t-nT)B_{t-nT}) \leqslant 2M\mu^{\frac{n}{r}} \quad \text{for } n \in \mathbb{N}.$$

Finally, for $s \geqslant T$, if $n \in \mathbb{N}$ is such that $\frac{s}{T} - 1 < n \leqslant \frac{s}{T}$, from the positive invariance of $\hat{B}$ we obtain

$$\begin{aligned}\kappa(S(t,t-s)B_{t-s}) &= \kappa\left(S\left(t,t-nT\right)S\left(t-nT,t-s\right)B_{t-s}\right)\\ &\leqslant \kappa\left(S\left(t,t-nT\right)B_{t-nT}\right) \leqslant 2M\mu^{\frac{n}{r}} \leqslant 2M\mu^{\frac{s}{rT}}\mu^{-\frac{1}{r}} \leqslant C\mu^{\omega s},\end{aligned}$$

where $C = 2M\mu^{-\frac{1}{r}}$ and $\omega = \frac{1}{rT}$. It follows from Proposition 2.9 that $S$ is $\varphi$-pullback $\kappa$-dissipative, with decay function $\varphi$ given by $\varphi(s) = \mu^s$, and therefore, from Theorem 2.8 there exists a generalized $\varphi$-pullback attractor $\hat{M}$ for $S$ with $\hat{M} \subset \hat{B}$. Since $\hat{B}$ is uniformly bounded, so is $\hat{M}$. □

## 2.4 PULLBACK ATTRACTORS

In this section we want to relate the notion of a generalized $\varphi$-pullback attractor and the one of a pullback attractor, just as in the exponential case. For a different and more detailed approach on this subject we refer to (CARVALHO; LANGA; ROBINSON, 2013). See also Definition 1.3 to recall the concept of the pullback attractor for an evolution process. More specifically, we state and prove the following result:

**Theorem 2.12.** *Let $S$ be a $\varphi$-pullback $\kappa$-dissipative evolution process in $X$ with a backwards bounded generalized $\varphi$-pullback attractor $\hat{M}$. Then $S$ has a pullback attractor $\hat{A}$, with $\hat{A} \subset \hat{M}$.*

As we know, in the autonomous theory of global attractors for semigroups (see (LADYZHENSKAYA, 2022), for instance) the *$\omega$-limit sets* are the fundamental tool. Here, things work analogously. Recall that for a nonempty subset $D$ of $X$ and $(t,s) \in \mathcal{P}$ we set

$S(t,s)D = \{S(t,s)x\colon x \in D\}$, and for a fixed $t \in \mathbb{T}$ we define the **pullback $\omega$-limit set** of the family $\hat{D} \in \mathfrak{F}$ at the time $t$ by

$$\omega(\hat{D},t) = \bigcap_{\sigma \leqslant t} \overline{\bigcup_{s \leqslant \sigma} S(t,s)D_s},$$

and the **pullback $\omega$-limit** of $\hat{D}$ is the family $\omega(\hat{D}) = \{\omega(\hat{D},t)\}_{t\in\mathbb{T}}$. If $D \subset X$ is any subset, we analogously define its pullback $\omega$-limit set at time $t \in \mathbb{T}$ by

$$\omega(D,t) = \bigcap_{\sigma \leqslant t} \overline{\bigcup_{s \leqslant \sigma} S(t,s)D},$$

and the pullback $\omega$-limit $\omega(D) = \{\omega(D,t)\}_{t\in\mathbb{T}}$. As in the autonomous case (see (LADYZHENSKAYA, 2022)), we have the following characterization:

**Proposition 2.13.** *Let $S$ be an evolution process in $X$, $\hat{D} \in \mathfrak{F}$ and $t \in \mathbb{T}$. Then $y \in \omega(\hat{D},t)$ iff there exist sequences $s_n \to -\infty$ and $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in D_{s_n}$ for each $n \in \mathbb{N}$ such that $S(t,s_n)x_n \to y$. Moreover the pullback $\omega$-limit $\omega(\hat{D})$ is closed.*

*Proof.* If $y \in \omega(\hat{D},t)$ then $y \in \overline{\bigcup_{s\leqslant\sigma} S(t,s)D_s}$ for all $\sigma \leqslant t$. Since $y \in \overline{\bigcup_{s\leqslant t-1} S(t,s)D_s}$, there exist $s_1 \leqslant t-1$ and $x_1 \in D_{s_1}$ such that $d(S(t,s_1)x_1,y) < 1$. In the same way, since $y \in \overline{\bigcup_{s\leqslant t-2} S(t,s)D_s}$, there exist $s_2 \leqslant t-2$ and $x_2 \in D_{s_2}$ such that $d(S(t,s_2)x_2,y) < \frac{1}{2}$. Continuing with this reasoning, we find sequences $s_n \to -\infty$ and $\{x_n\}_{n\in\mathbb{N}}$ where $x_n \in D_{s_n}$ satisfying $d(S(t,s_n)x_n,y) < \frac{1}{n}$ for each $n$, which implies $S(t,s_n)x_n \to y$.

On the other hand, take $y \in X$, sequences $s_n \to -\infty$ and $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in D_{s_n}$ for all $n \in \mathbb{N}$ and $S(t,s_n)x_n \to y$, and fix $\sigma \leqslant t$. Since $s_n \to -\infty$, there exists $n_0 \in \mathbb{N}$ such that $s_n \leqslant \sigma$ for all $n \geqslant n_0$. Therefore, $S(t,s_n)x_n \in \bigcup_{s\leqslant\sigma} S(t,s)D_s$ for all $n \geqslant n_0$. Since $S(t,s_n)x_n \to y$, it follows that $y \in \overline{\bigcup_{s\leqslant\sigma} S(t,s)D_s}$. Since $\sigma \leqslant t$ is arbitrary, we obtain

$$y \in \bigcap_{\sigma \leqslant t} \overline{\bigcup_{s \leqslant \sigma} S(t,s)D_s} = \omega(\hat{D},t).$$

□

We say that an evolution process $S$ is **pullback $\omega$-limit compact** if for given $D \subset X$ bounded and $t \in \mathbb{T}$ we have

$$\lim_{\sigma\to-\infty} \kappa\left(\bigcup_{s\leqslant\sigma} S(t,s)D\right) = 0.$$

It is clear that $S$ is a pullback $\omega$-limit compact evolution process if and only if given $D \subset X$ bounded, $t \in \mathbb{T}$ and $\varepsilon > 0$, there exists $\tau_0 > 0$ such that

$$\kappa\left(\bigcup_{\sigma\geqslant\tau} S(t,t-\sigma)D\right) < \varepsilon \text{ for all } \tau \geqslant \tau_0.$$

We have the following result:

**Proposition 2.14.** *If $S$ is a $\varphi$-pullback $\kappa$-dissipative evolution process in $X$, then $S$ is pullback $\omega$-limit compact. If $S$ is pullback $\omega$-limit compact, then for each $t \in \mathbb{T}$, a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ and $s_n \to -\infty$, we have $\{S(t,s_n)x_n\}_{n\in\mathbb{N}}$ relatively compact in $X$.*

*Proof.* Let $D \subset X$ a bounded subset, $t \in \mathbb{T}$ and $\varepsilon > 0$. Since $S$ is $\varphi$-pullback $\kappa$-dissipative, there exist constants $c > 0$ and $\tau_0 > 0$ such that $\kappa\left(\bigcup_{\sigma\geqslant\tau} S(t,t-\sigma)D\right) \leqslant c\varphi(\omega\tau)$ for all $\tau \geqslant \tau_0$. Since $\varphi$ is a decay function, there exists $\alpha > \omega\tau_0$ such that $c\varphi(\alpha) < \varepsilon$. Thus,

$$\kappa\Big(\bigcup_{\sigma\geqslant\frac{\alpha}{\omega}} S(t,t-\sigma)D\Big) \leqslant c\varphi\left(\omega\frac{\alpha}{\omega}\right) = c\varphi(\alpha) < \varepsilon,$$

which means that $S$ is pullback $\omega$-limit compact.

Now suppose that the process $S$ is pullback $\omega$-limit compact. Let $t \in \mathbb{T}$, $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence and $s_n \to -\infty$. Set $D := \{x_n \colon n \in \mathbb{N}\}$. Given $\varepsilon > 0$, there exists $s_0 \leqslant t$ such that $\kappa\left(\bigcup_{s\leqslant s_0} S(t,s)D\right) \leqslant \varepsilon$. Since $s_n \to -\infty$, there exist $N \in \mathbb{N}$ such that $s_n \leqslant s_0$ whenever $n \geqslant N$, which implies

$$\bigcup_{n\geqslant N} S(t,s_n)x_n \subset \bigcup_{n\geqslant N} S(t,s_n)D \subset \bigcup_{s\leqslant s_0} S(t,s)D.$$

Thus, $\kappa\left(\bigcup_{n\geqslant N} S(t,s_n)x_n\right) \leqslant \varepsilon$ and hence

$$\begin{aligned}\kappa\left(\bigcup_{n\in\mathbb{N}} S(t,s_n)x_n\right) &\leqslant \kappa\left(\bigcup_{n=1}^{N-1} S(t,s_n)x_n \cup \bigcup_{n\geqslant N} S(t,s_n)x_n\right)\\ &\leqslant \max\left\{\kappa\left(\bigcup_{n=1}^{N-1} S(t,s_n)x_n\right), \kappa\left(\bigcup_{n\geqslant N} S(t,s_n)x_n\right)\right\}\\ &= \max\left\{0, \kappa\left(\bigcup_{n\geqslant N} S(t,s_n)x_n\right)\right\} \leqslant \kappa\left(\bigcup_{n\geqslant N} S(t,s_n)x_n\right) \leqslant \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\kappa\left(\bigcup_{n\in\mathbb{N}} S(t,s_n)x_n\right) = 0$, which proves that the sequence $\{S(t,s_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

□

The concept of pullback absorption is closely related to the one of pullback attraction (see Definition 1.2). It is easy to verify that if $\hat{B}$ is pullback absorbing (attracting) and $\hat{B} \subset \hat{A}$ then $\hat{A}$ is pullback absorbing (attracting). Clearly, if $\hat{B}$ is a pullback absorbing family then $\hat{B}$ is a pullback attracting family. The converse does not always hold, but the following result gives us a partial converse. For $r > 0$ and a nonempty set $A \subset X$ we define the $r$-**neighborhood of** $A$ by

$$\mathcal{O}_r(A) = \{x \in X \colon d(x,a) < r \text{ for some } a \in A\}.$$

**Proposition 2.15.** *If $\hat{B}$ is a pullback attracting family for $S$, then for each $r > 0$ the family $\hat{A} = \{\mathcal{O}_r(B_t)\}_{t\in\mathbb{T}}$ is pullback absorbing for $S$.*

*Proof.* Let $D \subset X$ bounded and $t \in \mathbb{T}$. Since $\hat{B}$ is a pullback attracting family, we have

$$\lim_{s \to -\infty} d_H(S(t,s)D, B_t) = 0.$$

This implies clearly that there exists a number $s_0$ such that $S(t,s)D \subset \mathcal{O}_r(B_t)$ for all $s \leqslant s_0$, and $\hat{A}$ is a pullback absorbing family. □

**Lemma 2.16.** *Let $S$ be a pullback $\omega$-limit compact process on $X$. If $D \subset X$ is bounded, then for each $t \in \mathbb{T}$ we have $\lim_{s \to -\infty} d_H(S(t,s)D, \omega(D,t)) = 0$.*

*Proof.* Suppose the opposite, that is, there exist $\varepsilon_0 > 0$ and sequences $s_n \to -\infty$ and $\{x_n\}_{n \in \mathbb{N}} \subset D$ such that

$$d(S(t,s_n)x_n, \omega(D,t)) \geqslant \varepsilon_0 > 0 \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Since $S$ is pullback $\omega$-limit compact, it follows from Proposition 2.14 that the sequence $\{S(t,s_n)x_n\}_{n \in \mathbb{N}}$ is relatively compact, thus it has a convergent subsequence (to some $y \in X$), in other words, $y = \lim_{j \to \infty} S(t,s_{n_j})x_{n_j}$. Clearly $y \in \omega(D,t)$, which is a contradiction with (2.15). □

**Lemma 2.17.** *Let $S$ be a pullback $\omega$-limit compact evolution process in $X$ and $\hat{B} \in \mathfrak{F}$ a pullback absorbing family. Then $\omega(\hat{B})$ is pullback attracting.*

*Proof.* Firstly, for $D \subset X$ bounded, we prove that $\omega(D,t) \subset \omega(\hat{B},t)$ for all $t \in \mathbb{T}$. Let $x_0 \in \omega(D,t)$. Then there exist sequences $s_n \to -\infty$ and $\{x_n\}_{n \in \mathbb{N}} \subset D$ such that $S(t,s_n)x_n \to x_0$. Since $\hat{B}$ is pullback absorbing, given $\tau_n \to -\infty$ there exists $\{\sigma_n\}_{n \in \mathbb{N}}$ with $\sigma_n \leqslant \tau_n$ such that $S(\tau_n,s)D \subset B_{\tau_n}$ for all $s \leqslant \sigma_n$. We can take a subsequence of $\{s_n\}_{n \in \mathbb{N}}$, which we will denote the same, such that $s_n \leqslant \sigma_n$ for all $n \in \mathbb{N}$. Thus,

$$x_0 = \lim_{n \to \infty} S(t,s_n)x_n = \lim_{n \to \infty} S(t,\tau_n)S(\tau_n,s_n)x_n,$$

and, since $S(\tau_n,s_n)x_n \in B_{\tau_n}$, it follows that $x_0 \in \omega(\hat{B},t)$ and this first claim is complete.

From $\omega(D,t) \subset \omega(\hat{B},t)$ for all $t$ and the fact that $\lim_{s \to -\infty} d_H(S(t,s)D, \omega(D,t)) = 0$ we conclude that

$$\lim_{s \to -\infty} d_H(S(t,s)D, \omega(\hat{B},t)) = 0,$$

and the proof is complete. □

The next is a simple and useful topological lemma.

**Lemma 2.18.** *Let $K$ be a compact subset of $X$. If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $X$ with $d(x_n,K) \to 0$, then there exists a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ converging to a point of $K$.*

*Proof.* For each $n \in \mathbb{N}$ we can choose $z_n \in K$ with $d(x_n,z_n) < \frac{1}{n}$. Since $\{z_n\}_{n \in \mathbb{N}} \subset K$ and $K$ is compact, up to a subsequence we can assume that $z_n \to z \in K$. Hence it is clear that $x_n \to z$, and the proof is complete. □

**Lemma 2.19.** *Let $S$ be a pullback $\omega$-limit compact evolution process in $X$. Then for each backwards bounded family $\hat{B} \in \mathfrak{F}$ and $t \in \mathbb{T}$, we have $\omega(\hat{B},t)$ nonempty, compact,*

$$\lim_{s\to-\infty} d_H(S(t,s)B_s,\omega(\hat{B},t)) = 0 \tag{2.16}$$

*and the family $\omega(\hat{B})$ is invariant.*

*Proof.* Let $t \in \mathbb{T}$ and $\hat{B} \in \mathfrak{F}$ be a backwards bounded family. Let $D = \bigcup_{s\leqslant t} B_s$, which is bounded by hypothesis. If $s_n \to -\infty$ and $\{x_n\}_{n\in\mathbb{N}}$ are sequences with $x_n \in B_{s_n}$, then $\{x_n\}_{n\in\mathbb{N}} \subset D$ and thus $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Thus, from Proposition 2.14, it follows that, up to a subsequence, $S(t,s_n)x_n \to y$ for some $y \in X$. Clearly $y \in \omega(\hat{B},t)$, and thus $\omega(\hat{B},t)$ is nonempty.

To show that it is compact, let $\{y_n\}_{n\in\mathbb{N}} \subset \omega(\hat{B},t)$. Thus for each $n \in \mathbb{N}$ there exists $s_n < -n$ and $x_n \in B_{s_n}$ with $d(S(t,s_n)x_n,y_n) < \frac{1}{n}$. From Proposition 2.14, there exists $y \in X$ such that $S(t,s_n)x_n \to y$, up to a subsequence. Clearly $y \in \omega(\hat{B},t)$ and, along this subsequence, $y_n \to y$. This shows that $\omega(\hat{B},t)$ is compact.

Now we prove (2.16) for each fixed $t \in \mathbb{T}$. Suppose that this does not hold, that is, assume there exist $\varepsilon_0 > 0$ and sequences $s_n \to -\infty$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \in B_{s_n}$ for all $n$ such that

$$d(S(t,s_n)x_n, \omega(\hat{B},t)) \geqslant \varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \tag{2.17}$$

Since $S$ is pullback $\omega$-limit compact, it follows from Proposition 2.14 that the sequence $\{S(t,s_n)x_n\}_{n\in\mathbb{N}}$ is relatively compact, thus it has a convergent subsequence $\{s_{n_j}\}_{j\in\mathbb{N}}$ (to some $y \in X$) of $\{s_n\}_{n\in\mathbb{N}}$, in other words, $S(t,s_{n_j})x_{n_j} \to y$ as $j \to \infty$. Clearly $y \in \omega(\hat{B},t)$, which contradicts (2.17).

Lastly we prove that $\omega(\hat{B})$ is invariant. We start showing that if $t \geqslant s$ then $S(t,s)\omega(\hat{B},s) \subset \omega(\hat{B},t)$. Indeed, let $z \in S(t,s)\omega(\hat{B},s)$. Then there exists $y \in \omega(\hat{B},s)$ such that $z = S(t,s)y$. Since $y \in \omega(\hat{B},s)$, there are sequences $s_n \to -\infty$ and $x_n \in B_{s_n}$ for all $n \in \mathbb{N}$, such that $S(s,s_n)x_n \to y$. For $t \geqslant s \geqslant s_n$, we have

$$S(t,s_n)x_n = S(t,s)S(s,s_n)x_n \to S(t,s)y = z,$$

and thus it is clear that $z \in \omega(\hat{B},t)$. To show the negative invariance, let $z \in \omega(\hat{B},t)$ and fix $s \leqslant t$. There exist sequences $s_n \to -\infty$ (which we can assume that $s_n \leqslant s$ for all $n \in \mathbb{N}$) and $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \in B_{s_n}$ for all $n \in \mathbb{N}$ such that $S(t,s_n)x_n \to z$. From what we have just proved, $\lim_{n\to\infty} d(S(s,s_n)x_n,\omega(\hat{B},s)) = 0$ and Lemma 2.18 implies that, up to a subsequence, $\{S(s,s_n)x_n\}_{n\in\mathbb{N}}$ converges to some $y \in \omega(\hat{B},s)$. This gives us

$$z = \lim_{j\to\infty} S(t,s_{n_j})x_{n_j} = \lim_{j\to\infty} S(t,s)S(s,s_{n_j})x_{n_j} = S(t,s)y,$$

which means $z = S(t,s)y$, where $y \in \omega(\hat{B},s)$. Therefore, $z \in S(t,s)\omega(\hat{B},s)$ and the proof is complete. ∎

**Lemma 2.20.** *Let $S$ be an evolution process in $X$ and let $\hat{B} \in \mathfrak{F}$ be a backwards bounded family. If $\hat{C} \in \mathfrak{F}$ is a closed pullback attracting family, then $\omega(\hat{B}) \subset \hat{C}$.*

*Proof.* Fix $t \in \mathbb{T}$ and let $z \in \omega(\hat{B},t)$ and $D = \bigcup_{s \leqslant t} B_s$, which is a bounded subset of X. There exist sequences $s_n \to -\infty$ and $x_n \in B_{s_n}$ for all $n \in \mathbb{N}$ such that $S(t,s_n)x_n \to z$. Since $\hat{C}$ is pullback attracting, we have

$$d(S(t,s_n)x_n, C_t) \leqslant d_H(S(t,s_n)B_{s_n}, C_t) \leqslant d_H(S(t,s_n)D, C_t) \to 0 \text{ as } n \to \infty.$$

Since $S(t,s_n)x_n \to z$, it follows that $z \in \overline{C_t} = C_t$ and the result is proved. □

Joining Lemmas 2.16, 2.17, 2.19 and 2.20 we have sufficient conditions to obtain a pullback attractor.

**Proposition 2.21.** *Let $S$ be a $\varphi$-pullback $\kappa$-dissipative evolution process in $X$ with $\hat{B} \in \mathfrak{F}$ a backwards bounded and pullback absorbing family. Then $S$ has a pullback attractor $\hat{A}$ given by $\hat{A} = \omega(\hat{B})$.*

With that, we are able to present the proof of Theorem 2.12.

*Proof of Theorem 2.12.* Since $\hat{M}$ is a backwards bounded generalized $\varphi$-pullback attractor, given $r > 0$ the family $\hat{B}$ defined by $B_t = \mathcal{O}_r(M_t)$, is a backwards bounded and pullback absorbing family for $S$, by Proposition 2.15.

Hence, from Proposition 2.21 it follows that $S$ has a pullback attractor $\hat{A} = \omega(\hat{B})$. Lastly, since $\hat{M}$ is a closed pullback attracting family, it follows from Lemma 2.20 that $\hat{A} = \omega(\hat{B}) \subset \hat{M}$, and the proof is complete. □

**Note.** In the next chapter, we will apply all this theory to the study of a class of nonautonomous wave equations. The step-by-step of this application is as follows: first, we prove the existence of a uniformly pullback absorbing family $\hat{B}$ for the given process, which is closed and uniformly bounded. Next, we construct a family $\hat{C}$ that satisfies the same properties as $\hat{B}$, but with the additional property of being positively invariant. With this family $\hat{C}$ and suitable properties for the evolution process, we prove the existence of a generalized $\varphi$-pullback attractor $\hat{M}$. As a consequence, we ensure the existence of the pullback attractor $\hat{A}$.

This process leads us to a sequence of containments between these families, namely $\hat{A} \subset \hat{M} \subset \hat{C} \subset \hat{B}$, as depicted in the Figure 1. Note that the closer to the center, the stronger the properties of compactness and invariance become, but the weaker the information on attraction rate becomes. In this sense, the generalized $\varphi$-pullback attractor - the central theme of this work - represents the structure that still maintains good properties related to invariance and compactness, but with controlled and understood attraction rate (in a qualitative sense).

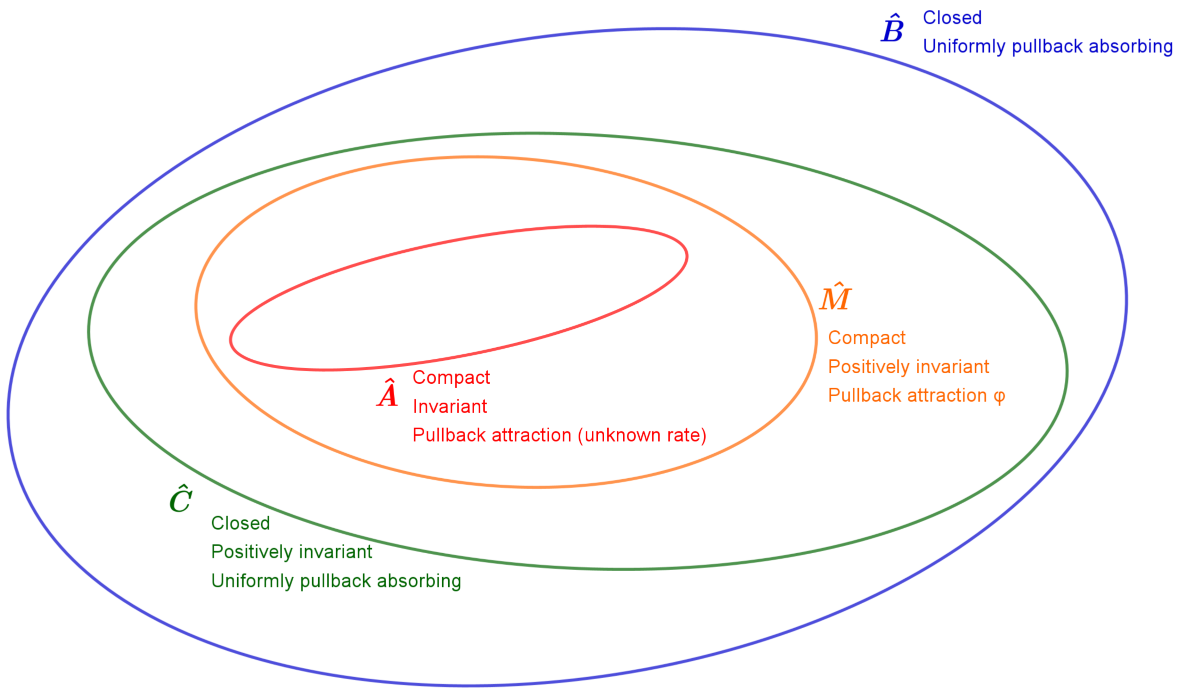


Figure 1 – Representation of the inclusion $\hat{A} \subset \hat{M} \subset \hat{C} \subset \hat{B}$.

## 3 APPLICATION TO A NONAUTONOMOUS WAVE EQUATION

Inspired by the works (ZHAO; ZHAO; ZHONG, 2020a), (ZHAO; ZHONG; YAN, 2022), and (YAN et al., 2023), and following their main ideas, we study the nonautonomous wave equation with non-local weak damping and anti-damping (NWE), and prove the existence of a *generalized polynomial pullback attractor* when $p > 0$ and the existence of a *generalized exponential pullback attractor* for the specific case $p = 0$. We recall that (NWE) is given by

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + k(t)\|u_t(t,\cdot)\|^p_{L^2(\Omega)} u_t(t,x) + f(t,u(t,x)) \\ \qquad\qquad\qquad = \displaystyle\int_\Omega K(x,y)u_t(t,y)dy + h(x), (t,x) \in [s,\infty) \times \Omega, \\ u(t,x) = 0, (t,x) \in [s,\infty) \times \partial\Omega, \\ u(s,x) = u_0(x),\ \ u_t(s,x) = u_1(x), x \in \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Here, $u$ represents the displacement of a wave in $\Omega$, subjected to a non-local damping $k(t)\|u_t(t,\cdot)\|^p_{L^2(\Omega)}$ and an anti-damping $\int_\Omega K(x,y)u_t(t,y)dy$. We assume the following:

**(H$_1$)** $p \geqslant 0$;

**(H$_2$)** $K \in L^2(\Omega \times \Omega)$ and we set $K_0 := \|K\|_{L^2(\Omega\times\Omega)}$;

**(H$_3$)** $k\colon \mathbb{R} \to (0,\infty)$ is a continuous function satisfying $0 < k_0 \leqslant k(t) \leqslant k_1$ for all $t \in \mathbb{R}$, where $k_0,k_1$ are constants, and when $p = 0$ we require that $k_0 > K_0$, where $K_0$ is the constant of **(H$_2$)**;

**(H$_4$)** $h \in L^2(\Omega)$ and we set $h_0 := \|h\|_{L^2(\Omega)}$;

**(H$_5$)** $f \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies

$$\liminf_{|v|\to\infty} \left( \inf_{t\in\mathbb{R}} \frac{\partial f}{\partial v}(t,v) \right) > -\lambda_1, \tag{3.1}$$

$f(t,0) = 0$ for all $t \in \mathbb{R}$, and there exists a constant $c_0 > 0$ such that for all $t,v \in \mathbb{R}$ we have

$$\left| \frac{\partial f}{\partial v}(t,v) \right| \leqslant c_0(1 + |v|^2) \tag{3.2}$$

$$\left| \frac{\partial f}{\partial t}(t,v) \right| \leqslant c_0, \tag{3.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian operator $-\Delta$ with Dirichlet boundary conditions in $\Omega$, that is, of the operator $A := -\Delta\colon H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$, which is positive and selfadjoint, with compact resolvent;

**(H$_6$)** For $t,v \in \mathbb{R}$ we define the function

$$F(t,v) = \int_0^v f(t,\xi)d\xi, \tag{3.4}$$

for which we assume that $c_0 > 0$ previously defined is such that for all $t \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \left| \frac{\partial F}{\partial t}(t,v) \right| dv \leqslant c_0.$$

The main result of this chapter is the following:

**Theorem 3.1.** *Assume that* $(\boldsymbol{H}_1)$-$(\boldsymbol{H}_6)$ *hold true. Then the evolution process* $S$ *associated with* (NWE) *in* $X := H_0^1(\Omega) \times L^2(\Omega)$ *possesses a uniformly bounded generalized* $\varphi$*-pullback attractor* $\hat{M}$ *in* $X$*, where*

(a) $\varphi(t) = t^{-\frac{1}{p}}$ *if* $p > 0$*;*

(b) $\varphi(t) = e^{-\alpha t}$ *if* $p = 0$*, where* $\alpha := \min\{\sqrt{\lambda_1}, k_0\}$*.*

*Furthermore,* $S$ *has a pullback attractor* $\hat{A}$*, with* $\hat{A} \subset \hat{M}$*.*

Our efforts from now on are dedicated to present the proof of this result. We point out that when $p > 0$, the damping term $k(t)\|u_t\|^p_{L^2(\Omega)} u_t$ is *effective*, that is, it overpowers the anti-damping term $\int_\Omega K(x,y)u_t(t,y)dy$, whatever the positive lower bound $k_0$ of $k(t)$ might be (since the damping $\|u_t\|^p u_t$ grows with power $p+1$, and the anti-damping has power 1). However, when $p = 0$, the damping term is $k(t)u_t$, which becomes effective only when it is stronger than the anti-damping term, which is why we require the condition $k_0 > K_0$, since both of them grow with power 1.

## 3.1 AUXILIARY ESTIMATES

Now we present a few estimates regarding the functions $f$ and $F$ that will help us in the sections to come. In what follows, unless stated otherwise, $C$ denotes an independent positive constant for which its value may vary from one result to another, from one line to another, and even in the same line. Also, from now on, we are assuming that conditions $(\mathbf{H}_1)$-$(\mathbf{H}_6)$ hold true. To simplify the notation, we will omit the $(\Omega)$ from the subscript of the norms. For instance, we write $\|\cdot\|_{L^2}$ instead of $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H_0^1}$ instead of $\|\cdot\|_{H_0^1(\Omega)}$, and henceforth. Also, for a Banach space $Z$ we will denote $\overline{B}^Z_R$ the closed ball with radius $R$ centered at 0 in $Z$.

**Proposition 3.2.** *For* $v \in L^2(\Omega)$ *we have*

$$\left\| \int_\Omega K(x,y)v(y)dy \right\|_{L^2} \leqslant K_0 \|v\|_{L^2}.$$

*Proof.* Using Hölder's inequality, we have

$$\left| \int_\Omega K(x,y)v(y)dy \right| \leqslant \int_\Omega |K(x,y)||v(y)|dy \leqslant \left( \int_\Omega |K(x,y)|^2 dy \right)^{\frac{1}{2}} \|v\|_{L^2}.$$

Thus

$$\left\| \int_\Omega K(x,y)v(y)dy \right\|^2_{L^2} = \int_\Omega \left| \int_\Omega K(x,y)v(y)dy \right|^2 dx \leqslant \int_\Omega \int_\Omega |K(x,y)|^2 dydx \|v\|^2_{L^2}$$

$$= \int_{\Omega\times\Omega} |K(x,y)|^2 dydx \|v\|_{L^2}^2 = K_0^2 \|v\|_{L^2}^2.$$

□

**Proposition 3.3.** *For all $t,v,w \in \mathbb{R}$ we have:*

*(i)* $|f(t,v)| \leqslant 2c_0(1+|v|^3)$;

*(ii)* $|f(t,v) - f(t,w)| \leqslant 2c_0(1+|v|^2+|w|^2)|v-w|$;

*(iii)* $|F(t,v)| \leqslant 4c_0(1+|v|^4)$;

*(iv)* $|F(t,v) - F(t,w)| \leqslant 8c_0(1+|v|^3+|w|^3)|v-w|$;

*(v)* $\left|\frac{\partial F}{\partial t}(t,v) - \frac{\partial F}{\partial t}(t,w)\right| \leqslant 2c_0|v-w|$.

*Proof.* Proof of (i). From $(\mathbf{H}_5)$, we have

$$|f(t,v)| - |f(t,0)| \leqslant |f(t,v) - f(t,0)| = \left|\int_0^v \frac{\partial f}{\partial r}(t,r)dr\right| \leqslant \left|\int_0^v \left|\frac{\partial f}{\partial r}(t,r)\right| dr\right|$$
$$\leqslant \left|\int_0^v c_0(1+|r|^2)dr\right| \leqslant c_0\left(\frac{|v|^3}{3} + |v|\right).$$

Therefore,

$$|f(t,v)| \leqslant |f(t,0)| + \frac{1}{3}c_0|v|^3 + c_0|v| \leqslant c_0 + c_0|v|^3 + c_0|v|,$$

and thus

$$|f(t,v)| \leqslant c_0(1+|v|+|v|^3) \quad \text{for } t,v \in \mathbb{R}.$$

Since $|v| \leqslant |v|^3 + 1$ we have

$$|f(t,v)| \leqslant 2c_0(|v|^3+1) \quad \text{for } t,v \in \mathbb{R}.$$

Proof of (ii). For $t,v,w \in \mathbb{R}$ there exists a point $r$ between $v$ and $w$ such that

$$|f(t,v) - f(t,w)| = \left|\frac{\partial f}{\partial v}(t,r)\right| |v-w| \leqslant c_0(1+|r|^2)|v-w|.$$

Since $|r| \leqslant |v| + |w|$, using Proposition A.6, we obtain

$$|f(t,v) - f(t,w)| \leqslant 2c_0(1+|v|^2+|w|^2)|v-w|.$$

Proof of (iii). Using (i) we have

$$|F(t,v)| \leqslant \left|\int_0^v |f(t,r)|dr\right| \leqslant 2c_0\left|\int_0^v (|r|^3+1)dr\right|$$
$$\leqslant 2c_0\left(\frac{|v|^4}{4} + |v|\right) \leqslant 2c_0(|v|^4+|v|) \leqslant 4c_0(1+|v|^4),$$

since $|v| \leqslant |v|^4 + 2$.

Proof of (iv). Again, using (i) and Proposition A.6, note that

$$
\begin{aligned}
|F(t,v) - F(t,w)| &\leqslant \left| \int_w^v f(t,r)dr \right| \leqslant \left| \int_w^v |f(t,r)|dr \right| \\
&\leqslant \left| \int_w^v 2c_0(1+|r|^3)dr \right| \leqslant 8c_0(1+|v|^3+|w|^3)|v-w|.
\end{aligned}
$$

Proof of (v). Since $\frac{\partial F}{\partial t}(t,v) - \frac{\partial F}{\partial t}(t,w) = \int_w^v \frac{\partial f}{\partial t}(t,r)dr$, this item follows easily. □

In what follows, we use $\hookrightarrow$ to denote continuous inclusions and $\hookrightarrow\!\!\hookrightarrow$ to denote compact inclusions.

**Lemma 3.4.** *There exists a constant $L_0 > 0$ such that*

$$
\|f(t,v) - f(t,w)\|_{L^2} \leqslant L_0(1 + \|v\|^2_{H^1_0} + \|w\|^2_{H^1_0})\|v-w\|_{H^1_0} \quad \textit{for all } v,w \in H^1_0(\Omega).
$$

*In particular*

$$
\|f(t,v) - f(t,w)\|_{L^2} \leqslant L_0(1+2R^2)\|v-w\|_{H^1_0} \quad \textit{for all } v,w \in \overline{B}_R^{H^1_0(\Omega)}.
$$

*Proof.* From Proposition 3.3, since $(a_1+a_2+a_3)^2 \leqslant 4(a_1^2+a_2^2+a_3^2)$ for $a_1,a_2,a_3 \geqslant 0$, we obtain

$$
|f(t,v) - f(t,w)|^2 \leqslant 16c_0^2(1+|v|^4+|w|^4)|v-w|^2,
$$

which, using Hölder's inequality, with exponents $\frac{3}{2}$ and 3, gives us

$$
\begin{aligned}
&\|f(t,v) - f(t,w)\|^2_{L^2} = \int_\Omega |f(t,v) - f(t,w)|^2 dx \\
&\leqslant 16c_0^2 \int_\Omega |v-w|^2 dx + 16c_0^2 \int_\Omega |v|^4|v-w|^2 dx + 16c_0^2 \int_\Omega |w|^4|v-w|^2 dx \\
&\leqslant 16c_0^2\|v-w\|^2_{L^2} + 16c_0^2\|v\|^4_{L^6}\|v-w\|^2_{L^6} + 16c_0^2\|w\|^4_{L^6}\|v-w\|^2_{L^6} \\
&\leqslant \tfrac{16c_0^2}{\lambda_1}\|v-w\|^2_{H^1_0} + 16c_0^2c^6\|v\|^4_{H^1_0}\|v-w\|^2_{H^1_0} + 16c_0^2c^6\|w\|^4_{H^1_0}\|v-w\|^2_{H^1_0},
\end{aligned}
$$

where in the last inequality we used the continuous inclusion $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$, with constant $c > 0$, and Poincaré's inequality $\|u\|_{L^2} \leqslant \frac{1}{\lambda_1}\|u\|_{H^1_0}$ for $u \in H^1_0(\Omega)$. This implies, for $L_0 := \left(\max\left\{\frac{16c_0^2}{\lambda_1}, 16c_0^2c^6\right\}\right)^{\frac{1}{2}}$, that

$$
\|f(t,v) - f(t,w)\|^2_{L^2} \leqslant L_0^2(1+\|v\|^4_{H^1_0} + \|w\|^4_{H^1_0})\|v-w\|^2_{H^1_0},
$$

and the result is proved, since $\sqrt{a_1+a_2+a_3} \leqslant \sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}$ for $a_1,a_2,a_3 \geqslant 0$. □

**Proposition 3.5.** *There exist $0 < \mu_0 < \lambda_1$ and $M = M(\mu_0) > 0$ such that*

$$
\inf_{t\in\mathbb{R}} \frac{\partial f}{\partial v}(t,v) > -\mu_0 \textit{ for all } |v| > M.
$$

*Proof.* This follows directly from the definition of lim inf. □

**Proposition 3.6.** *For each $M > 0$ we have $\int_{-M}^M |f(t,v)|dv \leqslant 8c_0(1+M^4)$ for all $t \in \mathbb{R}$.*

*Proof.* For $t \in \mathbb{R}$ we have

$$\int_{-M}^{M} |f(t,v)|dv \leqslant 2c_0 \int_{-M}^{M} (1+|v|^3)dv = 4c_0 \int_0^M (1+v^3)dv$$
$$= 4c_0 \left(M + \frac{M^4}{4}\right) \leqslant 8c_0(1+M^4),$$

since $M \leqslant M^4 + 2$. □

**Proposition 3.7.** *Fixing $0 < \mu_0 < \lambda_1$ and $M = M(\mu_0) > 0$ given by Proposition 3.5, we have*

$$F(t,v) \geqslant -\tfrac{\mu_0+\lambda_1}{4} v^2 - 8c_0(1+M^4) \quad \text{for } |v| > M \text{ and } t \in \mathbb{R},$$

*and*

$$|F(t,v)| \leqslant 8c_0(1+M^4) \quad \text{for } |v| \leqslant M \text{ and } t \in \mathbb{R}.$$

*Proof.* From Proposition 3.5 we know that $\frac{\partial f}{\partial v}(t,v) > -\mu_0 > -\frac{\mu_0+\lambda_1}{2}$ for $t \in \mathbb{R}$ and $|v| > M$. By the Fundamental Theorem of Calculus, for $v > M$ and $t \in \mathbb{R}$ we have $f(t,v) > -\frac{\mu_0+\lambda_1}{2} v$ and, consequently,

$$\int_M^v f(t,r)dr > -\int_M^v \tfrac{(\mu_0+\lambda_1)}{2} r dr \geqslant -\int_0^v \tfrac{(\mu_0+\lambda_1)}{2} r dr = -\tfrac{(\mu_0+\lambda_1)}{4} v^2.$$

Since from Proposition 3.6 we have

$$\left|\int_0^M f(t,r)dr\right| \leqslant \int_0^M |f(t,r)|\, dr \leqslant \int_{-M}^M |f(t,r)|\, dr \leqslant 8c_0(1+M^4),$$

we obtain

$$F(t,v) = \int_0^M f(t,r)dr + \int_M^v f(t,r)dr \geqslant -\tfrac{(\mu_0+\lambda_1)}{4} v^2 - 8c_0(1+M^4).$$

Analogously we prove the estimate for $v < -M$.

Now, for $|v| \leqslant M$ and $t \in \mathbb{R}$,

$$|F(t,v)| = \left|\int_0^v f(t,r)dr\right| \leqslant \int_{-M}^M |f(t,r)|\, dr \leqslant 8c_0(1+M^4),$$

and the proof is complete. □

**Remark 3.8.** The hypotheses $f(t,0) = 0$ is used to obtain the estimates as they appear in Proposition 3.7. We could, alternatively, require only that $|f(t,0)| \leqslant c_0$ for all $t \in \mathbb{R}$. This proposition (and all the subsequent results) would remain true, but with a small change in the constant $-8c_0(1+M^4)$.

**Proposition 3.9.** *Fixing $0 < \mu_0 < \lambda_1$ and $M = M(\mu_0) > 0$ given by Proposition 3.5, there exists a constant $e_0 > 0$ such that*

$$F(t,v) \leqslant vf(t,v) + \frac{\mu_0}{2} v^2 + e_0 \quad \text{for } |v| > M \text{ and } t \in \mathbb{R}.$$

*Proof.* From Proposition 3.5, we have $\frac{\partial f}{\partial v}(t,v) > -\mu_0$ for $|v| > M$ and $t \in \mathbb{R}$. Using integration by parts, we obtain

$$\int_0^v f(t,r)\ dr = vf(t,v) - \int_0^v r\frac{\partial f}{\partial r}(t,r)\ dr.$$

If $v > M$,

$$F(t,v) = \int_0^v f(t,r)\ dr = vf(t,v) - \int_0^M r\frac{\partial f}{\partial r}(t,r)\ dr - \int_M^v r\frac{\partial f}{\partial r}(t,r)\ dr.$$

We choose a constant $c > \frac{\mu_0 M^2}{2}$ sufficiently large such that $\int_0^M r\frac{\partial f}{\partial r}(t,r)\ dr > -c$. Furthermore,

$$\int_M^v r\frac{\partial f}{\partial r}(t,r)\ dr > -\int_M^v r\mu_0\ dr = -\frac{\mu_0 v^2}{2} + \frac{\mu_0 M^2}{2}.$$

Then,

$$F(t,v) \leqslant vf(t,v) + c + \frac{\mu_0}{2}v^2 - \frac{\mu_0 M^2}{2} = vf(t,v) + \frac{\mu_0}{2}v^2 + e_0,$$

where $e_0 := c - \frac{\mu_0 M^2}{2} > 0$. Analogously we prove the estimate for $v < -M$. □

**Translation of the problem**

Seeking to use our knowledge of the autonomous problem, presented in (ZHAO; ZHAO; ZHONG, 2020a; ZHAO; ZHONG; YAN, 2022), we use the translations of the original nonautonomous problem in order to obtain a problem defined in $[0,\infty)$ rather than on $[s,\infty)$. To be more specific, fixing $s \in \mathbb{R}$ and setting $v(t,x) := u(t+s,x)$ for $t \geqslant 0$ and $x \in \Omega$, we formally have

$$\begin{aligned} &v_{tt}(t,x) - \Delta v(t,x) + k_s(t)\,\|v_t(t,\cdot)\|_{L^2}^p\, v_t(t,x) + f_s(t,v(t,x)) \\ &\qquad - \int_\Omega K(x,y)v_t(t,y)dy - h(x) \\ &= u_{tt}(t+s,x) - \Delta u(t+s,x) + k(t+s)\,\|u_t(t+s,\cdot)\|_{L^2}^p\, u_t(t+s,x) \\ &\qquad + f(t+s,u(t+s,x)) - \int_\Omega K(x,y)u_t(t+s,y)dy - h(x) = 0, \end{aligned}$$

where the boundary and initial conditions become

$$\begin{aligned} &v(t,x) = u(t+s,x) = 0 \quad \text{for } (t,x) \in [0,\infty)\times\partial\Omega, \\ &v(0,x) = u(s,x) = u_0(x),\ v_t(0,x) = u_t(s,x) = u_1(x) \quad \text{for } x \in \Omega. \end{aligned}$$

Thus, we will study the boundary and initial conditions problem

$$\begin{cases} v_{tt}(t,x) - \Delta v(t,x) + k_s(t)\,\|v_t(t,\cdot)\|_{L^2}^p\, v_t(t,x) + f_s(t,v(t,x)) \\ \qquad\qquad = \displaystyle\int_\Omega K(x,y)v_t(t,y)dy + h(x),\ (t,x) \in [0,\infty)\times\Omega, \\ v(t,x) = 0,\ (t,x) \in [0,\infty)\times\partial\Omega, \\ v(0,x) = u_0(x),\ v_t(0,x) = u_1(x),\ x \in \Omega, \end{cases} \tag{tNWE}$$

instead of (NWE). This problem is equivalent to the initial one, but with the nonautonomous terms being $k_s(\cdot) = k(\cdot + s)$ and $f_s(\cdot,\cdot) = f(\cdot + s,\cdot)$ instead of $k$ and $f$. Additionally, we denote $F_s(\cdot,\cdot) = F(\cdot + s,\cdot)$.

## 3.2 WELL-POSEDNESS

We will use the classical Semigroups Theory to obtain the existence of local weak solutions, continuous dependence on initial data and a result regarding the continuation of solution. To that end, we transform (tNWE) into an abstract Cauchy problem in an appropriate phase space.

Taking $w = v_t$ in (tNWE), we obtain

$$w_t = v_{tt} = \Delta v - k_s(t)\|v_t\|_{L^2}^p v_t - f_s(t,v) + \int_\Omega K(x,y)v_t(t,y)dy + h(x),$$

and thus

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} &= \begin{bmatrix} v_t \\ w_t \end{bmatrix} = \begin{bmatrix} w \\ \Delta v - k_s(t)\|v_t(t,\cdot)\|_{L^2}^p v_t - f_s(t,v) + \int_\Omega K(x,y)v_t(t,y)dy + h(x) \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \int_\Omega K(x,y)w(t,y)dy + h(x) - f_s(t,v) - k_s(t)\|w\|_{L^2}^p w \end{bmatrix}.\end{aligned}$$

Setting $X := H_0^1(\Omega) \times L^2(\Omega)$, $V = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$, $V_0 = \left[\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right]$, $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ \Delta & 0 \end{smallmatrix}\right]$ and $\mathcal{G}(t,V) = \left[\begin{smallmatrix} 0 \\ G(t,V) \end{smallmatrix}\right]$, where

$$G(t,V) = \int_\Omega K(x,y)w(t,y)dy + h(x) - f_s(t,v) - k_s(t)\|w\|_{L^2(\Omega)}^p w,$$

we can represent (tNWE) by an abstract Cauchy problem

$$\begin{cases} \dfrac{dV}{dt} = \mathcal{A}V + \mathcal{G}(t,V), \ t > 0 \\ V(0) = V_0 \in X. \end{cases} \tag{ACP}$$

Clearly $X$ is a Hilbert space with the inner product defined by

$$\langle \left[\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} v_2 \\ w_2 \end{smallmatrix}\right] \rangle_X = \langle v_1, v_2\rangle_{H_0^1(\Omega)} + \langle w_1, w_2\rangle_{L^2(\Omega)},$$

with associated norm

$$\|\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]\|_X^2 = \|v\|_{H_0^1}^2 + \|w\|_{L^2}^2.$$

The operator $\mathcal{A}\colon D(\mathcal{A}) \subset X \to X$ has its usual domain $D(\mathcal{A}) = \{\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in X : \mathcal{A}\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in X\}$, and we can characterize it completely.

**Lemma 3.10.** *We have $D(\mathcal{A}) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.*

*Proof.* Let $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in D(\mathcal{A})$. Then $v \in H_0^1(\Omega)$, $\Delta v \in L^2(\Omega)$ and $w \in H_0^1(\Omega)$. Therefore $v \in H^2(\Omega)$ (by elliptic regularity), which proves that $D(\mathcal{A}) \subset [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.

Conversely, if $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, we have $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w \in H_0^1(\Omega) \subset L^2(\Omega)$. Clearly $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in X$ and since $v \in H^2(\Omega)$, we have $\Delta v \in L^2(\Omega)$. This proves the converse inclusion and completes the proof. □

### The linear problem

It follows directly from the characterization of $D(\mathcal{A})$ that it is densely defined. If we can prove that it is closed, dissipative and $\operatorname{Im}(I-\mathcal{A}) = X$, we can apply the Lumer-Philips Theorem (see (PAZY, 2012)) to show that $\mathcal{A}$ is the infinitesimal generator of a $C^0$-semigroup of contractions $\{e^{\mathcal{A}t} \colon t \geqslant 0\}$ in X.

**Proposition 3.11.** *$\mathcal{A}$ is closed, dissipative and* $\operatorname{Im}(I-A) = X$*.*

*Proof.* To show that $\mathcal{A}$ is closed consider the sequence $\{\left[\begin{smallmatrix} v_n \\ w_n \end{smallmatrix}\right]\} \subset D(\mathcal{A})$ such that $\left[\begin{smallmatrix} v_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ in $X$, and also $\mathcal{A}\left[\begin{smallmatrix} v_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} y \\ b \end{smallmatrix}\right]$ in $X$. This means that

$$\text{(i)}\ v_n \xrightarrow{H^1_0(\Omega)} v; \qquad \text{(ii)}\ w_n \xrightarrow{L^2(\Omega)} w; \qquad \text{(iii)}\ w_n \xrightarrow{H^1_0(\Omega)} y; \qquad \text{(iv)}\ \Delta v_n \xrightarrow{L^2(\Omega)} b.$$

From (ii) and (iii) is clear that $w = y \in H^1_0(\Omega)$. Now for a given test function $\phi \in C^\infty_0(\Omega)$ we have

$$\int_\Omega b\phi dx \leftarrow \int_\Omega \Delta v_n \phi dx = \int_\Omega v_n \Delta\phi dx \to \int_\Omega v\Delta\phi dx,$$

which shows that $b = \Delta v \in L^2(\Omega)$, and since $v \in H^1_0(\Omega)$, we obtain $v \in H^2(\Omega)$. Therefore $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in D(\mathcal{A})$ and $\mathcal{A}\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} y \\ b \end{smallmatrix}\right]$, which shows that $\mathcal{A}$ is closed.

Now if $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in D(\mathcal{A})$ we obtain

$$\begin{aligned}
\langle \mathcal{A}\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]\rangle_X = \langle \left[\begin{smallmatrix} w \\ \Delta v \end{smallmatrix}\right], \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]\rangle_X &= \langle w, v\rangle_{H^1_0(\Omega)} + \langle \Delta v, w\rangle_{L^2(\Omega)} \\
&= \int_\Omega \nabla w \nabla v dx + \int_\Omega w\Delta v dx \\
&= \int_\Omega \nabla v \nabla w dx - \int_\Omega \nabla v \nabla w dx = 0,
\end{aligned}$$

which shows that $\mathcal{A}$ is dissipative.

It remains to show that $\operatorname{Im}(I-\mathcal{A}) = X$, that is, given $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$, we want to find $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in D(A)$ such that $(I-\mathcal{A})\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$. This is equivalent to find $v \in H^2(\Omega) \cap H^1_0(\Omega)$ and $w \in H^1_0(\Omega)$ such that

$$\begin{cases} v - w = f, \\ w - \Delta v = g. \end{cases}$$

Adding both equations we obtain:

$$v - \Delta v = f + g. \tag{3.5}$$

If we can find $v \in H^2(\Omega) \cap H^1_0(\Omega)$ that solves (3.5), defining $w := v - f$ we have $w \in H^1_0(\Omega)$ and the problem is solved. Thus, our problem is that given $f \in H^1_0(\Omega)$ and $g \in L^2(\Omega)$, we want to find $v \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $v - \Delta v = g + f$.

Define the symmetrical bilinear form $a \colon H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ by

$$a(h,v) = \int_\Omega vh dx + \int_\Omega \nabla h \nabla v dx = \langle h,v\rangle_{L^2(\Omega)} + \langle h, v\rangle_{H^1_0(\Omega)}.$$

Using the Poincaré's inequality, we have

$$|a(h,v)| \leqslant (1+\lambda_1^{-1})\|h\|_{H_0^1}\|v\|_{H_0^1},$$

hence $a$ is continuous. Also

$$a(v,v) \geqslant \|v\|_{H_0^1}^2,$$

which means that $a$ is coercive.

Define the linear functional $\xi\colon H_0^1(\Omega) \to \mathbb{R}$ by $\xi(h) = \int_\Omega (g+f)\, h dx$. Again, using the Poincaré's inequality we obtain

$$|\xi(h)| \leqslant \lambda_1^{-1}(\|g\|_{L^2} + \|f\|_{L^2})\|h\|_{H_0^1},$$

which shows that $\xi$ is continuous. From the Lax-Milgram Theorem (see (BREZIS, 2011)) there exists a unique $v \in H_0^1(\Omega)$ such that $a(h,v) = \xi(h)$ for every $h \in H_0^1(\Omega)$, that is

$$\int_\Omega \nabla v \nabla h dx + \int_\Omega v h dx = \int_\Omega (g+f)\, h dx.$$

If $h \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \nabla v \nabla h dx = \int_\Omega (g+f-v) h dx,$$

which means that $\Delta v = v - f - g \in L^2(\Omega)$. Since $v \in H_0^1(\Omega)$ we obtain $v \in H^2(\Omega)$, and thus, $v - \Delta v = f + g$. Therefore, $\mathrm{Im}(I - \mathcal{A}) = X$ and the proof is complete. □

Thus, by the Lumer-Philips Theorem, $\mathcal{A}$ generates a $C^0-$semigroup of contractions $\{e^{\mathcal{A}t}\colon t \geqslant 0\} \subset \mathcal{L}(X)$. Hence, from the results of (PAZY, 2012), for each $V_0 \in X$, the map

$$[0,\infty) \ni t \mapsto e^{\mathcal{A}t} V_0 \in X$$

is the unique weak solution to the problem

$$\begin{cases} \dfrac{dV}{dt} = \mathcal{A}V, \ t > 0, \\ V(0) = V_0. \end{cases}$$

### The semilinear hyperbolic problem

Now we deal with problem (ACP), which is a semilinear problem of hyperbolic type. To that end, we look into the nonlinear term $\mathcal{G}(t,V)$.

**Proposition 3.12.** *Given $R > 0$ there exists $L_R \geqslant 0$ such that*

$$\|\mathcal{G}(t,V_1) - \mathcal{G}(t,V_2)\|_X \leqslant L_R \|V_1 - V_2\|_X \quad \text{for all } V_1,V_2 \in \overline{B}_R^X \text{ and } t \in \mathbb{R},$$

*that is $\mathcal{G}\colon \mathbb{R} \times X \to X$ is locally Lipschitz in the second variable, uniformly for $t \in \mathbb{R}$.*

*Proof.* Let $V_1 = \left[\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right]$ and $V_2 = \left[\begin{smallmatrix} v_2 \\ w_2 \end{smallmatrix}\right]$ taken in $\overline{B}_R^X$. We can assume, without loss of generality, that $\|w_1\|_{L^2} \geqslant \|w_2\|_{L^2}$. Note that

$$\begin{aligned}&\|G(t,V_1) - G(t,V_2)\|_{L^2} \\ &\leqslant \left\|\int_\Omega K(x,y)(w_1 - w_2)(t,y)dy\right\|_{L^2} + \|f_s(t,v_1) - f_s(t,v_2)\|_{L^2} \\ &\quad + k_s(t)\left\|\|w_1\|_{L^2}^p w_1 - \|w_2\|_{L^2}^p w_2\right\|_{L^2}.\end{aligned}$$

The first term on the right hand side is estimated using Proposition 3.2:

$$\left\|\int_\Omega K(x,y)(w_1 - w_2)(t,y)dy\right\|_{L^2} \leqslant K_0\|w_1 - w_2\|_{L^2}.$$

Applying Lemma 3.4, we estimate the second term:

$$\|f_s(t,v_1) - f_s(t,v_2)\|_{L^2} \leqslant L_0(1 + 2R^2)\|v_1 - v_2\|_{H_0^1},$$

since $v_1,v_2 \in \overline{B}_R^{H_0^1(\Omega)}$. For the third one, it follows from Proposition A.10 that

$$k_s(t)\left\|\|w_1\|_{L^2}^p w_1 - \|w_2\|_{L^2}^p w_2\right\|_{L^2} \leqslant k_1 R^p(1 + p)\|w_1 - w_2\|_{L^2}.$$

Thus, we obtain

$$\|G(t,V_1) - G(t,V_2)\|_{L^2} \leqslant L_0(1 + 2R^2)\|v_1 - v_2\|_{H_0^1} + (K_0 + k_1 R^p(1 + p))\|w_1 - w_2\|_{L^2}.$$

Setting

$$L_R := (2\max\{L_0^2(1 + 2R^2)^2,\ (K_0 + k_1 R^p(1 + p))^2\})^{\frac{1}{2}},$$

we obtain

$$\|G(t,V_1) - G(t,V_2)\|_{L^2}^2 \leqslant L_R^2(\|v_1 - v_2\|_{H_0^1}^2 + \|w_1 - w_2\|_{L^2}^2) = L_R^2\|V_1 - V_2\|_X^2,$$

and the result follows easily, since $\|\mathcal{G}(t,V_1) - \mathcal{G}(t,V_2)\|_X = \|G(t,V_1) - G(t,V_2)\|_{L^2}$. □

In the next result we will prove that $\mathcal{G}$ takes bounded subsets of $\mathbb{R}\times X$ into bounded subsets of $X$.

**Proposition 3.13.** *Given $R > 0$, for all $V \in \overline{B}_R^X$ and $t \in \mathbb{R}$, we have*

$$\|\mathcal{G}(t,V)\|_X \leqslant RL_R + h_0 + c_0|\Omega|^{\frac{1}{2}},$$

*where $|\Omega|$ denotes the 3-dimensional Lebesgue measure of $\Omega$.*

*Proof.* From the previous proposition, for $t \in \mathbb{R}$ and $V \in \overline{B}_R^X$, we have

$$\|\mathcal{G}(t,V) - \mathcal{G}(t,0)\|_X \leqslant L_R\|V\|_X \leqslant RL_R.$$

Thus

$$\|\mathcal{G}(t,V)\|_X \leqslant RL_R + \|\mathcal{G}(t,0)\|_X.$$

Since $G(t,0) = h - f_s(t,0)$, we have

$$\|\mathcal{G}(t,0)\|_X = \|G(t,0)\|_{L^2} \leqslant \|h\|_{L^2} + \|f_s(t,0)\|_{L^2} \leqslant h_0 + c_0|\Omega|^{\frac{1}{2}}.$$

□

All these properties, when put together, using the results of (PAZY, 2012), give us the following:

**Proposition 3.14.** *For any given $V_0 \in X$ there exists a unique maximal weak solution $V(\cdot,V_0)\colon [0,\tau_{max}) \to X$ of (ACP), that is, a continuous function such that*

$$V(t,V_0) = e^{\mathcal{A}t}V_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{G}(\tau,V(\tau,V_0))d\tau \quad \text{for } t \in [0,\tau_{max}),$$

*such that either $\tau_{max} = \infty$ or $\tau_{max} < \infty$ and $\limsup_{t\to\tau_{max}^-} \|V(t,V_0)\|_X = \infty$. This solution is such that for each $\xi^* \in D(\mathcal{A}^*)$ the map $(0,\tau_{max}) \ni t \mapsto \langle V(t,V_0),\xi^*\rangle_{X,X^*}$ is continuously differentiable and for all $t \in (0,\tau_{max})$ it satisfies*

$$\frac{d}{dt}\langle V(t,V_0),\xi^*\rangle_{X,X^*} = \langle V(t,V_0),\mathcal{A}^*\xi^*\rangle_{X,X^*} + \langle \mathcal{G}(t,V_0),\xi^*\rangle_{X,X^*}, \tag{3.6}$$

*where $\langle\cdot,\cdot\rangle_{X,X^*}$ denotes the duality of $X^*$ in $X$.*

Writing $V(t,V_0) = \left[\begin{smallmatrix} v(t) \\ w(t) \end{smallmatrix}\right]$ we can see that

$$v \in C([0,\tau_{max}),H_0^1(\Omega)) \text{ and } w \in C([0,\tau_{max}),L^2(\Omega)).$$

Since $w = v_t$, we conclude that

$$v \in C([0,\tau_{max}),H_0^1(\Omega)) \cap C^1([0,\tau_{max}),L^2(\Omega))$$

is the unique maximal weak solution to (tNWE). We will write (3.6) in the usual distributional sense and, to that end, we first note that $X^* = H^{-1}(\Omega) \times L^2(\Omega)$, where $H^{-1}(\Omega) = (H_0^1(\Omega))^*$. We must describe the dual operator $\mathcal{A}^*\colon D(\mathcal{A}^*) \subset H^{-1}(\Omega)\times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$ of $\mathcal{A}$. If $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in L^2(\Omega) \times D(A)$ then for each $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in D(\mathcal{A})$, since $A$ is self-adjoint, we have

$$\begin{aligned}\left\langle \mathcal{A}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]\right\rangle_{X,X^*} &= \left\langle \left[\begin{smallmatrix} y \\ Ax \end{smallmatrix}\right], \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]\right\rangle_{X,X^*} = \langle y,x^*\rangle_{H_0^1,H^{-1}} + \langle Ax,y^*\rangle \\ &= \langle y,x^*\rangle_{H_0^1,H^{-1}} + \langle x,Ay^*\rangle = \left\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & A \\ I & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]\right\rangle_{X,X^*}\end{aligned}$$

hence $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in D(\mathcal{A}^*)$ and $\mathcal{A}^*\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] = \left[\begin{smallmatrix} Ay^* \\ x^* \end{smallmatrix}\right]$. Conversely, if $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in D(\mathcal{A}^*)$ and $\mathcal{A}^*\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] = \left[\begin{smallmatrix} z^* \\ w^* \end{smallmatrix}\right]$ then for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in D(\mathcal{A})$ we have

$$\left\langle \mathcal{A}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]\right\rangle_{X,X^*} = \left\langle \left[\begin{smallmatrix} y \\ Ax \end{smallmatrix}\right], \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]\right\rangle_{X,X^*} = \langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} z^* \\ w^* \end{smallmatrix}\right]\rangle_{X,X^*}. \tag{3.7}$$

Taking $x \in D(A)$ and $y = 0$ we have $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in D(\mathcal{A})$ and (3.7) implies that

$$\langle Ax,y^*\rangle = \langle x,z^*\rangle,$$

which shows that $y^* \in D(A)$ and $Ay^* = z^*$. Analogously, taking $y \in H_0^1(\Omega)$ then $\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \in D(\mathcal{A})$ and (3.7) implies that

$$\langle y,x^*\rangle_{H_0^1,H^{-1}} = \langle y,w^*\rangle_{H_0^1,H^{-1}},$$

which proves that $w^* = x^* \in L^2(\Omega)$. Thus $D(\mathcal{A}^*) = L^2(\Omega) \times D(A)$ and for $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in D(\mathcal{A}^*)$ we have $\mathcal{A}^* \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] = \left[\begin{smallmatrix} Ay^* \\ x^* \end{smallmatrix}\right]$.

With this, (3.7) reads: for all $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in L^2(\Omega) \times D(A)$ we have the map

$$(0,\tau_{max}) \ni t \mapsto \langle v(t),x^* \rangle_{H_0^1,H^{-1}} + \langle v_t(t),y^* \rangle$$

continuously differentiable, and for all $t \in (0,\tau_{max})$ we have

$$\frac{d}{dt}\Big( \langle v(t),x^* \rangle_{H_0^1,H^{-1}} + \langle v_t(t),y^* \rangle \Big) = \langle v(t), Ay^* \rangle_{H_0^1,H^{-1}} + \langle v_t(t),x^* \rangle + \langle G(t,V_0), y^* \rangle.$$

Since $x^* \in L^2(\Omega)$, $Ay^* \in L^2(\Omega)$ and $v(t) \in H_0^1(\Omega)$ we have

$$\frac{d}{dt}\Big( \langle v(t),x^* \rangle + \langle v_t(t),y^* \rangle \Big) = \langle v(t), Ay^* \rangle + \langle v_t(t),x^* \rangle + \langle G(t,V_0), y^* \rangle.$$

This means that taking $x^* = 0$ and $y^* = \psi \in D(A)$, the solution $V(t,V_0) = \left[\begin{smallmatrix} v(t) \\ v_t(t) \end{smallmatrix}\right]$ satisfies

$$\begin{aligned} \frac{d}{dt}\int_\Omega v_t(t)\psi dx + \int_\Omega f_s(t,v(t))\psi dx + k_s(t)\|v_t(t)\|_{L^2}^p \int_\Omega v_t(t)\psi dx = \int_\Omega v(t)\Delta\psi dx \\ + \int_{\Omega\times\Omega} K(x,y)v_t(t,y)\psi(x)dydx + \int_\Omega h\psi dx, \end{aligned}$$

that is, for all $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega v_t(t)\psi dx + \int_\Omega \nabla v(t)\nabla\psi dx + k_s(t)\|v_t(t)\|_{L^2}^p \int_\Omega v_t(t)\psi dx + \int_\Omega f_s(t,v(t))\psi dx \\ = \int_{\Omega\times\Omega} K(x,y)v_t(t,y)\psi(x)dydx + \int_\Omega h\psi dx, \end{aligned} \tag{3.8}$$

and by density, this last equality holds for all $\psi \in H_0^1(\Omega)$, which is the *usual definition* of the weak solution for (tNWE).

**The function $v_{tt}$.**

For a weak maximal solution $V(t,V_0) = (v(t),v_t(t))$ of (ACP) and $t \in (0,\tau_{max})$ fixed, we define the linear functional $\lambda\colon H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} \lambda(\psi) = -\int_\Omega \nabla v(t)\nabla\psi dx - k_s(t)\|v_t(t)\|_{L^2}^p \int_\Omega v_t(t)\psi dx - \int_\Omega f_s(t,v(t))\psi dx \\ + \int_{\Omega\times\Omega} K(x,y)v_t(t,y)\psi(x)dydx + \int_\Omega h\psi dx. \end{aligned}$$

From the estimates we have and Poincaré's inequality we obtain

$$\begin{aligned} |\lambda(\psi)| \leqslant \|v(t)\|_{H_0^1}\|\psi\|_{H_0^1} + k_s(t)\lambda_1^{-\frac{1}{2}}\|v_t(t)\|_{L^2}^{p+1}\|\psi\|_{H_0^1} + \lambda_1^{-\frac{1}{2}}\|f_s(t,v(t))\|_{L^2}\|\psi\|_{H_0^1} \\ + K_0\lambda_1^{-\frac{1}{2}}\|v_t(t)\|_{L^2}\|\psi\|_{H_0^1} + \lambda_1^{-\frac{1}{2}}h_0\|\psi\|_{H_0^1}, \end{aligned}$$

which shows that $\lambda$ is continuous, that is, $\lambda \in H^{-1}(\Omega)$. From (3.8) we have

$$\frac{d}{dt}\langle v_t(t),\psi \rangle = \lambda(\psi),$$

for all $\psi \in H_0^1(\Omega)$. Thus, if we define $v_{tt}(t) = \lambda \in H^{-1}(\Omega)$, we can rewrite (3.8) in $H^{-1}(\Omega)$ as

$$\langle \psi, v_{tt}(t) \rangle_{H_0^1, H^{-1}} = \lambda(\psi) \quad \text{for all } \psi \in H_0^1(\Omega).$$

Thus, we can make sense of the function $v_{tt} \colon (0,\tau_{\max}) \to H^{-1}(\Omega)$ and we obtain

$$\begin{aligned} \|v_{tt}(t)\|_{H^{-1}} \leqslant \|v(t)\|_{H_0^1} + k_s(t)\lambda_1^{-\frac{1}{2}}\|v_t(t)\|_{L^2}^{p+1} + \lambda_1^{-\frac{1}{2}}\|f_s(t,v(t))\|_{L^2} \\ + K_0\lambda_1^{-\frac{1}{2}}\|v_t(t)\|_{L^2} + \lambda_1^{-\frac{1}{2}}h_0. \end{aligned} \tag{3.9}$$

Also, since $\frac{d}{dt}\langle \psi, v_t(t) \rangle_{H_0^1, H^{-1}} = \frac{d}{dt}\langle v_t(t), \psi \rangle = \langle \psi, v_{tt}(t) \rangle_{H_0^1, H^{-1}}$, $v_{tt}$ is the weak derivative of $v_t$ in the distributional sense in $H^{-1}(\Omega)$.

### Non-explosion in finite time

In order to prove that $\tau_{max} = \infty$ for all $V_0 \in X$, we need to show that the second condition cannot occur. This will be a consequence of some results we will present later (see Remark 3.23) and we know that for each given $V_0 \in X$ the solution of (ACP) is defined for all $t \geqslant 0$.

Fix $s \in \mathbb{R}$ and $V_0 := (u_0,u_1) \in X$. If $V(\cdot,V_0) \colon [0,\infty) \to X$ is the solution of (ACP), we define for $t \geqslant s$

$$S(t,s)(u_0,u_1) = V(t-s, V_0). \tag{3.10}$$

Note that if, for $t \geqslant s$, we set $(u(t),y(t)) = S(t,s)(u_0,u_1)$, then $u(t) = v(t-s)$ and $y(t) = v_t(t-s) = u_t(t)$. Thus $(u(t),u_t(t))$ is the weak solution of (NWE), and $\{S(t,s) \colon t \geqslant s\}$ defines an evolution process in $X$ associated with (NWE), provided we have continuity with respect with initial data (which will be proven in what follows).

### Continuous dependence on the initial data: a Lipschitz condition

Let $V_1,V_2 \in X$ be the initial data for (ACP), taken in the closed ball $\overline{B}_R^X$, for a given $R > 0$. Consider $v, w$ the corresponding solutions for (tNWE) related respectively to these initial data. Later, see Proposition 3.22, we will show that there exists a constant $c_R \geqslant 0$ such that for all $t \geqslant 0$ we have

$$\|V(t,V_1)\|_X \leqslant c_R \quad \text{and} \quad \|V(t,V_2)\|_X \leqslant c_R. \tag{3.11}$$

If $V(t,V_1) = (v(t),v_t(t))$ and $V(t,V_2) = (w(t),w_t(t))$, setting $z := v - w$, formally we obtain

$$z_{tt} - \Delta z + k_s(t)(\|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t) + f_s(t,v) - f_s(t,w) = \int_\Omega K(x,y) z_t(t,y)dy. \tag{3.12}$$

**Proposition 3.15.** *Given $R > 0$, there exists a constant $\gamma_0 = \gamma_0(R) \geqslant 0$ such that for $t \geqslant 0$ and $V_1,V_2 \in \overline{B}_R^X$, we have*

$$\|V(t,V_1) - V(t,V_2)\|_X \leqslant \|V_1 - V_2\|_X e^{\gamma_0 t}.$$

*Proof.* If $Z(t) = V(t,V_1) - V(t,V_2)$, then $Z(t) = (z(t),z_t(t))$ for each $t \geqslant 0$. Multiplying formally (3.12) by $z_t$ and integrating over $\Omega$, we obtain

$$\begin{aligned}\int_\Omega z_t z_{tt} dx - \int_\Omega \Delta z z_t dx + k_s(t)\|v_t\|_{L^2}^p \int_\Omega v_t z_t dx - k_s(t)\|w_t\|_{L^2}^p \int_\Omega w_t z_t dx \\ + \int_\Omega \left[f_s(t,v) - f_s(t,w)\right] z_t dx = \int_{\Omega\times\Omega} K(x,y) z_t(t,y) z_t(t,x) dy dx.\end{aligned} \tag{3.13}$$

Now observe that, if $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2(\Omega)$, (3.13) becomes

$$\begin{aligned}\langle z_t,z_{tt}\rangle - \langle \Delta z, z_t\rangle + k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \\ + \langle f_s(t,v) - f_s(t,w), z_t\rangle = \int_{\Omega\times\Omega} K(x,y) z_t(t,y) z_t(t,x) dy dx.\end{aligned} \tag{3.14}$$

Since

$$\frac{1}{2}\frac{d}{dt}\|Z(t)\|_X^2 = \frac{1}{2}\frac{d}{dt}\Big(\|z\|_{H_0^1}^2 + \|z_t\|_{L^2}^2\Big) = \langle z_t, z_{tt}\rangle - \langle \Delta z, z_t\rangle, \tag{3.15}$$

equation (3.14) writes

$$\begin{aligned}\frac{d}{dt}\|Z(t)\|_X^2 = 2\int_{\Omega\times\Omega} K(x,y) z_t(t,y) z_t(t,x) dy dx - 2\langle f_s(t,v) - f_s(t,w), z_t\rangle \\ - 2k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle.\end{aligned} \tag{3.16}$$

It follows from Proposition A.11, in particular, that

$$\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \geqslant 0,$$

and hence

$$\frac{d}{dt}\|Z(t)\|_X^2 \leqslant 2\int_{\Omega\times\Omega} K(x,y) z_t(t,y) z_t(t,x) dy dx - 2\langle f_s(t,v) - f_s(t,w), z_t\rangle.$$

Later (see (3.27)), we will see that

$$\int_{\Omega\times\Omega} K(x,y) z_t(t,y) z_t(t,x) dy dx \leqslant K_0\|z_t\|_{L^2}^2, \tag{3.17}$$

and, by Lemma 3.4 and (3.11), we have

$$\begin{aligned}-\langle f_s(t,v) - f_s(t,w), z_t\rangle &\leqslant \|f_s(t,v) - f_s(t,w)\|_{L^2}\|z_t\|_{L^2} \\ &\leqslant L_0(1 + \|v\|_{H_0^1}^2 + \|w\|_{H_0^1}^2)\|v - w\|_{H_0^1}\|z_t\|_{L^2} \leqslant L_0(1 + 2c_R^2)\|z\|_{H_0^1}\|z_t\|_{L^2}.\end{aligned} \tag{3.18}$$

Therefore, for $t \geqslant 0$, we obtain

$$\begin{aligned}\frac{d}{dt}\|Z(t)\|_X^2 &\leqslant 2K_0\|z_t\|_{L^2}^2 + 2L_0(1 + 2c_R^2)\|z\|_{H_0^1}\|z_t\|_{L^2} \\ &\overset{(1)}{\leqslant} 2K_0\|z_t\|_{L^2}^2 + L_0(1 + 2c_R^2)(\|z\|_{H_0^1}^2 + \|z_t\|_{L^2}^2) \leqslant (2K_0 + L_0(1 + 2c_R^2))\|Z(t)\|_X^2,\end{aligned}$$

where in (1) we use again the usual Young's inequality. Applying Grönwall's inequality (Lemma A.9), we have

$$\|Z(t)\|_X \leqslant \|Z(0)\|_X e^{\gamma_0 t},$$

where $\gamma_0 = K_0 + \frac{L_0}{2}(1 + 2c_R^2)$, and the proof is complete.

□

## 3.3 PROPERTIES OF UNIFORMLY PULLBACK ABSORPTION

In this section we prove the existence of a closed, positively invariant, uniformly bounded and uniformly pullback absorbing family for the process related to the problem (NWE) in the cases $p > 0$ and $p = 0$. This fact, together with the $\varphi$-pullback $\kappa$-dissipativity of the evolution process $S$ associated with (NWE) is essential to show the existence of a generalized polynomial pullback attractor (when $p > 0$) and of a generalized exponential pullback attractor (when $p = 0$). These facts, consequently, will imply the existence of a pullback attractor for both cases.

We fix $0 < \mu_0 < \lambda_1$ and $M = M(\mu_0) > 0$ given by Proposition 3.5. For a function $v \in L^2(\Omega)$ we set

$$\Omega_1 = \{x \in \Omega \colon |v(x)| > M\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega \colon |v(x)| \leqslant M\}. \tag{3.19}$$

From Proposition 3.7, for each fixed $t \in \mathbb{R}$ we obtain

$$\begin{aligned}\int_\Omega F(t,v)dx &= \int_{\Omega_1} F(t,v)dx + \int_{\Omega_2} F(t,v)dx \\ &\geqslant -\int_{\Omega_1} \left[\tfrac{\mu_0+\lambda_1}{4}|v|^2 + 8c_0(1+M^4)\right]dx - \int_{\Omega_2} 8c_0(1+M^4)dx \geqslant -\tfrac{\mu_0+\lambda_1}{4}\|v\|^2_{L^2} - C_0,\end{aligned}$$

where $C_0 := 8c_0(1+M^4)|\Omega| > 0$. That is, we obtained

$$\int_\Omega F(t,v)dx + \tfrac{\mu_0+\lambda_1}{4}\|v\|^2_{L^2} + C_0 \geqslant 0. \tag{3.20}$$

For each given initial data $V_0 = (u_0,u_1) \in X$, define the function $E_s(\cdot,V_0)\colon \mathbb{R}^+ \to \mathbb{R}$ given by

$$E_s(t,V_0) = \tfrac{1}{2}(\|v_t\|^2_{L^2} + \|v\|^2_{H^1_0}) + \int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|^2_{L^2} + C_0, \tag{3.21}$$

where $C_0$ is the constant obtained in (3.20), and $V(t,V_0) = (v(t),v_t(t))$ for $t \geqslant 0$. For simplicity, we write $E_s(t)$ instead of $E_s(t,V_0)$, but we must keep in mind that this function depends on the initial data $V_0 \in X$, since it depends on the solution $V(t,V_0)$.

**Existence of a uniformly pullback absorbing family for the process $S$ associated with (NWE) when $p > 0$**

In this section we will show that there exists a constant $r_0 > 0$ such that the family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$ given by

$$B_t = \overline{B}^X_{r_0} \quad \text{for each } t \in \mathbb{R} \tag{3.22}$$

is a uniformly pullback absorbing family for $S$. More precisely, we will prove the following theorem:

**Theorem 3.16** (Existence of a pullback absorbing family)**.** *There exists $r_0 > 0$ such that for each $R > 0$ there exists $\tau_0 = \tau_0(R) \geqslant 0$ with*

$$\|S(t,t-\tau)(u_0,u_1)\|_X \leqslant r_0,$$

*for all $\tau \geqslant \tau_0$, $t \in \mathbb{R}$ and $(u_0,u_1) \in X$ with $\|(u_0,u_1)\|_X \leqslant R$.*

Note that for $t \geqslant s$ and $V_0 = (u_0,u_1) \in X$, from (3.20) and (3.21) we have

$$\|S(t,s)(u_0,u_1)\|_X^2 = \|V(t-s,V_0)\|_X^2 \leqslant 2E_s(t-s).$$

Hence, the study of the function $E_s$ is paramount, since this function bounds the norm of $S(t,s)(u_0,u_1)$ for $t \geqslant s$. The proof of Theorem 3.16 is not trivial, and to facilitate its comprehension, we present a scheme that will help us with this goal.

**Scheme.**

(I) Define an auxiliary function $V_\varepsilon^s$ and prove that it is *equivalent* to $E_s$ in the sense of (3.24), which is done in Proposition 3.17;

(II) Using the function $V_\varepsilon^s$ obtained in (I), prove that the function $E_s$ remains bounded for all times $t \geqslant 0$, uniformly for initial data in bounded subsets of $X$. This result is achieved in Proposition 3.22;

(III) Improve the result obtained in (II), and in Proposition 3.28 show that there exists a fixed bounded subset of $X$ such that $E_s(t)$ is cointained in this subset if $t$ is sufficiently large, uniformly for the initial data in bounded subsets of $X$;

(IV) Use the result from (III) to prove Theorem 3.16.

Development of (I).

For a fixed $\varepsilon > 0$ and $(v,v_t) = V(t,V_0)$ a solution to (ACP) we define $V_\varepsilon^s(\cdot,V_0)\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$V_\varepsilon^s(t,V_0) = \tfrac{1}{2}\Big(\|v\|_{H_0^1}^2 + \|v_t\|_{L^2}^2\Big) + \int_\Omega F_s(t,v)dx - \int_\Omega hvdx + \varepsilon \int_\Omega v_t v dx. \tag{3.23}$$

As we did for the function $E_s$, to simplify the notation we simply write $V_\varepsilon^s(t)$ instead of $V_\varepsilon^s(t,V_0)$. However, we must keep in mind that this function depends on the initial data $V_0 \in X$, and all the comparisons that are done are using the same initial data $V_0$.

Our goal at this stage is to prove the following proposition.

**Proposition 3.17.** *There exist $\varepsilon_0 > 0$ and a constant $d_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $s \in \mathbb{R}$ and $t \geqslant 0$ we have*

$$\tfrac{1}{4}\left(1 - \tfrac{\mu_0}{\lambda_1}\right) E_s(t) - d_0 \leqslant V_\varepsilon^s(t) \leqslant \tfrac{5}{4}E_s(t) + d_0. \tag{3.24}$$

The proof of this result is lengthy, and we firstly prove a few auxiliary lemmas.

**Lemma 3.18.** *There exists a constant $g_0 > 0$ such that for all $s \in \mathbb{R}$ and $t \geqslant 0$ we have*

$$-\int_\Omega f_s(t,v)vdx \leqslant -\int_\Omega F_s(t,v)dx - \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 - C_0 + \tfrac{1}{4}\left(\tfrac{3\mu_0}{\lambda_1}+1\right)\|v\|_{H_0^1}^2 + g_0,$$

*where $C_0 > 0$ is given in (3.20).*

*Proof.* Recalling the definitions for $\Omega_1$ and $\Omega_2$ in (3.19), from Proposition 3.9 we obtain

$$-\int_\Omega f_s(t,v)vdx = -\int_{\Omega_1} f_s(t,v)vdx - \int_{\Omega_2} f_s(t,v)vdx$$

$$
\begin{aligned}
&\leqslant -\int_{\Omega_1} F_s(t,v)dx + \frac{\mu_0}{2}\int_{\Omega_1} v^2dx + \int_{\Omega_1} e_0 dx - \int_{\Omega_2} f_s(t,v)vdx \\
&= -\int_{\Omega} F_s(t,v)dx + \int_{\Omega_2} F_s(t,v)dx + \tfrac{\mu_0}{2}\|v\|_{L^2}^2 - \tfrac{\mu_0}{2}\int_{\Omega_2} v^2dx + e_0|\Omega_1| - \int_{\Omega_2} f_s(t,v)vdx \\
&= -\int_{\Omega} F_s(t,v)dx + \tfrac{\mu_0}{2}\|v\|_{L^2}^2 + e_0|\Omega_1| + \int_{\Omega_2} F_s(t,v)dx - \tfrac{\mu_0}{2}\int_{\Omega_2} v^2dx - \int_{\Omega_2} f_s(t,v)vdx.
\end{aligned}
$$

Obviously $-\frac{\mu_0}{2}\int_{\Omega_2} v^2dx \leqslant 0$. Using Proposition 3.7 we have

$$
\int_{\Omega_2} F_s(t,v)dx \leqslant \int_{\Omega_2} |F_s(t,v)|\,dx \leqslant 8c_0(1+M^4)|\Omega_2|,
$$

and using Proposition 3.6 we obtain

$$
-\int_{\Omega_2} f_s(t,v)vdx \leqslant \int_{\Omega_2} |f_s(t,v)||v|dx \leqslant 8Mc_0(1+M^4),
$$

since $|v| \leqslant M$ in $\Omega_2$. Therefore, setting $g_0 := e_0|\Omega_1| + 8c_0(1+M^4)|\Omega_2| + 8Mc_0(1+M^4) + C_0 > 0$, we obtain

$$
-\int_{\Omega} f_s(t,v)vdx \leqslant -\int_{\Omega} F_s(t,v)dx + \tfrac{\mu_0}{2}\|v\|_{L^2}^2 + g_0 - C_0.
$$

Using Poincaré's inequality we obtain

$$
\begin{aligned}
-\int_{\Omega} f_s(t,v)vdx &\leqslant -\int_{\Omega} F_s(t,v)dx - \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 - C_0 + \tfrac{\mu_0}{2}\|v\|_{L^2}^2 + g_0 + \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 \\
&\leqslant -\int_{\Omega} F_s(t,v)dx - \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 - C_0 + \tfrac{1}{4}\left(\tfrac{3\mu_0}{\lambda_1}+1\right)\|v\|_{H_0^1}^2 + g_0.
\end{aligned}
$$

□

**Lemma 3.19.** *There exists a constant $K^* > 0$ such that for all $s \in \mathbb{R}$, $t \geqslant 0$ and $\varepsilon > 0$ we have*

$$
\begin{aligned}
&-k_s(t)\|v_t\|_{L^2}^{p+2} - \varepsilon k_s(t)\|v_t\|_{L^2}^{p}\int_{\Omega} v_t v dx \\
&\qquad \leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1 - \tfrac{\varepsilon K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right) + \tfrac{\varepsilon}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2;
\end{aligned}
\tag{3.25}
$$

*Also, for all $s \in \mathbb{R}$, $t \geqslant 0$ and $\varepsilon > 0$ we have*

$$
\left|\int_{\Omega\times\Omega} K(x,y)v_t(t,y)v(t,x)dydx\right| \leqslant \tfrac{1}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 + \tfrac{3K_0^2}{\lambda_1-\mu_0}\|v_t\|_{L^2}^2; \tag{3.26}
$$

$$
\left|\int_{\Omega\times\Omega} K(x,y)v_t(t,y)v_t(t,x)dydx\right| \leqslant K_0\|v_t\|_{L^2}^2. \tag{3.27}
$$

*Proof.* Proof of (3.25). Using Hölder's and Poincaré's inequalities we have

$$
\begin{aligned}
&-k_s(t)\|v_t\|_{L^2}^{p}\int_{\Omega} v_t v dx \leqslant k_1\|v_t\|_{L^2}^{p}\int_{\Omega}|v_t v|dx \leqslant k_1\|v_t\|_{L^2}^{p+1}\|v\|_{L^2} \leqslant \tfrac{k_1}{\sqrt{\lambda_1}}\|v_t\|_{L^2}^{p+1}\|v\|_{H_0^1} \\
&= \underbrace{\tfrac{k_1}{\sqrt{\lambda_1}}\left[\tfrac{p+2}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\right]^{-\frac{1}{p+2}}\|v_t\|_{L^2}^{p+1}\|v\|_{H_0^1}^{\frac{p}{p+2}}}_{=a}\underbrace{\left[\tfrac{p+2}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\right]^{\frac{1}{p+2}}\|v\|_{H_0^1}^{\frac{2}{p+2}}}_{=b}.
\end{aligned}
$$

Using Young's inequality $ab \leqslant \frac{a^{q_1}}{q_1} + \frac{b^{q_2}}{q_2}$ with $q_1 := \frac{p+2}{p+1}$ and $q_2 := p+2$, for which $\frac{1}{q_1}+\frac{1}{q_2} = 1$, we obtain

$$
-k_s(t)\|v_t\|_{L^2}^{p}\int_{\Omega} v_t v dx \leqslant \tfrac{p+1}{p+2}\left\{\tfrac{k_1}{\sqrt{\lambda_1}}\left[\tfrac{p+2}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\right]^{-\frac{1}{p+2}}\|v_t\|_{L^2}^{p+1}\|v\|_{H_0^1}^{\frac{p}{p+2}}\right\}^{\frac{p+2}{p+1}}
$$

$$+\tfrac{1}{p+2}\left\{\left[\tfrac{p+2}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\right]^{\frac{1}{p+2}}\|v\|_{H_0^1}^{\frac{2}{p+2}}\right\}^{p+2}$$
$$=\tfrac{p+1}{p+2}\left(\tfrac{k_1}{\sqrt{\lambda_1}}\right)^{\frac{p+2}{p+1}}\left[\tfrac{p+2}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\right]^{-\frac{1}{p+1}}\|v_t\|_{L^2}^{p+2}\|v\|_{H_0^1}^{\frac{p}{p+1}}+\tfrac{1}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2$$
$$=K^*k_1\|v_t\|_{L^2}^{p+2}\|v\|_{H_0^1}^{\frac{p}{p+1}}+\tfrac{1}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2,$$

with $K^*:=k_1^{\frac{1}{p+1}}\frac{p+1}{p+2}\left(\frac{1}{\sqrt{\lambda_1}}\right)^{\frac{p+2}{p+1}}\left[\frac{p+2}{12}\left(1-\frac{\mu_0}{\lambda_1}\right)\right]^{-\frac{1}{p+1}}$. Thus, we obtain

$$-k_s(t)\|v_t\|_{L^2}^{p+2}-\varepsilon k_s(t)\|v_t\|_{L^2}^{p}\int_\Omega v_t v\,dx$$
$$\leqslant -k_0\|v_t\|_{L^2}^{p+2}+\varepsilon K^*k_1\|v_t\|_{L^2}^{p+2}\|v\|_{H_0^1}^{\frac{p}{p+1}}+\tfrac{\varepsilon}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2$$
$$\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1-\tfrac{\varepsilon K^*k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right)+\tfrac{\varepsilon}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2.$$

Proof of (3.26). From $(\mathbf{H}_2)$, using Hölder's and Poincaré's inequalities, and using Proposition 3.2, we have

$$\left|\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v(t,x)dydx\right|\leqslant\int_\Omega\left|v(t,x)\int_\Omega K(x,y)v_t(t,y)\,dy\right|dx$$
$$\leqslant\|v\|_{L^2}\left\|\int_\Omega K(x,y)v_t(t,y)\,dy\right\|_{L^2}\leqslant K_0\|v_t\|_{L^2}\|v\|_{L^2}\leqslant\tfrac{K_0}{\sqrt{\lambda_1}}\|v_t\|_{L^2}\|v\|_{H_0^1}$$
$$=\underbrace{\tfrac{1}{\sqrt{6}}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{\frac12}\|v\|_{H_0^1}}_{=a}\underbrace{\tfrac{\sqrt{6}}{\sqrt{\lambda_1}}K_0\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{-\frac12}\|v_t\|_{L^2}}_{=b}.$$

Using the usual Young's inequality $ab\leqslant\frac{a^2}{2}+\frac{b^2}{2}$, we obtain

$$\left|\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v(t,x)dydx\right|\leqslant\tfrac{1}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+\tfrac{3K_0^2}{\lambda_1-\mu_0}\|v_t\|_{L^2}^2.$$

Proof of (3.27). This is trivial. □

**Lemma 3.20.** *Fix* $\varepsilon_0:=\frac{\sqrt{\lambda_1}}{8}\left(1-\frac{\mu_0}{\lambda_1}\right)>0$. *Then, for all* $0<\varepsilon\leqslant\varepsilon_0$ *we have*

$$\left|\varepsilon\int_\Omega v_t v\,dx\right|\leqslant\tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2+\|v\|_{H_0^1}^2).$$

*Proof.* Indeed, for $0<\varepsilon\leqslant\varepsilon_0$, applying Hölder's and Poincaré's inequalities we obtain

$$\left|\varepsilon\int_\Omega v_t v\,dx\right|\leqslant\varepsilon\int_\Omega|v_t.v|\,dx\leqslant\varepsilon\|v_t\|_{L^2}\,\|v\|_{L^2}\leqslant\frac{\varepsilon}{\sqrt{\lambda_1}}\|v_t\|_{L^2}\|v\|_{H_0^1}$$
$$\leqslant\frac{\varepsilon_0}{2\sqrt{\lambda_1}}(\|v_t\|_{L^2}^2+\|v\|_{H_0^1}^2)=\tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2+\|v\|_{H_0^1}^2).$$

□

**Lemma 3.21.** *We have*

$$\left|\int_\Omega hv\,dx\right|\leqslant\tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+\tfrac{4}{\lambda_1-\mu_0}h_0^2.$$

*Proof.* Note that using Hölder's, Poincaré's and Young's inequalities, we obtain

$$\begin{aligned}\left|\int_\Omega hv dx\right| &\leqslant \int_\Omega |hv| dx \leqslant \|h\|_{L^2}\|v\|_{L^2} \leqslant \tfrac{1}{\sqrt{\lambda_1}} h_0 \|v\|_{H_0^1} \\ &= \left(\tfrac{\sqrt{8}}{\sqrt{\lambda_1}}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{-\frac{1}{2}} h_0\right)\left(\tfrac{1}{\sqrt{8}}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{\frac{1}{2}}\|v\|_{H_0^1}\right) \\ &\leqslant \tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|^2_{H_0^1} + \tfrac{4}{\lambda_1-\mu_0} h_0^2.\end{aligned}$$

□

Joining all these lemmas, we are able to present the proof of Proposition 3.17.

*Proof of Proposition 3.17.* Recall that

$$E_s(t) = \frac{1}{2}\left(\|v_t\|^2_{L^2} + \|v\|^2_{H_0^1}\right) + \int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|^2_{L^2} + C_0.$$

We have $\|v\|^2_{H_0^1} \leqslant 2E_s(t)$ from (3.20), and

$$\tfrac{1}{2}(\|v_t\|^2_{L^2} + \|v\|^2_{H_0^1}) + \int_\Omega F_s(t,v)dx = E_s(t) - \tfrac{\lambda_1+\mu_0}{4}\|v\|^2_{L^2} - C_0 \leqslant E_s(t).$$

This, together with Lemmas 3.20 and 3.21, implies that

$$\begin{aligned}V_\varepsilon^s(t) &= \tfrac{1}{2}(\|v_t\|^2_{L^2} + \|v\|^2_{H_0^1}) + \int_\Omega F_s(t,v)dx - \int_\Omega hv dx + \varepsilon\int_\Omega v_t v dx \\ &\leqslant E_s(t) - \int_\Omega hv dx + \varepsilon \int_\Omega v_t v dx \\ &\leqslant E_s(t) + \tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\underbrace{\|v\|^2_{H_0^1}}_{\leqslant 2E_s(t)} + \tfrac{4}{\lambda_1-\mu_0}h_0^2 + \tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\underbrace{(\|v_t\|^2_{L^2} + \|v\|^2_{H_0^1})}_{\leqslant 2E_s(t)} \\ &\leqslant \tfrac{5}{4}E_s(t) + \tfrac{4}{\lambda_1-\mu_0}h_0^2 \leqslant \tfrac{5}{4}E_s(t) + \tfrac{4}{\lambda_1-\mu_0}h_0^2 + C_0.\end{aligned}$$

For the converse inequality, again applying Lemmas 3.20 and 3.21, we have

$$\begin{aligned}V_\varepsilon^s(t) &= E_s(t) - \tfrac{\lambda_1+\mu_0}{4}\|v\|^2_{L^2} - C_0 - \int_\Omega hv dx + \varepsilon\int_\Omega v_t v dx \\ &\geqslant E_s(t) - \tfrac{1}{4}\left(1+\tfrac{\mu_0}{\lambda_1}\right)\|v\|^2_{H_0^1} - C_0 - \tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|^2_{H_0^1} - \tfrac{4}{\lambda_1-\mu_0}h_0^2 \\ &\quad - \tfrac{1}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|^2_{L^2} + \|v\|^2_{H_0^1}) \\ &\geqslant E_s(t) - \tfrac{1}{2}\left(1+\tfrac{\mu_0}{\lambda_1}\right)E_s(t) - \tfrac{1}{8}\left(1-\tfrac{\mu_0}{\lambda_1}\right)E_s(t) - \tfrac{1}{8}\left(1-\tfrac{\mu_0}{\lambda_1}\right)E_s(t) - C_0 - \tfrac{4}{\lambda_1-\mu_0}h_0^2 \\ &= \tfrac{1}{4}\left(1-\tfrac{\mu_0}{\lambda_1}\right)E_s(t) - C_0 - \tfrac{4}{\lambda_1-\mu_0}h_0^2,\end{aligned}$$

which completes the proof, choosing $d_0 := C_0 + \frac{4}{\lambda_1-\mu_0}h_0^2$. □

This concludes the proof of the goal established in item (I) of our scheme.

Development of (II).

Following our scheme, we want to prove the following result:

**Proposition 3.22.** *Given $R > 0$ there exists a constant $c_R \geqslant 0$ such that*

$$E_s(t) \leqslant c_R \quad \textit{for all } V_0 \in \overline{B}_R^X \textit{ and } t \geqslant 0.$$

As we did above, we will prove auxiliary lemmas that will come together to prove Proposition 3.22. However, before that, we will present a remark to justify the existence of global solutions.

**Remark 3.23** (Solutions to (ACP) are global)**.** Assume that $V_0 \in X$ is given and consider the unique maximal weak solution $V(\cdot,V_0)\colon [0,\tau_{max}) \to X$ of (ACP). The computations that will be presented, up until the end of the proof of Proposition 3.22, remain true for $V(t,V_0)$, or for $E_s(t,V_0)$, when $0 \leqslant t < \tau_{max}$. Hence, Proposition 3.22 shows that for some constant $c > 0$, we have $E_s(t,V_0) \leqslant c$ for all $0 \leqslant t < \tau_{max}$. Therefore, $\|V(t,V_0)\|_X \leqslant c$ for all $0 \leqslant t < \tau_{max}$ and shows that if we assume $\tau_{max} < \infty$ we obtain a contradiction with Proposition 3.14. Hence $\tau_{max} = \infty$ for all initial data $V_0 \in X$.

**Lemma 3.24.** *The function $V_\varepsilon^s$ satisfies:*

$$\begin{aligned}\frac{d}{dt}V_\varepsilon^s(t) = &-k_s(t)\|v_t\|_{L^2}^{p+2} - \varepsilon k_s(t)\|v_t\|_{L^2}^p \int_\Omega v_t v dx + \int_\Omega \frac{\partial F_s}{\partial t}(t,v)dx \\ &+ \int_{\Omega\times\Omega} K(x,y)v_t(t,y)v_t(t,x)dydx - \varepsilon\|v\|_{H_0^1}^2 + \varepsilon\|v_t\|_{L^2}^2 - \varepsilon\int_\Omega f_s(t,v)vdx \\ &+ \varepsilon\int_{\Omega\times\Omega} K(x,y)v_t(t,y)v(t,x)dydx + \varepsilon\int_\Omega hvdx.\end{aligned}$$

*Proof.* Formally multiplying (tNWE) by $v_t + \varepsilon v$ and integrating over $\Omega$ we obtain

$$\begin{aligned}&\int_\Omega v_{tt}v_t dx - \int_\Omega v_t\Delta v dx + k_s(t)\|v_t\|_{L^2}^p \int_\Omega v_t v_t dx + \int_\Omega f_s(t,v)v_t dx \\ &\quad + \varepsilon\int_\Omega v_{tt}vdx - \varepsilon\int_\Omega v\Delta v dx + \varepsilon k_s(t)\|v_t\|_{L^2}^p\int_\Omega v_t v dx + \varepsilon\int_\Omega f_s(t,v)vdx \\ &= \int_\Omega v_t(t,x)\int_\Omega K(x,y)v_t(t,y)dydx + \int_\Omega hv_t dx \\ &\quad + \varepsilon\int_\Omega v(t,x)\int_\Omega K(x,y)v_t(t,y)dydx + \varepsilon\int_\Omega hvdx.\end{aligned}$$

Now, observe that

(i) $\displaystyle\int_\Omega v_{tt}v_t dx = \frac{1}{2}\frac{d}{dt}\|v_t\|_{L^2}^2$;

(ii) $\displaystyle -\int_\Omega v_t\Delta v dx = \frac{1}{2}\frac{d}{dt}\|v\|_{H_0^1}^2$;

(iii) Since

$$\frac{d}{dt}\int_\Omega v_t v dx = \frac{d}{dt}\langle v_t, v\rangle = \langle v_{tt}, v\rangle + \langle v_t, v_t\rangle = \int_\Omega v v_{tt} dx + \|v_t\|_{L^2}^2$$

and

$$-\int_\Omega v\Delta v dx = \|v\|_{H_0^1}^2,$$

we have

$$\varepsilon\int_\Omega v_{tt}v dx-\varepsilon\int_\Omega v\Delta v dx=\varepsilon\frac{d}{dt}\int_\Omega v_t v dx-\varepsilon\|v_t\|_{L^2}^2+\varepsilon\|v\|_{H_0^1}^2;$$

(iv) $\displaystyle\int_\Omega v_t(t,x)\int_\Omega K(x,y)v_t(t,y)dydx=\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v_t(t,x)dydx;$

(v) $\displaystyle\varepsilon\int_\Omega v(t,x)\int_\Omega K(x,y)v_t(t,y)dydx=\varepsilon\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v(t,x)dydx;$

(vi) $\displaystyle\int_\Omega hv_t dx=\frac{d}{dt}\int_\Omega hv dx;$

(vii) $\displaystyle\int_\Omega f_s(t,v)v_t dx=\frac{d}{dt}\int_\Omega F_s(t,v)dx-\int_\Omega\frac{\partial F_s}{\partial t}(t,v)dx;$

Using (i)-(vii) together we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|v_t\|_{L^2}^2+\frac12\frac{d}{dt}\|v\|_{H_0^1}^2+k_s(t)\|v_t\|_{L^2}^{p+2}+\frac{d}{dt}\int_\Omega F_s(t,v)dx-\int_\Omega\frac{\partial F_s}{\partial t}(t,v)dx\\&\qquad+\varepsilon\frac{d}{dt}\int_\Omega v_t v dx-\varepsilon\|v_t\|_{L^2}^2+\varepsilon\|v\|_{H_0^1}^2+\varepsilon k_s(t)\|v_t\|_{L^2}^p\int_\Omega v_t v dx+\varepsilon\int_\Omega f_s(t,v)v dx\\&=\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v_t(t,x)dydx+\frac{d}{dt}\int_\Omega hv dx\\&\qquad+\varepsilon\int_{\Omega\times\Omega}K(x,y)v_t(t,y)v(t,x)dydx+\varepsilon\int_\Omega hv dx,\end{aligned}$$

which concludes the proof. □

**Lemma 3.25.** *If* $\varepsilon_0>0$ *is given as in Lemma 3.20, there exist constants* $\delta_0,\delta_1>0$ *such that for all* $0<\varepsilon\leqslant\varepsilon_0$*,* $s\in\mathbb{R}$ *and* $t\geqslant 0$*, we have*

$$\frac{d}{dt}V_\varepsilon^s(t)\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(\tfrac12-\delta_0\varepsilon(V_\varepsilon^s(t)+d_0)^{\frac{p}{2(p+1)}}\right)-\tfrac45\left(1-\tfrac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t)+\delta_1.$$

*Proof.* Since, from $(\mathbf{H}_6)$, $\int_\Omega\left|\frac{\partial F_s}{\partial t}(t,x)\right|dx\leqslant c_0$, joining Lemmas 3.18, 3.19, 3.21 and 3.24, for $0<\varepsilon\leqslant\varepsilon_0$, we obtain

$$\begin{aligned}\frac{d}{dt}V_\varepsilon^s(t)&\leqslant -k_0\|v_t\|_{L^2}^{p+2}\Big(1-\tfrac{\varepsilon K^*k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\Big)+\tfrac{\varepsilon}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+c_0+K_0\|v_t\|_{L^2}^2-\varepsilon\|v\|_{H_0^1}^2\\&\quad+\varepsilon\|v_t\|_{L^2}^2-\varepsilon\int_\Omega F_s(t,v)dx-\varepsilon\tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2-\varepsilon C_0+\tfrac{\varepsilon}{4}\left(\tfrac{3\mu_0}{\lambda_1}+1\right)\|v\|_{H_0^1}^2+\varepsilon g_0\\&\quad+\tfrac{\varepsilon}{12}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+\tfrac{3\varepsilon K_0^2}{\lambda_1-\mu_0}\|v_t\|_{L^2}^2+\tfrac{\varepsilon}{16}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+\tfrac{4\varepsilon}{\lambda_1-\mu_0}h_0^2\\&\overset{(1)}{\leqslant} -k_0\|v_t\|_{L^2}^{p+2}\Big(1-\tfrac{\varepsilon K^*k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\Big)+C_1\|v_t\|_{L^2}^2-\tfrac{25\varepsilon}{48}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2\\&\quad-\tfrac{\varepsilon}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2-\varepsilon\int_\Omega F_s(t,v)dx-\varepsilon\tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2+c_0-\varepsilon C_0+\varepsilon_0 g_0+\tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2\\&\overset{(2)}{\leqslant} -k_0\|v_t\|_{L^2}^{p+2}\Big(1-\tfrac{\varepsilon K^*k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\Big)+\tfrac{k_0}{2}\|v_t\|_{L^2}^{p+2}+C_2-\tfrac{\varepsilon}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2+\|v\|_{H_0^1}^2)\\&\quad-\varepsilon\Big[\int_\Omega F_s(t,v)dx+\tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2+C_0\Big]+c_0+\varepsilon_0 g_0+\tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2,\end{aligned}\tag{3.28}$$

where in (1) we added and subtracted the term $\frac{\varepsilon}{2}(1-\frac{\mu_0}{\lambda_1})\|v_t\|_{L^2}^2$, used the estimate

$$K_0\|v_t\|_{L^2}^2+\varepsilon\|v_t\|_{L^2}^2+\tfrac{3\varepsilon K_0^2}{\lambda_1-\mu_0}\|v_t\|_{L^2}^2+\tfrac{\varepsilon}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2\leqslant C_1\|v_t\|_{L^2}^2,$$

for $C_1 := K_0 + \varepsilon_0 + \frac{3\varepsilon_0 K_0^2}{\lambda_1-\mu_0} + \frac{\varepsilon_0}{2}\left(1-\frac{\mu_0}{\lambda_1}\right)$, and Poincaré's inequality. In (2) we applied Young's inequality $ab \leqslant \frac{a^{q_1}}{q_1} + \frac{b^{q_2}}{q_2}$ with $q_1 = \frac{p+2}{2}$ and $q_2 = \frac{p+2}{p}$, noting that $\frac{1}{q_1}+\frac{1}{q_2} = 1$, in order to obtain

$$\begin{aligned} C_1\|v_t\|_{L^2}^2 &= \left(\tfrac{k_0(p+2)}{4}\right)^{\frac{2}{p+2}} \|v_t\|_{L^2}^2 \left(\tfrac{4}{k_0(p+2)}\right)^{\frac{2}{p+2}} C_1 \\ &\leqslant \tfrac{k_0}{2}\|v_t\|_{L^2}^{p+2} + \tfrac{p}{p+2}\left(\tfrac{4}{k_0(p+2)}\right)^{\frac{2}{p}} C_1^{\frac{p+2}{p}} = \tfrac{k_0}{2}\|v_t\|_{L^2}^{p+2} + C_2, \end{aligned}$$

for $C_2 := \frac{p}{p+2}\left(\frac{4}{k_0(p+2)}\right)^{\frac{2}{p}} C_1^{\frac{p+2}{p}}$.

From Proposition 3.17 we have $2E_s(t) \leqslant 8(1-\frac{\mu_0}{\lambda_1})^{-1}(V_\varepsilon^s(t)+d_0)$, which implies

$$(2E_s(t))^{\frac{p}{2(p+1)}} \leqslant \left(\tfrac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}} (V_\varepsilon^s(t)+d_0)^{\frac{p}{2(p+1)}}.$$

Hence,

$$\begin{aligned} &-k_0\|v_t\|_{L^2}^{p+2}\Big(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}(2E_s(t))^{\frac{p}{2(p+1)}}\Big) \\ &\qquad \leqslant -k_0\|v_t\|_{L^2}^{p+2}\Big(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}\left(\tfrac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}} (V_\varepsilon^s(t)+d_0)^{\frac{p}{2(p+1)}}\Big), \end{aligned}$$

which, since (3.21) holds, we have that $\|v\|_{H_0^1} \leqslant (2E_s(t))^{\frac{1}{2}}$, and then

$$\begin{aligned} -k_0\|v_t\|_{L^2}^{p+2}\left(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right) &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\Big(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}(2E_s(t))^{\frac{p}{2(p+1)}}\Big) \\ &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\Big(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}\left(\tfrac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}} (V_\varepsilon^s(t)+d_0)^{\frac{p}{2(p+1)}}\Big). \end{aligned} \tag{3.29}$$

Also observe that

$$\begin{aligned} &\tfrac{1}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) + \int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 + C_0 \\ &= \tfrac{1}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) + E_s(t) - \tfrac{1}{2}(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) \\ &= -\tfrac{\mu_0}{2\lambda_1}(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) + E_s(t) \geqslant -\tfrac{\mu_0}{\lambda_1}E_s(t) + E_s(t) = E_s(t)\left(1-\tfrac{\mu_0}{\lambda_1}\right), \end{aligned}$$

implying that

$$\begin{aligned} &-\tfrac{\varepsilon}{2}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\left(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2\right) - \varepsilon\left(\int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 + C_0\right) \\ &\leqslant -\left(1-\tfrac{\mu_0}{\lambda_1}\right)\varepsilon E_s(t) \overset{(3)}{\leqslant} -\tfrac{4}{5}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t) + \tfrac{4}{5}\varepsilon_0(1-\tfrac{\mu_0}{\lambda_1})d_0, \end{aligned} \tag{3.30}$$

where in (3) we used Proposition 3.17. From (3.28), (3.29) and (3.30) we obtain

$$\begin{aligned} \frac{d}{dt}V_\varepsilon^s(t) &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\Big(\tfrac{1}{2} - \tfrac{\varepsilon K^* k_1}{k_0}\left(\tfrac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}} (V_\varepsilon^s(t)+d_0)^{\frac{p}{2(p+1)}}\Big) \\ &\quad - \tfrac{4}{5}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t) + \tfrac{4}{5}\varepsilon_0(1-\tfrac{\mu_0}{\lambda_1})d_0 + c_0 + \varepsilon_0 g_0 + \tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2 + C_2, \end{aligned}$$

and the result is proven, choosing

$$\delta_0 := \frac{K^* k_1}{k_0}\left(\tfrac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}}$$

and

$$\delta_1 := \tfrac{4}{5}\varepsilon_0(1-\tfrac{\mu_0}{\lambda_1})d_0 + c_0 + \varepsilon_0 g_0 + \tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2 + C_2.$$

□

As a consequence of the previous result, we have the following lemma, which tell us, in general terms, that a suitable estimate for the function $V_\varepsilon^s$ at a time $\tau$ gives us a similar estimate for $V_\varepsilon^s$ for all $t \geqslant \tau$.

**Lemma 3.26.** *Assume that* $0 < \varepsilon \leqslant \varepsilon_0$. *If for* $\tau \geqslant 0$ *we have*

$$V_\varepsilon^s(\tau) \leqslant (2\delta_0\varepsilon)^{-\frac{2(p+1)}{p}} - \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\varepsilon^{-1} - d_0,$$

*then for all* $t \geqslant \tau$ *we have*

$$V_\varepsilon^s(t) \leqslant (2\delta_0\varepsilon)^{-\frac{2(p+1)}{p}} - d_0.$$

*Proof.* We fix $0 < \varepsilon \leqslant \varepsilon_0$ and set

$$\Psi(\varepsilon) := (2\delta_0\varepsilon)^{-\frac{2(p+1)}{p}} - d_0, \quad \text{and} \tag{3.31}$$

$$\Phi(\varepsilon) := (2\delta_0\varepsilon)^{-\frac{2(p+1)}{p}} - \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\varepsilon^{-1} - d_0. \tag{3.32}$$

Note that $V_\varepsilon^s(\tau) \leqslant \Phi(\varepsilon) < \Psi(\varepsilon)$. By the continuity of $[0,\infty) \ni t \mapsto V_\varepsilon^s(t)$, there exists $T > \tau$ such that $V_\varepsilon^s(t) \leqslant \Psi(\varepsilon)$ for $\tau \leqslant t < T$. Let $T_0 = \inf\{t\colon t \geqslant \tau \text{ and } V_\varepsilon^s(t) > \Psi(\varepsilon)\}$ and assume that $T_0 < \infty$. It is clear that $T_0 \geqslant T > \tau$, $V_\varepsilon^s(t) \leqslant \Psi(\varepsilon)$ for $\tau \leqslant t \leqslant T_0$ and $V_\varepsilon^s(T_0) = \Psi(\varepsilon)$.

From Proposition 3.17 we know that $V_\varepsilon^s(t) + d_0 \geqslant \frac{1}{4}\left(1 - \frac{\mu_0}{\lambda_1}\right) E_s(t) \geqslant 0$. Then, for $\tau \leqslant t \leqslant T_0$ we have

$$(V_\varepsilon^s(t) + d_0)^{\frac{p}{2(p+1)}} \leqslant (\Phi(\varepsilon) + d_0)^{\frac{p}{2(p+1)}} \leqslant \frac{1}{2\delta_0\varepsilon},$$

which implies that

$$\tfrac{1}{2} - \delta_0\varepsilon(V_\varepsilon^s(t) + d_0)^{\frac{p}{2(p+1)}} \geqslant 0. \tag{3.33}$$

Using Lemma 3.25, for $\tau \leqslant t \leqslant T_0$ and $0 < \varepsilon \leqslant \varepsilon_0$, we obtain

$$\frac{d}{dt}V_\varepsilon^s(t) + \tfrac{4}{5}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t) \leqslant \delta_1.$$

Using Grönwall's inequality (Lemma A.9), for $\tau \leqslant t \leqslant T_0$ we obtain

$$\begin{aligned} V_\varepsilon^s(t) &\leqslant V_\varepsilon^s(\tau)e^{-\frac{4}{5}\left(1-\frac{\mu_0}{\lambda_1}\right)\varepsilon(t-\tau)} + \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\varepsilon^{-1} < V_\varepsilon^s(\tau) + \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\varepsilon^{-1} \\ &\leqslant \Phi(\varepsilon) + \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\varepsilon^{-1} = \Psi(\varepsilon). \end{aligned}$$

Taking $t = T_0$ we obtain $V_\varepsilon^s(T_0) < \Psi(\varepsilon)$, which is a contradiction. Thus $T_0 = \infty$ and the proof is complete. □

**Remark 3.27.** One aspect that we want to draw attention to is that we concluded that if $V_\varepsilon^s(\tau) \leqslant \Phi(\varepsilon)$ then (3.33) holds for all $t \geqslant \tau$.

Now we can prove Proposition 3.22.

*Proof of Proposition 3.22.* From Proposition 3.3 and the continuous inclusion $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$, with constant $c > 0$, we obtain

$$\left|\int_\Omega F(s,v)dx\right| \leqslant \int_\Omega |F(s,v)|dx \leqslant \int_\Omega 4c_0(1+|v|^4)dx$$
$$= 4c_0|\Omega| + 4c_0\|v\|^4_{L^4} \leqslant 4c_0|\Omega| + 4c_0c^4\|v\|^4_{H^1_0} \leqslant \alpha_0(1+\|v\|^4_{H^1_0}),$$

where $\alpha_0 = 4c_0\max\{|\Omega|,c^4\}$. Fix $R > 0$ and let $V_0 = (u_0,u_1) \in X$ be such that $V_0 \in \overline{B}^X_R$. From the inequality above and Poincaré's inequality, we deduce

$$\begin{aligned} E_s(0) &= \tfrac{1}{2}\left(\|u_1\|^2_{L^2} + \|u_0\|^2_{H^1_0}\right) + \int_\Omega F(s,u_0)dx + \tfrac{\lambda_1+\mu_0}{4}\|u_0\|^2_{L^2} + C_0 \\ &\leqslant \tfrac{1}{2}\|V_0\|^2_X + \alpha_0(1+\|u_0\|^4_{H^1_0}) + \tfrac{\lambda_1+\mu_0}{4\lambda_1}\|u_0\|^2_{H^1_0} + C_0 \\ &\leqslant \tfrac{1}{2}R^2 + \alpha_0 R^4 + \tfrac{\lambda_1+\mu_0}{4\lambda_1}R^2 + \alpha_0 + C_0. \end{aligned}$$

By Proposition 3.17 we obtain

$$V^s_\varepsilon(0) \leqslant \tfrac{5}{4}E_s(0) + d_0 \leqslant \tfrac{5}{4}\left(\tfrac{1}{2}R^2 + \alpha_0R^4 + \tfrac{\lambda_1+\mu_0}{4\lambda_1}R^2 + \alpha_0 + C_0\right) + d_0 := \gamma_R. \tag{3.34}$$

Consider the maps $\Psi,\Phi\colon (0,\infty) \to \mathbb{R}$ defined in (3.31) and (3.32), respectively. Note that

$$\Phi'(\varepsilon) = -\frac{1}{\varepsilon^2}\left(\tfrac{2(p+1)}{p}(2\delta_0)^{\frac{-2(p+1)}{p}}\varepsilon^{-\frac{p+2}{p}} - \tfrac{5}{4}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1\right). \tag{3.35}$$

Setting

$$\varepsilon_1 := \left(\tfrac{5p}{8(p+1)}\tfrac{\lambda_1}{\lambda_1-\mu_0}\delta_1(2\delta_0)^{\frac{2(p+1)}{p}}\right)^{-\frac{p}{p+2}},$$

we can see that $\varepsilon_1 > 0$, that $\Phi'(\varepsilon) < 0$ for $0 < \varepsilon < \varepsilon_1$ and $\Phi'(\varepsilon) > 0$ for $\varepsilon > \varepsilon_1$. Hence $\Phi$ is strictly decreasing in the interval $(0,\varepsilon_1]$ and strictly increasing in $[\varepsilon_1,\infty)$. It is clear that $\lim\limits_{\varepsilon\to\infty}\Phi(\varepsilon) = -d_0$. On the other hand, since $\frac{2(p+1)}{p} > 1$, we have $\lim\limits_{\varepsilon\to 0^+}\Phi(\varepsilon) = \infty$. Hence $\Phi(\varepsilon_1) < -d_0 < 0$ and, from the Intermediate Value Theorem and the monotonicity of $\Phi$ in $(0,\varepsilon_1]$, there exists a unique point $0 < \varepsilon_2 < \varepsilon_1$ such that $\Phi(\varepsilon_2) = 0$ and $\Phi(\varepsilon) > 0$ for all $0 < \varepsilon < \varepsilon_2$.

Observe that $\Phi\colon (0,\varepsilon_2] \to [0,\infty)$ is bijective and take $\alpha_R := \min\{\varepsilon_0, \Phi^{-1}(\gamma_R)\} \leqslant \varepsilon_2$. Since $\alpha_R \leqslant \Phi^{-1}(\gamma_R)$ and $\Phi$ is decreasing in $(0,\varepsilon_2]$, it follows from (3.34) that $V^s_\varepsilon(0) \leqslant \gamma_R \leqslant \Phi(\alpha_R)$. Thus, from Lemma 3.26 we obtain $V^s_\varepsilon(t) \leqslant \Psi(\alpha_R)$ for all $t \geqslant 0$. From this fact and Proposition 3.17, for all $t \geqslant 0$ we obtain

$$E_s(t) \leqslant \tfrac{4\lambda_1}{\lambda_1-\mu_0}(V^s_\varepsilon(t) + d_0) \leqslant \tfrac{4\lambda_1}{\lambda_1-\mu_0}(\Psi(\alpha_R) + d_0),$$

which completes the proof, defining $c_R := \frac{4\lambda_1}{\lambda_1-\mu_0}(\Psi(\alpha_R) + d_0)$. □

Development of (III).

Continuing our scheme, our goal for now is to prove the following result.

**Proposition 3.28.** *There exists a constant $R_0 > 0$ such that for any $R > 0$ and $s \in \mathbb{R}$, we have*

$$\limsup_{\tau\to\infty}\left(\sup_{\|V_0\|_X\leqslant R} E_s(\tau)\right) \leqslant R_0.$$

*Proof.* Consider the function $\Theta\colon [0,\infty) \to (0,\infty)$ defined by

$$\Theta(\sigma) = \min\left\{\varepsilon_0, \Phi^{-1}\left(\tfrac{5}{4}\sigma + d_0\right)\right\} \qquad \text{for } \sigma \geqslant 0.$$

It is clear that $\Theta$ is a continuous and strictly decreasing function.

Now fix $r \geqslant 0$ and define $\varepsilon := \Theta(E_s(r))$. Note that $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon < \varepsilon_2$ and

$$\varepsilon \leqslant \Phi^{-1}\left(\tfrac{5}{4}E_s(r) + d_0\right).$$

Since $\Phi$ is decreasing in $(0,\varepsilon_2]$, we have $\Phi(\varepsilon) \geqslant \frac{5}{4}E_s(r) + d_0 \geqslant V_\varepsilon^s(r)$. From Remark 3.27, for all $t \geqslant r$ we obtain

$$\tfrac{1}{2} - \delta_0\varepsilon(V_\varepsilon^s(t) + d_0)^{\frac{p}{2(p+1)}} \geqslant 0.$$

This information and Lemma 3.25, for $t \geqslant r$, yield

$$\begin{aligned}\frac{d}{dt}V_\varepsilon^s(t) &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(\tfrac{1}{2} - \varepsilon\delta_0(V_\varepsilon^s(t) + d_0)^{\frac{p}{2(p+1)}}\right) - \tfrac{4}{5}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t) + \delta_1\\ &\leqslant -\tfrac{4}{5}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\Theta(E_s(r))V_\varepsilon^s(t) + \delta_1.\end{aligned}$$

Applying Grönwall's inequality (Lemma A.9) we obtain

$$V_\varepsilon^s(t) \leqslant V_\varepsilon^s(r)e^{-\frac{4}{5}\left(1-\frac{\mu_0}{\lambda_1}\right)\Theta(E_s(r))(t-r)} + \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\left[\Theta(E_s(r))\right]^{-1},$$

for $t \geqslant r$. Using Proposition 3.17 we obtain

$$\begin{aligned}\tfrac{1}{4}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)E_s(t) - d_0 &\leqslant V_\varepsilon^s(r)e^{-\frac{4}{5}\left(1-\frac{\mu_0}{\lambda_1}\right)\Theta(E_s(r))(t-r)} + \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\left[\Theta(E_s(r))\right]^{-1}\\ &\leqslant \left(\tfrac{5}{4}E_s(r) + d_0\right)e^{-\frac{4}{5}\left(1-\frac{\mu_0}{\lambda_1}\right)\Theta(c_R)(t-r)} + \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\left[\Theta(E_s(r))\right]^{-1},\end{aligned}$$

where we have used that, for a fixed $R > 0$, since $E_s(r) \leqslant c_R$ for all $r \geqslant 0$ and $\Theta$ is decreasing, $\Theta(E_s(r)) \geqslant \Theta(c_R) > 0$. Hence, for this given $R > 0$, setting $\gamma_{R,s}(t) := \sup_{\|V_0\|_X\leqslant R} E_s(t)$ we obtain

$$\tfrac{1}{4}\left(1-\tfrac{\mu_0}{\lambda_1}\right)\gamma_{R,s}(t) - d_0 \leqslant \left(\tfrac{5}{4}\gamma_{R,s}(r) + d_0\right)e^{-\frac{4}{5}\left(1-\frac{\mu_0}{\lambda_1}\right)\Theta(c_R)(t-r)} + \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\left[\Theta\left(\gamma_{R,s}(r)\right)\right]^{-1}.$$

Consequently, if $w_{R,s} := \limsup_{t\to\infty}\gamma_{R,s}(t)$, we obtain

$$\tfrac{1}{4}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)w_{R,s} \leqslant \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\left[\Theta\left(w_{R,s}\right)\right]^{-1} + d_0,$$

that is,

$$\frac{1}{w_{R,s}}\left(\tfrac{5}{4d_0}\tfrac{\lambda_1}{\lambda_1-\mu_0}\left[\Theta(w_{R,s})\right]^{-1} + 1\right) \geqslant \tfrac{1}{4d_0}\left(1 - \tfrac{\mu_0}{\lambda_1}\right). \qquad (3.36)$$

Now, for $z > 0$, consider the function

$$G(z) = \tfrac{1}{z}\left(\tfrac{5}{4d_0}\tfrac{\lambda_1}{\lambda_1-\mu_0}\left[\Theta(z)\right]^{-1} + 1\right).$$

Since $\Phi^{-1}$ is a positive decreasing function approaching to zero at infinity, we have $\lim\limits_{z\to\infty}\Theta(z) = \lim\limits_{z\to\infty}\min\left\{\varepsilon_0, \Phi^{-1}(\tfrac{5}{4}z + d_0)\right\} = \lim\limits_{z\to\infty}\Phi^{-1}(\tfrac{5}{4}z + d_0) = 0$, and thus

$$\begin{aligned}\lim_{z\to\infty} G(z) &= \lim_{z\to\infty}\frac{\tfrac{5}{4d_0}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{-1}\left[\Phi^{-1}(\tfrac{5}{4}z + d_0)\right]^{-1} + 1}{z}\\ &= \lim_{\sigma\to\infty}\frac{\tfrac{5}{4d_0}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{-1}\sigma + 1}{\tfrac{4}{5}\left(\Phi\left(\tfrac{1}{\sigma}\right) - d_0\right)} \overset{(1)}{=} \lim_{\sigma\to\infty}\frac{\tfrac{5}{4d_0}\left(1-\tfrac{\mu_0}{\lambda_1}\right)^{-1}}{-\tfrac{4}{5}\Phi'\left(\tfrac{1}{\sigma}\right)\tfrac{1}{\sigma^2}} \overset{(2)}{=} 0,\end{aligned}$$

where in (1) we applied the L'Hospital Rule and for (2) we note that

$$\lim_{\sigma\to\infty} -\tfrac{4}{5}\Phi'\left(\tfrac{1}{\sigma}\right)\tfrac{1}{\sigma^2} = \lim_{\sigma\to\infty}\tfrac{4}{5}\left[\tfrac{2(p+1)}{p}(2\delta_0)^{\frac{-2(p+1)}{p}}\sigma^{\frac{p+2}{p}} - \tfrac{5}{4}\tfrac{\lambda_1\delta_1}{\lambda_1-\mu_0}\right] = \infty.$$

Since $G(w_{R,s}) \geqslant \tfrac{1}{4d_0}\left(1-\tfrac{\mu_0}{\lambda_1}\right)$, this implies that there exists $R_0 > 0$ such that $w_{R,s} \leqslant R_0$ for all $R > 0$ (noting that the constant $\tfrac{1}{4d_0}(1-\tfrac{\mu_0}{\lambda_1})$ does not depend on $R$). □

Development of (IV).

With all the work done in items (I), (II) and (III), now the proof of Theorem 3.16 follows easily.

*Proof of Theorem 3.16.* Let $R > 0$. From Proposition 3.28 there exists $\tau_0 = \tau_0(R) \geqslant 0$ such that for $\tau \geqslant \tau_0$ we have $\sup\limits_{\|(u_0,u_1)\|_X\leqslant R} E_{t-\tau}(\tau) \leqslant 2R_0$ for all $t \in \mathbb{R}$. Therefore, if $\tau \geqslant \tau_0$ and $V_0 = (u_0,u_1)$, we have

$$\|S(t,t-\tau)(u_0,u_1)\|_X^2 = \|V(\tau,V_0)\|_X^2 \leqslant 2E_{t-\tau}(\tau) \leqslant 4R_0,$$

and the proof is complete taking $r_0 := 2R_0^{\frac{1}{2}}$. □

This implies, in particular, that the family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$, with $B_t = \overline{B}_{r_0}^X$ for all $t \in \mathbb{R}$, is a uniformly pullback absorbing family for $S$.

**Existence of a uniformly pullback absorbing family for the process $S$ associated with (NWE) when $p = 0$**

We consider again the energy function

$$E_s(t,V_0) = \tfrac{1}{2}(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) + \int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 + C_0, \tag{3.37}$$

where $C_0$ is the constant obtained in (3.20), and the auxiliary function

$$V_\varepsilon^s(t,V_0) = \tfrac{1}{2}\left(\|v\|_{H_0^1}^2 + \|v_t\|_{L^2}^2\right) + \int_\Omega F_s(t,v)dx - \int_\Omega hvdx + \varepsilon\int_\Omega v_tvdx, \tag{3.38}$$

where $\varepsilon > 0$ is a fixed constant. Remember that we denote $V_0 := (u_0,u_1)$ and $V(t,V_0) = (v(t),v_t(t))$ for $t \geqslant 0$.

All the calculations we did in Lemmas 3.18, 3.19, 3.20 and 3.21 to obtain the Proposition 3.17 remain identical when we assume $p = 0$ and we conclude from them that there exist $\varepsilon_0 > 0$ and a constant $d_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $s \in \mathbb{R}$ and $t \geqslant 0$ we have

$$\frac{1}{4}\left(1 - \frac{\mu_0}{\lambda_1}\right) E_s(t) - d_0 \leqslant V_\varepsilon^s(t) \leqslant \frac{5}{4} E_s(t) + d_0. \tag{3.39}$$

The calculations in the proof of Lemma 3.24 also remain the same when we fix $p = 0$. From this result we obtain

$$\begin{aligned}\frac{d}{dt}V_\varepsilon^s(t) &= -k_s(t)\|v_t\|_{L^2}^2 - \varepsilon k_s(t)\int_\Omega v_t v dx + \int_\Omega \frac{\partial F_s}{\partial t}(t,v)dx \\ &+ \int_{\Omega\times\Omega} K(x,y)v_t(t,y)v_t(t,x)dydx - \varepsilon\|v\|_{H_0^1}^2 + \varepsilon\|v_t\|_{L^2}^2 - \varepsilon\int_\Omega f_s(t,v)vdx \\ &+ \varepsilon\int_{\Omega\times\Omega} K(x,y)v_t(t,y)v(t,x)dydx + \varepsilon\int_\Omega hvdx.\end{aligned}$$

**Lemma 3.29.** *There exist a constant $\delta_1 > 0$ and a fixed $\varepsilon \in (0,\varepsilon_0]$ sufficiently small such that for $s \in \mathbb{R}$ and $t \geqslant 0$ we have*

$$\frac{d}{dt}V_\varepsilon^s(t) \leqslant -\frac{4}{5}\left(1 - \frac{\mu_0}{\lambda_1}\right)\varepsilon V_\varepsilon^s(t) + \delta_1. \tag{3.40}$$

*Proof.* Following the proof of Lemma 3.25 we obtain

$$\begin{aligned}\frac{d}{dt}V_\varepsilon^s(t) &\leqslant -k_0\|v_t\|_{L^2}^2\Big(1 - \tfrac{\varepsilon K^* k_1}{k_0}\Big) + \tfrac{\varepsilon}{12}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 + c_0 + K_0\|v_t\|_{L^2}^2 - \varepsilon\|v\|_{H_0^1}^2 \\ &+ \varepsilon\|v_t\|_{L^2}^2 - \varepsilon\int_\Omega F_s(t,v)dx - \varepsilon\tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 - \varepsilon C_0 + \tfrac{\varepsilon}{4}\left(\tfrac{3\mu_0}{\lambda_1} + 1\right)\|v\|_{H_0^1}^2 + \varepsilon g_0 \\ &+ \tfrac{\varepsilon}{12}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 + \tfrac{3\varepsilon K_0^2}{\lambda_1-\mu_0}\|v_t\|_{L^2}^2 + \tfrac{\varepsilon}{16}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 + \tfrac{4\varepsilon}{\lambda_1-\mu_0}h_0^2 \\ &\overset{(1)}{\leqslant} -k_0\|v_t\|_{L^2}^2\left[1 - \tfrac{K_0}{k_0} - \varepsilon\left(\tfrac{K^*k_1}{k_0} + \tfrac{1}{k_0} + \tfrac{3K_0^2}{(\lambda_1-\mu_0)k_0} + \tfrac{\lambda_1-\mu_0}{2k_0\lambda_1}\right)\right] - \tfrac{25\varepsilon}{48}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 \\ &- \tfrac{\varepsilon}{2}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2 - \varepsilon\int_\Omega F_s(t,v)dx - \varepsilon\tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 + c_0 - \varepsilon C_0 + \varepsilon_0 g_0 + \tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2 \\ &\overset{(2)}{\leqslant} -\tfrac{\varepsilon}{2}\left(1 - \tfrac{\mu_0}{\lambda_1}\right)(\|v_t\|_{L^2}^2 + \|v\|_{H_0^1}^2) - \varepsilon\Big[\int_\Omega F_s(t,v)dx + \tfrac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2 + C_0\Big] \\ &+ c_0 + \varepsilon_0 g_0 + \tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2 \\ &\overset{(3)}{\leqslant} -\tfrac{4}{5}\Big(1 - \tfrac{\mu_0}{\lambda_1}\Big)\varepsilon V_\varepsilon^s(t) + \tfrac{4}{5}\varepsilon_0\Big(1 - \tfrac{\mu_0}{\lambda_1}\Big)d_0 + c_0 + \varepsilon_0 g_0 + \tfrac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2\end{aligned}$$

where in (1) we added and subtracted the term $\frac{\varepsilon}{2}(1 - \frac{\mu_0}{\lambda_1})\|v_t\|_{L^2}^2$ and used Poincaré's inequality. In (2) we choose a fixed $\varepsilon \in (0,\varepsilon_0]$ sufficiently small such that

$$\varepsilon\left(\tfrac{K^*k_1}{k_0} + \tfrac{1}{k_0} + \tfrac{3K_0^2}{(\lambda_1-\mu_0)k_0} + \tfrac{\lambda_1-\mu_0}{2k_0\lambda_1}\right) \leqslant 1 - \tfrac{K_0}{k_0}. \tag{3.41}$$

Here it is important to note that for the case $p = 0$ we are taking $k_0 > K_0$, and then, $1 - \frac{K_0}{k_0}$ is a strictly positive number. In (3) we used (3.30). The proof is done if we take $\delta_1 = \frac{4}{5}\varepsilon_0\Big(1 - \frac{\mu_0}{\lambda_1}\Big)d_0 + c_0 + \varepsilon_0 g_0 + \frac{4\varepsilon_0}{\lambda_1-\mu_0}h_0^2$. □

The following lemma is an immediate consequence of an application of the Lemma A.9 to the inequality (3.40).

**Lemma 3.30.** *Fix $\varepsilon \in (0, \varepsilon_0]$ satisfying (3.41). There exist constants $\beta_1, \beta_2 > 0$ such that for every $s \in \mathbb{R}$ and $t \geqslant 0$ we have*

$$V_\varepsilon^s(t) \leqslant V_\varepsilon^s(0)e^{-\beta_1 t} + \beta_2.$$

Now, let $R > 0$ and take $V_0 = (u_0,u_1)$ such that $\|V_0\|_X \leqslant R$. From (3.34), there exists $\gamma_R > 0$ (depending on $R$) such that $V_\varepsilon^s(0) \leqslant \gamma_R$. Then, for $\tau \geqslant 0$ and $s \in \mathbb{R}$,

$$V_\varepsilon^s(\tau) \leqslant \gamma_R e^{-\beta_1 \tau} + \beta_2.$$

If $\tau \geqslant \tau_0(R) := \max\left\{0, \frac{\ln \gamma_R - \ln \beta_2}{\beta_1}\right\}$, we have

$$\begin{aligned}\|S(t,t-\tau)(u_0,u_1)\|_X^2 = \|V(\tau, V_0)\|_X^2 \leqslant 2E_{t-\tau}(\tau) &\overset{(1)}{\leqslant} 8\left(1 - \tfrac{\mu_0}{\lambda_1}\right)^{-1}(V_\varepsilon^{t-\tau}(\tau) + d_0) \\ &\leqslant 8\left(1 - \tfrac{\mu_0}{\lambda_1}\right)^{-1}(2\beta_2 + d_0)\end{aligned}$$

where in (1) we applied (3.39).

This implies, in particular, that the family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$, with $B_t = \overline{B}_{r_0}^X$ for all $t \in \mathbb{R}$, where

$$r_0 = \sqrt{8\left(1 - \frac{\mu_0}{\lambda_1}\right)^{-1}(2\beta_2 + d_0)},$$

is a uniformly pullback absorbing family for $S$.

### Existence of a closed, uniformly bounded and positively invariant uniformly pullback absorbing family

Using any of the families $\hat{B}$ obtained above, we will construct a respective family $\hat{C}$ that is closed, uniformly bounded, positively invariant and uniformly pullback absorbing for $S$. Since $\overline{B}_{r_0}^X$ is bounded, there exists $\tau_1 \geqslant 0$ such that $S(t,t-\tau)\overline{B}_{r_0}^X \subset \overline{B}_{r_0}^X$ for all $\tau \geqslant \tau_1$ and $t \in \mathbb{R}$. Consider the family $\hat{C} = \{C_t\}_{t\in\mathbb{R}}$ defined by

$$C_t = \overline{\bigcup_{\tau \geqslant \tau_1} S(t,t-\tau)\overline{B}_{r_0}^X} \quad \text{for each } t \in \mathbb{R}.$$

**Theorem 3.31.** *The family $\hat{C}$ is closed, uniformly bounded, positively invariant and uniformly pullback absorbing for $S$.*

*Proof.* Clearly $\hat{C}$ is closed, and $C_t \subset \overline{B}_{r_0}^X$ for all $t \in \mathbb{R}$, which implies $\bigcup_{t\in\mathbb{R}} C_t \subset \overline{B}_{r_0}^X$, that is, $\hat{C}$ is uniformly bounded. Now, let $t \geqslant s$. Then we can write $s = t - \sigma$ for some $\sigma \geqslant 0$. Note that

$$S(t,s)C_s = S(t,t-\sigma)C_{t-\sigma} = S(t,t-\sigma)\overline{\bigcup_{\tau\geqslant\tau_1} S(t-\sigma,t-\sigma-\tau)\overline{B}_{r_0}^X}$$

$$
\begin{aligned}
&\subset \overline{S(t,t-\sigma)\bigcup_{\tau\geqslant\tau_1} S(t-\sigma,t-\sigma-\tau)\overline{B}^X_{r_0}}\\
&= \overline{\bigcup_{\tau\geqslant\tau_1} S(t,t-\sigma)S(t-\sigma,t-\sigma-\tau)\overline{B}^X_{r_0}}\\
&= \overline{\bigcup_{\tau\geqslant\tau_1} S(t,t-(\sigma+\tau))\overline{B}^X_{r_0}} \subset \overline{\bigcup_{\tau\geqslant\sigma+\tau_1} S(t,t-\tau)\overline{B}^X_{r_0}}\\
&\subset \overline{\bigcup_{\tau\geqslant\tau_1} S(t,t-\tau)\overline{B}^X_{r_0}} = C_t,
\end{aligned}
$$

which prove the positively invariance of the family $\hat{C}$.

To see that $\hat{C}$ is uniformly pullback absorbing, let $D \subset H^1_0(\Omega)\times L^2(\Omega)$ be a bounded set. We know that there exist $\tau_0,\tau_1 \geqslant 0$ such that $S(t,t-\tau)D \subset \overline{B}^X_{r_0}$ for every $t\in\mathbb{R}$ and $\tau\geqslant\tau_0$, and $S(t,t-\tau)\overline{B}^X_{r_0} \subset \overline{B}^X_{r_0}$ for every $t\in\mathbb{R}$ and $\tau\geqslant\tau_1$. If $t\in\mathbb{R}$ and $\tau\geqslant\tau_0+\tau_1$ then $\tau-\tau_1\geqslant\tau_0$ and we obtain

$$
\begin{aligned}
S(t,t-\tau)D &= S(t,t-\tau_1)S(t-\tau_1,t-\tau)D\\
&= S(t,t-\tau_1)S(t-\tau_1,t-\tau_1-(\tau-\tau_1))D \subset S(t,t-\tau_1)\overline{B}^X_{r_0} \subset C_t,
\end{aligned}
$$

and the proof is complete. □

## 3.4 EXISTENCE OF A GENERALIZED POLYNOMIAL PULLBACK ATTRACTOR WHEN $p > 0$

In this subsection we aim to apply Theorem 2.10 in order to establish the existence of a generalized polynomial pullback attractor for our problem in the case $p>0$. As we can see, this theorem is somehow technical and requires the verification of several hypotheses involving precompact pseudometrics and contractive functions. With the goal of obtaining such pseudometrics and functions, we initiate our work as follows.

Consider the closed, uniformly bounded, positively invariant and uniformly pullback absorbing family $\hat{C}$ obtained in Subsection 3.3. If $s\in\mathbb{R}$ and $V_1,V_2\in C_s\subset\overline{B}^X_{r_0}$, from Proposition 3.22 there exists a constant $c_{r_0}>0$ such that

$$
\|V(t,V_1)\|_X \leqslant c_{r_0} \quad\text{and}\quad \|V(t,V_2)\|_X \leqslant c_{r_0} \quad\text{for all } t\geqslant 0.
$$

If $V(t,V_1)=(v(t),v_t(t))$ and $V(t,V_2)=(w(t),w_t(t))$, setting $Z(t)=(z(t),z_t(t))=V(t,V_1)-V(t,V_2)$ we have $z=v-w$, and (3.12) holds. Defining

$$
\mathcal{E}_s(t) = \frac{1}{2}\|Z(t)\|^2_X = \frac{1}{2}\left(\|z\|^2_{H^1_0} + \|z_t\|^2_{L^2}\right),
$$

we have the following result.

**Lemma 3.32.** *For $T > 0$ we have*

$$
\begin{aligned}
T\mathcal{E}_s(T) = &-\frac{1}{2}\langle z_t,z\rangle\Big|_0^T + \int_0^T \|z_t\|^2_{L^2}dt - \frac{1}{2}\int_0^T k_s(t)\langle\|v_t\|^p_{L^2}v_t - \|w_t\|^p_{L^2}w_t, z\rangle dt \\
&- \frac{1}{2}\int_0^T \langle f_s(t,v) - f_s(t,w), z\rangle dt + \frac{1}{2}\int_0^T\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z(t,x)dydxdt \\
&- \int_0^T\int_t^T k_s(\tau)\langle\|v_t\|^p_{L^2}v_t - \|w_t\|^p_{L^2}w_t, z_t\rangle d\tau dt \\
&- \int_0^T\int_t^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau dt \\
&+ \int_0^T\int_t^T\int_{\Omega\times\Omega} K(x,y)z_t(\tau,y)z_t(\tau,x)dydxd\tau dt.
\end{aligned} \tag{3.42}
$$

*Proof.* Formally multiplying (3.12) by $z_t$, as in (3.13), we obtain

$$
\begin{aligned}
&\int_\Omega z_t z_{tt}dx - \int_\Omega \Delta z z_t dx + k_s(t)\,\|v_t\|^p_{L^2}\int_\Omega v_t z_t dx - k_s(t)\|w_t\|^p_{L^2}\int_\Omega w_t z_t dx \\
&\quad + \int_\Omega [f_s(t,v) - f_s(t,w)]z_t dx = \int_{\Omega\times\Omega} K(x,y)z_t(t,y)z_t(t,x)dydx.
\end{aligned} \tag{3.43}
$$

For $T > 0$, integrating from $t$ to $T$, we obtain

$$
\begin{aligned}
&\int_t^T \langle z_t, z_{tt}\rangle d\tau - \int_t^T \langle \Delta z, z_t\rangle d\tau + \int_t^T k_s(\tau)\langle\|v_t\|^p_{L^2}v_t - \|w_t\|^p_{L^2}w_t, z_t\rangle d\tau \\
&+ \int_t^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau - \int_t^T\int_{\Omega\times\Omega} K(x,y)z_t(\tau,y)z_t(\tau,x)dydxd\tau = 0.
\end{aligned}
$$

Note that

$$
\mathcal{E}_s(T) - \mathcal{E}_s(t) = \int_t^T \frac{d}{d\tau}\mathcal{E}_s(\tau)d\tau = \int_t^T \langle z_t, z_{tt}\rangle d\tau - \int_t^T \langle \Delta z, z_t\rangle d\tau,
$$

and hence we obtain

$$
\begin{aligned}
\mathcal{E}_s(t) = \mathcal{E}_s(T) &+ \int_t^T k_s(\tau)\langle\|v_t\|^p_{L^2}v_t - \|w_t\|^p_{L^2}w_t, z_t\rangle d\tau \\
&+ \int_t^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau - \int_t^T\int_{\Omega\times\Omega} K(x,y)z_t(\tau,y)z_t(\tau,x)dydxd\tau.
\end{aligned} \tag{3.44}
$$

Integrating (3.44) from 0 to $T$ we obtain

$$
\begin{aligned}
T\mathcal{E}_s(T) = &\int_0^T \mathcal{E}_s(t)dt - \int_0^T\int_t^T k_s(\tau)\langle\|v_t\|^p_{L^2}\,v_t - \|w_t\|^p_{L^2}w_t, z_t\rangle d\tau dt \\
&- \int_0^T\int_t^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau dt \\
&+ \int_0^T\int_t^T\int_{\Omega\times\Omega} K(x,y)z_t(\tau,y)z_t(\tau,x)dydxd\tau dt.
\end{aligned} \tag{3.45}
$$

On the other hand, we can formally multiply (3.12) by $z$, integrate in $\Omega$ and integrate from 0 to $T$ in order to obtain

$$
\begin{aligned}
&\int_0^T \langle z_{tt},z\rangle dt - \int_0^T \langle \Delta z, z\rangle dt + \int_0^T k_s(t)\langle\|v_t\|^p_{L^2}v_t - \|w_t\|^p_{L^2}w_t, z\rangle dt \\
&+ \int_0^T \langle f_s(t,v) - f_s(t,w), z\rangle dt = \int_0^T\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z(t,x)dydxdt.
\end{aligned} \tag{3.46}
$$

Claim. $\int_0^T \langle z_{tt},z\rangle dt - \int_0^T \langle \Delta z, z\rangle dt = \langle z_t,z\rangle\Big|_0^T - 2\int_0^T \|z_t\|_{L^2}^2 dt + 2\int_0^T \mathcal{E}_s(t)dt.$

Indeed, since $\langle \Delta z,z\rangle = -\|z\|_{H_0^1}^2$ and $\frac{d}{dt}\langle z_t, z\rangle = \|z_t\|_{L^2}^2 + \langle z_{tt},z\rangle$, we obtain

$$\begin{aligned}\langle z_{tt},z\rangle - \langle \Delta z, z\rangle &= \frac{d}{dt}\langle z_t, z\rangle - \|z_t\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 \\ &= \frac{d}{dt}\langle z_t, z\rangle - 2\|z_t\|_{L^2}^2 + \|z_t\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 = \frac{d}{dt}\langle z_t, z\rangle - 2\|z_t\|_{L^2}^2 + 2\mathcal{E}_s(t),\end{aligned}$$

which proves the claim.

Using this claim in (3.46), we obtain

$$\begin{aligned}\langle z_t,z\rangle\Big|_0^T - 2\int_0^T \|z_t\|_{L^2}^2\, dt + 2\int_0^T \mathcal{E}_s(t)dt + \int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle dt \\ + \int_0^T \langle f_s(t,v) - f_s(t,w), z\rangle dt = \int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(t,y) z(t,x) dydxdt,\end{aligned}$$

which implies

$$\begin{aligned}\int_0^T \mathcal{E}_s(t)dt = -\frac{1}{2}\langle z_t,z\rangle\Big|_0^T + \int_0^T \|z_t\|_{L^2}^2 dt - \frac{1}{2}\int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle dt \\ - \frac{1}{2}\int_0^T \langle f_s(t,v) - f_s(t,w), z\rangle dt + \frac{1}{2}\int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(t,y) z(t,x) dydxdt.\end{aligned} \tag{3.47}$$

Joining (3.45) and (3.47), we obtain (3.42). □

**Proposition 3.33.** *For $T > 0$ there exist constants $\Gamma_{T,1},\Gamma_{T,2} > 0$ such that*

$$\begin{aligned}\mathcal{E}_s(T) \leqslant \Gamma_{T,1} \sup_{t\in[0,T]} \|z(t)\|_{L^2} + \Gamma_{T,2}\Big(\mathcal{E}_s(0) - \mathcal{E}_s(T) \\ + \left|\int_0^T \langle f_s(\tau, v) - f_s(\tau, w), z_t\rangle d\tau\right| + 2c_{r_0}\int_0^T \left\|\int_\Omega K(x,y) z_t(t,y) dy\right\|_{L^2} dt\Big)^{\frac{2}{p+2}} \\ + \tfrac{1}{T}\left|\int_0^T \int_t^T \langle f_s(\tau, w) - f_s(\tau, v), z_t\rangle d\tau dt\right| + 2c_{r_0}\int_0^T \left\|\int_\Omega K(x,y) z_t(t,y) dy\right\|_{L^2} dt.\end{aligned} \tag{3.48}$$

*Proof.* Using Proposition A.11 we have

$$\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \geqslant 2^{-p}\|z_t\|_{L^2}^{p+2}, \tag{3.49}$$

which implies that

$$\|z_t\|_{L^2}^2 \leqslant 2^{\frac{2p}{p+2}} \langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle^{\frac{2}{p+2}}.$$

Using the fact that $[0,\infty) \ni r \mapsto r^{\frac{2}{2+p}}$ is a continuous concave function, Proposition A.12 implies that

$$\begin{aligned}\int_0^T \|z_t\|_{L^2}^2 dt &\leqslant T2^{\frac{2p}{p+2}}\frac{1}{T}\int_0^T \langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle^{\frac{2}{p+2}} dt \\ &\leqslant T2^{\frac{2p}{p+2}}\left(\frac{1}{T}\int_0^T \langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt\right)^{\frac{2}{p+2}} \\ &= (4T)^{\frac{p}{p+2}}\left(\int_0^T \langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt\right)^{\frac{2}{p+2}} \\ &\overset{(*)}{\leqslant} (4T)^{\frac{p}{p+2}} k_0^{\frac{-2}{p+2}}\left(\int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt\right)^{\frac{2}{p+2}},\end{aligned} \tag{3.50}$$

where in $(*)$ we used the fact that $\frac{k_s(t)}{k_0} \geqslant 1$ for all $t,s \in \mathbb{R}$.

Now, using (3.44) and (3.49) , we obtain

$$\begin{aligned} 0 &\leqslant \int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt = \mathcal{E}_s(0) - \mathcal{E}_s(T) \\ &\quad - \int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau + \int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(\tau,y) z_t(\tau,x)\, dydxdt \\ &\leqslant \mathcal{E}_s(0) - \mathcal{E}_s(T) + \left| \int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau \right| \\ &\quad + \int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(\tau,y) z_t(\tau,x)\, dydxdt. \end{aligned} \tag{3.51}$$

Since $\|z_t\|_{L^2} \leqslant 2c_{r_0}$ for all $t \geqslant 0$, using Hölder's inequality we have

$$\begin{aligned} \int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(\tau,y) z_t(\tau,x)\, dydxd\tau &\leqslant \int_0^T \left\| \int_\Omega K(x,y) z_t(\tau,y) dy \right\|_{L^2} \|z_t(\tau,x)\|_{L^2} d\tau \\ &\leqslant 2c_{r_0} \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt, \end{aligned} \tag{3.52}$$

plugging (3.51) and (3.52) into (3.50) we obtain

$$\begin{aligned} \int_0^T \|z_t\|_{L^2}^2\, dt \leqslant (4T)^{\frac{p}{p+2}} k_0^{\frac{-2}{p+2}} \Bigg( &\mathcal{E}_s(0) - \mathcal{E}_s(T) + \left| \int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau \right| \\ &+ 2c_{r_0} \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt \Bigg)^{\frac{2}{p+2}}. \end{aligned} \tag{3.53}$$

Furthermore

$$-\tfrac{1}{2}\langle z_t, z\rangle \Big|_0^T \leqslant 2c_{r_0} \sup_{t\in[0,T]} \|z(t)\|_{L^2}, \tag{3.54}$$

and

$$\begin{aligned} -\tfrac{1}{2} \int_0^T k_s(t) \langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle dt &\leqslant \tfrac{k_1}{2} \int_0^T |\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle| dt \\ \leqslant \tfrac{k_1}{2} \int_0^T \left\| \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t \right\|_{L^2} \|z(t)\|_{L^2} dt &\leqslant k_1 c_{r_0}^{p+1} T \sup_{t\in[0,T]} \|z(t)\|_{L^2}. \end{aligned} \tag{3.55}$$

Using Lemma 3.4 we have

$$\begin{aligned} -\tfrac{1}{2} \int_0^T \langle f_s(t,v) - f_s(t,w), z\rangle dt &\leqslant \tfrac{1}{2} \int_0^T \|f_s(t,v) - f_s(t,w)\|_{L^2} \|z(t)\|_{L^2} dt \\ \leqslant \tfrac{L_0(1+2c_{r_0}^2)}{2} \int_0^T \|z(t)\|_{H_0^1} \|z(t)\|_{L^2} dt &\leqslant \tfrac{L_0 c_{r_0}(1+2c_{r_0}^2)T}{2} \sup_{t\in[0,T]} \|z(t)\|_{L^2}. \end{aligned} \tag{3.56}$$

Using Hölder's inequality we obtain

$$\begin{aligned} \tfrac{1}{2} \int_0^T \int_{\Omega\times\Omega} K(x,y) z_t(t,y) z(t,x)\, dydxdt &\leqslant \tfrac{K_0}{2} \int_0^T \|z_t\|_{L^2} \|z\|_{L^2} dt \\ &\leqslant K_0 c_{r_0} T \sup_{t\in[0,T]} \|z(t)\|_{L^2}, \end{aligned} \tag{3.57}$$

and

$$\int_0^T \int_t^T \int_{\Omega\times\Omega} K(x,y) z_t(\tau,y) z_t(\tau,x)\, dydxd\tau dt$$

$$\leqslant \int_0^T \int_t^T \left\| \int_\Omega K(x,y) z_t(\tau,y) dy \right\|_{L^2} \|z_t(\tau,y)\|_{L^2} \, d\tau dt$$
$$\leqslant 2c_{r_0} T \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt.$$

Using all these inequalities in (3.42), we conclude that

$$\begin{aligned} T\mathcal{E}_s(T) \leqslant\ & 2c_{r_0} \sup_{t\in[0,T]} \|z(t)\|_{L^2} + (4T)^{\frac{p}{p+2}} k_0^{\frac{-2}{p+2}} \Big( \mathcal{E}_s(0) - \mathcal{E}_s(T) \\ & + \left| \int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t \rangle d\tau \right| + 2c_{r_0} \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt \Big)^{\frac{2}{p+2}} \\ & + k_1 c_{r_0}^{p+1} T \sup_{t\in[0,T]} \|z(t)\|_{L^2} + \tfrac{L_0 c_{r_0}(1+2c_{r_0}^2)T}{2} \sup_{t\in[0,T]} \|z(t)\|_{L^2} + K_0 c_{r_0} T \sup_{t\in[0,T]} \|z(t)\|_{L^2} \\ & + \left| \int_0^T \int_t^T \langle f_s(\tau,v) - f_s(\tau,w), z_t \rangle d\tau dt \right| + 2c_{r_0} T \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt, \end{aligned}$$

which, naming

$$\Gamma_{T,1} = \tfrac{1}{T} \left( 2c_{r_0} + k_1 c_{r_0}^{p+1} T + \tfrac{L_0 c_{r_0}(1+2c_{r_0}^2)T}{2} + K_0 c_{r_0} T \right)$$

and $\Gamma_{T,2} = 4^{\frac{p}{p+2}} (Tk_0)^{-\frac{2}{p+2}}$, gives us (3.48). □

**Corollary 3.34.** *We have*

$$\mathcal{E}_s(T) \leqslant \mathcal{E}_s(0) + \left| \int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t \rangle d\tau \right| + 2c_{r_0} \int_0^T \left\| \int_\Omega K(x,y) z_t(\tau,y) dy \right\|_{L^2} dt.$$

*Proof.* It follows directly from (3.53) and the fact that $\int_0^T \|z_t\|_{L^2}^2 dt \geqslant 0$. □

Our goal for now is to use Theorem 2.10 to prove that the evolution process $S$ associated with (NWE) is polynomially pullback $\kappa$-dissipative. So, from this point on, we verify that we are in condition to use Theorem 2.10.

**The pseudometrics $\rho_1$ and $\rho_2$.**

For $V_1,V_2 \in X$, setting $Z(t) = (z(t),z_t(t)) = V(t,V_1) - V(t,V_2)$, define the maps $\rho_1,\rho_2 \colon X \times X \to \mathbb{R}^+$ as

$$\rho_1(V_1,V_2) = 4c_{r_0} \int_0^T \left\| \int_\Omega K(x,y) z_t(t,y) dy \right\|_{L^2} dt, \tag{3.58}$$

and

$$\rho_2(V_1,V_2) = 2\Gamma_{T,1} \sup_{t\in[0,T]} \|z(t)\|_{L^2}. \tag{3.59}$$

This part of our work is dedicated to prove the following result.

**Proposition 3.35.** *The functions $\rho_1,\rho_2$ are pseudometrics in $X$, which are precompact in $\overline{B}^X_{r_0}$.*

It is clear that both $\rho_1$ and $\rho_2$ are pseudometrics. It remains to prove that they are precompact in $\overline{B}^X_{r_0}$. Recall that $A = -\Delta\colon H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ denotes the negative Laplacian with Dirichlet boundary conditions, which is a positive self-adjoint operator, with compact resolvent. Define the bounded linear operator $\Lambda\colon L^2(\Omega) \to L^2(\Omega)$ given by

$$\Lambda v = \int_\Omega K(x,y)v(y)dy \quad \text{for } v \in L^2(\Omega),$$

and let $V$ be the completion of $L^2(\Omega)$ with respect to the norm

$$\|v\|_V = \|\Lambda v\|_{L^2} + \|v\|_{H^{-1}}.$$

Since both $\Lambda$ is a bounded linear operator in $L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we obtain $L^2(\Omega) \hookrightarrow V \hookrightarrow H^{-1}(\Omega)$. Furthermore, since $A$ has compact resolvent, we have $L^2(\Omega) \hookrightarrow\hookrightarrow V$.

For each $V_0 \in \overline{B}^X_{r_0}$, if $V(t,V_0) = (v(t),v_t(t))$, we denote

$$\mathcal{F} = \left\{v_t\colon (0,T) \to L^2(\Omega) \middle| \, V_0 \in \overline{B}^X_{r_0}\right\}.$$

Note that for each $v_t \in \mathcal{F}$, we have $v_{tt}\colon (0,T) \to H^{-1}(\Omega)$, and hence

$$\frac{d\mathcal{F}}{dt} = \left\{v_{tt}\colon (0,T) \to H^{-1}(\Omega) | \, V_0 \in \overline{B}^X_{r_0}\right\}.$$

**Lemma 3.36.** *$\mathcal{F}_s$ is relatively compact in $L^1(0,T;V)$.*

*Proof.* If $v_t \in \mathcal{F}$ we have

$$\|v_t\|_{L^1(0,T;L^2(\Omega))} = \int_0^T \|v_t(t)\|_{L^2} dt \leqslant c_{r_0} T,$$

hence $\mathcal{F}$ is bounded in $L^1(0,T;L^2(\Omega))$. Now for $V_0 \in \overline{B}^X_{r_0}$, from (3.9) we have

$$\|v_{tt}\|_{H^{-1}} \leqslant \|v\|_{H^1_0} + \lambda_1^{-\frac{1}{2}}(k_1\|v_t\|^{p+1}_{L^2} + K_0\|v_t\|_{L^2} + \|f_s(t,v)\|_{L^2} + h_0). \tag{3.60}$$

Since

$$\begin{aligned} \|f_s(t,v)\|_{L^2} &\leqslant \|f_s(t,v) - f_s(t,0)\|_{L^2} + \|f_s(t,0)\|_{L^2} \\ &\leqslant L_0(1+2c^2_{r_0})\|v\|_{H^1_0} + c_0|\Omega|^{\frac{1}{2}} \leqslant L_0(1+2c^2_{r_0})c_{r_0} + c_0|\Omega|^{\frac{1}{2}}, \end{aligned} \tag{3.61}$$

we obtain

$$\|v_{tt}\|_{L^1(0,T;H^{-1})} = \int_0^T \|v_{tt}(t)\|_{H^{-1}} dt \leqslant cT,$$

for some constant $c > 0$. This means that $\frac{d\mathcal{F}}{dt}$ is bounded in $L^1(0,T;H^{-1}(\Omega))$. From (SIMON, 1986, Corolary) we obtain $\mathcal{F}$ relatively compact in $L^1(0,T;V)$. □

*Proof of Proposition 3.35.* We begin with $\rho_1$. From Lemma 3.36, $\mathcal{F}$ is relatively compact in $L^1(0,T;V)$ and given $\delta > 0$ there exist $V_1, \cdots, V_n \in \overline{B}^X_{r_0}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^n B^{L^1(0,T;V)}_{\frac{\delta}{4c_{r_0}}}(v^{(i)}_t)$$

where $V(t,V_i) = (v^{(i)}(t),v_t^{(i)}(t))$ for each $i = 1,\cdots,n$. If $V_0 \in \overline{B}_{r_0}^X$ and $V(t,V_0) = (v(t),v_t(t))$, then $v_t \in \mathcal{F}$ and thus $v_t \in B_{\frac{\delta}{4c_{r_0}}}^{L^1(0,T;V)}(v_t^{(i)})$ for some $i \in \{1,\cdots,n\}$. This implies that

$$\rho_1(V_0,V_i) = 4c_{r_0}\int_0^T \left\|\int_\Omega K(x,y)[v_t(t,y) - v_t^{(i)}(t,y)]dy\right\|_{L^2} dt$$
$$\leqslant 4c_{r_0}\int_0^T \|v_t(t) - v_t^{(i)}(t)\|_V dt = 4c_{r_0}\|v_t - v_t^{(i)}\|_{L^1(0,T;V)} < \delta,$$

that is,

$$\overline{B}_{r_0}^X \subset \bigcup_{i=1}^n B_\delta^{\rho_1}(V_i),$$

which proves that $\rho_1$ is precompact in $\overline{B}_{r_0}^X$.

Now we prove for $\rho_2$. If $\{V_n\}_{n\in\mathbb{N}}$ is a sequence in $\overline{B}_{r_0}^X$ and $\{v^{(n)}\}_{n\in\mathbb{N}}$ is the sequence of its corresponding solutions (that is, $V(t,V_n) = (v^{(n)}(t),v_t^{(n)}(t))$), we have $\{v^{(n)}\}_{n\in\mathbb{N}} \subset C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$. From the Arzelà-Ascoli Theorem we have

$$C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)) \hookrightarrow\hookrightarrow C(0,T;L^2(\Omega)),$$

which implies that $\{v^{(n)}\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{v^{(n_k)}\}_{k\in\mathbb{N}}$ in $C(0,T;L^2(\Omega))$ (consequently a Cauchy subsequence in $C(0,T;L^2(\Omega))$). This means that given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho_2(V_{n_k},V_{n_j}) = 2\Gamma_{T,1}\sup_{t\in[0,T]}\|v^{(n_k)}(t) - v^{(n_j)}(t)\|_{L^2} < \varepsilon, \tag{3.62}$$

for all $k,j \geqslant n_0$, that is, $\{V_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to $\rho_2$. From Proposition A.13, $\rho_2$ is the precompact on $\overline{B}_{r_0}^X$. □

**The contractive maps $\psi_1$ and $\psi_2$.**

As in the definition of $\rho_1$ and $\rho_2$, for $V_1,V_2 \in \overline{B}_{r_0}^X$, if $V(t,V_1) = (v(t),v_t(t))$, $V(t,V_2) = (w(t),w_t(t))$, setting $Z(t) = (z(t),z_t(t)) = V(t,V_1) - V(t,V_2)$ then $z = v - w$, $z_t = v_t - w_t$, and we define $\psi_1,\psi_2\colon X \times X \to \mathbb{R}^+$ by

$$\psi_1(V_1,V_2) = 2\left|\int_0^T \langle f_s(\tau,v) - f_s(\tau,w), z_t\rangle d\tau\right|, \tag{3.63}$$

$$\psi_2(V_1,V_2) = \frac{2}{T}\left|\int_0^T\int_t^T \langle f_s(\tau,v) - f_s(\tau,w), v_t - w_t\rangle d\tau dt\right|. \tag{3.64}$$

Our goal for now is to prove the following result.

**Proposition 3.37.** $\psi_1,\psi_2 \in \operatorname{contr}(\overline{B}_{r_0}^X)$.

As we proceeded in previous sections, we will prove a sequence of auxiliary lemmas to help us prove Proposition 3.37.

**Lemma 3.38.** *Let $\{V_n\}_{n\in\mathbb{N}} \subset \overline{B}^X_{r_0}$ and denote $V(t,V_n) = (v^{(n)}(t),v^{(n)}_t(t))$. For a fixed $\gamma \in (0,1)$, up to a subsequence, we have*

$$\begin{cases} (v^{(n)},v_t^{(n)}) \overset{*}{\rightharpoonup} (v,v_t) \text{ in } L^\infty(0,T;X), \\ v^{(n)} \to v \text{ in } C([0,T];H^\gamma(\Omega)). \end{cases} \tag{3.65}$$

*Proof.* Since $\{(v^{(n)},v_t^{(n)})\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;X) = [L^1(0,T;H^{-1}(\Omega)\times L^2(\Omega))]^*$, where $H^{-1}(\Omega) = (H^1_0(\Omega))^*$, it follows from Alaoglu's Theorem that, up to a subsequence which we name the same, there exists $(v,z) \in L^\infty(0,T;X)$ such that $(v^{(n)},v_t^{(n)}) \overset{*}{\rightharpoonup} (v,z)$ in $L^\infty(0,T;X)$, that is, for every $\phi \in L^1(0,T;H^{-1})$ and $\psi \in L^1(0,T;L^2(\Omega))$ we have

$$\begin{aligned} &\int_0^T \langle v^{(n)}(t) - v(t), \phi(t)\rangle_{H^1_0,H^{-1}} dt \to 0 \text{ as } n \to \infty, \\ &\int_0^T \langle v_t^{(n)}(t) - z(t), \psi(t)\rangle dt \to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.66}$$

Let $\phi \in C_0^\infty([0,T];C_0^\infty(\Omega))$. We have $\phi' \in L^1(0,T;L^2(\Omega)) \cap L^1(0,T;H^{-1}(\Omega))$ and, for each $t\in[0,T]$, we obtain

$$\langle v^{(n)}(t) - v(t),\phi'(t)\rangle_{H^1_0,H^{-1}} = \langle v^{(n)}(t) - v(t),\phi'(t)\rangle,$$

since $\langle v,\eta\rangle_{H^1_0,H^{-1}} = \langle v,\eta\rangle$ when $v \in H^1_0(\Omega)$ and $\eta \in L^2(\Omega)$. Therefore, using integration by parts we have

$$\begin{aligned} \int_0^T \langle v^{(n)}(t) - v(t), \phi'(t)\rangle_{H^1_0,H^{-1}} dt &= \int_0^T \langle v^{(n)}(t) - v(t), \phi'(t)\rangle dt \\ &= \int_0^T \langle v^{(n)}(t), \phi'(t)\rangle dt - \int_0^T \langle v(t), \phi'(t)\rangle dt = -\int_0^T \langle v_t^{(n)}(t), \phi(t)\rangle dt - \int_0^T \langle v(t),\phi'(t)\rangle dt \\ &\overset{n\to\infty}{\longrightarrow} -\int_0^T \langle z(t),\phi(t)\rangle dt - \int_0^T \langle v(t),\phi'(t)\rangle dt. \end{aligned}$$

On the other hand, since $\phi' \in L^1(0,T;H^{-1}(\Omega))$, we obtain

$$\int_0^T \langle v^{(n)}(t) - v(t), \phi'(t)\rangle_{H^1_0,H^{-1}} dt \overset{n\to\infty}{\longrightarrow} 0,$$

thus

$$\int_0^T \langle v(t),\phi'(t)\rangle dt = -\int_0^T \langle z(t),\phi(t)\rangle dt.$$

which means that $z = v_t$ in $L^1(0,T;L^2(\Omega))$. Therefore, $(v^{(n)},v_t^{(n)}) \overset{*}{\rightharpoonup} (v,v_t)$ in $L^\infty(0,T;X)$.

With this subsequence we want to show that $v^{(n)} \to v$ in $C(0,T;H^\gamma(\Omega))$. Consider the family $\mathcal{G} = \{v^{(n)}\colon n\in\mathbb{N}\}$, which is bounded in $L^\infty(0,T;H^1_0(\Omega))$. Since $\frac{d\mathcal{G}}{dt} = \{v_t^{(n)}\colon n \in \mathbb{N}\}$ is bounded in $L^2(0,T;L^2(\Omega))$, from (SIMON, 1986, Corollary 4) and the fact that $H^1_0(\Omega) \hookrightarrow\hookrightarrow H^\gamma(\Omega) \hookrightarrow L^2(\Omega)$, it follows that $\mathcal{G}$ is relatively compact in $C([0,T];H^\gamma(\Omega))$. Hence, up to a subsequence which we name the same, there exists $y \in C(0,T;H^\gamma(\Omega))$ such that $v^{(n)} \to y$ in $C([0,T];H^\gamma(\Omega))$. Therefore, for any $\phi \in C_0^\infty([0,T];C_0^\infty(\Omega))$ we have

$$\left|\int_0^T \langle v^{(n)}(t) - y(t), \phi(t)\rangle_{H^1_0,H^{-1}} dt\right| = \left|\int_0^T \langle v^{(n)}(t) - y(t),\phi(t)\rangle dt\right|$$

$$\leqslant \int_0^T \|v^{(n)}(t) - y(t)\|_{L^2} \|\phi(t)\|_{L^2} dt \leqslant C \int_0^T \|v^{(n)}(t) - y(t)\|_{H^\gamma} \|\phi(t)\|_{L^2} dt$$
$$\leqslant CT \sup_{t\in[0,T]} \left\|v^{(n)}(t) - y(t)\right\|_{H^\gamma} \overset{n\to\infty}{\longrightarrow} 0,$$

which ensures that $y = v$ in $L^1(0,T;H^{-1}(\Omega))$. Since they are continuous, $y = v$ in $C([0,T];H^\gamma(\Omega))$ and, consequently, $v^{(n)} \to v$ in $C([0,T];H^\gamma(\Omega))$. □

**Lemma 3.39.** *There exists a constant $C > 0$ such that for all $t \geqslant 0$ and $n \in \mathbb{N}$ we have*

$$\left|\int_\Omega F_s(t,v^{(n)}(t))dx - \int_\Omega F_s(t,v(t))dx\right| \leqslant C\|v^{(n)}(t) - v(t)\|_{H^\gamma},$$

*and*

$$\left|\int_\Omega \frac{\partial F_s}{\partial t}(t,v^{(n)}(t))dx - \int_\Omega \frac{\partial F_s}{\partial t} t,v(t))dx\right| \leqslant C\|v^{(n)}(t) - v(t)\|_{H^\gamma}.$$

*Proof.* From Proposition 3.3, using Hölder's inequality and the continuous inclusions $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$ and $H^\gamma(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$\left|\int_\Omega F_s(t,v^{(n)}(t))dx - \int_\Omega F_s(t,v(t))dx\right| \leqslant \int_\Omega \left|F_s(t,v^{(n)}(t)) - F_s(t,v(t))\right| dx$$
$$\leqslant 8c_0 \int_\Omega (1 + |v^{(n)}(t)|^3 + |v(t)|^3)|v^{(n)}(t) - v(t)|dx$$
$$\leqslant C \left(\int_\Omega (1 + |v^{(n)}(t)|^6 + |v(t)|^6)dx\right)^{\frac{1}{2}} \|v^{(n)}(t) - v(t)\|_{L^2}$$
$$\leqslant C(1 + \|v^{(n)}\|^3_{L^6} + \|v\|^3_{L^6})\|v^{(n)}(t) - v(t)\|_{L^2}$$
$$\leqslant C(1 + \|v^{(n)}\|^3_{H^1_0} + \|v\|^3_{H^1_0})\|v^{(n)}(t) - v(t)\|_{H^\gamma} \leqslant C\|v^{(n)}(t) - v(t)\|_{H^\gamma},$$

where the constant $C > 0$ changed from one line to another, and we used the fact that $\|v^{(n)}(t)\|_{H^1_0}, \|v(t)\|_{H^1_0} \leqslant c_{r_0}$ for all $t \geqslant 0$ (see Proposition 3.22).

For the second estimate, note that from Proposition 3.3 and Hölder's inequality we obtain

$$\left|\int_\Omega \frac{\partial F_s}{\partial t}(t,v^{(n)}(t))dx - \int_\Omega \frac{\partial F_s}{\partial t}(t,v(t))dx\right| \leqslant 2c_0 \int_\Omega |v^{(n)}(t) - v(t)|dx$$
$$\leqslant 2c_0|\Omega|^{\frac{1}{2}} \|v^{(n)}(t) - v(t)\|_{L^2} \leqslant C\|v^{(n)}(t) - v(t)\|_{H^\gamma},$$

for some constant $C > 0$. □

From Lemmas 3.38 and 3.39 we obtain the following result.

**Corollary 3.40.** *We have*

$$\int_\Omega F_s(t, v^{(n)}(t))dx \overset{n\to\infty}{\longrightarrow} \int_\Omega F_s(t, v(t))dx,$$

*and*

$$\int_\Omega \frac{\partial F_s}{\partial t}(t, v^{(n)}(t))dx \overset{n\to\infty}{\longrightarrow} \int_\Omega \frac{\partial F_s}{\partial t}(t, v(t))dx,$$

*uniformly for $t \in [0,T]$.*

**Lemma 3.41.** *For each $t \in [0,T]$ and $\xi \in L^1(0,T; H^3(\Omega) \cap H_0^1(\Omega))$ it holds that*

$$\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \xi(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} 0.$$

*Proof.* First we note that since $H^3(\Omega) \hookrightarrow L^\infty(\Omega)$, we have $L^1(\Omega) \hookrightarrow (L^\infty(\Omega))^\star \hookrightarrow H^{-3}(\Omega)$. Using Proposition 3.3 we obtain

$$\begin{aligned}
&\|f_s(t, v^{(n)}(t)) - f_s(t, v(t))\|_{H^{-3}} \leqslant C\|f_s(t, v^{(n)}(t)) - f_s(t, v(t))\|_{L^1}\\
&\quad \leqslant C \int_\Omega (1 + |v^{(n)}(t)|^2 + |v(t)|^2)|v^{(n)}(t) - v(t)|dx\\
&\quad \leqslant C \left(\int_\Omega (1 + |v^{(n)}(t)|^4 + |v(t)|^4)dx\right)^{\frac{1}{2}} \|v^{(n)}(t) - v(t)\|_{L^2}\\
&\quad \leqslant C(1 + \|v^{(n)}(t)\|_{L^4}^2 + \|v(t)\|_{L^4}^2)\|v^{(n)}(t) - v(t)\|_{L^2}\\
&\quad \leqslant C(1 + \|v^{(n)}(t)\|_{H_0^1}^2 + \|v(t)\|_{H_0^1}^2)\|v^{(n)}(t) - v(t)\|_{H^\gamma} \leqslant C\|v^{(n)}(t) - v(t)\|_{H^\gamma},
\end{aligned}$$

since $H^\gamma(\Omega) \hookrightarrow L^2(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and the fact that $\|v^{(n)}(t)\|_{H_0^1}, \|v(t)\|_{H_0^1} \leqslant c_{r_0}$ for all $t \geqslant 0$.

From Lemma 3.38 we conclude that

$$\sup_{t\in[0,T]} \|f_s(t, v^{(n)}(t)) - f_s(t, v(t))\|_{H^{-3}} \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, if $\xi \in L^1(0, T, H^3(\Omega) \cap H_0^1(\Omega))$ we have

$$\begin{aligned}
&\left|\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \xi(\tau)\rangle d\tau\right|\\
&\qquad \leqslant \sup_{t\in[0,T]} \|f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v(\tau))\|_{H^{-3}} \int_0^T \|\xi(\tau)\|_{H^3} d\tau,
\end{aligned}$$

and the result is proven. ☐

**Corollary 3.42.** *For each $t \in [0,T]$ we have*

$$f_s(\tau, v^{(n)}(\tau)) \overset{*}{\rightharpoonup} f_s(\tau, v(\tau)) \text{ in } L^\infty(t, T; L^2(\Omega)) \text{ as } n \to \infty.$$

*Proof.* Let $\psi \in L^1(t,T; L^2(\Omega))$ and consider $\varepsilon > 0$ arbitrary. Then, from Proposition A.14 we have $L^1(t,T; H^3(\Omega) \cap H_0^1(\Omega)) \hookrightarrow L^1(t,T; L^2(\Omega)$, with dense inclusion, and thus there exists $\xi \in L^1(t,T; H^3(\Omega) \cap H_0^1(\Omega))$ such that

$$\int_t^T \|\psi(\tau) - \xi(\tau)\|_{L^2} d\tau < \varepsilon.$$

For such $\xi$, it follows from Lemma 3.41 that there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ we have

$$\left|\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \xi(\tau)\rangle d\tau\right| < \varepsilon.$$

Therefore, if $n \geqslant n_0$, using Lemma 3.4, we have

$$\left|\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \psi(\tau)\rangle d\tau\right|$$

$$\leqslant \left|\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \psi(\tau) - \xi(\tau)\rangle d\tau\right|$$
$$+ \left|\int_t^T \langle f_s(\tau, v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \xi(\tau)\rangle d\tau\right|$$
$$\leqslant \int_t^T \|f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v(\tau))\|_{L^2}\|\psi(\tau) - \xi(\tau)\|_{L^2} d\tau + \varepsilon$$
$$\leqslant C\int_t^T \|\psi(\tau) - \xi(\tau)\|_{L^2} d\tau + \varepsilon \leqslant (C+1)\varepsilon,$$

which proves that

$$\int_t^T \langle f_s(\tau, w^{(n)}(\tau)) - f_s(\tau, w(\tau)), \psi(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} 0,$$

and completes the proof. □

**Lemma 3.43.** *For each $t \in [0,T]$ and $n \in \mathbb{N}$ we have*

$$\int_t^T \langle f_s(\tau, v^{(n)}(\tau)), v_t^{(m)}(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} \int_t^T \langle f_s(\tau, v^{(n)}(\tau)), v_t(\tau)\rangle d\tau,$$

*and*

$$\int_t^T \langle f_s(\tau, v^{(m)}(\tau)), v_t^{(n)}(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} \int_t^T \langle f_s(\tau, v(\tau)), v_t^{(n)}(\tau)\rangle d\tau.$$

*Proof.* Define $\xi\colon [0,T] \to L^2(\Omega)$ by $\xi(\tau) = f_s(\tau, v^{(n)}(\tau))$ for each $\tau \in [0,T]$. Note that $\xi \in L^1(0,T;L^2(\Omega))$, since

$$\int_0^T \|\xi(\tau)\|_{L^2} d\tau = \int_0^T \left\|f_s(\tau,v^{(n)}(\tau))\right\|_{L^2} d\tau \leqslant CT,$$

for some constant $C > 0$. Since $v_t^{(m)} \overset{*}{\rightharpoonup} v_t$ in $L^\infty(0,T;L^2(\Omega))$ as $m \to \infty$ we have

$$\int_0^T \langle \xi(\tau), v_t^{(m)}(\tau) - v_t(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} 0,$$

and the proof of the first item is complete.

For the second item, define $\xi\colon [0,T] \to L^2(\Omega)$ by $\xi(\tau) = v_t^{(n)}(\tau)$ for each $\tau \in [0,T]$. Clearly $\xi \in L^1(0,T;L^2(\Omega))$, since for some constant $C > 0$ we have

$$\int_0^T \|\xi(\tau)\|_{L^2} d\tau = \int_0^T \|v_t^{(n)}(\tau)\|_{L^2} d\tau \leqslant CT,$$

and hence, since $f_s(\tau,v^{(m)}(\tau)) \overset{*}{\rightharpoonup} f_s(\tau,v(\tau))$ in $L^\infty(t,T;L^2(\Omega))$ as $m \to \infty$ we have

$$\int_t^T \langle f_s(\tau,v^{(m)}(\tau)) - f_s(\tau,v(\tau)),\xi(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} 0,$$

and the result is proven. □

**Lemma 3.44.** *For each $t \in [0,T]$ we have*

$$\int_t^T \langle f_s(\tau, v^{(n)}(\tau)), v_t(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} \int_t^T \langle f_s(\tau, v(\tau)), v_t(\tau)\rangle d\tau,$$

*and*

$$\int_t^T \langle f_s(\tau, v(\tau)), v_t^{(n)}(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} \int_t^T \langle f_s(\tau, v(\tau)), v_t(\tau)\rangle d\tau.$$

*Proof.* Define $\xi(\tau) = v_t(\tau)$. It is clear that $\xi \in L^1(0,T;L^2(\Omega))$, since for some constant $C > 0$ we have

$$\int_0^T \|\xi(\tau)\|_{L^2} d\tau = \int_0^T \|v_t(\tau)\|_{L^2} d\tau \leqslant CT.$$

From Corollary 3.42, we have

$$\int_t^T \langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v(\tau)),\xi(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} 0,$$

which proves the first item. For the second item define $\xi(\tau) = f_s(\tau,v(\tau))$, and note that $\xi \in L^1(0,T;L^2(\Omega))$. Since $v_t^{(n)} \overset{*}{\rightharpoonup} v_t$ in $L^\infty(0,T;L^2(\Omega))$ as $n \to \infty$, we have

$$\int_t^T \langle \xi(\tau),v_t^{(n)}(\tau) - v_t(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} 0,$$

and the proof is complete. □

**Lemma 3.45.** *For each $t \in [0,T]$ we have*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_t^T \langle f_s(\tau,v^{(n)}(\tau)),v_t^{(m)}(\tau)\rangle d\tau$$
$$= \int_\Omega F_s(T,v(T))dx - \int_\Omega F_s(t,v(t))dx - \int_\Omega\int_t^T \frac{\partial}{\partial t}F_s(\tau,v(\tau))d\tau dx,$$

*and*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_t^T \langle f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau)\rangle d\tau$$
$$= \int_\Omega F_s(T,v(T))dx - \int_\Omega F_s(t,v(t))dx - \int_\Omega\int_t^T \frac{\partial}{\partial t}F_s(\tau,v(\tau))d\tau dx.$$

*Proof.* Since

$$\int_\Omega |f_s(\tau,v(\tau))v_t(\tau)|dx \leqslant \|f_s(\tau,v(\tau))\|_{L^2}\|v_t(\tau)\|_{L^2} \leqslant C,$$

for some constant $C > 0$, we can apply Fubini's Theorem to obtain

$$\int_t^T\int_\Omega f_s(\tau,v(\tau))v_t(\tau)dxd\tau = \int_\Omega\int_t^T f_s(\tau,v(\tau))v_t(\tau)d\tau dx$$
$$= \int_\Omega F_s(T,v(T))dx - \int_\Omega F_s(t,v(t))dx - \int_\Omega\int_t^T \frac{\partial}{\partial t}F_s(\tau,v(\tau))d\tau dx.$$

The result is now a simple consequence of Lemmas 3.43 and 3.44. The proof of the second item is analogous. □

**Corollary 3.46.** *We have*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T \langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau) - v_t^{(m)}(\tau)\rangle d\tau = 0.$$

*Proof.* Note that

$$\langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau) - v_t^{(m)}(\tau)\rangle$$
$$= \langle f_s(\tau,v^{(n)}(\tau)),v_t^{(n)}(\tau)\rangle - \langle f_s(\tau,v^{(n)}(\tau)),v_t^{(m)}(\tau)\rangle$$

$$- \langle f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau)\rangle + \langle f_s(\tau,v^{(m)}(\tau)),v_t^{(m)}(\tau)\rangle.$$

Using Fubini's Theorem we have

$$\begin{aligned}\int_0^T \langle f_s(\tau,v^{(n)}(\tau)),v_t^{(n)}(\tau)\rangle d\tau &= \int_\Omega\int_0^T f_s(\tau,v^{(n)}(\tau))v_t^{(n)}(\tau)d\tau dx\\ &= \int_\Omega F_s(T,v^{(n)}(T))dx - \int_\Omega F_s(0,v^{(n)}(0))dx - \int_\Omega\int_0^T \frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx.\end{aligned}$$

Again, using Fubini's Theorem we have

$$\int_\Omega\int_0^T \frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx = \int_0^T\int_\Omega \frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx,$$

and from Lemma 3.39 and Lebesgue's Dominated Convergence Theorem we have

$$\int_\Omega\int_0^T \frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx \stackrel{n\to\infty}{\longrightarrow} \int_\Omega\int_0^T \frac{\partial F_s}{\partial t}(\tau,v(\tau))d\tau dx.$$

Hence, again from Lemma 3.39 we have

$$\begin{aligned}&\int_0^T \langle f_s(\tau,v^{(n)}(\tau)),v_t^{(n)}(\tau)\rangle d\tau\\ &\quad\stackrel{n\to\infty}{\longrightarrow} \int_\Omega F_s(T,v(T))dx - \int_\Omega F_s(0,v(0))dx - \int_\Omega\int_0^T \frac{\partial F_s}{\partial t}(\tau,v(\tau))d\tau.\end{aligned}$$

Using this fact and Lemma 3.45 with $t=0$, the result is proven. □

**Corollary 3.47.** *For each $t\in[0,T]$ we have*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_t^T \langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau)-v_t^{(m)}(\tau)\rangle d\tau dt = 0.$$

*Proof.* The proof of the previous corollary works exactly the same with $\int_t^T$ instead of $\int_0^T$, and the result follows by applying once again Lebesgue's Dominated Convergence Theorem. □

With all these lemmas, the proof of Proposition 3.37 is quite trivial.

*Proof of Proposition 3.37.* The contractiveness of $\psi_1$ follows from Corollary 3.46, and the contractiveness of $\psi_2$ follows from Corollary 3.47. □

**The generalized polynomial pullback attractor: the proof of Theorem 3.1(a)**

*Proof.* Consider the functions $g_1,g_2\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ given by $g_1(\alpha,\beta)=\alpha$ and $g_2(\alpha,\beta)=\alpha+\beta$. It is clear that $g_1,g_2$ are non-decreasing with respect to each variable, $g_1(0,0)=g_2(0,0)=0$ and they are continuous at $(0,0)$. Also, we observe that $2\mathcal{E}_s(0)=\|V_1-V_2\|_X^2=d_X(V_1,V_2)^2$ and for any $s\in\mathbb{R}$ we have

$$\begin{aligned}2\mathcal{E}_s(\tau) &= \|Z(\tau)\|_X^2 = \|V(\tau,V_1)-V(\tau,V_2)\|_X^2 = \|S(\tau+s,s)V_1 - S(\tau+s,s)V_2\|_X^2\\ &= d_X(S(\tau+s,s)V_1, S(\tau+s,s)V_2)^2.\end{aligned}$$

From Corollary 3.34 it follows that

$$\begin{aligned}
d_X(S(T+s,s)V_1, S(T+s,s)V_2)^2 &= 2\mathcal{E}_s(T)\\
&\leqslant 2\mathcal{E}_s(0) + \psi_1(V_1,V_2) + \rho_1(V_1,V_2)\\
&= 2\mathcal{E}_s(0) + \psi_1(V_1,V_2) + g_1(\rho_1(V_1,V_2),\rho_2(V_1,V_2))\\
&= d_X(V_1,V_2)^2 + \psi_1(V_1,V_2) + g_1(\rho_1(V_1,V_2),\rho_2(V_1,V_2)),
\end{aligned}$$

and from Proposition 3.33 it follows that

$$\begin{aligned}
&d_X(S(T+s,s)V_1, S(T+s,s)V_2)^2 = 2\mathcal{E}_s(T)\\
&\quad\leqslant 2\Gamma_{T,2}\Big(\mathcal{E}_s(0) - \mathcal{E}_s(T) + \tfrac{1}{2}\rho_1(V_1,V_2) + \tfrac{1}{2}\psi_1(V_1,V_2)\Big)^{\frac{2}{p+2}}\\
&\qquad + \rho_1(V_1,V_2) + \rho_2(V_1,V_2) + \psi_2(V_1,V_2)\\
&\quad= 2\Gamma_{T,2}\Big[\tfrac{1}{2}\Big(2\mathcal{E}_s(0) - 2\mathcal{E}_s(T) + \rho_1(V_1,V_2) + \psi_1(V_1,V_2)\Big)\Big]^{\frac{2}{p+2}}\\
&\qquad + \rho_1(V_1,V_2) + \rho_2(V_1,V_2) + \psi_2(V_1,V_2)\\
&\quad\leqslant 2^{\frac{p}{p+2}}\Gamma_{T,2}\Big(2\mathcal{E}_s(0) - 2\mathcal{E}_s(T) + g_1(\rho_1(V_1,V_2),\rho_2(V_1,V_2)) + \psi_1(V_1,V_2)\Big)^{\frac{2}{p+2}}\\
&\qquad + g_2(\rho_1(V_1,V_2), \rho_2(V_1,V_2)) + \psi_2(V_1,V_2)\\
&\quad\leqslant 2^{\frac{p}{p+2}}\Gamma_{T,2}\Big(d_X(V_1, V_2)^2 - d_X(S(T+s,s)V_1, S(T+s,s)V_2)^2\\
&\qquad + g_1(\rho_1(V_1,V_2),\rho_2(V_1,V_2)) + \psi_1(V_1,V_2)\Big)^{\frac{2}{p+2}} + g_2(\rho_1(V_1,V_2), \rho_2(V_1,V_2)) + \psi_2(V_1,V_2).
\end{aligned}$$

Finally, Proposition 3.15 gives the last hypotheses for us to apply Theorem 2.10 and guarantee that the evolution process $S$, associated with (NWE), is $\varphi$-pullback $\kappa$-dissipative, with decay function $\varphi(t) = t^{-\frac{1}{p}}$, and that $S$ has a bounded generalized $\varphi$-pullback attractor $\hat{M}$. Furthermore, from Theorem 2.12, $S$ has a pullback attractor $\hat{A}$, with $\hat{A} \subset \hat{M}$. □

NOTE

The theoretical results on generalized $\varphi$-pullback attractors, as well as their applications in order to obtain a generalized polynomial pullback attractor for the nonautonomous wave equation, both arising from this thesis, are highlighted in the work

> M.C. Bortolan, T. Caraballo, and C. Pecorari Neto. Generalized $\varphi$-pullback attractors for evolution processes and application to a nonautonomous wave equation. arXiv, 2023. https://doi.org/10.48550/arXiv.2311.15630.

which is in the process of publication up to the completion of this thesis.

## 3.5 EXISTENCE OF A GENERALIZED EXPONENTIAL PULLBACK ATTRACTOR WHEN $p = 0$

Similarly to what was done in the previous subsection, here our goal is to verify the hypothesis for the application of Theorem 2.11 in order to guarantee the existence of a generalized exponential pullback attractor for our problem in the case $p = 0$. We proceed as follows.

Consider the closed, uniformly bounded, positively invariant and uniformly pullback absorbing family $\hat{C}$ previously obtained. If $s \in \mathbb{R}$ and $V_1,V_2 \in C_s \subset \overline{B}^X_{r_0}$, we already know that there exists a constant $c_{r_0} > 0$ such that

$$\|V(t,V_1)\|_X \leqslant c_{r_0} \quad \text{and} \quad \|V(t,V_2)\|_X \leqslant c_{r_0} \quad \text{for all } t \geqslant 0.$$

If $V(t,V_1) = (v(t),v_t(t))$ and $V(t,V_2) = (w(t),w_t(t))$, setting again $Z(t) = (z(t),z_t(t)) = V(t,V_1) - V(t,V_2)$ we have $z = v - w$ and we formally obtain from (tNWE) that

$$z_{tt} - \Delta z + k_s(t)z_t + f_s(t,v) - f_s(t,w) = \int_\Omega K(x,y)z_t(t,y)dy. \tag{3.67}$$

Our next goal is to use Theorem 2.11 to prove that the process $S$ associated with (NWE) possesses a generalized exponential pullback attractor. Note that for $s \in \mathbb{R}$ and $T > 0$

$$\begin{aligned} d(S(T+s,s)V_1, S(T+s,s)V_2)^2 &= \|S(T+s,s)V_1 - S(T+s,s)V_2\|^2_X \\ &= \|V(T,V_1) - V(T,V_2)\|^2_X = \|Z(T)\|^2_X . \end{aligned} \tag{3.68}$$

Then it is natural to study estimates for the function $\mathcal{E}_s(\cdot, Z_0)\colon \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\mathcal{E}_s(t, Z_0) = \frac{1}{2}\|Z(t)\|^2_X = \frac{1}{2}\left(\|z\|^2_{H^1_0} + \|z_t\|^2_{L^2}\right)$$

where $Z(t,Z_0) = (z(t),z_t(t))$ for $t \geqslant 0$. Here, $Z_0 = V_1 - V_2$. It is clear that $\mathcal{E}_s(0, Z_0) = \frac{1}{2}d(V_1,V_2)^2$. Again, we write $\mathcal{E}_s(t)$ instead of $\mathcal{E}_s(t, Z_0)$ for simplicity, but keeping in mind that this function depends on the initial data $Z_0 \in X$.

In order to obtain a suitable estimate for $\mathcal{E}_s$, we will use the auxiliary function $\mathcal{V}_s$ defined by

$$\mathcal{V}_s(t) = \mathcal{E}_s(t) + \tfrac{\varepsilon_0}{2}\langle z, z_t\rangle,$$

where $\varepsilon_0 := \min\left\{\sqrt{\lambda_1}, k_0\right\} > 0$. We first present some lemmas.

**Lemma 3.48.** *For $t \geqslant 0$ we have*

$$\tfrac{1}{2}\mathcal{E}_s(t) \leqslant \mathcal{V}_s(t) \leqslant \tfrac{3}{2}\mathcal{E}_s(t).$$

*Proof.* It follows immediately from the fact that

$$\begin{aligned} \left|\frac{\varepsilon_0}{2}\langle z, z_t\rangle\right| &\leqslant \frac{\varepsilon_0}{2}\|z\|_{L^2}\|z_t\|_{L^2} \overset{(1)}{\leqslant} \frac{\varepsilon_0}{2\sqrt{\lambda_1}}\|z\|_{H^1_0}\|z_t\|_{L^2} \overset{(2)}{\leqslant} \frac{1}{2}\|z\|_{H^1_0}\|z_t\|_{L^2} \\ &\leqslant \frac{1}{2}\left[\frac{1}{2}(\|z\|^2_{H^1_0} + \|z_t\|^2_{L^2})\right] = \frac{1}{2}\mathcal{E}_s(t), \end{aligned}$$

where in (1) we applied Poincaré's inequality and in (2) we used that $\varepsilon_0 \leqslant \sqrt{\lambda_1}$. □

**Lemma 3.49.** *There exists a constant $C_{r_0} > 0$ such that, for $t \geqslant 0$,*

$$\frac{d}{dt}\mathcal{V}_s(t) + \varepsilon_0\mathcal{V}_s(t) \leqslant C_{r_0}\|z\|_{L^2} - \langle f_s(t,v) - f_s(t,w), z_t\rangle + C_{r_0}\left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2}.$$

*Proof.* Formally multiplying (3.67) by $z_t + \frac{\varepsilon_0}{2}z$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega z_{tt}z_t dx - \int_\Omega z_t\Delta z dx + \frac{\varepsilon_0}{2}\int_\Omega zz_{tt}dx - \frac{\varepsilon_0}{2}\int_\Omega z\Delta z dx + k_s(t)\|z_t\|^2_{L^2}\\ &\quad + \frac{\varepsilon_0}{2}k_s(t)\int_\Omega z_t z dx + \int_\Omega [f_s(t,v) - f_s(t,w)]z_t dx + \frac{\varepsilon_0}{2}\int_\Omega [f_s(t,v) - f_s(t,w)]z dx\\ &\quad = \int_{\Omega\times\Omega} K(x,y)z_t(t,y)z_t(t,x)dydx + \frac{\varepsilon_0}{2}\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z(t,x)dydx.\end{aligned} \tag{3.69}$$

Since, as we already know, we have

(i) $\displaystyle\int_\Omega z_{tt}z_t dx - \int_\Omega z_t\Delta z dx = \frac{d}{dt}\left(\frac{1}{2}\|z\|^2_{H^1_0} + \frac{1}{2}\|z_t\|^2_{L^2}\right) = \frac{d}{dt}\mathcal{E}_s(t)$ and

(ii) $\displaystyle\frac{\varepsilon_0}{2}\int_\Omega zz_{tt}dx - \frac{\varepsilon_0}{2}\int_\Omega z\Delta z dx = \frac{\varepsilon_0}{2}\frac{d}{dt}\int_\Omega z_t z dx - \frac{\varepsilon_0}{2}\|z_t\|^2_{L^2} + \frac{\varepsilon_0}{2}\|z\|^2_{H^1_0}$,

adding $\varepsilon_0\|z_t\|^2_{L^2} + \frac{\varepsilon_0^2}{2}\langle z, z_t\rangle$ to both sides of equation (3.69), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{V}_s(t) + \varepsilon_0\mathcal{V}_s(t) = &-k_s(t)\|z_t\|^2_{L^2} + \varepsilon_0\|z_t\|^2_{L^2} + \frac{\varepsilon_0^2}{2}\langle z,z_t\rangle - \frac{\varepsilon_0}{2}k_s(t)\langle z, z_t\rangle\\ &- \langle f_s(t,v) - f_s(t,w), z_t\rangle - \frac{\varepsilon_0}{2}\langle f_s(t,v) - f_s(t,w), z\rangle\\ &+ \int_{\Omega\times\Omega} K(x,y)z_t(t,y)z_t(t,x)dydx\\ &+ \frac{\varepsilon_0}{2}\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z(t,x)dydx.\end{aligned} \tag{3.70}$$

Now observe that

(i) $-k_s(t)\|z_t\|^2_{L^2} + \varepsilon_0\|z_t\|^2_{L^2} \leqslant (\underbrace{\varepsilon_0 - k_0}_{\leqslant 0})\|z_t\|^2_{L^2} \leqslant 0$;

(ii) $\displaystyle\frac{\varepsilon_0^2}{2}\langle z,z_t\rangle \leqslant \frac{\varepsilon_0^2}{2}\|z_t\|_{L^2}\|z\|_{L^2} \leqslant \frac{\varepsilon_0^2}{2}2c_{r_0}\|z\|_{L^2} = c_{r_0}\varepsilon_0^2\|z\|_{L^2}$;

(iii) $\displaystyle-\frac{\varepsilon_0}{2}k_s(t)\langle z, z_t\rangle \leqslant \frac{\varepsilon_0}{2}k_1\|z_t\|_{L^2}\|z\|_{L^2} \leqslant \frac{\varepsilon_0}{2}k_1 2c_{r_0}\|z\|_{L^2} = \varepsilon_0 k_1 c_{r_0}\|z\|_{L^2}$;

(iv) $\displaystyle-\frac{\varepsilon_0}{2}\langle f_s(t,v) - f_s(t,w), z\rangle \leqslant \frac{\varepsilon_0}{2}\|f_s(t,v) - f_s(t,w)\|_{L^2}\|z\|_{L^2} \leqslant \varepsilon_0 L_0 c_{r_0}(1 + 2c_{r_0}^2)\|z\|_{L^2}$, where $L_0$ is the constant from Lemma 3.4;

(v) $\displaystyle\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z_t(t,x)dydx \leqslant 2c_{r_0}\left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2}$;

(vi) $\displaystyle\frac{\varepsilon_0}{2}\int_{\Omega\times\Omega} K(x,y)z_t(t,y)z(t,x)dydx \leqslant \frac{\varepsilon_0}{2}K_0 2c_{r_0}\|z\|_{L^2} = \varepsilon_0 K_0 c_{r_0}\|z\|_{L^2}$.

Taking $C_{r_0} = \max\left\{c_{r_0}\varepsilon_0^2 + \varepsilon_0 k_1 c_{r_0} + \varepsilon_0 L_0 c_{r_0}(1 + 2c_{r_0}^2) + \varepsilon_0 K_0 c_{r_0}, 2c_{r_0}\right\}$ and inserting the previous estimates into (3.70), we conclude

$$\frac{d}{dt}\mathcal{V}_s(t) + \varepsilon_0\mathcal{V}_s(t) \leqslant C_{r_0}\|z\|_{L^2} - \langle f_s(t,v) - f_s(t,w), z_t\rangle + C_{r_0}\left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2}.$$

□

**Lemma 3.50.** *Let $T > 0$ fixed. Then,*

$$\mathcal{E}_s(T) \leqslant 3e^{-\varepsilon_0 T}\mathcal{E}_s(0) + 2C_{r_0}T \sup_{t\in[0,T]} \|z(t)\|_{L^2} + 2C_{r_0}\int_0^T \left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2} dt$$
$$+ 2\left|\int_0^T e^{\varepsilon_0(t-T)}\langle f_s(t,v) - f_s(t,w), z_t(t)\rangle dt\right|.$$

*Proof.* For every $t \geqslant 0$ we have

$$\frac{d}{dt}[e^{\varepsilon_0 t}\mathcal{V}_s(t)] = e^{\varepsilon_0 t}\left(\frac{d}{dt}\mathcal{V}_s(t) + \varepsilon_0\mathcal{V}_s(t)\right)$$
$$\overset{(1)}{\leqslant} e^{\varepsilon_0 t}C_{r_0}\|z\|_{L^2} - e^{\varepsilon_0 t}\langle f_s(t,v) - f_s(t,w), z_t\rangle + e^{\varepsilon_0 t}C_{r_0}\left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2},$$

where in (1) we applied Lemma 3.49. Now, an integration from 0 to $T$ provides

$$e^{\varepsilon_0 T}\mathcal{V}_s(T) - \mathcal{V}_s(0) \leqslant e^{\varepsilon_0 T}C_{r_0}\int_0^T \|z(t)\|_{L^2} dt + e^{\varepsilon_0 T}C_{r_0}\int_0^T \left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2} dt$$
$$+ \left|\int_0^T e^{\varepsilon_0 t}\langle f_s(t,v) - f_s(t,w), z_t(t)\rangle dt\right|,$$

which implies

$$\mathcal{V}_s(T) \leqslant e^{-\varepsilon_0 T}\mathcal{V}_s(0) + C_{r_0}\int_0^T \|z(t)\|_{L^2} dt + C_{r_0}\int_0^T \left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2} dt$$
$$+ \left|\int_0^T e^{\varepsilon_0(t-T)}\langle f_s(t,v) - f_s(t,w), z_t(t)\rangle dt\right|,$$

and the proof is done if we apply Lemma 3.48 to the previous inequality and consider that $\int_0^T \|z(t)\|_{L^2} dt \leqslant T \sup_{t\in[0,T]} \|z(t)\|_{L^2}$. □

**The generalized exponential pullback attractor: the proof of Theorem 3.1(b)**

From (3.68) and Lemma 3.50, we conclude that for $s \in \mathbb{R}$ and a fixed $T > 0$,

$$d_X(S(T+s,s)V_1, S(T+s,s)V_2)^2 \leqslant 3e^{-\varepsilon_0 T}d_X(V_1,V_2)^2 + 4C_{r_0}T \sup_{t\in[0,T]} \|z(t)\|_{L^2}$$
$$+ 4C_{r_0}\int_0^T \left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2} dt + 4\left|\int_0^T e^{\varepsilon_0(t-T)}\langle f_s(t,v) - f_s(t,w), z_t(t)\rangle dt\right|.$$

Note that $\mu := 3e^{-\varepsilon_0 T} \in (0,1)$ if we choose a constant $T > \varepsilon_0^{-1}\ln 3$. For instance, taking $T = 1 + \varepsilon_0^{-1}\ln 3$, we have

$$\mu = 3e^{-\varepsilon_0(\varepsilon_0^{-1}ln3+1)} = 3e^{-ln3}e^{-\varepsilon_0} = e^{-\min\{\sqrt{\lambda_1},k_0\}}.$$

Consider the function $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by $g(\alpha,\beta) = \alpha + \beta$. It is clear that $g$ is non-decreasing with respect to each variable, $g(0,0) = 0$ and $g$ is continuous at $(0,0)$. Additionally, define the maps $\rho_1,\rho_2,\psi\colon X \times X \to \mathbb{R}^+$ by

$$\rho_1(V_1,V_2) = 4C_{r_0}\int_0^T \left\|\int_\Omega K(x,y)z_t(t,y)dy\right\|_{L^2} dt,$$

$$\rho_2(V_1,V_2) = 4C_{r_0}T \sup_{t\in[0,T]} \|z(t)\|_{L^2},$$

$$\psi(V_1,V_2) = 4\left|\int_0^T e^{\varepsilon_0(t-T)}\langle f_s(t,v) - f_s(t,w), z_t(t)\rangle dt\right|,$$

where $(v(t),v_t(t)) = V(t,V_1)$, $(w(t),w_t(t)) = V(t,V_2)$ and $(z(t),z_t(t)) = Z(t) = V(t,V_1) - V(t,V_2)$.

We already know from Proposition 3.35 that $\rho_1$ and $\rho_2$ are precompact in $\overline{B}^X_{r_0}$ and that the process $S$ associated with (NWE) in the case $p = 0$ satisfies the Lipschitz condition required in Theorem 2.11. We just need to verify that $\psi \in \operatorname{contr}(\overline{B}^X_{r_0})$, for which the proof is slightly different from Proposition 3.37. Let $\{V_n\}_{n\in\mathbb{N}} \subset \overline{B}^X_{r_0}$ and denote $V(t,V_n) = (v^{(n)}(t),v_t^{(n)}(t))$. Similarly to Lemmas 3.43 and 3.44 we have the following two results.

**Lemma 3.51.** *For $n \in \mathbb{N}$ we have*

$$\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(n)}(\tau)),v_t^{(m)}(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(n)}(\tau)),v_t(\tau)\rangle d\tau,$$

*and*

$$\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(m)}(\tau)), v_t^{(n)}(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v(\tau)),v_t^{(n)}(\tau)\rangle d\tau.$$

*Proof.* Note that defining $\xi\colon [0,T] \to L^2(\Omega)$ by $\xi(\tau) = e^{\varepsilon_0(\tau-T)}f_s(\tau, v^{(n)}(\tau))$ for each $\tau \in [0,T]$, we have $\xi \in L^1(0,T;L^2(\Omega))$, since

$$\begin{aligned}\int_0^T \|\xi(\tau)\|_{L^2}d\tau &= \int_0^T \left\|e^{\varepsilon_0(\tau-T)}f_s(\tau,v^{(n)}(\tau))\right\|_{L^2} d\tau\\ &= \int_0^T \underbrace{\left|e^{\varepsilon_0(\tau-T)}\right|}_{\leqslant 1} \left\|f_s(\tau,v^{(n)}(\tau))\right\|_{L^2} d\tau \leqslant CT,\end{aligned}$$

for some constant $C > 0$. Since, by Lemma 3.38, $v_t^{(m)} \overset{*}{\rightharpoonup} v_t$ in $L^\infty(0,T;L^2(\Omega))$ as $m \to \infty$ we have

$$\int_0^T \langle\xi(\tau), v_t^{(m)}(\tau) - v_t(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} 0.$$

For the second item, define $\xi\colon [0,T] \to L^2(\Omega)$ by $\xi(\tau) = e^{\varepsilon_0(\tau-T)}v_t^{(n)}(\tau)$ for each $\tau \in [0,T]$. Clearly $\xi \in L^1(0,T;L^2(\Omega))$, since for some constant $C > 0$ we have

$$\int_0^T \|\xi(\tau)\|_{L^2}d\tau = \int_0^t \|e^{\varepsilon_0(\tau-T)}v_t^{(n)}(\tau)\|_{L^2}d\tau = \int_0^t \underbrace{|e^{\varepsilon_0(\tau-T)}|}_{\leqslant 1} \|v_t^{(n)}(\tau)\|_{L^2}d\tau \leqslant CT,$$

and hence, since $f_s(\tau,v^{(m)}(\tau)) \overset{*}{\rightharpoonup} f_s(\tau,v(\tau))$ in $L^\infty(0,T;L^2(\Omega))$ as $m \to \infty$ by Corollary 3.42, we have

$$\int_t^T \langle f_s(\tau,v^{(m)}(\tau)) - f_s(\tau,v(\tau)),\xi(\tau)\rangle d\tau \overset{m\to\infty}{\longrightarrow} 0,$$

and the result is proven. □

**Lemma 3.52.**

$$\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(n)}(\tau)),v_t(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v(\tau)),v_t(\tau)\rangle d\tau,$$

*and*

$$\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v(\tau)),v_t^{(n)}(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v(\tau)),v_t(\tau)\rangle d\tau.$$

*Proof.* For the first item, define $\xi(\tau) = e^{\varepsilon_0(\tau-T)}v_t(\tau)$. Since $\xi \in L^1(0,T;L^2(\Omega))$, we have

$$\int_0^T \langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau, v(\tau)), \xi(\tau)\rangle d\tau \overset{n\to\infty}{\longrightarrow} 0.$$

For the second item, define $\xi(\tau) = e^{\varepsilon_0(\tau-T)}f_s(\tau,v(\tau))$, and note that $\xi \in L^1(0,T;L^2(\Omega))$. □

Now we have

**Lemma 3.53.**

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(n)}(\tau)),v_t^{(m)}(\tau)\rangle d\tau \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(m)}(\tau)),v_t^{(n)}(\tau)\rangle d\tau \\
&= \int_\Omega F_s(T,v(T))dx - e^{-\varepsilon_0 T}\int_\Omega F_s(0,v(0))dx \\
&\quad - \varepsilon_0\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))d\tau dx - \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial}{\partial t}F_s(\tau, v(\tau))d\tau dx.
\end{aligned}$$

*Proof.* From the previous two lemmas we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(n)}(\tau)),v_t^{(m)}(\tau)\rangle d\tau \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau, v^{(m)}(\tau)),v_t^{(n)}(\tau)\rangle d\tau \\
&= \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v(\tau)), v_t(\tau)\rangle d\tau.
\end{aligned}$$

Since for $\tau \in [0,T]$ we have

$$\int_\Omega |e^{\varepsilon_0(\tau-T)}f_s(\tau, v(\tau))v_t(\tau)|dx \leqslant \|f_s(\tau, v(\tau))\|_{L^2}\|v_t(\tau)\|_{L^2} \leqslant C,$$

for some constant $C > 0$, we can apply Fubini's Theorem to obtain

$$\int_0^T\int_\Omega e^{\varepsilon_0(\tau-T)}f_s(\tau, v(\tau))v_t(\tau)dxd\tau = \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}f_s(\tau, v(\tau))v_t(\tau)d\tau dx.$$

Note that

$$\begin{aligned}
\frac{d}{d\tau}\left[\frac{1}{\varepsilon_0}e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))\right] &= e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau)) + \frac{1}{\varepsilon_0}e^{\varepsilon_0(\tau-T)}\frac{\partial}{\partial t}F_s(\tau,v(\tau)) \\
&\quad + \frac{1}{\varepsilon_0}e^{\varepsilon_0(\tau-T)}f_s(\tau,v(\tau))v_t(\tau),
\end{aligned}$$

which, integrating from 0 to $T$, implies

$$\int_0^T e^{\varepsilon_0(\tau-T)} f_s(\tau,v(\tau))v_t(\tau)d\tau = \int_0^T \frac{d}{d\tau}[e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))]d\tau - \varepsilon_0 \int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))d\tau - \int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial}{\partial t}F_s(\tau,v(\tau))d\tau.$$

Now the conclusion follows immediately. □

**Corollary 3.54.** *We have*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau) - v_t^{(m)}(\tau)\rangle d\tau = 0.$$

*Proof.* Note that

$$\begin{aligned} &e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)) - f_s(\tau,v^{(m)}(\tau)),v_t^{(n)}(\tau) - v_t^{(m)}(\tau)\rangle \\ &\quad = e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)),v^{(n)}(\tau)\rangle - e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)),v^{(m)}(\tau)\rangle \\ &\quad\quad - e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(m)}(\tau)),v^{(n)}(\tau)\rangle + e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(m)}(\tau)),v^{(m)}(\tau)\rangle. \end{aligned}$$

Using Fubini's Theorem we have

$$\begin{aligned} \int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)),v^{(n)}(\tau)\rangle d\tau &= \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)} f_s(\tau,v^{(n)}(\tau)),v^{(n)}(\tau)d\tau dx \\ &= \int_\Omega F_s(T,v^{(n)}(T))dx - e^{-\varepsilon_0 T}\int_\Omega F_s(0,v^{(n)}(0))dx \\ &\quad - \varepsilon_0\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v^{(n)}(\tau))d\tau dx - \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial}{\partial t}F_s(\tau,v^{(n)}(\tau))d\tau dx. \end{aligned}$$

Again, using Fubini's Theorem we have

$$\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx = \int_0^T\int_\Omega e^{\varepsilon_0(\tau-T)}\frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx,$$

and from Lemma 3.39 and Lebesgue's Dominated Convergence Theorem we have

$$\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial F_s}{\partial t}(\tau,v^n(\tau))d\tau dx \overset{n\to\infty}{\longrightarrow} \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial F_s}{\partial t}(\tau,v(\tau))d\tau dx.$$

W can also apply Lemma 3.39, Fubini's Theorem and Lebesgue's Dominated Convergence Theorem to obtain

$$\varepsilon_0\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v^{(n)}(\tau))d\tau dx \overset{n\to\infty}{\longrightarrow} \varepsilon_0\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))d\tau dx.$$

Hence, again from Lemma 3.39 we have

$$\begin{aligned} &\int_0^T e^{\varepsilon_0(\tau-T)}\langle f_s(\tau,v^{(n)}(\tau)),v^{(n)}(\tau)\rangle d\tau \\ &\quad\overset{n\to\infty}{\longrightarrow} \int_\Omega F_s(T,v(T))dx - e^{-\varepsilon_0 T}\int_\Omega F_s(0,v(0))dx \\ &\quad\quad - \varepsilon_0\int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}F_s(\tau,v(\tau))d\tau dx - \int_\Omega\int_0^T e^{\varepsilon_0(\tau-T)}\frac{\partial}{\partial t}F_s(\tau,v(\tau))d\tau dx. \end{aligned}$$

Using this fact and Lemma 3.53 the result is proven. □

With this previous result we conclude the contractiveness of $\psi$ in $\overline{B}^X_{r_0}$. Now, Theorem 2.11 ensures the existence of a generalized $\varphi$-pullback attractor for the process $S$ associated with (NWE) when $p=0$, with the decay function $\varphi(t)=e^{-\alpha t}$, where $\alpha=\min\{\sqrt{\lambda_1},k_0\}$.

NOTE

In the work

> M.C. Bortolan, T. Caraballo, and C. Pecorari Neto. Generalized exponential pullback attractor for a nonautonomous wave equation. arXiv, 2024. https://doi.org/10.48550/arXiv.2401.06631.

which is in the process of publication up to the completion of this thesis, we define the concept of generalized exponential $\mathcal{D}$-pullback attractors, where $\mathcal{D}$ is a universe of families of sets in $X$. We prove that for a specific universe, denoted by $\mathcal{D}_{\mathcal{C}^*}$, the nonautonomous wave equation studied in this thesis (with $p = 0$) has a generalized exponential $\mathcal{D}_{\mathcal{C}^*}$-pullback attractor. This, in turn, also implies the existence of the $\mathcal{D}_{\mathcal{C}^*}$-pullback attractor for such problem.

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# APPENDIX A – AUXILIARY RESULTS

The following appendix contains technical results that are complementary to the main theory presented in this work. While these results are important for a deeper understanding of the subject matter, they are not necessary for the main argument. As such, they have been included here as supplementary material. Readers interested in delving deeper into the technical details of the analysis may find the contents of this appendix informative and insightful. However, for those who are primarily interested in the main theory, the appendix can be safely skipped without affecting the coherence or understanding of the main text.

## A.1 KURATOWSKI MEASURE OF NON-COMPACTNESS

**Proposition A.1.** *For $B \subset X$ bounded, let*

$$\widetilde{\kappa}(B) = \inf\{\delta > 0\colon B \text{ admits a finite cover by sets of diameter less than } \delta\}$$

*and*

$$\widetilde{\beta}(B) = \inf\{r > 0\colon B \text{ admits a finite cover by closed balls of radius } r\}.$$

*Then $\widetilde{\kappa}(B) = \kappa(B)$ and $\beta(B) = \widetilde{\beta}(B)$.*

*Proof.* Clearly $\kappa(B) \leqslant \widetilde{\kappa}(B)$. Now fix $\varepsilon > 0$ and assume that $\delta > 0$ is such that $B \subset \cup_{i=1}^{n} C_i$ with $\operatorname{diam}(C_i) \leqslant \delta < \delta + \varepsilon$. Hence $\widetilde{\kappa}(B) \leqslant \delta + \varepsilon$, which implies that $\widetilde{\kappa}(B) \leqslant \kappa(B) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\widetilde{\kappa}(B) = \kappa(B)$.

We have directly that $\widetilde{\beta}(B) \leqslant \beta(B)$. Fix $\varepsilon > 0$ and assume that $r > 0$ is such that $B \subset \cup_{j=1}^{n} \overline{B}_r(x_j)$. Thus $B \subset \cup_{j=1}^{n} B_{r+\varepsilon}(x_j)$, and hence, $\beta(B) \leqslant r + \varepsilon$. This implies that $\beta(B) \leqslant \widetilde{\beta}(B) + \varepsilon$, and since $\varepsilon > 0$ is arbitrary we obtain the equality. □

In what follows we present the main properties of the Kuratowski and ball measures of non-compactness, and unless clearly stated otherwise, $(X,d)$ is a *complete metric space*.

**Proposition A.2.** *We have the following:*

*(i) if $B_1, B_2 \subset X$ and $B_1 \subset B_2$ then $\kappa(B_1) \leqslant \kappa(B_2)$;*

*(ii) $\kappa(B) = \kappa(\overline{B})$;*

*(iii) $\kappa(B) = 0$ if and only if $\overline{B}$ is compact;*

*(iv) $\kappa(B_1 \cup B_2) \leqslant \max\{\kappa(B_1),\kappa(B_2)\}$;*

*(v) if $(X,\|\cdot\|)$ is a Banach space and $B_1,B_2 \subset X$, we have $\kappa(B_1 + B_2) \leqslant \kappa(B_1) + \kappa(B_2)$.*

*Proof.* Item (i) follows directly from the definition of $\kappa$.

(ii) From (i) it is clear that $\kappa(B) \leqslant \kappa(\overline{B})$. Now if $\delta > 0$ and $B \subset \cup_{i=1}^{n} C_i$ with $\operatorname{diam}(C_i) \leqslant \delta$ for $i = 1, \cdots ,n$, then since $\operatorname{diam}(C_i) = \operatorname{diam}(\overline{C_i})$ and $\overline{B} \subset \cup_{i=1}^{n} \overline{C_i}$, we obtain $\kappa(\overline{B}) \leqslant \delta$. Thus $\kappa(\overline{B}) \leqslant \kappa(B)$ and this item is proved.

(iii) Since we are in a complete metric space, compactness equals to totally boundedness and closedness. Hence $\overline{B}$ is compact iff $\overline{B}$ is totally bounded, and it is not difficult to see that $\overline{B}$ is totally bounded iff $\kappa(\overline{B}) = 0$. Using (ii), the proof of this item is complete.

(iv) Assume, without loss of generality, that $\kappa(B_1) \leqslant \kappa(B_2)$. Given $\varepsilon > 0$ we have $B_2 \subset \cup_{i=1}^n C_i$ with $\operatorname{diam}(C_i) \leqslant \kappa(B_2) + \varepsilon$ for $i = 1, \cdots, n$, and also $B_1 \subset \cup_{j=1}^m D_j$ with $\operatorname{diam} D_j \leqslant \kappa(B_1) + \varepsilon \leqslant \kappa(B_2) + \varepsilon$. Hence $B_1 \cup B_2 \subset \cup_{i=1}^n C_i \cup \cup_{j=1}^m D_j$, which proves that

$$\kappa(B_1 \cup B_2) \leqslant \kappa(B_2) + \varepsilon,$$

and completes the proof, since $\varepsilon > 0$ is arbitrary.

(v) Firstly we note that if $C, D \subset X$ and $X$ is a Banach space then $\operatorname{diam}(C + D) \leqslant \operatorname{diam}(C) + \operatorname{diam}(D)$, since for $c_1, c_2 \in C$ and $d_1, d_2 \in D$ we have

$$\|c_1 + d_1 - (c_2 + d_2)\| \leqslant \|c_1 - c_2\| + \|d_1 - d_2\| \leqslant \operatorname{diam}(C) + \operatorname{diam}(D).$$

Now given $\varepsilon > 0$, we have $B_1 \subset \cup_{i=1}^n C_i$ with $\operatorname{diam}(C_i) \leqslant \kappa(B_1) + \varepsilon$ and $B_2 \subset \cup_{j=1}^m D_j$ with $\operatorname{diam}(D_j) \leqslant \kappa(B_2) + \varepsilon$. Thus $\{C_i + D_j : 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ is a finite cover of $B_1 + B_2$ for which $\operatorname{diam}(C_i + D_j) \leqslant \operatorname{diam}(C_i) + \operatorname{diam}(D_j) \leqslant \kappa(B_1) + \kappa(B_2) + 2\varepsilon$. Since $\varepsilon > 0$ was taken arbitrarily, it follows that $\kappa(B_1 + B_2) \leqslant \kappa(B_1) + \kappa(B_2)$. □

**Proposition A.3.** *For each bounded set $B \subset X$ we have $\beta(B) \leqslant \kappa(B) \leqslant 2\beta(B)$.*

*Proof.* Given $\varepsilon > 0$, there exists a family $\{C_i\}_{i=1}^n$ of balls of radius $r$ that covers $B$, with $r \leqslant \beta(B) + \varepsilon$. Since $\operatorname{diam}(C_i) \leqslant 2r$ for $i = 1, \ldots, n$, $\{C_i\}_{i=1}^n$ is a finite cover of $B$ by sets of diameter less than or equal to $2r$, which means $\kappa(B) \leqslant 2r$. Thus, $\kappa(B) \leqslant 2(\beta(B) + \varepsilon)$ and by the arbitrariness of $\varepsilon > 0$ we conclude that $\kappa(B) \leqslant 2\beta(B)$.

Now, given $\varepsilon > 0$, let $\{U_i\}_{i=1}^n$ be a finite cover of $B$ such that $\delta_i := \operatorname{diam}(U_i) \leqslant \kappa(B) + \varepsilon$ for $i = 1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, take $x_i \in U_i$. Then, $U_i \subset \overline{B}_{\delta_i}(x_i) \subset \overline{B}_{\kappa(B)+\varepsilon}(x_i)$ and thus $\{\overline{B}_{\kappa(B)+\varepsilon}(x_i)\}_{i=1}^n$ is a finite cover of $B$ by balls of radius $\kappa(A) + \varepsilon$ which implies $\beta(B) \leqslant \kappa(B) + \varepsilon$. Again, by the arbitrariness of $\varepsilon$, we conclude that $\beta(B) \leqslant \kappa(B)$. □

We recall that for nonempty subsets $U, V \subset X$, the *Hausdorff semidistance* between $U$ and $V$ is defined by

$$d_H(U, V) = \sup_{u \in U} \inf_{v \in V} d(u, v).$$

**Proposition A.4.** *Let $K$ and $B$ nonempty subsets of $X$, with $K$ compact. If $d_H(B, K) \leqslant \gamma$ then $B$ is bounded and $\kappa(B) \leqslant 2\gamma$.*

*Proof.* Let $\varepsilon > 0$. Given $x \in B$, there exists $y_x \in K$ such that $d(x, y_x) < \gamma + \frac{\varepsilon}{4}$. Therefore,

$$B \subset \bigcup_{x \in B} B_{\gamma + \frac{\varepsilon}{4}}(y_x). \tag{A.1}$$

Let $C = \{y_x : x \in B\}$. Since $C \subset K$ and $K$ is compact, $\overline{C}$ is compact. Moreover

$$\overline{C} \subset \overline{\bigcup_{x\in B} B_{\frac{\varepsilon}{8}}(y_x)} \subset \bigcup_{x\in B} B_{\frac{\varepsilon}{4}}(y_x),$$

and thus, there exist $y_1,\dots,y_m \in K$ in such a manner that

$$C \subset \overline{C} \subset \bigcup_{i=1}^{m} B_{\frac{\varepsilon}{4}}(y_i). \tag{A.2}$$

We claim that

$$\bigcup_{x\in B} B_{\gamma+\frac{\varepsilon}{4}}(y_x) \subset \bigcup_{i=1}^{m} B_{\gamma+\frac{\varepsilon}{2}}(y_i). \tag{A.3}$$

Indeed, let $z \in B_{\gamma+\frac{\varepsilon}{4}}(y_x)$. From (A.2) there exists $i \in \{1,\cdots,m\}$ for which $y_x \in B_{\frac{\varepsilon}{4}}(y_i)$. Thus, $d(z,y_i) \leqslant d(z,y_x)+d(y_x,y_i) < \gamma+\varepsilon/4+\varepsilon/4 = \gamma+\varepsilon/2$ which shows that $z \in B_{\gamma+\frac{\varepsilon}{2}}(y_i)$ and proves (A.3).

Now, from (A.1) and (A.3) we have $B \subset \bigcup_{i=1}^{m} B_{\gamma+\frac{\varepsilon}{2}}(y_i)$, which shows that $\kappa(B) \leqslant 2\gamma + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\kappa(B) \leqslant 2\gamma$. □

**Corollary A.5.** *If $K \subset X$ is compact and $r > 0$ then $\kappa(\mathcal{O}_r(K)) \leqslant 2r$.*

*Proof.* Since $d_H(\mathcal{O}_r(K),K) \leqslant r$, the result follows from the previous proposition. □

## A.2 SOME USEFUL INEQUALITIES

**Proposition A.6.** *Let $a,b \geqslant 0$. For $\alpha \geqslant 1$ we have*

$$a^\alpha + b^\alpha \leqslant (a+b)^\alpha \leqslant 2^{\alpha-1}(a^\alpha + b^\alpha),$$

*and for $0 < \alpha < 1$ we have*

$$2^{\alpha-1}(a^\alpha + b^\alpha) \leqslant (a+b)^\alpha \leqslant a^\alpha + b^\alpha.$$

*Proof.* Let $\alpha \geqslant 1$. These inequalities are trivial if $b = 0$ or $\alpha = 1$. For $b > 0$ and $\alpha > 1$, taking $t = \frac{a}{b}$, they are equivalent to

$$1 \leqslant \frac{(1+t)^\alpha}{1+t^\alpha} \leqslant 2^{\alpha-1} \quad \text{for } t \geqslant 0.$$

Setting $f(t) = \frac{(1+t)^\alpha}{1+t^\alpha}$ we see that $f(0) = 1$ and $f(t) \to 1$ as $t \to \infty$. Hence $f$ attains its global maximum at its critical point $t = 1$, where it takes the value $2^{\alpha-1}$, since $f$ is increasing in the interval $[0,1]$ and decreasing in $[1,\infty)$. Hence $f(t) \in [1,2^{\alpha-1}]$ for all $t \geqslant 0$.

For $0 < \alpha < 1$, the inequalities are trivial if $b = 0$. For $b > 0$, taking $t = \frac{a}{b}$, they are equivalent to

$$2^{\alpha-1} \leqslant \frac{(1+t)^\alpha}{1+t^\alpha} \leqslant 1 \quad \text{for } t \geqslant 0.$$

Analogously as in the previous case, we set $f(t) = \frac{(1+t)^\alpha}{1+t^\alpha}$. Note that $f(0) = 1$ and $f(t) \to 1$ as $t \to \infty$. Hence $f$ attains its global minimum at its critical point $t = 1$, where it takes the value $2^{\alpha-1}$, since $f$ is decreasing in the interval $[0,1]$ and increasing in $[1,\infty)$. Hence $f(t) \in [2^{\alpha-1},1]$ for all $t \geqslant 0$. □

Before the next result, we need a simple lemma.

**Lemma A.7.** *Let $\beta \in (0,1)$. If $b > a > 0$, there exists $\theta \in (0,1)$ such that*

$$b^{1-\frac{1}{\beta}} - a^{1-\frac{1}{\beta}} = \left(1 - \tfrac{1}{\beta}\right) [\theta a + (1-\theta) b]^{-\frac{1}{\beta}} (b-a).$$

*Proof.* Consider the function $f(t) = t^{1-\frac{1}{\beta}}$ defined for $t > 0$. From the Mean Value Theorem, there exists a number $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$. Since $f'(t) = \left(1-\frac{1}{\beta}\right) t^{-\frac{1}{\beta}}$, we obtain $b^{1-\frac{1}{\beta}} - a^{1-\frac{1}{\beta}} = \left(1-\frac{1}{\beta}\right) c^{-\frac{1}{\beta}}(b-a)$. The results follows by taking $\theta = \frac{b-c}{b-a}$. □

**Proposition A.8.** *Consider the function $u \colon \mathbb{R}^+ \to \mathbb{R}^+$ given by $u(t) = (3C)^{-1/\beta} t^{1/\beta} + t$, where $C > 0$ and $0 < \beta < 1$ are constants. Let $v \colon \mathbb{R}^+ \to \mathbb{R}^+$ be its inverse function and fix a real number $t_0 \geqslant 0$. Then the sequence $\{t_n\}$ defined by $t_n = v^n(t_0)$ for each $n \in \mathbb{N}$ satisfies*

*(i)* $\{t_n\}$ *is non-increasing;*

*(ii)* $t_n - t_{n+1} = (3C)^{-1/\beta}(t_{n+1})^{1/\beta}$;

*(iii)* $t_n \to 0$ *when* $n \to \infty$;

*(iv) there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have*

$$(t_n)^{1-1/\beta} \geqslant \left(\tfrac{1}{\beta} - 1\right)(1+3C)^{-1/\beta} + (t_{n-1})^{1-1/\beta};$$

*(v) with $n_0$ as in (iv) if $n \geqslant n_0$ and $k \in \mathbb{N}$ we have*

$$(t_{n_0+k})^{1-1/\beta} \geqslant k\left(\tfrac{1}{\beta} - 1\right)(1+3C)^{-1/\beta} + (t_{n_0})^{1-1/\beta};$$

*(vi) with $n_0$ as in (iv) if $n \geqslant n_0$ we have*

$$t_n = v^n(t_0) \leqslant \left[(n-n_0)\left(\tfrac{1}{\beta} - 1\right)(1+3C)^{-\frac{1}{\beta}} + (t_0)^{1-\frac{1}{\beta}}\right]^{\frac{\beta}{\beta-1}}.$$

*Proof.* (i) Since $u'(t) = \frac{1}{\beta(3C)^{1/\beta}} t^{\frac{1}{\beta}-1} + 1 > 0$ for all $t \geqslant 0$, $u$ is an increasing function and, consequently, its inverse function $v$ is also increasing. Then, since $u(t) \geqslant t$ for all $t \geqslant 0$, we have $t = v(u(t)) \geqslant v(t)$ for all $t \geqslant 0$. Now, it is clear that $t_0 \geqslant v(t_0) = t_1 \geqslant v(t_1) = t_2 \geqslant v(t_2) = t_3 \geqslant \cdots$

(ii) $t_n = u(t_{n+1}) = (3C)^{-1/\beta}(t_{n+1})^{1/\beta} + t_{n+1}$.

(iii) Since $\{t_n\}$ is non-increasing we have $t_n \to \alpha$. Assume that $\alpha > 0$. We have $t_n - t_{n+1} \to 0$ and, from (ii), we have $t_n - t_{n+1} \to (3C)^{-1/\beta}\alpha^{1/\beta} > 0$, which is a contradiction.

(iv) Since $\{t_n\}$ is non-increasing and $t_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that $t_{n-1} - t_n \in (0,1)$ for all $n \geqslant n_0$. Furthermore, from (ii), we have $t_n = (3C)(t_{n-1} - t_n)^\beta$ for all $n \geqslant 1$. Now, if $n \geqslant n_0$ we have

$$t_{n-1} = t_n + (t_{n-1} - t_n) = (3C)(t_{n-1} - t_n)^\beta + (t_{n-1} - t_n)$$

$$\leqslant 3C\left(t_{n-1}-t_n\right)^\beta+\left(t_{n-1}-t_n\right)^\beta=(1+3C)\left(t_{n-1}-t_n\right)^\beta .$$

Then, using Proposition A.6 and Lemma A.7, for $n \geqslant n_0$ we have

$$\begin{aligned}
(t_n)^{1-\frac{1}{\beta}}-(t_{n-1})^{1-\frac{1}{\beta}} &= \underbrace{\left(1-\tfrac{1}{\beta}\right)}_{<0}\underbrace{\left[\theta t_n+(1-\theta)t_{n-1}\right]^{-\frac{1}{\beta}}}_{\geqslant (t_{n-1})^{-\frac{1}{\beta}}}\underbrace{(t_n-t_{n-1})}_{<0} \\
&\geqslant \left(1-\tfrac{1}{\beta}\right)(t_{n-1})^{-\frac{1}{\beta}}(t_n-t_{n-1}) \geqslant \left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}},
\end{aligned}$$

concluding the proof of item (iv).

(v) The cases $k=0$ and $k=1$ are trivial from (iv). Suppose that (v) is valid for a $k \in \mathbb{N}$. For $k+1$ we have

$$\begin{aligned}
\left(t_{n_0+(k+1)}\right)^{1-\frac{1}{\beta}} &\geqslant \left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+(t_{n_0+k})^{1-\frac{1}{\beta}} \\
&\geqslant \left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+k\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+(t_{n_0})^{1-\frac{1}{\beta}} \\
&= (k+1)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+(t_{n_0})^{1-\frac{1}{\beta}}.
\end{aligned}$$

(vi) Since $\frac{\beta}{\beta-1}<0$, it is clear from (v) that if $n \geqslant n_0$ we have

$$\begin{aligned}
t_n = v^n(t_0) &\leqslant \left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+(t_{n_0})^{1-\frac{1}{\beta}}\right]^{\frac{\beta}{\beta-1}} \\
&\leqslant \left[(n-n_0)\left(\tfrac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}+(t_0)^{1-\frac{1}{\beta}}\right]^{\frac{\beta}{\beta-1}}.
\end{aligned}$$

□

**Lemma A.9** (Grönwall's inequality)**.** *Suppose $\phi\colon [a,b] \to \mathbb{R}$ a continuous function, continuously differentiable in $(a,b)$, $\alpha \neq 0$, $\beta \in \mathbb{R}$ such that*

$$\phi'(t)+\alpha\phi(t) \leqslant \beta \text{ for all } t \in (a,b).$$

*Then,*

$$\phi(t) \leqslant \phi(a)e^{-\alpha(t-a)}+\frac{\beta}{\alpha}(1-e^{-\alpha(t-a)}).$$

*Additionally, if $\alpha$ is assumed to be positive we have*

$$\phi(t) \leqslant \phi(a)e^{-\alpha(t-a)}+\frac{\beta}{\alpha}.$$

*Proof.* We have for all $t \in (a,b)$

$$(e^{\alpha t}\phi(t))' = e^{\alpha t}\phi'(t)+\alpha e^{\alpha t}\phi(t) \leqslant \beta e^{\alpha t}.$$

Fix $c \in (a,b)$ and $t \in (c,b)$. Integrating this inequality from $c$ to $t$ we have

$$e^{\alpha t}\phi(t)-e^{\alpha c}\phi(c) \leqslant \frac{\beta}{\alpha}(e^{\alpha t}-e^{\alpha c}),$$

hence, we obtain

$$\phi(t) \leqslant \phi(c)e^{-\alpha(t-c)}+\frac{\beta}{\alpha}(1-e^{-\alpha(t-c)}),$$

and the result follows by making $c \to a^+$. □

**Proposition A.10.** *If $H$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associate norm $\|\cdot\|$, then for $p \geqslant 0$ and $u,v \in \overline{B}_R^H$ we have*

$$\| \|u\|^p u - \|v\|^p v\| \leqslant (p+1)R^p \|u-v\|. \tag{A.4}$$

*Proof.* When $p = 0$, (A.4) is a trivial equality. When $v = 0$ then (A.4) is also straightforward. Thus, we consider $p > 0$ and we can assume, without loss of generality, that $0 < \|v\| \leqslant \|u\|$. Observe that (A.4) is equivalent to

$$\|u\|^{2p+2} - 2\|u\|^p\|v\|^p\langle u,v\rangle + \|v\|^{2p+2} \leqslant (p+1)^2R^{2p}(\|u\|^2 - 2\langle u,v\rangle + \|v\|^2). \tag{A.5}$$

Also note that $\|u\|^2 - 2\langle u,v\rangle + \|v\|^2 = 0$ iff $\|u-v\| = 0$ iff $u = v$, in which case the inequality is also trivial. Thus we can also assume that $u \neq v$. Setting $x = \|u\|$, $y = \|v\|$ and $z = \langle u,v\rangle$, we have $0 < y \leqslant x \leqslant R$, $-xy \leqslant z \leqslant xy$, $x^2 - 2z + y^2 \neq 0$ and (A.5) becomes

$$x^{2p+2} - 2x^py^pz + y^{2p+2} \leqslant (p+1)^2R^{2p}(x^2 - 2z + y^2). \tag{A.6}$$

Case I. $-xy \leqslant z \leqslant 0$.

In this case, since $z \leqslant 0$, $x^p \leqslant (p+1)R^p$, and $y^p \leqslant (p+1)R^p$, we have

$$\begin{aligned} &x^{2p+2} - 2x^py^pz + y^{2p+2} - (p+1)^2R^{2p}(x^2 - 2z + y^2) \\ &\quad = x^2[x^p - (p+1)R^p][x^p + (p+1)R^p] + y^2[y^p - (p+1)R^p][y^p + (p+1)R^p] \\ &\quad\quad - 2z(x^py^p - (p+1)^2R^{2p}) \leqslant 0, \end{aligned}$$

which proves (A.6).

Case II. $0 < z \leqslant xy$.

Define

$$f(x,y,z) = \frac{x^{2p+2} - 2x^py^pz + y^{2p+2}}{x^2 - 2z + y^2}.$$

We have

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{(-2x^py^p)(x^2 - 2z + y^2) + 2(x^{2p+2} - 2x^py^pz + y^{2p+2})}{(x^2 - 2z + y^2)^2},$$

and since

$$\begin{aligned} &(-2x^py^p)(x^2 - 2z + y^2) + 2(x^{2p+2} - 2x^py^pz + y^{2p+2}) \\ &\quad = 2x^{p+2}(x^p - y^p) - 2y^{p+2}(x^p - y^p) = 2(x^p - y^p)(x^{p+2} - y^{p+2}) \geqslant 0, \end{aligned}$$

we have $\frac{\partial f}{\partial z}(x,y,z) \geqslant 0$. Thus, for each pair $(x,y)$ satisfying the conditions mentioned above, the function $(0,xy] \ni z \mapsto f(x,y,z)$ is increasing and, consequently, $f(x,y,z) \leqslant f(x,y,xy)$. Set

$$g(x,y) := f(x,y,xy) = \frac{x^{2p+2} - 2x^{p+1}y^{p+1} + y^{2p+2}}{x^2 - 2xy + y^2} = \frac{(x^{p+1} - y^{p+1})^2}{(x-y)^2}.$$

A direct application of the Mean Value Theorem to the function $h(t) = t^{p+1}$ shows that there exists $\xi \in (y,x)$ such that

$$x^{p+1} - y^{p+1} = (p+1)\xi^p(x-y) \leqslant (p+1)x^p(x-y) \leqslant (p+1)R^p(x-y),$$

which implies $f(x,y,z) \leqslant g(x,y) \leqslant (p+1)^2 R^{2p}$, and completes the proof. □

The next result is taken from (ZHAO; ZHAO; ZHONG, 2020b, Lemma 2.2), and we present a slightly adapted proof.

**Proposition A.11.** *If $H$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associate norm $\|\cdot\|$, then for $p \geqslant 0$ and $u,v \in H$ we have*

$$\langle \|u\|^p u - \|v\|^p v, u-v\rangle \geqslant 2^{-p}\|u-v\|^{p+2}. \tag{A.7}$$

*Proof.* When $p = 0$, the inequality is trivial and $C_0 = 1$. Hence we assume $p > 0$. Clearly (A.7) is trivial when either $u = 0$ or $v = 0$. We can assume, without loss of generality, that $0 < \|v\| \leqslant \|u\|$ and $u \neq 0$. Note that (A.7) is equivalent to

$$\|u\|^{p+2} + \|v\|^{p+2} - (\|u\|^p + \|v\|^p)\langle u,v\rangle \geqslant 2^{-p}(\|u\|^2 + \|v\|^2 - 2\langle u,v\rangle)^{\frac{p+2}{2}}.$$

Dividing both sides by $\|u\|^{p+2}$ we obtain

$$1 + \frac{\|v\|^{p+2}}{\|u\|^{p+2}} - \left(1 + \frac{\|v\|^p}{\|u\|^p}\right)\frac{\|v\|}{\|u\|}\frac{\langle u,v\rangle}{\|u\|\|v\|} \geqslant 2^{-p}\left(1 + \frac{\|v\|^2}{\|u\|^2} - 2\frac{\|v\|}{\|u\|}\frac{\langle u,v\rangle}{\|u\|\|v\|}\right)^{\frac{p+2}{2}}.$$

Setting $t = \frac{\|v\|}{\|u\|} \in [0,1]$ and $s = \frac{\langle u,v\rangle}{\|u\|\|v\|} \in [-1,1]$ we obtain

$$1 + t^{p+2} - (t + t^{p+1})s \geqslant 2^{-p}(1 + t^2 - 2ts)^{\frac{p+2}{2}}. \tag{A.8}$$

Since $1 + t^2 - 2ts = (t-s)^2 + 1 - s^2 = 0$ if and only if $t = s = 1$, we can prove (A.8) for $t \in [0,1)$ and $s \in [-1,1)$, where $1 + t^2 - 2ts \neq 0$. Define $h\colon [0,1) \times [-1,1) \to \mathbb{R}$ by

$$h(t,s) = \frac{1 + t^{p+2} - (t + t^{p+1})s}{(1 + t^2 - 2ts)^{\frac{p+2}{2}}}.$$

We have

$$\frac{\partial h}{\partial s}(t,s) = \frac{t[(p+2)(1+t^{p+2}) - (1+t^p)(1+t^2) - p(t+t^{p+1})s]}{(1+t^2-2ts)^{\frac{p+4}{2}}},$$

and for $s < 1$ we have

$$\begin{aligned}(p+2)(1+t^{p+2}) &- (1+t^p)(1+t^2) - p(t+t^{p+1})s \\ &\geqslant (p+2)(1+t^{p+2}) - (1+t^p)(1+t^2) - p(t+t^{p+1}).\end{aligned}$$

Note that for all $t \in [0,1]$ we have

$$(p+2)(1+t^{p+2}) - (1+t^p)(1+t^2) - p(t+t^{p+1})$$

$$= p(1 - t^{p+1})(1 - t) + (1 - t^p)(1 - t^2) \geqslant 0$$

thus $\frac{\partial h}{\partial s}(t,s) \geqslant 0$ for $s \in [-1,1)$ and $t \in [0,1)$. Thus, for each fixed $t \in [0,1)$, $s \mapsto h(t,s)$ is increasing and thus $h(t,s) \geqslant h(t,-1)$ for each $s \in [-1,1)$ and $t \in [0,1)$. Set

$$g(t) = h(t,-1) = \frac{t^{p+2} + t^{p+1} + t + 1}{(1+t)^{p+2}},$$

and note that for $t \in [0,1]$ we have

$$g'(t) = \frac{(p+1)(t+1)(t^p - 1)}{(1+t)^{p+3}} \leqslant 0.$$

Thus $g$ is decreasing and hence

$$h(t,s) \geqslant h(t,-1) \geqslant g(t) \geqslant g(1) = 2^{-p},$$

which proves (A.8) and completes the result. □

**Proposition A.12** (Jensen's inequality for concave functions)**.** *Let $g\colon [0,\infty) \to \mathbb{R}$ a concave and upper semicontinuous function. If $T > 0$ and $h\colon [0,T] \mapsto \mathbb{R}$ is an integrable function then $g \circ h$ is integrable and*

$$\frac{1}{T}\int_0^T g \circ h(\tau)d\tau \leqslant g\left(\frac{1}{T}\int_0^T h(\tau)d\tau\right).$$

*Proof.* Define $\phi\colon [0,\infty) \to \mathbb{R}$ by $\phi(x) = -g(x)$, $S = [0,1]$ and $f\colon [0,1] \to \mathbb{R}$ by $f(x) = h(Tx)$. Then $\phi$ is convex and lower semicontinuous, $|S| = 1$ (where $|S|$ denotes the 1-dimensional Lebesgue measure), and $f$ is integrable. Using (HYTÖNEN et al., 2016, Proposition 1.2.11) we have

$$\phi\left(\int_0^1 f(\xi)d\xi\right) \leqslant \int_0^1 \phi \circ f(\xi)d\xi,$$

and since $\phi = -g$, we have

$$\int_0^1 g \circ h(T\xi)d\xi \leqslant g\left(\int_0^1 h(T\xi)d\xi\right),$$

and the change of variables $\tau = T\xi$ completes the proof. □

## A.3 PRECOMPACT PSEUDOMETRICS

**Proposition A.13.** *Let $X$ be a complete metric space, $\varnothing \neq B \subset X$ and $\rho$ a pseudometric on $X$. Then $\rho$ is precompact on $B$ if and only if any sequence $\{x_n\} \subset B$ has a Cauchy subsequence $\left\{x_{n_j}\right\}$ with respect to $\rho$, that is, given $\varepsilon > 0$, there exists $N > 0$ such that $\rho(x_{n_i},x_{n_j}) < \varepsilon$ for all $i,j \geqslant N$.*

*Proof.* Suppose $\rho$ is precompact on $B$ and let $\{x_n\}_{n\in\mathbb{N}} \subset B$. There exists a finite set of points $\{w_1^{(1)}, \dots, w_{r_1}^{(1)}\} \subset B$ such that $B \subset \cup_{j=1}^{r_1} B_1^\rho(w_j^{(1)})$. This implies that there exists $j_1 \in \{1, \cdots, r_1\}$ for which the ball $B_1^\rho(w_{j_1})$ contains infinitely many terms of $\{x_n\}_{n\in\mathbb{N}}$. Let $\{x_{1,n}\}_{n\in\mathbb{N}}$ be the subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that lies in $B_1^\rho(w_{j_1})$.

Again there exists $\{w_1^{(2)},\dots,w_{r_2}^{(2)}\} \subset B$ such that $\{x_{1,n}\} \subset B \subset \bigcup_{j=1}^{r_2} B_{1/2}^\rho(w_j^{(2)})$, which implies that there exists $j_2 \in \{1, \dots, r_2\}$ for which the ball $B_{1/2}^\rho(w_{j_2})$ contains infinitely many terms of $\{x_{1,n}\}_{n\in\mathbb{N}}$, which we name $\{x_{2,n}\}_{n\in\mathbb{N}}$.

Continuing this process inductively, we obtain a subsequence $\{y_n\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ where $y_n = x_{n,n}$ and for each $n \in \mathbb{N}$ we have

$$y_n \in B_1^\rho(w_{j_1}) \cap B_{1/2}^\rho(w_{j_2}) \cap \cdots \cap B_{1/n}^\rho(w_{j_n}).$$

Let $\varepsilon > 0$ given and $N > \frac{2}{\varepsilon}$. If $m,n \geqslant N$ we have $y_n, y_m \in B_{1/N}^\rho(w_{j_N})$, which ensures that

$$\rho(y_m, y_n) \leqslant \rho(y_n, w_{j_N}) + \rho(w_{j_N}, y_m) < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon,$$

that is, $\{x_n\}_{n\in\mathbb{N}}$ has a Cauchy subsequence with respect to $\rho$.

On the other hand, assume that any sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ has a Cauchy subsequence with respect to $\rho$. Let $\delta > 0$ given. Since $B \neq \varnothing$, we can pick a point $y_1 \in B$. If $\rho(y_1, z) < \delta$ for all $z \in B$ we have

$$B \subset B_\delta^\rho(y_1),$$

and the result is done. Otherwise, let $y_2 \in B$ such that $\rho(y_1,y_2) \geqslant \delta$. If $\rho(z,y_1) < \delta$ or $\rho(z,y_2) < \delta$ for all $z \in B$, we have

$$B \subset B_\delta^\rho(y_1) \cup B_\delta^\rho(y_2),$$

and, again, the result is done. Otherwise, let $y_3 \in B$ such that $\rho(y_1,y_3) \geqslant \delta$ and $\rho(y_2,y_3) \geqslant \delta$. We can continue this process only until a finite number of iterations. In fact, if this is not the case, we can find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset B$ such that $\rho(y_i,y_j) \geqslant \delta$ for every $i \neq j$. Of course such sequence does not have a Cauchy subsequence, which is a contradiction with our initial assumption, and the result is complete. □

## A.4 DENSITY IN BOCHNER SPACES

**Proposition A.14.** *Let $X$,$Y$ be two Banach spaces such that $X \hookrightarrow Y$, with dense inclusion. If $I \subset \mathbb{R}$ is an interval and $1 \leqslant p < \infty$ then $L^p(I,X) \hookrightarrow L^p(I,Y)$, with dense inclusion.*

*Proof.* It is clear that if $\xi \in L^p(I,X)$, since $X \hookrightarrow Y$, then

$$\int_I \|\xi(\tau)\|_Y^p d\tau \leqslant C \int_I \|\xi(\tau)\|_X^p d\tau,$$

thus $L^p(I,X) \hookrightarrow L^p(I,Y)$.

Now let $\phi = \sum_{i=1}^{n} y_i \chi_{A_i}$ be a simple function in $L^p(S,Y)$, where $y_i \in Y$, $A_i \subset I$ for each $i = 1, \ldots ,n$, $A_i \cap A_j = \varnothing$ if $i \neq j$, and $\cup_{i=1}^{n} A_i = I$. Given $\varepsilon > 0$, there exists $x_i \in X$, $i = 1, \ldots ,n$ such that $\|x_i - y_i\|_Y^p \leqslant \frac{\varepsilon}{|I|}$, where $|I|$ denotes the one-dimensional Lebesgue measure of $I$. Consider $\eta = \sum_{i=1}^{n} x_i \chi_{A_i}$, which is a simple function with values in $X$, hence $\eta \in L^p(I,X)$. Note that

$$\int_I \|\eta(\tau) - \phi(\tau)\|_Y^p d\tau < \varepsilon.$$

Since the $Y$-valued simple functions are dense in $L^p(I,Y)$ (see (HYTÖNEN et al., 2016, Lemma 1.2.19), for instance), the result is complete. □

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