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Continuous Families of $\mathbb{Z}_2$ Monopoles

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ABSTRACT

In this thesis $\mathbb{Z}_2$ monopole solutions were studied in Yang–Mills–Higgs theories. We considered a theory with gauge symmetry $su(4)$ being spontaneously broken to $so(4)$ by a scalar field in the symmetric part of the 4×4 representation. We showed that $\mathbb{Z}_2$ monopoles are solutions of this theory and proceeded to construct their asymptotic form. These monopoles belong to continuous families of up to five parameters containing the discrete family previously known. We also explicitly determined homotopies between solutions, thus classifying those as type $\mathbb{Z}_2$.

Palavras-chave: Magnetic Monopoles. Yang–Mills–Higgs Theories. Lie Algebras. Classical Field Theories.

RESUMO

Neste trabalho estudamos monopolos $\mathbb{Z}_2$ em teorias de Yang–Mills–Higgs. Consideramos uma teoria com simetria $su(4)$ espontaneamente quebrada em $so(4)$ por um campo escalar na parte simétrica da representação 4×4. Mostramos que esta teoria apresenta monopolos $\mathbb{Z}_2$ e construímos as formas assintóticas destas soluções. Estes monopolos apresentam–se em famílias contínuas de até cinco parâmetros, contendo a família discreta previamente conhecida. Também encontramos explicitamente as homotopias entre as soluções as quais classificam os monopolos na classe $\mathbb{Z}_2$.

Palavras-chave: Monopolos Magnéticos, Teorias de Yang–Mills–Higgs. Álgebras de Lie. Teoria Clássica de Campos.

RESUMO EXPANDIDO

0.1 INTRODUÇÃO

O monopolo magnético é um dos mais antigos tópicos de discussão na física que não possui validação experimental. Este interesse prolongado pode inicialmente parecer sem fundamento, no entanto este campo de pesquisa provou-se muito frutífero do ponto de vista teórico. A beleza de certos aspectos da teoria, como simetria eletromagnética e quantização eletromagnética além da sua relevância para problemas mais modernos em teorias de grande unificação como confinamento de quarks tornaram este tópico um foco de interesse teórico.

Monopolos do tipo $\mathbb{Z}_n$ surgem em modelos em que o campo de Higgs pertence à uma representação diferente da adjunta. Em (KNEIPP; LIEBGOTT, 2010) e (KNEIPP; LIEBGOTT, 2013) os autores obtêm soluções de monopolo $\mathbb{Z}_2$ em teoria Yang-Mills-Higgs com simetria de calibre $su(n)$ quebrada por um campo de Higgs na representação $n \times n$ simétrica. De maneira geral foram descritas famílias discretas de soluções não fundamentais.

0.2 OBJETIVOS

Como uma extensão deste trabalho buscaremos novas soluções assintóticas de monopolo do tipo $\mathbb{Z}_2$ no modelo de Yang-Mills-Higgs com campo escalar na representação $n \times n$ do $su(n)$. Buscando inclusões genéricas de subálgebras $su(2)$ será possível descrever monopolos genéricos deste modelo e categoriza-los em famílias de vários parâmetros contínuos. Estas famílias conteriam, em particular, as famílias discretas já conhecidas deste modelo. Por fim devemos analisar se estes novos monopolos preservam o grupo de homotopia $\mathbb{Z}_2$ de seus predecessores.

0.3 METODOLOGIA

Afim de cumprir com o objetivo propomos uma subálgebra $su(2)$ de $su(n)$ arbitrária e, sem perda de generalidade, fixamos um de seus geradores na subalgebra de Cartan de $su(n)$ na representação simétrica $n \times n$. Partindo disto e de suas relações de comutação, os geradores restantes devem satisfazer um sistema de equações quadráticas. Para solucioná-las, particionamos as suas matrizes em blocos 2×2 e deduzimos condições para os coeficientes de um gerador, ditos pesos magnéticos. Fixando pesos magnéticos admissíveis, foi possível solucionar o sistema de equações, no caso $n = 4$, e assim observamos diversas famílias de soluções distintas.

0.4 RESULTADOS E DISCUSSÃO

Nesta dissertação desenvolvemos um método para a descrição de novas soluções de monopolos $\mathbb{Z}_2$ em teorias de Yang-Mills-Higgs. Por simplicidade nos focamos na quebra de simetria $su(4) \to so(4)$ com campo escalar na parte simétrica da representação 4×4. Um método geral foi desenvolvido para se encontrar subálgebras $su(2)$ das quais é possível escrever a forma explícita dos campos assintóticos destes monopolos $\mathbb{Z}_2$.

Observamos que para os pesos magnéticos $\beta = (1, 0)$ e $\beta = (0, 1)$ nossas soluções se encontram numa família de dimensão assim como o monopolo de Hooft–Polyakov original.

Quando consideramos inclusões isoclínicas, no entanto, observamos famílias de dimensão quatro. Enquanto que as inclusões três-para-um $\beta = (1, \pm 3)$ geram famílias de monopolos de três parâmetros.

Ademais caracterizamos homotopias explicitas entre soluções assintóticas. Assim pode-se verificar a existência de duas classes de homotopia, a trivial e a fundamental, confirmando que o segundo grupo de homotopia do vácuo para estes novos casos é de fato $\mathbb{Z}_2$.

0.5 CONSIDERAÇÕES FINAIS

Em projetos futuros seria possível generalizar esta ideia para diferentes quebras de simetria, e se elucidar a contagem do número de graus de liberdade restantes destas soluções. Inclusive possivelmente determinar condições de calibre que permitam soluções dinâmicas de monopolo análogas ao dyon para modelos em grupos de calibre maiores com quebra em grupos não abelianos.

CONTENTS

1 INTRODUCTION

The magnetic monopole is arguably one of the oldest topic of discussion in theoretical Physics which bears no experimental validation. This huge interest might therefore seem unfounded at first. Nevertheless it is the vast playground of ideas in this field of study that sparks peoples interest. The beauty in certain aspects of the theory, such as the concept of symmetry and the eventual relevance to fundamental problems, such as quark confinement and electric charge quantization, make it a topic of great theoretical interest.

Throughout History magnetic monopoles have been conceived to exist many times, for instance by Pierre Curie in 1894. But their theoretical possibility becomes a lot more apparent in the context of Maxwell's equations which makes apparent the symmetric character of the theory. In the Lorentz covariant formalism they read

$$\partial^\mu F_{\mu\nu} = \mu_0 j_\nu,$$
$$\partial^\mu * F_{\mu\nu} = 0,$$

where $\mu, \nu = 0, \ldots, 3$ denote space-time indices, $F_{\mu\nu}$ the electromagnetic tensor, j_ν the electric four-current and $*$ denotes the Hodge dual. When no charges are present, $j_\nu = 0$, the above equations become symmetric in the sense that it is possible to rename

$$F_{\mu\nu} \to *F_{\mu\nu},$$

and the equations are still satisfied. This transformation exchanges the roles of electric and magnetic fields. The existence of only electric charges nevertheless breaks this symmetry. The introduction of a magnetic four–current k_ν would thus restore the symmetry so as long as the transformation above also exchanges electric and magnetic fluxes accordingly.

If we write the magnetic field $\mathbf{B}$ as the curl of some potential

$$\mathbf{B} = \nabla \times \mathbf{A},$$

then it may seem incompatible with non-vanishing magnetic sources for

$$\rho_m = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{A}) = 0,$$

wherever this vector potential is continuous and well-defined. Dirac was able to reconcile (4) with non-vanishing charges by considering a string-like singularity carrying some magnetic field all the way from the spacial asymptotic to a single point, as though it is made up of an infinitesimally thin solenoid. So long as this string is undetectable this furnishes the theory with means of describing a magnetic monopole using only the vector potential $\mathbf{A}$, essential for quantum mechanics. Experimentally this string could be detected via the Aharonov-Bohm effect (AHARONOV; BOHM, 1959, 3), which is sensitive to phase changes in the wave function. So requiring the string to be unphysical translates to the equation

$$eg = 2\pi\hbar n,$$

called Dirac's quantization condition. Here e and g are the electric and magnetic charges, $\hbar$ Planck's constant and n an integer. This states that, provided there is at least one magnetic monopole in the Universe with magnetic charge g, all other electrically charged particles must have an electric charge which is an integer multiple of $2\pi\hbar/g$. Later the concept of gauge invariance on compact gauge groups (YANG; MILLS, 1954, 1) allowed for an alternative explanation for the quantization of electric charge which doesn't require the existence of monopoles: Put simply an electric charge operator can now be defined such that its eigenvalues enumerate the allowed electric charges for particles in the theory. Electric charge is then quantized in a similar fashion as angular momentum.

The new paradigm of symmetry breaking developed, among others by Nambu (NAMBU, 1960), Goldstone (GOLDSTONE, 1961) and Higgs (HIGGS, 1964) allowed the rediscovery of the subject of magnetic monopoles. Now it is manifested as a soliton-like solution holding itself together by means of a self-interaction potential. Which is, in turn, responsible for symmetry breaking. As first noted by 't Hooft ('T HOOFT, 1974) and Polyakov (POLYAKOV, 1974), some Yang-Mills-Higgs theories admit non-trivial classical solutions which are both localized in space and topologically stable. Localization in space together with their assignment of energy and momentum enables the interpretation of those particular field solutions as particles. These solutions have their stability guaranteed by the existence of conserved charges given by topological invariants which remains unchanged under continuous deformations of the field. Therefore if a field solution and a vacuum solution, for instance, have differing invariants, then the former cannot be deformed into the latter and it is said to be topologically stable. In contrast to Noether charges, these conserved charges are not related to continuous symmetry of the action. They are related to the homotopic degree of the fields which map space time regions into internal space regions (MANTON; SUTCLIFFE, 2004).

The possibility for non-trivial invariants, on the other hand, relies on the gauge group considered by the theory and on the pattern of symmetry breaking. Many grand unification theories predict the existence of magnetic monopoles and that they would have been synthesized in beginning of the universe (COLEMAN, 1982). Their apparent absence, known as the monopole problem, can serve the purpose of constraining parameters, ruling theories out altogether or even proposing new ones(GUTH, 1981).

Magnetic monopoles can be relevant for the so-called Confinement Problem in QCD. There is no satisfying answer to the question why quarks and gluons appear only as color singlets in nature. One of the ideas, due to 't Hooft (HOOFT, 1975) and Mandelstam (MANDELSTAM, 1976) conjectures that electric charge confinement in QCD is a dual phenomenon to the magnetic charge confinement in a type II superconductor ('T HOOFT, 1982). In this model, there is an electromagnetic duality (FIGUEROA-O'FARRILL, 1998) mapping monopoles into electric charged particles and vice versa.

The idea of electromagnetic duality itself is not completely established. There are some works based on tests leading to a conjecture of an electromagnetic duality in non-Abelian models (P. GODDARD; NUYTS; D. OLIVE, 1977), (MONTONEN; D. OLIVE, 1977). However, there is no proof that this type of duality actually holds. There are studies to be done in this topic in order to obtain a better understanding of this duality and this is related to a better understanding of monopoles themselves. In particular, it is necessary a better knowledge of the so-called $\mathbb{Z}_n$ monopoles.

The $\mathbb{Z}_n$ monopoles appear when the scalar field is not in the adjoint representation. In (KNEIPP; LIEBGOTT, 2010) and (KNEIPP; LIEBGOTT, 2013) the authors obtained $\mathbb{Z}_2$ monopoles in Yang-Mills theories, and in particular they obtained a discrete family of non fundamental monopoles. The general aim of this thesis is to overview the construction of magnetic monopoles in gauge field theories and to describe a method of distinguishing $su(2)$ subalgebras from which one can construct distinct $\mathbb{Z}_2$ monopole solutions belonging to multi-parameter continuous families. We focus on the $su(4) \to so(4)$ case but the methods used can in principle be extended to $\mathfrak{su}(n) \to \mathfrak{so}(n)$.

In chapter 2 some mathematical background has been reviewed. We discussed results from Lie algebra theory, Lie groups and their homotopy groups, as well as a brief examination of the group $SO(4)$ in particular.

In chapter 3 different monopole configurations were considered, from its original description by Dirac, to the 't Hooft–Polyakov solution in the Georgi–Glashow model and an asymptotic equivalence between the two. The BPS equations were also defined as well as the concept of the vacuum manifold and the manifold of collective coordinates known as the moduli space.

In chapter 4 we review monopole solutions for $su(n) \to so(n)$ and develop a general method for embedding $su(2)$ subalgebras. We then move on to focus on the case $su(4) \to so(4)$ in order to describe all families of embeddings from which one generates monopole solutions. By the end of the chapter we write down the explicit form for the Higgs field and determine homotopies between distinct solutions.

In chapter 5 we discuss what we concluded from this thesis and the outlook for further research.

In the appendix A we explore the connection between topological degree of the two–sphere and the magnetic flux surface integral, as well as defining the generators for $SO(4)$ and a useful invariant in this context known as the pfaffian.

2 MATHEMATICAL REVIEW

2.1 LIE ALGEBRAS

A *Lie group* is a group defined over a differential manifold. Given a Lie group G of dimension $\dim G = d$, the *generators* X_a, $a = 1, \ldots, d$ form a basis of the vector space tangent to the identity $T_I G$. This vector space is called the *Lie algebra* $\mathrm{Lie}(G) = \mathfrak{g}$ (FULTON; W. HARRIS; J. HARRIS, 1991). This algebra inherits a product from its parent group called the *Lie bracket*. This operation can be constructed as follows: Let $x : (0, \varepsilon) \to G$ be a regular path on G starting at $x(0) = I$ and satisfying $x'(0) = X \in \mathfrak{g}$. Take $Y \in \mathfrak{g}$, then the vector tangent to the new path $x(t)Yx(t)^{-1}$ also belongs to $\mathfrak{g}$. That is, given $X, Y \in \mathfrak{g}$ and a path, for concreteness the exponential $x(t) = e^{itX} \in G$, then

$$[X, Y] = \frac{d}{idt}\left(e^{itX} Y e^{-itX}\right)\Big|_{t=0} \in \mathfrak{g},$$

defines a bilinear product $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

$$\begin{aligned}
&[X, Y] + [Y, X] = 0, \\
&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$. A *linear representation* R of a Lie group G is a group homomorphism $R : G \to \mathrm{GL}(V)$, $GL(V)$ the group of general linear transformations on some vector space W. This representation of G induces a representation of its algebra $R : \mathfrak{g} \to \mathrm{gl}(W)$, acting on our basis like $R(X_a) = T_a$. Then the representation of the lie bracket yields $R([X_a, X_b]) = T_a T_b - T_b T_a$ also known as the *commutator*. Besides it is now possible to endow $\mathfrak{g}$ with a symmetric form $\langle .,. \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by taking the trace

$$\langle X_a, X_b \rangle = \mathrm{tr}(T_a T_b),$$

the representation for which the vector space W is $\mathfrak{g}$ itself is called the *adjoint* representation $R = \mathrm{Ad}$ of $\mathfrak{g}$. In this case the above trace is called the *Killing form* on $\mathfrak{g}$.

A *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace closed under the bracket operation. An *Abelian subalgebra* $\mathfrak{a}$ of $\mathfrak{g}$ is a subalgebra for which all its elements are mutually commuting. Furthermore it is said to be *maximal* if it is not contained in any larger Abelian subalgebra. This is also called a Cartan subalgebra $\mathrm{CSA}(\mathfrak{g})$ and it is of central importance for classifying Lie algebras. Its dimension defines $\mathrm{rank}\, \mathfrak{g} = r$.

An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. It is also solvable if the sequence $[\mathfrak{h}, \ldots, [\mathfrak{h}, \mathfrak{h}]] \subseteq \{0\}$ eventually. A Lie algebra $\mathfrak{g}$ is said to be *simple* if it is neither Abelian nor does it contain a proper ideal, meaning its only ideals are $\mathfrak{g}$ itself and $\{0\}$. It is said *semisimple* if does not contain any solvable proper ideal (KNAPP, 1988). A few notable ideals are the *centralizer* of a subset $s \subset \mathfrak{g}$ defined by $Z_{\mathfrak{g}}(s) = \{X \in \mathfrak{g} : [X, Y] = 0,\ \forall Y \in s\}$ and the *normalizer* $N_{\mathfrak{g}}(s) = \{X \in \mathfrak{g} : [X, Y] \in s,\ \forall Y \in s\}$.

In the so called *Cartan–Weyl basis* a Cartan subalgebra is spanned by Hermitian generators $\mathrm{CSA}(\mathfrak{g}) = \mathrm{span}\{H_1, \ldots, H_r\}$. Given a representation $R : \mathfrak{g} \to \mathrm{gl}(V)$, one can always write states of the representation space V in a basis where all H_i are diagonal: $H_i\,|\mu\rangle = \mu_i\,|\mu\rangle$. Their eigenvectors are called *weights* and the r–tuple $\mu = (\mu_1, \ldots, \mu_n)$ is a *weight vector*. The remaining $d - r$ generators are *step operators* E_α defined as simultaneous eigenvectors of H_i in the adjoint representation (GEORGI, 2018). That is

$$H_i\,|E_\alpha\rangle = \alpha_i\,|E_\alpha\rangle\,,$$

for each $i = 1, \ldots, r$. In terms of commutators this means $[H_i, E_\alpha] = \alpha_i E_\alpha$. Note that, because H_i are all Hermitian, $E_\alpha^\dagger = E_{-\alpha}$. Now, we can think of the eigenvalues α_i as images of a linear functional $\alpha : \mathrm{CSA}(\mathfrak{g}) \to \mathbb{R}$ defined by $\alpha(H_i) = \alpha_i$ (B. HALL; B.C. HALL, 2003). Their values are called *roots* and the functional α is a vector of the dual space $\mathrm{CSA}(\mathfrak{g})^*$ which we write as the r–tuple $\alpha = (\alpha_1, \ldots, \alpha_r)$ called a *root vector*. The state $E_\alpha\,|\mu\rangle$ has weight $\alpha + \mu$. Because of this $E_\alpha\,|E_{-\alpha}\rangle$ has weight vector 0 and thus lies in $\mathrm{CSA}(\mathfrak{g})$. More precisely it yields $[E_\alpha, E_{-\alpha}] = \alpha_i H_i = \alpha \cdot H$. This allows one to construct an $su(2)_\alpha = \mathrm{span}\{T_1, T_2, T_3\}$ subalgebra generated by the Hermitian operators

$$\begin{aligned}
T_1 &= \frac{1}{\sqrt{2}|\alpha|}(E_\alpha + E_{-\alpha}),\\
T_2 &= \frac{1}{\sqrt{2}i|\alpha|}(E_\alpha - E_{-\alpha}),\\
T_3 &= \frac{1}{2|\alpha|^2}\alpha \cdot H,
\end{aligned}$$

satisfying $[T_i, T_j] = i\varepsilon_{ijk}T_k$. Therefore the generator T_3 may only attain integer or half–integer eigenvalues, depending on its representation. Because of this

$$2\,\frac{\beta \cdot \alpha}{\alpha^2} \in \mathbb{Z}, \tag{2.1.1}$$

for any roots $\alpha, \beta \in \Phi(\mathfrak{g})$. Furthermore, exchanging the roles of α and β in (2.1.1) and multiplying both results gives us the strong geometrical constraint

$$4\,\frac{(\alpha \cdot \beta)^2}{\alpha^2\beta^2} = 4\cos^2\theta \in \mathbb{Z},$$

for any two roots α and β. This implies that the angle between them must satisfy $\cos\theta = \pm\frac{1}{2}\sqrt{n}$ for some integer $0 \le n \le 4$.

Given a basis for our weight space, a weight μ is said to be *positive* if its first non–zero component is positive. A weight μ is said to be higher than a weight ν if $\mu - \nu$ is positive. This allows us to speak of the *highest weight* Λ of a representation, which specifies it completely. It satisfies $E_\alpha\,|\Lambda\rangle = 0$ for all positive roots α. A positive root is said to be *simple* if it cannot be written as a proper sum of positive roots. The angle between simple roots satisfies $\frac{\pi}{2} \le \theta < \pi$ and there are always $r = \mathrm{rank}\ \mathfrak{g}$ linearly independent roots spanning the whole $\mathrm{CSA}(g)^*$.

Their relative angles are used to classify all semisimple Lie algebras. Diagrammatically we denote simple roots by vertices and their relative angles $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ by $0, 1, 2, 3$ lines between them respectively. These are called *Dynkin diagrams*. Labeling a base of simple roots α_i, $i = 1, \ldots, r$, we may define *coroots*

$$\alpha_i^\vee = 2\,\frac{\alpha_i}{\alpha_i^2},$$

and their relative inner product

$$K_{ij} = \alpha_i \cdot \alpha_j^\vee \in \mathbb{Z},$$

is called the *Cartan matrix* of $\mathfrak{g}$.

The Dynkin labels for a weight λ are the coefficients

$$\lambda_i = \lambda \cdot \alpha_i^\vee \in \mathbb{Z},$$

the *fundamental weights* Λ_i are the ones forming a base orthornormal to the coroots, that is,

$$\Lambda_i \cdot \alpha_j^\vee = \delta_{ij},$$

by reflecting a root β with respect to another root α the resulting vector

$$\sigma_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha,$$

is again a root of $\mathfrak{g}$. This is called a *Weyl reflection*, and the set of all such reflections σ_α is named the *Weyl group*.

2.2 HOMOTOPY GROUPS

Given a division algebra A and integer n the group of invertible linear tranformations

$$\mathcal{O}_n(A) = \{X \in \mathrm{GL}_n(A) \mid X^*X = 1\}, \tag{2.2.1}$$

defines a manifold embedded in A^{n^2}, one dimension for each matrix entry. The specific cases

$$O(n) = \mathcal{O}_n(\mathbb{R}), \qquad U(n) = \mathcal{O}_n(\mathbb{C}), \qquad Sp(n) = \mathcal{O}_n(\mathbb{H}),$$

are called the *orthogonal groups*, *unitary groups* and *symplectic groups* respectively. The additional constraint $\det X = +1$ defines the *special* groups

$$SO(n) = \{X \in \mathcal{O}_n(\mathbb{R}) \mid \det X = +1\}, \qquad SU(n) = \{X \in \mathcal{O}_n(\mathbb{C}) \mid \det X = +1\},$$

All groups above, except for $O(n)$, are *path–connected* which means that, given any two points $g, h \in G$, there exists a continuous path $\gamma : [0,1] \to G$ connecting them both, starting at $\gamma(0) = g$ and ending at $\gamma(1) = h$. In practice these paths are exponentials of generators for the group. For instance let $G = SO(2)$ the group of proper rotations on $\mathbb{R}^2$.

Because $\dim SO(2) = 1$ the algebra $\mathfrak{g} = so(2)$ contains only one Hermitian generator, namely $T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$, So the path $\gamma : [0, \tau] \longrightarrow SO(2)$

$$\gamma(\theta) = e^{i\theta T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

connects the identity I to a rotation by τ, so any two rotations can be connected by some continuous path and $SO(2)$ is indeed path–connected.

We can also speak of closed paths starting from and ending at the identity $\gamma(0) = \gamma(1) = I$ also called *loops* of G, whose set we'll denote $\mathrm{loop}(G)$. By defining a product which glues the images of two different loops γ, η like

$$\gamma \oplus \eta : [0,1] \longrightarrow G,$$
$$t \longmapsto \begin{cases} \gamma(2t), & t \leq 1/2, \\ \eta(2t-1), & t > 1/2, \end{cases}$$

we will be able to furnish $\mathrm{loop}(G)$ with a group structure. The only major thing remaining is identifying all loops that can be continuously deformed into one another. To do this we think of loops as points in $\mathrm{loop}(G)$, and we would like to describe paths connecting different points in *this* space. That is, given $\gamma, \eta \in \mathrm{loop}(G)$ if there exists a continuous path

$$H : [0,1] \longrightarrow \mathrm{loop}(G),$$
$$s \longmapsto H(s),$$

starting at $H(0) = \gamma$ and ending at $H(1) = \eta$ then γ and η are said to be *homotopic* and the map H is called a *homotopy*. Now we can use the existance of this map to define an equivalence relation $\sim$ and divide $\mathrm{loop}(G)$ into equivalence classes $[\gamma] = \{\eta \in \mathrm{loop}(G) : \gamma \sim \eta\}$ of mutually homotopic loops. These classes, together with the product above, define the *first homotopy group* of G denoted $\pi_1(G) = (\mathrm{loop}(G)/\sim\,,\ \oplus)$.

We may go further and think of the space of homotopies themselves starting from and ending at the identity $[I] \in \pi_1(G)$, call it $\mathrm{loop}_2(G)$. We can then apply the previous reasoning to define the *second homotopy group* of G, $\pi_2(G) = (\mathrm{loop}_2(G)/\sim\,,\ \oplus)$. Note that by varying both parameters the map $\phi(t,s) = H(s)(\gamma(t))$ is the image of a 2–sphere in G. So we can also think of $\pi_2(G)$ as the group of 2–dimensional loops in G, up to continuous deformations.

This procedure can be extended to define the *n–th homotopy group* $\pi_n(G)$ for $n \geq 3$. Working backwards even $\pi_0(G)$ makes sense: Here the equivalence classes are comprised of 0–spheres, i.e. points, which are path–connected. So there exists one and only one element in $\pi_0(G)$ for each connected component of G. That is, G is path–connected if and only if $\pi_0(G) \cong 1$. Similarly G is said to be *simply–connected* if $\pi_1(G) \cong 1$, meaning any path can be contracted down to a point.

For instance every element of the group $G = O(3)$ satisfies $\det X = \pm 1$. Because every loop in G is continuous its points must also vary in determinant continuously. Thus $O(3)$ falls apart into two disconnected components: $\pi_0(O(3)) \cong \mathbb{Z}_2$ labeled by the determinant of its elements.[1]

For an example of a group which is connected but not simply–connected we can take $G = SO(3)$. Here, there are loops which cannot be contracted to the identity, for example the path of rotations around the z–axis ending at the rotation of 2π; $\gamma : [0, 2\pi] \to SO(3)$,

$$\gamma(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If, however, we were to go around twice; $\gamma : [0, 4\pi] \to SO(3)$ then the resulting loop *would* be homotopic to the identity. This is the case because $SO(3)$ is topologically identical to the three–dimensional ball B^3 with antipodes identified. It can be shown that any loop in $SO(3)$ is homotopic to one of the two loops above so we have $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

It is possible, however, to view any lie group G as a subset of a larger simply–connected group $\tilde{G}$ called its *universal covering group*[2]. This defines the spin groups as $Spin(n) = \widetilde{SO(n)}$. For the previous case $\widetilde{SO(3)} \cong Spin(3) \cong SU(2)$.

Given a Lie group G and subgroup $H \subset G$, one may measure the homotopy groups of the homogeneous quotient space G/H, the short exact sequence of homomorphisms (ALBERT SCHWARZ, 1994)

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1,$$

provides the isomorphism $\pi_2(G/H) \cong \ker(\pi_1(H) \to \pi_1(G))$. When G is simply–connected we have

$$\pi_2(G/H) \cong \pi_1(H). \tag{2.2.2}$$

Similarly, if G is connected we have

$$\pi_1(G/H) \cong \pi_0(H). \tag{2.2.3}$$

This result will prove useful for determining the second homotopy class of a solution in a Yang–Mills gauge theory. In particular we will look at the symmetry breakdown $su(4) \to so(4)$, so it will prove useful to understand the defining representation of $so(4)$.

2.3 SO(4) GROUP STRUCTURE

When considering the symmetry break $su(4) \to so(4)$ it proves useful to analyze the group structure of $H = SO(4)$ in order to determine whether or not topologically stable monopoles appear in this model.

[1] $\mathbb{Z}_n$ stands for the finite cyclic group of order n

[2] In fact there is always some subgroup K of $\tilde{G}$ such that $G \cong \tilde{G}/K$.

Every element $g \in SO(4)$ in the defining, four–dimensional, representation corresponds to a rotation in $\mathbb{R}^4$. By writing g in its block diagonal form, also called its normal form,

$$D(g) = \begin{pmatrix} \begin{matrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{matrix} & 0 \\ 0 & \begin{matrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{matrix} \end{pmatrix},$$

it can be described as a composition of two independent rotations acting on a pair orthogonal planes. In general these are the only planes left invariant by g. The exception is the case where both rotations have the same absolute angle, i.e. $\theta_1 = \pm\theta_2$, then there are an infinite number of invariant planes all rotating by $\pm\theta_1$. These special rotations are called *isoclinic*, meaning same angle, and are further distinguished by orientation: The ones for which both angles have the same sign, $\theta_1 = \theta_2$, are said to be *left–isoclinic* whereas when they have opposite signs, $\theta_1 = -\theta_2$, they are said to be *right–isoclinic*.

The identity, I, and central inversion, $-I$, are both left and right–isoclinic and it can be shown that the product of left(right)–isoclinic rotations is again left(right)–isoclinic which makes them subgroups $SU(2)^L$ and $SU(2)^R$ of $SO(4)$. Furthermore both subgroups are normal and the homomorphism

$$\begin{aligned} \rho : SU(2)^L \times SU(2)^R &\to SO(4), \\ (g_L, g_R) &\mapsto g_L g_R, \end{aligned}$$

is two–to–one. This is because every rotation $g \in SO(4)$ is a product of isoclinic rotations, $g = g_L g_R$, and there are exactly two ways of decomposing g, namely

$$\rho(-g_L, -g_R) = \rho(g_L, g_R) = g_L g_R = g.$$

Therefore we find

$$SO(4) \cong \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}.$$

Because $SU(2)$ is simply connected, so is $SU(2) \times SU(2)$, therefore this homomorphism allows one to compute the first homotopy group

$$\begin{aligned} \pi_1(SO(4)) &\cong \pi_1\left(\frac{SU(2) \times SU(2)}{\mathbb{Z}_2}\right), \\ &\cong \pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2. \end{aligned}$$

A discussion about the generators for these subgroups can be found in (A.2)

3 MONOPOLE CONFIGURATIONS

3.1 DIRAC MONOPOLE

Studied by Paul Dirac (DIRAC, 1931) the now called Dirac monopole offered an explanation for the quantization of electric charge in nature. Suppose, in the classical setting, there is a point source of magnetic field with strengh g. No continuous vector potential $\mathbf{A}$ could ever describe this field simply because of Gauss' law

$$g = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S = \varnothing} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = 0,$$

S standing for a smooth surface enclosing g. To circumvent this difficulty $\mathbf{A}$ must be singular at some point in S for every choice of S. This describes some curve connecting the charge to the spacial asymptotic. In order to determine a potential $\mathbf{A}$ fitting this requirement it helps to imagine an infinitesimally thin solenoid coming from $(0, 0, -\infty)$ and ending at $(0, 0, 0)$, the so–called *Dirac string*. It produces a magnetic field described by

$$\mathbf{B}_s = \frac{g}{4\pi r^2}\hat{r} + g\delta(x)\delta(y)\Theta(-z)\hat{z},$$

Θ standing for the Heaviside step function, see figure 3.1. The flux of the field pointing radially outwards always cancels the flux going up the solenoid so we have $\nabla \cdot \mathbf{B}_s = 0$. Thus we may look for a potential satisfying

$$\nabla \times \mathbf{A} = \frac{g}{4\pi r^2}\hat{r} + g\theta(-z)\delta(x)\delta(y)\hat{z}.$$

Integrating the flux going through a spherical cap of radius r and angle θ with respect to $\hat{z}$,

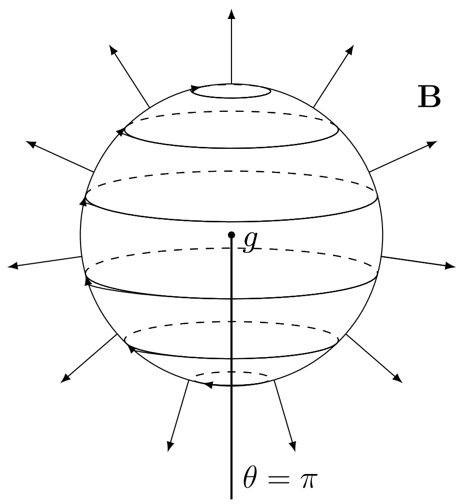

Figure 1 – The magnetic field of a Dirac monopole

The radial magnetic field $\mathbf{B}$ emanating from a Dirac monopole of charge g may be described by the curl of an azimuthal vector potential $\mathbf{A}$. Both fields are singular at the negative z semi-axis.

also assuming $\mathbf{A}$ in the $\hat{\varphi}$ direction we gather[1]

$$\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\theta} r^2 \sin\theta \mathrm{d}\theta \frac{g}{4\pi r^2}, \tag{3.1.1}$$

$$A 2\pi r \sin\theta = \frac{g}{2}(1-\cos\theta), \tag{3.1.2}$$

$$\mathbf{A} = \frac{g}{4\pi r} \tan\left(\frac{\theta}{2}\right)\hat{\varphi}. \tag{3.1.3}$$

Note that this diverges when $\theta \to \pi$, precisely the singularity designed for $\mathbf{A}$. The Aharanov–Bohm effect could, nevertheless, be used to detect this string. The additional requirement that the effect always vanishes translates to

$$\exp\left(\frac{ie}{\hbar}\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{l}\right) = 1,$$

for any curve C. In particular for a surface intersecting the string, whose boundary is C, the integral yields g and therefore

$$eg = 2\pi n\hbar, \tag{3.1.4}$$

for some integer n. This is Dirac's original quantization of the electric charge. In terms of topology, this reflects the fact that the first homotopy group of the gauge group $G = U(1)$ is

$$\pi_1(U(1)) \cong \mathbb{Z},$$

which is nontrivial because $U(1)$ is not simply connected. The integer in (3.1.4) specifies in which sector a loop in $U(1)$, and therefore the configuration for $\mathbf{A}$, lies. In the following constructions the gauge groups will always be simply connected so we turn to the second homotopy group in order to search for topological configurations.

3.2 T' HOOFT–POLYAKOV MONOPOLE

Yang–Mills theories where G is non–Abelian generally allow for magnetic monopoles as solutions of the equations of motion. Here we have a scalar field Φ in the adjoint representation of $\mathfrak{g} = su(2)$, that is $G = SO(3)$, breaking the symmetry down to an $H = SO(2)$ subgroup generated by Φ itself. The Lagrangian density is

$$L = -\frac{1}{4l}\operatorname{tr} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2l}\operatorname{tr} D^{\mu}\Phi D_{\mu}\Phi - V(\Phi),$$

F is the non–Abelian field strength, D stands for the covariant derivative in G

$$D\Phi = \mathrm{d}\Phi + ie[A,\Phi],$$
$$F = DA = \mathrm{d}A + ie[A,A],$$

[1] the last expression follows from double-angle trigonometric identities

A is the connection on G, the non–Abelian generalization of the electromagnetic potential A^μ.

Choosing a base of generators T_a we have

$$[T_a, T_b] = i\varepsilon_{abc}T_c,$$
$$\operatorname{tr} T_a T_b = l\delta_{ab},$$

Where l stands for the index of the representation for Φ. The quartic scalar potential V given by

$$V(\Phi) = \lambda \left(|\Phi|^2 - v^2\right)^2, \tag{3.2.1}$$

here the norm for Φ is defined by

$$|\Phi|^2 = \frac{1}{l} \operatorname{tr} \Phi^2 = \Phi^a \Phi^a.$$

In terms of space–time and isospin components we have

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - e\varepsilon_{abc}A_b^\mu A_c^\nu, \tag{3.2.2}$$
$$(D^\mu\Phi)_a = \partial^\mu\Phi_a - e\varepsilon_{abc}A_b^\mu\Phi_c. \tag{3.2.3}$$

The equations of motion yield a non–linear second order system of differential equations

$$D^\nu F_{\mu\nu}^a = -e\varepsilon^{abc}\Phi^b D_\mu\Phi^c, \tag{3.2.4}$$
$$D^\mu D_\mu \Phi^a = \lambda\Phi^a(\Phi^b\Phi^b - v^2), \tag{3.2.5}$$

The symmetric energy–momentum tensor is given by

$$T_{\mu\nu} = F_{\mu\rho}^a F_v^{\rho a} + D_\mu\Phi^a D^\mu\Phi^a - g_{\mu\nu}L, \tag{3.2.6}$$
$$T_{00} = \frac{1}{2}\left((\mathbf{E}_a)^2 + (\mathbf{B}_a)^2 + (D^0\Phi)^2 + (\mathbf{D}\Phi)^2\right) + V(\Phi). \tag{3.2.7}$$

Because physical field configurations carry finite energy we must look for fields with vanishing energy density in the spacial asymptotic. Because (3.2.7) is positive definite, this means $F = 0$, $\mathbf{D}\Phi = 0$ and $|\Phi^a|^2 = v^2$ as $r \to +\infty$. From this we may infer $\mathbf{A}$ in terms of Φ:

$$\mathbf{D}\Phi = 0, \tag{3.2.8}$$
$$\partial_i\Phi^a = e\varepsilon^{abc}A_i^b\Phi^c, \tag{3.2.9}$$
$$\varepsilon^{ade}\partial_i\Phi^a = e\varepsilon^{abc}\varepsilon^{ade}A_i^b\Phi^c, \tag{3.2.10}$$
$$\varepsilon^{abc}\partial_i\Phi^a = e(A_i^b\Phi^c - A_i^c\Phi^b). \tag{3.2.11}$$

Because Φ generates an Abelian subgroup $H = SO(2)$ we call

$$\mathbf{\Lambda} = \frac{\operatorname{tr}(\mathbf{A}\Phi)}{l|\Phi|} = \frac{1}{v}\mathbf{A}^a\Phi^a,$$

the *Abelian projection* of $\mathbf{A}$ (SHNIR, 2006). Now contracting the expression above with Φ^c we find

$$\begin{aligned}\varepsilon^{abc}\Phi^c\partial_i\Phi^a &= veA_i^b - v\Lambda_i\Phi^b,\\ A_i^a &= \frac{1}{ve}\varepsilon^{abc}\Phi^b\partial_i\Phi^c + \frac{1}{e}\Lambda_i\Phi^a.\end{aligned}$$

From this expression one can derive the field strength

$$\begin{aligned}F_{\mu\nu} = &\frac{2}{iev^2}[\partial_\mu\Phi, \partial_\nu\Phi] + \frac{1}{v}\Phi(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu)+\\ &+\frac{1}{v}(\partial_\mu\Phi\Lambda_\nu - \partial_\nu\Phi\Lambda_\mu) + \frac{1}{v^3}[[\Phi, \partial_\mu\Phi], \Phi]\Lambda_\nu + \frac{1}{v^3}[\Phi, [\Phi, \partial_\nu\Phi]]\Lambda_\mu.\end{aligned}$$

Taking the Abelian projection of the field strength $\mathcal{F}_{\mu\nu} = \frac{1}{lv}\operatorname{tr} F_{\mu\nu}\Phi$ the last three terms vanish and we arrive at

$$\mathcal{F}_{\mu\nu} = \frac{1}{v^3}\varepsilon^{abc}\Phi^a\partial_\mu\Phi^b\partial_\nu\Phi^c + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu.$$

Where we can see that the Abelian projection Λ_μ acts like the four–potential of Abelian electrodynamics. The corresponding magnetic field yields

$$\mathcal{B}^i = \frac{1}{2}\varepsilon^{ijk}\mathcal{F}_{jk} = \frac{1}{2v^3}\varepsilon^{ijk}\varepsilon^{abc}\Phi^a\partial_j\Phi^b\partial_k\Phi^c + \varepsilon^{ijk}\partial_j\Lambda_k.$$

And the magnetic four–current is given by

$$k_\mu = \partial^\sigma\tilde{\mathcal{F}}_{\sigma\mu} = \frac{1}{2ev^3}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\mu\Phi^a\partial^\nu\Phi^b\partial^\sigma\Phi^c.$$

Notice that, with the assumption of continuity, the dependence of k_μ on the gauge field completely vanishes. Because the partial derivatives commute this current is conserved, $\partial^\mu k_\mu = 0$. Finally the magnetic charge contained in a sphere of radius approaching infinity is

$$\begin{aligned}g &= \int \mathrm{d}^3x\, k_0 = \frac{1}{2ev^3}\int \mathrm{d}^3x\, \varepsilon_{abc}\varepsilon_{mnk}\partial^m(\Phi^a\partial^n\Phi^b\partial^k\Phi^c)\\ &= \frac{1}{2ev^3}\int \mathrm{d}^2S^m\, \varepsilon_{abc}\varepsilon_{mnk}\Phi^a\partial^n\Phi^b\partial^k\Phi^c.\end{aligned}$$

This integral can be rewritten as the topological degree of a map $\hat{\Phi} : S^2_{\text{int}} \to S^2_\infty$, between the internal and asymptotic spheres:

$$g = \pm\frac{1}{e}\int \mathrm{d}^2\xi\sqrt{\det\left(\partial_\alpha\hat{\Phi}^a\partial_\beta\hat{\Phi}^a\right)}.$$

Where $\hat{\Phi}^a = \Phi^a/\sqrt{\Phi^b\Phi^b} = \Phi^a/v$ denotes the normalized field. A proof of this is presented in section A.1. Taking, for instance, the representative, normalized configuration of winding number q,

$$\hat{\Phi} = \sin\theta\cos q\varphi\hat{x} + \sin\theta\sin q\varphi\hat{y} + \cos\theta\hat{z}. \tag{3.2.12}$$

The Jacobian $J_{\alpha\beta} = \partial_\alpha \hat{\Phi}^a \partial_\beta \hat{\Phi}^a$ yields

$$\partial_\theta \Phi = \cos\theta \cos q\varphi \hat{x} + \cos\theta \sin q\varphi \hat{y} - \sin\theta \hat{z},$$
$$\partial_\varphi \Phi = -q \sin\theta \cos\varphi \hat{x} + q \sin\theta \cos q\varphi \hat{y}.$$

Their inner product gives $J_{\theta\theta} = 1$, $J_{\theta\varphi} = J_{\varphi\theta} = 0$, $J_{\varphi\varphi} = q^2 \sin^2\theta$, so

$$\det J_{\alpha\beta} = J_{\theta\theta} J_{\varphi\varphi} - J_{\theta\varphi} J_{\varphi\theta} = q^2 \sin^2\theta.$$

The magnetic charge of this configuration is thus

$$g = \pm \frac{1}{e} \int \mathrm{d}\theta \mathrm{d}\varphi \, q \sin\theta = \pm \frac{4\pi q}{e}.$$

Because $\hat{\Phi} : S^2_{\text{int}} \to S^2_\infty$ is a map from the unit sphere in internal space to the asymptotic sphere in physical space, q must be an integer, the topological invariant labeling the second homotopy group elements of S^2. This can also be interpreted as the second homotopy group of the homogeneous space $G/H = SO(3)/SO(2)$, that is,

$$\pi_2(SO(3)/SO(2)) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

This homogeneous space defines the vacuum manifold which will come to discuss shortly.

3.3 BPS EQUATIONS

The equations of motion (3.2.5) become easier to solve by completing the square of the Hamiltonian. Assuming a static, purely magnetic $A_0 = 0$ configuration its total mass is given by

$$\begin{aligned} M &= \int \mathrm{d}^3 x \left(\frac{1}{2} B^a_k B^a_k + D_k \Phi^a D_k \Phi^a + V(\Phi) \right), \\ &= \frac{1}{2} \int \mathrm{d}^3 x \, (B^a_k \mp D_k \Phi^a)^2 \pm \int \mathrm{d}^3 x \, B^a_k D_k \Phi^a + \int \mathrm{d}^3 x \, V(\Phi). \end{aligned}$$

Integrating by parts and using $D_k B^a_k = 0$, the second terms gives

$$M_0 = \int \mathrm{d}^3 x \, B^a_k D_k \Phi^a = \int \mathrm{d}^2 S_k \, B^a_k \Phi^a,$$

which depends only on the spacial asymptotic. Therefore, in the limit $\lambda \to 0$, the energy density in the bulk of space is minimized by fields satisfying either of

$$B^a_k = \pm D_k \Phi^a. \tag{3.3.1}$$

These are the *BPS equations*. Assuming the limits at $r \to \infty$ to be of the form

$$B^a_k(\theta, \varphi) = \frac{Q^a}{4\pi r^2} \hat{r}_k,$$
$$\Phi^a(\theta, \varphi) = \Phi^a_0,$$

for some generator $Q \in \mathrm{Lie}(G)$ called the non–Abelian magnetic charge, and some constant vacuum state Φ_0 which we may assume to lie in the same direction as Q. The total mass is bounded bellow by

$$M_0 = \pm \int \mathrm{d}^2 S_k \, B^a_k \Phi^a = \pm Q^a \Phi^a_0.$$

Depending on the sign chosen for (3.3.1). We arrive at the bound

$$M \geq v|g|.$$

Where $g = \mathrm{tr}(Q\hat{\Phi}_0)$ stands for the abelian projection of Q onto Φ_0. This lower bound means that the minimum energy achievable at any given sector g is $M = v|g|$ being achieved precisely when the BPS conditions (3.3.1) are satisfied everywhere in space.

3.4 VACUUM MANIFOLD

We've encountered scenarios where $G = U(1)$ for the Dirac monopole and $G = SO(3)$ for the 't Hooft–Polyakov monopole. We'd like to generalize the previous procedure to an arbitrary symmetry group G acting on a scalar field $|\phi\rangle$ in a likewise arbitrary representation. Let's start from the basic assumption that any physical field must reach some minimum energy state $V(\Phi) = D^\mu \Phi = 0$ at points far enough from the origin in such a way that its total energy remains finite. One defines the *vacuum manifold* given by such states (GODDARD; D. I. OLIVE, 1978)

$$\mathcal{V} = \{\Phi \in W : V(\Phi) = 0\}.$$

Here W stands for the state space of the gauge group G in some representation R. The self–interaction $V(\Phi)$ being gauge invariant implies the group action of G on every $\Phi \in \mathcal{V}$ remains in $\mathcal{V}$. Moreover when each point $\Phi_1 \in \mathcal{V}$ can be reached by any other $\Phi_2 \in \mathcal{V}$ through some action $g \in G$, that is,

$$\forall \Phi_1, \Phi_2 \in G, \; \exists g_{12} \in G : \quad g_{12} \cdot \Phi_2 = \Phi_1.$$

G is said to act *transitively* on $\mathcal{V}$, or equivalently $\mathcal{V}$ is said *homogeneous* for G. Likewise, for each point in the vacuum, the set of orbits $G \cdot \Phi = \{g \cdot \Phi : R(g) \in G\}$, sweeps out all of $\mathcal{V}$.

By fixing a marked vacuum state Φ_0 we may therefore label each point $\Phi \in \mathcal{V}$ by the appropriate action $g \in G$ which sends Φ_0 to Φ. This becomes a one–to–one correspondence provided we quotient G by those elements which act trivially on our initial choice, i.e. the elements which fix Φ_0. Their set define the *unbroken gauge group* also known as the *stabilizer subgroup*

$$H(\Phi_0) = \{h \in G : h \cdot \Phi_0 = \Phi_0\}.$$

Thus the Higgs vacuum is isomorphic to the space of right cosets of $H(\Phi_0)$ in G:

$$\mathcal{V} \cong G/H(\Phi_0). \tag{3.4.1}$$

Topological solutions in 3–space require a nontrivial asymptotic map $\hat{\Phi} : \mathcal{V} \to S^2_\infty$, this translates to a nontrivial second homotopy group $\pi_2(\mathcal{V})$. By virtue of (3.4.1) and (2.2.2) we have

$$\pi_2(\mathcal{V}) \cong \pi_2(G/H(\Phi_0)) \cong \pi_1(H(\Phi_0)).$$

For instance when the Higgs field belongs to the adjoint representation of $G = SU(n)$, depending on the multiplicities n_i of eigenvalues of Φ_0 (HORVATHY; RAWNSLEY, 1985) the unbroken group yields [2]

$$H(\Phi_0) = S\left(U(n_1) \times \cdots \times U(n_p)\right).$$

For the special case where all eigenvalues are distinct from one another and zero, known as *maximal symmetry breakdown*, this becomes $S\left(U(1) \times \cdots \times U(1)\right) \cong U(1)^{n-1}$ and the second homotopy group of the Higgs vacuum is given by

$$\pi_2(\mathcal{V}) \cong \pi_1\left(U(1)^{n-1}\right) \cong \mathbb{Z}^{n-1}.$$

That is, each field configuration is labeled by a set of $n-1$ integers. On the other hand in, if the Higgs field belongs to a symmetric $n \times n$ representation of $G = SU(n)$, $n \geq 4$, all its eigenvalues being identical, then

$$\pi_2(\mathcal{V}) \cong \pi_1\left(\frac{Spin(n)}{\mathbb{Z}_2}\right) \cong \mathbb{Z}_2.$$

This is the case approached by this thesis.

3.5 ASYMPTOTIC EQUIVALENCE

From a distance 't Hooft–Polyakov monopoles look like Dirac monopoles. The reason for this is that in the spacial asymptotic there is a gauge transformations which trivialize the scalar field while simultaneously introducing a string singularity to the gauge field.

Let $E = su(2)$ be a subalgebra of the gauge symmetry algebra $\mathfrak{g}$ and T_i a set of Hermitian generators for E such that $T_3 \in \mathfrak{h}$ the unbroken algebra while $T_1, T_2 \notin \mathfrak{h}$. Then define the asymptotic gauge transformation in terms of Euler angles θ and φ and winding number $q \in \mathbb{Z}$: (ARAFUNE; FREUND; GOEBEL, 1975), (WEINBERG; LONDON; ROSNER, 1984).

$$U(\theta, \varphi) = \exp(-iq\varphi T_3)\exp(-i\theta T_2)\exp(iq\varphi T_3). \tag{3.5.1}$$

[2] S stands for the requirement that only the total determinant be one, i.e. $\det(U_1 \ldots U_p) = 1$

This transformation is able to translate the Dirac configuration into a 't Hooft–Polyakov configuration. Note that for the winding number $q = 1$, U is singular at $\theta = \frac{\pi}{2}$. We start by choosing the adjoint Higgs field to be constant and gauge fields to asymptotically resemble the Dirac monopole potential.

$$\begin{aligned} \Phi_0 &= vT_3, \\ A_\mu(x) &= \frac{q}{er}\tan\left(\frac{\theta}{2}\right)\hat{\varphi}\; T_3. \end{aligned}$$

Because of this, this choice of fields will be referred to be in the *string gauge*, see Figure **??**.

Notice that because all fields point in the same internal direction, T_3, this configuration is effectively abelian:

$$\begin{aligned} F_1^{\mu\nu} &= F_2^{\mu\nu} = 0, \\ F_3^{\mu\nu} &= \partial^\mu A_3^\nu - \partial^\nu A_3^\mu - e\varepsilon_{3bc}A_b^\mu A_c^\nu = \partial^\mu A_3^\nu - \partial^\nu A_3^\mu, \\ (D^\mu\phi)_a &= \partial^\mu \phi_a - e\varepsilon_{abc}A_b^\mu \phi_c = 0. \end{aligned}$$

Now, by acting on Φ_0 with (3.5.1) we have

$$\Phi' = U\Phi_0 U^\dagger, \tag{3.5.2}$$

$$\Phi'(x) = v\exp(-iq\varphi T_3)\exp(-i\theta T_2)\, T_3\, \exp(i\theta T_2)\exp(i\varphi T_3). \tag{3.5.3}$$

Which we may determine by means of the identity

$$\exp(A)B\exp(-A) = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots + \frac{1}{n!}\overbrace{[A,\cdots[A}^{n},B]\cdots] + \cdots .$$

Yielding

$$\exp(-i\theta T_2)T_3\exp(i\theta T_2) = \sum_{k=0}^{\infty}\frac{(-i\theta)^{2k}}{(2k)!}T_3 + \sum_{k=0}^{\infty}\frac{(-i\theta)^{2k+1}}{(2k+1)!}iT_1 = \cos\theta T_3 + \sin\theta T_1.$$

Substituting back into (3.5.3) we get

$$\begin{aligned} \Phi(x) &= v\exp(-iq\phi T_3)(\cos\theta T_3 + \sin\theta T_1)\exp(iq\phi T_3), \\ &= v\cos\theta T_3 + \sin\theta(\cos q\varphi T_1 + \sin q\varphi T_2). \end{aligned}$$

This is the configuration considered in (3.2.12). Provided the gauge field is continuous outside the origin we conclude that this transformed field configuration carries charge $g = 4\pi q/e$. To make sure we must also transform the field

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{e}\partial_\mu U U^{-1}.$$

Because $A_0 = \partial_0 U = 0$ we have

$$\mathbf{A}' = U\mathbf{A}U^{-1} - \frac{i}{e}\boldsymbol{\nabla} U U^{-1}.$$

In spherical coordinates the gradient reads

$$\boldsymbol{\nabla} U = \partial_r U \hat{r} + \frac{1}{r}\partial_\theta U \hat{\theta} + \frac{1}{r\sin\theta}\partial_\varphi U \hat{\varphi}.$$

In the convention where $\hat{r}^2 = \hat{\theta}^2 = \hat{\varphi}^2 = 1$.

$$\begin{aligned}
\partial_r U &= 0, \\
\partial_\theta U &= i\exp(-iq\varphi T_3) T_2 \exp(i\theta T_2)\exp(iq\varphi T_3), \\
\partial_\theta U U^{-1} &= i\exp(-iq\varphi T_3) T_2 \exp(iq\varphi T_3), \\
&= i(\cos q\varphi T_2 + \sin q\varphi T_1), \\
\partial_\varphi U &= \partial_\varphi \left(\exp(-iq\varphi T_3)\exp(i\theta T_2)\exp(iq\varphi T_3)\right), \\
&= -iq(T_3 U - U T_3), \\
\partial_\varphi U U^{-1} &= -iq(T_3 - U T_3 U^{-1}).
\end{aligned}$$

Substituting back in the expressions for $\mathbf{A}$:

$$\begin{aligned}
A'_t &= A'_r = 0, \\
A'_\theta &= -\frac{i}{er}\partial_\theta U U^{-1} = \frac{1}{er}(\cos q\varphi T_2 + \sin q\varphi T_1), \\
A'_\varphi &= U A_\varphi U^{-1} - \frac{i}{e}\partial_\varphi U U^{-1}, \\
&= -\frac{q}{er}\frac{1-\cos\theta}{\sin\theta} U T_3 U^{-1} - \frac{iq}{er\sin\theta}(iT_3 - iUT_3U^{-1}), \\
&= \frac{q}{er\sin\theta}[T_3 - \cos\theta U T_3 U^{-1}], \\
&= \frac{q}{er\sin\theta}[T_3 - \cos^2\theta T_3 - \cos\theta\sin\theta\cos q\varphi T_1 - \cos\theta\sin\theta\sin q\varphi T_2], \\
&= \frac{q}{er}[\sin\theta T_3 - \cos\theta\cos q\varphi T_1 + \cos\theta\sin q\varphi T_2].
\end{aligned}$$

This field is manifestly continuous on the surface of any sphere containing the origin. Hence the Higgs field Φ is the only source of magnetic four–current in this gauge. We might call it the *smooth gauge* and we will be working on it through out the whole thesis. Furthermore notice that by applying an additional global transformation

$$U = \exp(i\chi T_3),$$

one achieves a one parameter family of solutions

$$\Phi(x) = v\cos\theta T_3 + \sin\theta(\cos q\varphi T'_1 + \sin q\varphi T'_2),$$

where

$$T'_1 = \cos\chi T_1 + \sin\chi T_2, \qquad T'_2 = \cos\chi T_2 - \sin\chi T_1.$$

In the 't Hooft–Polyakov monopole, $\mathfrak{g} = su(2)$, this is the only free parameter, or internal degree of freedom, available.

3.6 MODULI SPACE

The unit charge monopole standing at the origin is only one of the possible solutions for the equations of motion (3.2.5). Other possibilities arise when one considers spacial translations, boosts as well as a time–dependent phases. In order to encompass the broadest set of solutions those variations just mentioned are labeled by a set of what are called *collective coordinates* q^i. Given a topological class n, The manifold $\mathcal{M}_n$ swept out by those coordinates is called the *moduli space* of class n. It is possible to determine its dimension by slightly varying fields (Φ, A^μ) which we know to be solutions: $(\Phi, A^\mu) \to (\Phi + \delta\Phi, A^\mu + \delta A^\mu)$, and requiring that it remains a solution. This means that the variation $(\delta\Phi, \delta A^\mu)$ must satisfy linearized BPS equations.

We must also decide in which directions those deformed fields result in physically distinct solutions. To do this one must factor out deformations arising from small (local) gauge transformations, since those are not physical. Given two variations $(\delta_1\Phi, \delta_1 A^\mu)$ and $(\delta_2\Phi, \delta_2 A^\mu)$ one defines their inner product by

$$\langle(\delta_1\Phi, \delta_1 A^\mu), (\delta_2\Phi, \delta_2 A^\mu)\rangle = \int \mathrm{d}^3x \ \mathrm{tr}\left(\delta_1\Phi\delta_2\Phi + \delta_1 A^\mu \delta_2 A_\mu\right).$$

The requirement that zero modes must be orthogonal to gauge transformations leads to

$$D_i\delta A_i + ie[\Phi, \delta\Phi] = 0. \tag{3.6.1}$$

Which are called the *background gauge conditions*.

3.6.1 Dyon

Aside for spatial translations the last zero mode remaining for the 't Hooft–Polyakov monopole is an internal degree of freedom generated by $H(\Phi_0) \cong U(1)$ as previously discussed. This allows for a solution which, besides carrying magnetic charge also carries an electric charge, called a *Dyon* (JULIA; ZEE, 1975, 8). One way of constructing dyons is by applying a time–dependent gauge transformation

$$U(t) = \exp\left(\frac{ie}{v}\chi(t)\Phi\right),$$

to the asymptotic fields of a purely magnetic monopole. The angular velocity $\omega = \dot{\chi}$ gives rise to a nonzero electric potential A_0 and consequently a proportional electric charge. This choice of time dependence is such that the background gauge condition (3.6.1) remains satisfied by this perturbation. Also, because this degree of freedom is angular, the moduli space of charge one in this model yields a manifold isometric to

$$\mathcal{M}_1 \cong \mathbb{R}^3 \times S^1,$$

where each point in $\mathcal{M}_1$ corresponds to a static solution centered at some point $x \in \mathbb{R}^3$ with phase $\chi \in S^1$. Low–energy dynamics further supply a curvature for $\mathcal{M}_1$ in such a way that geodesics of this manifold describe dyons of constant internal angular velocity traveling at a constant speed in space.

When the unbroken gauge group H is also non-Abelian, it seems that the monopole can transform under new generators of the corresponding non-Abelian global symmetry. One might therefore suspect that additional collective coordinates could be introduced in a similar fashion to the dyonic phase. This turns out to not to be the case because it is not always possible to find zero modes satisfying the linearized field equations while simultaneously preserving the background gauge conditions (DOREY et al., 1996). This thesis is nevertheless concerned in labeling the different families of monopoles in $\mathfrak{g} = su(4)$ without regard to whether or not these internal parameters give rise to collective coordinates.

4 CONTINUOUS FAMILIES OF BPS $\mathbb{Z}_2$ MONOPOLES

4.1 SU(N)→ SO(N)

As an extension to the work done in (KNEIPP; LIEBGOTT, 2010), we will analyze $\mathbb{Z}_2$ monopole solutions obtained in a Yang–Mills–Higgs theory having gauge symmetry $\mathfrak{g} = \mathfrak{su}(n)$ broken by a scalar field $|\phi\rangle$ in the symmetric $D \otimes D$ representation[1]. One defines the *Cartan automorphism* of $\mathfrak{g}$ by the homomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that

$$\sigma(H_a) = -H_a,$$
$$\sigma(E_\alpha) = -E_{-\alpha}.$$

The unbroken subalgebra is then defined by the invariants of σ, namely the subalgebra

$$\mathfrak{h}(\sigma) = \{E_\alpha - E_{-\alpha} \mid \alpha \in \Phi^+\},$$

α simple roots of $\mathfrak{g}$. We choose

$$|\phi\rangle_0 = \frac{v}{\sqrt{2}} \sum_{k=1}^{n} |kk\rangle \,, \tag{4.1.1}$$

as our marked point in some unspecified connected vacuum manifold $\mathcal{V}$ which we take to be homogeneous for G. One then checks that the invariants of σ annihilate the vacuum and form an $\mathfrak{so}(n)$ subgroup of $\mathfrak{g}$, $\mathfrak{h}(\sigma) = \mathfrak{h}(|\phi\rangle_0) = \mathfrak{so}(n)$. The second homotopy group of $\mathcal{V}$ has been determined by the general method

$$\pi_2(G/H) \cong \pi_1(H) \cong \pi_1\left(\frac{\tilde{H}}{\ker R}\right) \cong \ker R,$$

where $\ker R$ stands for the kernel of the representation of $\mathfrak{h}$ which is a finite subgroup of $\mathcal{Z}(Spin(n))$, the center of $Spin(n)$. For $n \geq 5$ it was shown that

$$\pi_2(G/H) \cong \pi_2\left(\frac{SU(n)}{Spin(n)/\mathbb{Z}_2}\right) \cong \mathbb{Z}_2.$$

And the elements of $\ker R$ may be written explicitly as

$$\ker R \cong \mathbb{Z}_2 \cong \{\exp(2\pi i \alpha^\vee \cdot h),\ \exp(2\pi i(\lambda_1^\vee + \alpha^\vee) \cdot h)\},$$

here $\alpha^\vee$ and $\lambda_1^\vee$ stand for co–roots and first fundamental co–weight of $\mathfrak{g}$.

In order to obtain the asymptotic form of the Higgs field, one applies the asymptotic gauge transformation (3.5.1) considered before

$$U(\theta, \varphi) = \exp(-iq\varphi T_3)\exp(-i\theta T_2)\exp(iq\varphi T_3),$$

to the vacuum state (4.9.1). Notice, however, that now we have as many choices for generators T_i as $su(2)$ embeddings in $su(n)$. This prompts us to label different monopole configurations by the embeddings from which they are generated.

[1] D stands for the defining, n–dimensional state space representation

4.2 SU(4)→ SO(4)

In this thesis we will consider a Higgs field $|\phi\rangle$ in the representation space of G, with algebra $\mathrm{Lie}(G) = su(4)$, in the symmetric $D \otimes D$ representation as before. We choose

$$|\phi\rangle_0 = \frac{v}{\sqrt{2}} \sum_{k=1}^{4} |kk\rangle \, .$$

The algebra for the unbroken gauge group yields $\mathfrak{h}(|\phi\rangle_0) \cong so(4)$. This is because, for every generator

$$M_{ij} = -i(E_{ij} - E_{ji}) \in so(4),$$

we have

$$\begin{aligned}
D \otimes D(M_{ij}) &= M_{ij} \otimes I + I \otimes M_{ij}, \\
D \otimes D(M_{ij}) \, |\phi\rangle &= \frac{v}{\sqrt{2}} \sum_{k=1}^{4} (M_{ij} \otimes I + I \otimes M_{ij}) \, |kk\rangle \, , \\
&= -\frac{iv}{\sqrt{2}} \sum_{k=1}^{4} (\delta_{jk} \, |i,k\rangle + \delta_{jk} \, |k,i\rangle - \delta_{ik} \, |j,k\rangle - \delta_{ik} \, |k,j\rangle), \\
&= -\frac{iv}{\sqrt{2}} (|i,j\rangle + |j,i\rangle - |i,j\rangle - |j,i\rangle) = 0,
\end{aligned}$$

proving $so(4) \subset \mathfrak{h}(|\phi\rangle_0)$. Conversely any generator $X \in su(4)$ which annihilates this vaccuum state must be antisymmetric:

$$\begin{aligned}
0 &= D \otimes D(X) \, |\phi_0\rangle \, , \\
&= \frac{1}{\sqrt{2}} \sum_{k=4}^{n} (X \otimes I + I \otimes X) \, |k\rangle \, |k\rangle \, , \\
&= \frac{1}{\sqrt{2}} \sum_{ijk} (|i\rangle \, X_{ij} \, \langle j|k\rangle \, |k\rangle + |k\rangle \, |i\rangle \, X_{ij} \, \langle j|k\rangle), \\
&= \frac{1}{\sqrt{2}} \sum_{ij} X_{ij} (|ij\rangle + |ji\rangle), \\
&= \frac{1}{\sqrt{2}} \sum_{ij} (X_{ij} + X_{ji}) \, |i,j\rangle \, .
\end{aligned}$$

Since $|i,j\rangle$ are linearly independent we get $X_{ij} = -X_{ji}$ and X generates elements of $so(4)$ as proposed.

Nevertheless in order to figure out the group $H(|\phi\rangle_0)$ one must analyze how the representation $\mathrm{Sym}^2(D)$ of $su(4)$ branches to, or induces, a representation R' restricted to $so(4)$. This is not straightforward thus, in order to determine the second homotopy group of the vacuum manifold we use the following result

Given two representations R_1 and R_2 of $\mathfrak{g}$ their respective restrictions R_1' and R_2' of $\mathfrak{h}$ we have

$$\pi_2(G_1/H_1) \cong \pi_2(G_2/H_2).$$

Where the groups G_1, G_2 and subgroups $H_1 \subset G_1$, $H_2 \subset G_2$ are defined as the images of R_1, R_2 and R_1', R_2' respectively.

Knowing this we can choose the defining, four–dimensional, representation as $R_2 = D$ and check how this one branches to $so(4)$. By adopting the base

$$\Sigma_{ab} = \frac{1}{2}\sigma_a \otimes \sigma_b \in su(4),$$

$a, b = 0, 1, 2, 3$; $\sigma_0 = I$. Excluding $a = b = 0$ so that they remain traceless, we achieve a complete base of fifteen linearly independent generators for $SU(4)$

Their commutation relations read

$$\begin{aligned}[\Sigma_{ab}, \Sigma_{cd}] =& i\varepsilon_{bde}(\delta_{ac}\Sigma_{0e} + \delta_{0a}\Sigma_{ce} + \delta_{c0}\Sigma_{ae})+ && (4.2.1)\\ &+i\varepsilon_{ace}(\delta_{bd}\Sigma_{0e} + \delta_{0b}\Sigma_{de} + \delta_{d0}\Sigma_{be}). && (4.2.2)\end{aligned}$$

Out of these elements, six represent generators of $\mathfrak{h}(\langle\phi\rangle_0) = so(4)$, the antisymmetric ones, namely,

$$so(4) = \mathrm{span}\{\Sigma_{ab} \mid a = 2, b \neq 2 \text{ or } a \neq 2, b = 2\}.$$

Because these represent the generators of isoclinic rotations

$$\begin{aligned}
M_1^L &= \frac{1}{2}(M_{12} + M_{34}) = \Sigma_{02}, & M_1^R &= \frac{1}{2}(M_{12} - M_{34}) = \Sigma_{32},\\
M_2^L &= \frac{1}{2}(M_{13} - M_{24}) = \Sigma_{23}, & M_2^R &= \frac{1}{2}(M_{13} + M_{24}) = \Sigma_{20},\\
M_3^L &= \frac{1}{2}(M_{14} + M_{23}) = \Sigma_{21}, & M_3^R &= \frac{1}{2}(M_{14} - M_{23}) = \Sigma_{12}.
\end{aligned}$$

We find that the restricted representation R_2' yields the defining representation of $so(4)$, that is, $H_2 = SO(4)$. Applying the fact that $SU(4)$ is simply connected we find

$$\begin{aligned}
\pi_2(\mathcal{V}) &\cong \pi_2(G/H),\\
&\cong \pi_2(G_2/H_2),\\
&\cong \pi_2(SU(4)/SO(4)),\\
&\cong \pi_1(SO(4)),\\
&\cong \pi_1((SU(2) \times SU(2))/\mathbb{Z}_2),\\
&\cong \pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2.
\end{aligned}$$

We conclude that in this model there are only two homotopy classes for the Higgs field

$$\pi_2(\mathcal{V}) \cong \mathbb{Z}_2.$$

In order to construct them explicitly we start off by embedding $su(2)$ subalgebras in G, as was done in (KNEIPP; LIEBGOTT, 2010). Among its generators we would like for one of them, namely T_3, to be in the broken subalgebra $\mathfrak{h} = so(n)$, while $T_1, T_2 \notin so(n)$. To do this it is easier to work with generators in the defining representation D. We may do this with no loss of generality as we are working at the level of algebras and it is straightforward to go from generators of D to $\mathrm{Sym}^2(D)$:

$$D \otimes D(X) = I \otimes D(X) + D(X) \otimes I,$$

for all $X \in su(n)$.

4.3 SETTING THE BASIS

We will focus on the even dimensional case $n = 2m$. In the $2m$–dimensional representation, D, the lie algebra generators for $\mathfrak{g} = su(2m)$ and $\mathfrak{h} = so(2m)$ may all be written in terms of the canonical basis $E_{ij} = |i\rangle\langle j|$, $i = 1, \ldots, 2m$, where $|i\rangle$, $\langle j|$, are the weight and coweight states, respectively, of D. The canonical basis for $\mathfrak{h} = so(2m)$ is given by the antisymmetric generators

$$M_{ij} = -i(E_{ij} - E_{ij}).$$

Among those are the block diagonal ones $h_a = M_{2a-1,2a}$ with $a = 1, \ldots, m = \mathrm{rank}(h)$. Notice that these are all mutually commuting and so may be chosen as the generators for the Cartan subalgebra $\mathrm{CSA}(\mathfrak{h}) = \mathrm{span}\{h_1, \ldots, h_m\}$. More generally we will employ indices $a, b, c, \ldots$ to run from 1 to m, whereas $i, j, k, \ldots$ run from 1 to $2m$. Like the construction of (WEINBERG; LONDON; ROSNER, 1984) we pick out generators T_i of an $su(2)$ subalgebras. Among these generators we would like for one of them, namely T_3, to be in the broken subalgebra $\mathfrak{h} = so(n)$, while $T_1, T_2 \notin so(n)$. In particular we take, without loss of generality, $T_3 \in \mathrm{CSA}(\mathfrak{g})$,

$$T_3 = \frac{1}{2}\,\beta_a h_a. \tag{4.3.1}$$

Here[2] $\beta = \beta_a e_a$ stands for a root in the dual vector space $\mathrm{CSA}(\mathfrak{h})^*$. It is also known as a *magnetic weight*. The non–Abelian quantization condition $\exp(4\pi i T^3) = I$ implies

$$\exp(2\pi i \beta_a h_a) = \prod_{a=1}^{m} (\cos(2\pi\beta_a) + i\sin(2\pi\beta_a) h_a) = I,$$

since h_a are all independent we conclude $\beta_a \in \mathbb{Z}$ must be integer coefficients. In order to classify all embeddings it proves useful to factor h_a as the following tensor product [3]

$$h_a = \mathrm{diag}(e_a) \otimes \sigma_2 = -i\mathrm{diag}(e_a) \otimes J,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore (4.3.1) may be rewritten as

$$T_3 = \frac{1}{2i}\mathrm{diag}(\beta) \otimes J.$$

[2] Where $e_a = (0, \ldots, 1, \ldots 0)$, the a–th canonical vector

[3] We'll make use of the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Take $m = 2$ for instance, here we have

$$T_3 = \frac{1}{2i}\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} \otimes J = \frac{1}{2i}\begin{pmatrix}\beta_1 J & 0\\ 0 & \beta_2 J\end{pmatrix},$$
$$= \frac{1}{2i}\begin{pmatrix}0 & \beta_1 & 0 & 0\\ -\beta_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta_2\\ 0 & 0 & -\beta_2 & 0\end{pmatrix}.$$

We may only factor T_3 like this in even dimensional cases. Even though it is possible to embed solutions found here in $su(2m+1)$ by simply disregarding the last row and column of each generator, we are not able to describe the most general set of solutions this way. Therefore we will only treat the even dimensional cases.

4.4 SOLVING FOR $su(2)$ SUBALGEBRAS

To find a generic $su(2)$ subalgebra we start from the system of equations

$$[T_1, T_2] = iT_3,$$
$$[T_2, T_3] = iT_1,$$
$$[T_3, T_1] = iT_2,$$

and eliminate T_2 from the first and second equations. After rearranging a few terms we arrive at the second order system

$$[T_3, [T_3, T_1]] = T_1, \tag{4.4.1}$$
$$[[T_3, T_1], T_1] = T_3, \tag{4.4.2}$$

which we wish to solve for T_1 being given

$$T_3 = \frac{1}{2i}\mathrm{diag}(\beta) \otimes J, \tag{4.4.3}$$

$\beta_a \in \mathbb{Z}$ to be determined. To do this we must also write T_1 in terms of 2×2 blocks, namely x_{ab},

$$T_1 = \frac{1}{2}\sum_{a,b=1}^{m} e_{ab} \otimes x_{ab},$$

where e_{ab} stands for the canonical basis for the $m \times m$ matrices. Notice that T_1 being Hermitian translates to

$$x_{ab}^{\dagger} = x_{ba}. \tag{4.4.4}$$

Therefore we have $m(m+1)/2$ independent blocks to solve for. Substituting these definitions we have

$$[T_3, T_1] = \frac{1}{4i}\sum_{ab} \mathrm{diag}(\beta)e_{ab} \otimes Jx_{ab} - e_{ab}\mathrm{diag}(\beta) \otimes x_{ab}J, \tag{4.4.5}$$

$$= \frac{1}{4i}\sum_{ab} e_{ab} \otimes \beta_a Jx_{ab} - e_{ab} \otimes \beta_b x_{ab}J, \tag{4.4.6}$$

$$= \frac{1}{4i}\sum_{ab} e_{ab} \otimes (\beta_a Jx_{ab} - \beta_b x_{ab}J). \tag{4.4.7}$$

Now we analyze equations (4.4.1) and (4.4.2) separately.

4.4.1 Determining T_3

Upon substitution of (4.4.3) and (4.4.7), equation (4.4.1) becomes

$$\begin{aligned} T_1 = [T_3, [T_3, T_1]] = -\frac{1}{8}\sum_{ab} &(\mathrm{diag}(\beta) \otimes J)(e_{ab} \otimes (Jx_{ab} - \beta_b x_{ab}J)) + \\ &- (e_{ab} \otimes (\beta_a Jx_{ab} - \beta_b x_{ab}J))(\mathrm{diag}(\beta) \otimes J), \\ = -\frac{1}{8}\sum_{ab} &e_{ab} \otimes (-\beta_a^2 x_{ab} - 2\beta_a\beta_b Jx_{ab}J - \beta_b^2 x_{ab}). \end{aligned}$$

Because e_{ab} are linearly independent we get, for each a, b,

$$x_{ab} = \frac{1}{4}(\beta_a^2 x_{ab} + 2\beta_a\beta_b Jx_{ab}J + \beta_b^2 x_{ab}), \tag{4.4.8}$$

$$(4 - \beta_a^2 - \beta_b^2)x_{ab} = 2\beta_a\beta_b Jx_{ab}J. \tag{4.4.9}$$

The following simple proposition enables us to further constrain the integers β_a.

Proposition: Let x be a complex–valued 2×2 matrix and J as before. Given

$$JxJ = kx, \tag{4.4.10}$$

for some $k \in \mathbb{C}$, then either $k = \pm 1$ or $x = 0$ identically.

Proof: Multiplying both sides of this equation by J on the left and right, using $J^2 = -1$, we get $x = kJxJ$. Now, apply (4.4.10) to the right–hand side yielding $x = k^2 x$. So $(1 - k^2)x = 0$, meaning that if some component in x is nonzero then $k^2 = 1$, otherwise $x = 0$. The result follows.

For now we shall assume both $\beta_a, \beta_b \neq 0$ for some pair a, b. Using this, equation (4.4.9) becomes:

$$Jx_{ab}J = \frac{4 - \beta_a^2 - \beta_b^2}{2\beta_a\beta_b} x_{ab}. \tag{4.4.11}$$

So for each block x_{ab} that we further assume to be non vanishing the previous proposition yields

$$\frac{4 - \beta_a^2 - \beta_b^2}{2\beta_a\beta_b} = k_{ab},$$

for some sign $k_{ab} = \pm 1$. It is worth pointing out that $k_{ba} = k_{ab}$ by virtue of the symmetry of this expression. Following through with this equation we can complete squares to get

$$4 - \beta_a^2 - \beta_b^2 = 2k_{ab}\beta_a\beta_b, \tag{4.4.12}$$
$$4 = \beta_a^2 + 2k_{ab}\beta_a\beta_b + \beta_b^2, \tag{4.4.13}$$
$$4 = (\beta_a + k_{ab}\beta_b)^2, \tag{4.4.14}$$
$$2l_{ab} = \beta_a + k_{ab}\beta_b, \tag{4.4.15}$$

which introduces a new set of signs $l_{ab} = \pm 1$. The previous set k_{ab} can now be inferred solely in terms of the new set. To do this, exchange a and b in (4.4.15) and multiply by k_{ba}:

$$2l_{ba} = \beta_b + k_{ba}\beta_a,$$
$$2k_{ba}l_{ba} = k_{ba}\beta_b + \beta_a.$$

Subtract this from (4.4.15) yielding

$$2(l_{ab} - k_{ba}l_{ab}) = (k_{ab} - k_{ba})\beta_b, \tag{4.4.16}$$
$$l_{ab} - k_{ba}l_{ba} = 0, \tag{4.4.17}$$
$$k_{ab} = l_{ab}l_{ba}. \tag{4.4.18}$$

The dependence on β_b vanishes because k_{ab} is symmetric, and our goal was achieved. One can therefore restate (4.4.15) as the symmetric expression

$$2 = l_{ab}\beta_a + l_{ba}\beta_b. \tag{4.4.19}$$

In summary, for each pair $\beta_a, \beta_b \neq 0$ and block $x_{ab} \neq 0$, condition (4.4.2) only holds when equation (4.4.19) is satisfied by a pair of signs labeled l_{ab} and l_{ba}.

If, however, some $\beta_a = 0$, take $\beta_2 = 0$ for definiteness, and $\beta_1 \neq 0$. We then refer back to (4.4.9) to find three independent conditions

$$(4 - 2\beta_1^2)x_{11} = 2\beta_1^2 J x_{11} J,$$
$$(4 - \beta_1^2)x_{12} = 0,$$
$$4x_{22} = 0.$$

The general procedure undertaken was to assume T_1 mostly zero and then to gradually constrain β by setting blocks $x_{ab} \neq 0$. We may consider, for instance, $x_{11} = 0$, $x_{12} \neq 0$. This turns out not to lead to a solution under further scrutiny. Taking $x_{11} \neq 0$ and $x_{12} = 0$ does, on the other hand, lead to a solution, called the *fundamental embedding*. In order to describe it we must first see what conditions the blocks x_{ab} must satisfy.

4.4.2 Determining T_1

We turn to equation (4.4.2):

$$\begin{aligned}
[[T_3, T_1], T_1] &= \frac{1}{8i} \sum_{abcd} (e_{ab} \otimes (\beta_a J x_{ab} - \beta_b x_{ab} J))(e_{cd} \otimes x_{cd}) - (e_{cd} \otimes x_{cd})(e_{ab} \otimes (\beta_a J x_{ab} - \beta_b x_{ab} J)), \\
&= \frac{1}{8i} \sum_{abcd} \delta_{bc} e_{ad} \otimes (\beta_a J x_{ab} x_{cd} - \beta_b x_{ab} J x_{cd}) - \delta_{da} e_{cb} \otimes (\beta_a x_{cd} J x_{ab} - \beta_b x_{cd} x_{ab} J), \\
&= \frac{1}{8i} \sum_{abc} e_{ab} \otimes (\beta_a J x_{ac} x_{cb} - \beta_c x_{ac} J x_{cb} - \beta_c x_{ac} J x_{cb} + \beta_b x_{ac} x_{cb} J).
\end{aligned}$$

Comparing this to T_3

$$T_3 = \frac{1}{2i} \sum_{ab} \beta_a \delta_{ab} e_{ab} \otimes J,$$

blockwise we have

$$4\beta_a \delta_{ab} J = \sum_c \beta_a J x_{ac} x_{cb} - 2\beta_c x_{ac} J x_{cb} + \beta_b x_{ac} x_{cb} J.$$

Multiplying on the right by $-J$

$$\begin{aligned}
4\beta_a \delta_{ab} &= \sum_c \beta_a J x_{ac} (-JJ) x_{cb} (-J) + 2\beta_c x_{ac} J x_{cb} J + \beta_b x_{ac} x_{cb} J(-J), \\
&= \sum_c (\beta_a k_{ac} k_{cb} + 2\beta_c k_{cb} + \beta_b) x_{ac} x_{cb}.
\end{aligned}$$

Here we used (4.4.11), so we should only consider the sum going over indices c for which neither x_{ac} nor x_{cb} are zero. Notice, however, that if we did sum over them this expression would remain true, with the caveat of neither k_{ac} or k_{cb} being previously defined. We shall adopt, nonetheless, the more general sum for the sake of brevity.

By virtue of (4.4.15) we can further simplify things by factoring out k_{cb}

$$\begin{aligned}
4\beta_a \delta_{ab} &= \sum_c k_{cb} (\beta_a k_{ac} + \beta_c + k_{cb} \beta_b + \beta_c) x_{ac} x_{cb}, \\
&= \sum_c k_{cb} (2l_{ac} + 2l_{bc}) x_{ac} x_{cb}.
\end{aligned}$$

Which in turn, because of $k_{ab} = l_{ab} l_{ba}$, may be restated as

$$2\beta_a \delta_{ab} = \sum_c (l_{ac} l_{cb} l_{bc} + l_{cb}) x_{ac} x_{cb}. \tag{4.4.20}$$

If we instead chose to multiply by $(-J)$ on the left we would have arrived at a similar equation

$$2\beta_a \delta_{ab} = \sum_c (l_{ca} + l_{ac} l_{ca} l_{bc}) x_{ac} x_{cb}. \tag{4.4.21}$$

Nevertheless, because $x_{ab}^\dagger = x_{ba}$, (4.4.20) and (4.4.21) are in fact equivalent.

Now, depending on which blocks we assume to be nonzero we may get different classes of solutions. The following definition will prove useful.

4.5 A NORM FOR BLOCKS

For every pair $\beta_a, \beta_b \neq 0$, we have

$$Jx_{ab}J = k_{ab}x_{ab},$$

$k_{ab} = \pm 1$ as before. So each block must be of the form

$$x_{ab} = \begin{pmatrix} u_{ab} & v_{ab}, \\ k_{ab}v_{ab} & -k_{ab}u_{ab} \end{pmatrix},$$

for some $u_{ab}, v_{ab} \in \mathbb{C}$. Because T_1 is Hermitian we get

$$x_{ab}x_{ba} = x_{ab}x_{ab}^{\dagger} = \begin{pmatrix} |u_{ab}|^2 + |v_{ab}|^2 & k_{ab}(u_{ab}v_{ab}^* - u_{ab}^*v_{ab}), \\ -k_{ab}(u_{ab}v_{ab}^* - u_{ab}^*v_{ab}) & |u_{ab}|^2 + |vab|^2 \end{pmatrix}.$$

Notice that the first entry satisfies the following norm properties

$$(x_{ab}x_{ba})_{11} \in \mathbb{R}_{\geq 0},$$
$$(x_{ab}x_{ba})_{11} = 0 \iff x_{ab} = 0.$$

So we may adopt the shorthand

$$||x_{ab}||^2 = (x_{ab}x_{ba})_{11}.$$

4.6 EXPLICIT EMBEDDINGS

4.6.1 Fundamental Embeddings

Assuming only $x_{11} \neq 0$ and $\beta_2 = 0$, equation (4.4.9) leads to

$$(2 - \beta_1^2)x_{11} = Jx_{11}J, \tag{4.6.1}$$
$$x_{11} = Jx_{11}J, \tag{4.6.2}$$

therefore $\beta_1 = \pm 1$ which, by (4.4.19), is labeled by $\beta_1 = l_{11}$. Substituting this in (4.4.20) yields

$$x_{11}x_{11} = 1. \tag{4.6.3}$$

Because of conditions (4.4.4) and (4.6.3) we have the form

$$x_{11} = \begin{pmatrix} u & v \\ v & -u \end{pmatrix} = v\sigma_1 + u\sigma_3. \tag{4.6.4}$$

Where $u, v \in \mathbb{R}$. But, because of (4.6.2), $u^2 + v^2 = 1$. Thus the general solution restricted to this block is given by

$$x_{11} = \cos\chi\sigma_1 + \sin\chi\sigma_3,$$

$\chi \in [0, 2\pi]$ is an arbitrary angle. By computing $T_2 = -i[T_3, T_1]$ we arrive at

$$T_1 = \frac{1}{2}\begin{pmatrix} \cos\chi\sigma_3 + \sin\chi\sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \pm\frac{1}{2}\begin{pmatrix} \cos\chi\sigma_1 - \sin\chi\sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \pm\frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.6.5}$$

$l_{11} = \pm 1$. Notice that we can rewrite

$$x_{11} = \exp\left(i\frac{\chi}{2}\sigma_2\right)\sigma_3 \exp\left(-i\frac{\chi}{2}\sigma_2\right).$$

So we may interpret (4.6.5) as the action of the element $U \in H$

$$U = \begin{pmatrix} \exp(i\chi\sigma_2/2) & 0 \\ 0 & I \end{pmatrix} = \exp\left(i\frac{\chi}{2}h_1\right) \in H,$$

over the standard embedding:

$$T_1 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_2 = \pm\frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_3 = \pm\frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This will be referred to as the $\beta = (\pm 1, 0)$ fundamental embedding. Likewise the $\beta = (0, \pm 1)$ fundamental embedding is defined by:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad T_2 = \pm\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad T_3 = \pm\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

4.6.2 Excluding One Possibility

Although no solutions exist in what follows, suppose for the sake of completeness that $x_{11} = x_{22} = 0$ and $\beta_2 = 0$. Then (4.4.9) for $a = b = 1$ and $a = b = 0$ are immediately satisfied. Setting $a = 1$, $b = 2$ yields

$$(4 - \beta_1^2)x_{12} = 0,$$

Since $x_{21} = x_{12}^\dagger$, for $T_1 \neq 0$ we require $\beta_1 = \pm 2$. Given these conditions

$$T_3 = \mp i\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & x_{12} \\ x_{12}^\dagger & 0 \end{pmatrix},$$

condition (4.4.2) nonetheless can never be fulfilled. The reason for this is that equation

$$[T_1, [T_1, T_3]] = \pm i\begin{pmatrix} [x_{12}x_{12}^\dagger, J] & 0 \\ 0 & 0 \end{pmatrix} = T_3,$$

translates to $[x_{12}x_{12}^\dagger, J] = -J$. But by taking the Hermitian of both sides: $[x_{12}x_{12}^\dagger, J] = +J$. We arrive at a contradiction, so there are no $\beta = (\pm 2, 0)$ or $\beta = (0, \pm 2)$ embeddings.

If, however, $\beta_2 \neq 0$ then solutions confined to x_{12} do exist, though they are part of a larger family of embeddings which we come to examine next.

4.6.3 Isoclinic Embedding

Starting from the assumption that all β_a and blocks x_{ab} are nonzero we find an embedding such that $\beta_1 = \pm 1$ and $\beta_2 = \pm 1$. Therefore T_3 generates some isoclinic rotation which motivates naming its corresponding embedding an isoclinic embedding. This comes from equations (4.4.9), for $a = b$, giving us $\beta_1 = l_{11}$ and $\beta_2 = l_{22}$. For $a \neq b$ we get

$$\begin{aligned} 2l_{12} &= l_{11} + k_{12} l_{22}, \\ 2l_{21} &= l_{22} + k_{12} l_{11}. \end{aligned}$$

Since k is symmetric. Focusing on the case $k_{12} = +1$ we get $l_{11} = l_{22} = l_{21} = l_{12}$. Substituting these in (4.4.15) yields

$$1 = x_{11}x_{11} + x_{12}x_{21}, \tag{4.6.6}$$

$$1 = x_{21}x_{12} + x_{22}x_{22}, \tag{4.6.7}$$

$$0 = x_{11}x_{12} + x_{22}x_{21}, \tag{4.6.8}$$

$$0 = x_{21}x_{11} + x_{22}x_{21}. \tag{4.6.9}$$

Using the fact that x_{11} and x_{22} are real and of the form (4.6.4)

$$x_{aa} = \begin{pmatrix} u_{aa} & v_{aa} \\ v_{aa} & -u_{aa} \end{pmatrix},$$

we get

$$x_{aa}x_{aa} = \begin{pmatrix} u_{aa}^2 + v_{aa}^2 & 0 \\ 0 & u_{aa}^2 + v_{aa}^2 \end{pmatrix} = \rho_a^2,$$

$\rho_a \in \mathbb{R}$. Writing x_{12} in a similar way:

$$x_{12} = \begin{pmatrix} u_{12} & v_{12} \\ v_{12} & -u_{12} \end{pmatrix}.$$

We get

$$x_{12}x_{21} = x_{12}x_{12}^\dagger = \begin{pmatrix} |u_{12}|^2 + |v_{12}|^2 & u_{12}v_{12}^* - u_{12}^* v_{12} \\ u_{12}^* v_{12} - u_{12} v_{12}^* & |u_{12}|^2 + |v_{12}|^2 \end{pmatrix},$$

by setting $\rho_{12}^2 = |u_{12}|^2 + |v_{12}|^2$, (4.6.6) becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho_{11}^2 + \rho_{12}^2 & u_{12}v_{12}^* - v_{12}^* v_{12} \\ u_{12}^* v_{12} - u_{12} v_{12}^* & \rho_{11}^2 + \rho_{12}^2 \end{pmatrix}.$$

Similarly for the second equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho_{22}^2 + \rho_{12}^2 & u_{12}^* v_{12} - u_{12} v_{12}^* \\ u_{12} v_{12}^* - u_{12}^* v_{12} & \rho_{22}^2 + \rho_{12}^2 \end{pmatrix}.$$

From the diagonal elements we can thus write

$$\rho_{11}^2 = \rho_{22}^2 = \cos^2\eta, \qquad \rho_{12}^2 = \sin^2\eta.$$

We gather that $\rho_{11} = \cos\eta$ absorbing the possible sign into the definition of η. Now $\rho_{22} = \pm\cos\eta$ and $\rho_{12} = \pm\sin\eta$. The off-diagonal elements are zero only if u_{12} and v_{12} have the same phase which we'll call ψ. From the definition of each ρ_{ab} we can define two new angles ζ and ψ. The last two equations (4.6.8),(4.6.9) provide us a coupling between angles:

$$0 = e^{i\psi}\sin\eta\begin{pmatrix}\cos(\chi_1-\zeta)-\cos(\zeta-\chi_2) & \sin(\chi_1-\zeta)-\sin(\zeta-\chi_2)\\ -\sin(\chi_1-\zeta)+\sin(\zeta-\chi_2) & \cos(\chi_1-\zeta)-\cos(\zeta-\chi_2)\end{pmatrix}.$$

This condition is only satisfied when $\zeta = \frac{1}{2}(\chi_1+\chi_2)$. We assume $l_{11} = 1$ with no loss of generality.[4] The general solution for $k_{12} = +1$ is therefore [5]

$$T_1 = \frac{1}{2}\begin{pmatrix}\cos\eta(\cos\chi_1\sigma_3+\sin\chi_1\sigma_1) & e^{i\psi}\sin\eta(\cos\zeta\sigma_3+\sin\zeta\sigma_1)\\ e^{-i\psi}\sin\eta(\cos\zeta\sigma_3+\sin\zeta\sigma_1) & \cos\eta(\cos\chi_2\sigma_3+\sin\chi_2\sigma_1)\end{pmatrix}, \tag{4.6.10}$$

$$T_2 = \frac{1}{2}\begin{pmatrix}\cos\eta(\cos\chi_1\sigma_1-\sin\chi_1\sigma_3) & e^{i\psi}\sin\eta(\cos\zeta\sigma_1-\sin\zeta\sigma_3)\\ e^{-i\psi}\sin\eta(\cos\zeta\sigma_1-\sin\zeta\sigma_3) & \cos\eta(\cos\chi_2\sigma_1-\sin\chi_2\sigma_3)\end{pmatrix}, \tag{4.6.11}$$

$$T_3 = \frac{1}{2}\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}. \tag{4.6.12}$$

A similar computation [6] for $k_{12} = -1$ yields

$$T_1 = \frac{1}{2}\begin{pmatrix}\cos\eta(\cos\chi_1\sigma_3+\sin\chi_1\sigma_1) & e^{i\psi}\sin\eta(\cos\xi\sigma_0+i\sin\xi\sigma_2)\\ e^{-i\psi}\sin\eta(\cos\xi\sigma_0-i\sin\xi\sigma_2) & \cos\eta(\cos\chi_2\sigma_3+\sin\chi_2\sigma_1)\end{pmatrix},$$

$$T_2 = \frac{1}{2}\begin{pmatrix}\cos\eta(\cos\chi_1\sigma_1-\sin\chi_1\sigma_3) & e^{i\psi}\sin\eta(-i\cos\xi\sigma_2+\sin\xi\sigma_0)\\ e^{-i\psi}\sin\eta(i\cos\xi\sigma_2+\sin\xi\sigma_0) & \cos\eta(-\cos\chi_2\sigma_1+\sin\chi_2\sigma_3)\end{pmatrix},$$

$$T_3 = \frac{1}{2}\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}.$$

Here $\xi = \frac{1}{2}(\chi_1-\chi_2)$. In both cases the requirement that neither T_1 nor T_2 annihilates the vacuum state implies $T_1^T \neq -T_1$ and $T_2^T \neq -T_2$. This in turn excludes the possibilities where $\eta = \pi/2$ and $\psi = \pi/2,\ 3\pi/2$, for all χ_1, χ_2.

4.6.4 Three–to–One Embedding

There is one nontrivial possibility remaining which is all $x_{ab} \neq 0$ except for a single diagonal block $x_{aa} = 0$. We'll take it to be the second one for definiteness. Equation (4.4.15) tells us that $\beta_1 = l_{11}$ and $x_{12} \neq 0$ implies

$$2l_{21} = \beta_2 + k_{21}\beta_1,$$
$$\beta_2 = 2l_{21} - k_{21}l_{11}.$$

[4] For $l_{11} = -1$ simply multiply both T_2 and T_3 by -1.
[5] The second generator follows from $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$
[6] This time identities $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\sigma_0$ and $\{\sigma_i, \sigma_0\} = 2\sigma_i$ were employed.

In this case, in order to avoid falling back to one of the previous families we'll pick $\beta_2 \neq \pm 1$:

$$\begin{aligned} 2l_{21} - k_{21}l_{11} &\neq \pm 1, \\ l_{21}(2 - l_{12}l_{11}) &\neq \pm 1, \\ l_{12}l_{11} &= -1. \end{aligned}$$

Where we used the identity $k_{ab} = l_{ab}l_{ba}$. We conclude that $l_{12} = -l_{11}$, upon substituting back,

$$\begin{aligned} \beta_2 &= 2l_{21} - k_{21}l_{11}, \\ &= -2k_{21}l_{11} - k_{21}l_{11}, \\ &= +3k_{12}l_{12}, \\ &= +3l_{21}, \end{aligned}$$

This fixes $\beta = \mathrm{diag}(l_{11}, 3l_{21})$. Equation (4.4.19) yields

$$\begin{aligned} \beta_1 &= l_{11}x_{11}x_{11} + l_{12}x_{12}x_{21}, \\ 0 &= 2k_{12}(l_{11} + l_{12})x_{12}x_{21}, \\ 0 &= 2k_{12}(l_{11} + l_{12})x_{12}x_{21}, \\ \beta_2 &= l_{21}x_{21}x_{12}, \end{aligned}$$

But $l_{12} + l_{11} = 0$, so the second and third conditions are satisfied automatically. Substituting the remaining values we get

$$\begin{aligned} 1 &= x_{11}x_{11} - x_{12}x_{21}, \\ 3 &= x_{21}x_{12}. \end{aligned}$$

Using the fact that $Jx_{12}J = k_{12}x_{12}$ the block has the following form

$$x_{12} = \begin{pmatrix} u_{12} & v_{12} \\ k_{12}v_{12} & -k_{12}u_{12} \end{pmatrix}.$$

And $x_{21} = x_{12}^\dagger$ so the last equation is solved by

$$x_{12} = \sqrt{3}e^{i\psi} \begin{pmatrix} \cos\chi_2 & \sin\chi_2 \\ k_{12}\sin\chi_2 & -k_{12}\cos\chi_2 \end{pmatrix}.$$

In this case $x_{12}x_{21} = x_{21}x_{12}$, so the first equation becomes $4 = x_{11}x_{11}$, and since x_{11} must be real, $k_{11} = +1$, we conclude

$$x_{11} = 2(\cos\chi_1\sigma_3 + \sin\chi_1\sigma_1).$$

We assume $l_{11} = 1$ as before. For $k_{12} = +1$ the complete solution becomes

$$\begin{aligned}
T_1 &= \frac{1}{2}\begin{pmatrix} 2(\cos\chi_1\sigma_3 + \sin\chi_1\sigma_1) & \sqrt{3}e^{i\psi}(\cos\chi_2\sigma_3 + \sin\chi_2\sigma_1) \\ \sqrt{3}e^{-i\psi}(\cos\chi_2\sigma_3 + \sin\chi_2\sigma_1) & 0 \end{pmatrix}, \\
T_2 &= \frac{1}{2}\begin{pmatrix} 2(\cos\chi_1\sigma_1 - \sin\chi_1\sigma_3) & \sqrt{3}e^{i\psi}(\cos\chi_2\sigma_1 - \sin\chi_2\sigma_3) \\ \sqrt{3}e^{-i\psi}(\cos\chi_2\sigma_1 - \sin\chi_2\sigma_3) & 0 \end{pmatrix}, \\
T_3 &= \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -3\sigma_2 \end{pmatrix}.
\end{aligned}$$

Meanwhile for $k_{12} = -1$ we arrive at

$$\begin{aligned}
T_1 &= \frac{1}{2}\begin{pmatrix} 2(\cos\chi_1\sigma_3 + \sin\chi_1\sigma_1) & \sqrt{3}e^{i\psi}(\cos\chi_2\sigma_0 + i\sin\chi_2\sigma_2) \\ \sqrt{3}e^{-i\psi}(\cos\chi_2\sigma_0 - i\sin\chi_2\sigma_2) & 0 \end{pmatrix}, \\
T_2 &= \frac{1}{2}\begin{pmatrix} 2(\cos\chi_1\sigma_1 - \sin\chi_1\sigma_3) & \sqrt{3}e^{i\psi}(i\cos\chi_2\sigma_2 - \sin\chi_2\sigma_0) \\ \sqrt{3}e^{-i\psi}(-i\cos\chi_2\sigma_2 - \sin\chi_2\sigma_0) & 0 \end{pmatrix}, \\
T_3 &= \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 3\sigma_2 \end{pmatrix}.
\end{aligned}$$

We shall refer to these two embeddings as *three–to–one* because T_3 generates a simultaneous rotation of two orthogonal planes, the second plane rotating three times for each turn of the first.

4.6.5 Diagrammatic Classification

One way to classify different embeddings is by deciding for which pairs a, b equations (4.4.19)

$$2 = l_{ab}\beta_a + l_{ba}\beta_b, \qquad l_{ab}, l_{ba} \in \{-1, +1\},$$

hold. To visualize this let us define a graph whose vertices, labeled 1 through m, may or may not be filled in, depending on whether or not equations in the diagonal, $a = b$, are satisfied. Notice that for the diagonal these equations reduce to $\beta_a^2 = 1$. Meanwhile an edge between vertices a and b implies the equation for the pair a, b holds. That is,

$\overset{a}{\bullet}$ stands for $\beta_a^2 = 1$, while $\overset{a}{\circ}$ stands for $\beta_a^2 \neq 1$, and

$\overset{a\qquad b}{\overline{\qquad\quad}}$ means there are signs l_{ab}, l_{ba} such that $l_{ab}\beta_a + l_{ba}\beta_b = 2$.

For $m = 1$ only one embedding is possible: $\bullet$. For $m = 2$ we have found three distinct possibilities:

$$\begin{aligned}
\text{Fundamental}: &\quad \bullet \qquad \circ; \\
\text{Isoclinic}: &\quad \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet; \\
\text{Three} - \text{to} - \text{One}: &\quad \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ.
\end{aligned}$$

The advantage of these diagrams is that they help clarify whether each possibility has been considered. Notice for instance that the only non–trivial[7] diagram missing is $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ and it did not yield an embedding, as it lead to a contradiction in section 4.6.2. Therefore we may assert that our list is exhaustive.

4.7 GENERATING MONOPOLE FAMILIES

The goal of this section is to restate the previous families of embeddings as some action of G over a simpler embedding.

Like for the fundamental embedding these parameters arise from internal degrees of freedom of the solution. These define elements of G which fix T_3, but would otherwise act non trivially on either T_1 or T_2. To be more precise we would like to find which generators of the stabilizer

$$G(T_3) = \{U \in H : UT_3U^\dagger = T_3\},$$

do not commute with either T_1 or T_2. To do this explicitly we will adopt the following base

$$\Sigma_{ab} = \frac{1}{2}\sigma_a \otimes \sigma_b \in su(4),$$

$a, b = 0, 1, 2, 3$. Excluding $a = b = 0$ so that $\operatorname{tr}\Sigma_{ab} = 0$. This yields a complete base of fifteen linearly independent generators of $su(4) = \operatorname{span}\{\Sigma_{ab}\}$. Their commutation relations read

$$[\Sigma_{ab}, \Sigma_{cd}] = i\varepsilon_{bde}(\delta_{ac}\Sigma_{0e} + \delta_{0a}\Sigma_{ce} + \delta_{c0}\Sigma_{ae}) + \tag{4.7.1}$$
$$+ i\varepsilon_{ace}(\delta_{bd}\Sigma_{0e} + \delta_{0b}\Sigma_{de} + \delta_{d0}\Sigma_{be}). \tag{4.7.2}$$

Out of these elements, six generate $\mathfrak{h}(|\phi\rangle_0) = so(4)$, namely,

$$so(4) = \operatorname{span}\{\Sigma_{ab} : a = 2, b \neq 2 \text{ or } a \neq 2, b = 2\},$$

Let us set $T_3 = \Sigma_{02} = \frac{1}{2}(M_{12} + M_{34}) = \frac{1}{2}(h_1 + h_2)$. Upon inspection of the commutation relations we have

$$\mathfrak{g}(\Sigma_{02}) = \operatorname{span}\{\Sigma_{02}, \Sigma_{32}, \Sigma_{20}, \Sigma_{12}, \Sigma_{10}, \Sigma_{30}, \Sigma_{22}\} \cong u(1) \oplus so(4). \tag{4.7.3}$$

Taking $\eta = \chi_1 = \chi_2 = 0$ on (4.6.10) gives $T_1 = \Sigma_{03}$. By checking which generators of (4.7.3) do not commute with Σ_{03} we arrive at the subset

$$\Sigma_{02}, \Sigma_{12}, \Sigma_{22}, \Sigma_{32} \in \mathfrak{g}(\Sigma_{02}).$$

Exponentiating each of these we construct unitary transformations taking Σ_{03} to

$$\Sigma_{01}, \Sigma_{11}, \Sigma_{21}, \Sigma_{31} \in G(\Sigma_{02}) \cdot \Sigma_{03} \subset \mathfrak{g}, \tag{4.7.4}$$

[7] The diagram $\circ \quad \circ$ readily implies $T_1 = 0$.

respectively. Now we repeat the process by checking which further elements of $\mathfrak{g}$ can be achieved by $G(\Sigma_{02})$ actions on (4.7.4), and so on until we complete the orbit of Σ_{03} under $G(\Sigma_{02})$. The general solution, when $k_{12} = +1$, (4.6.10)–(4.6.12) can, after some examination, be expressed as

$$T_1 = U\Sigma_{03}U^\dagger, \qquad T_2 = U\Sigma_{01}U^\dagger, \qquad T_3 = \Sigma_{02},$$

where U is given by

$$U = \exp(i(\theta_1 - \theta_2)\Sigma_{32}) \exp(i(\theta_1 + \theta_2)\Sigma_{02}) \exp(-i\psi\Sigma_{30}) \exp(i\eta\Sigma_{20}).$$

Similarly, when $k_{12} = -1$,

$$T_1 = V\Sigma_{03}V^\dagger, \qquad T_2 = V\Sigma_{31}V^\dagger, \qquad T_3 = \Sigma_{32},$$

where V is given by

$$V = \exp(i(\theta_1 - \theta_2)\Sigma_{32}) \exp(i(\theta_1 + \theta_2)\Sigma_{02})) \exp(-i\psi\Sigma_{30}) \exp(i\eta\Sigma_{13}).$$

4.8 DIAGONAL EMBEDDING IN $su(2m)$

In this section we take $\mathfrak{g} = su(2m)$ and assume $x_{aa} \neq 0$ for all a and $x_{ab} = 0$ if $a \neq b$. Equation (4.4.9) further simplifies

$$2l_{aa} = \beta_a + k_{aa}\beta_a = 2\beta_a.$$

Since $l_{aa} = \pm 1$, $k_{aa} = +1$ and we get $\beta_a = l_{aa} = \pm 1$.The remaining equation simplifies to

$$1 = x_{aa}x_{aa}.$$

So that $||x_{aa}||^2 = 1$. The fact that each x_{aa} is Hermitian and their $k_{aa} = +1$ gives us the form

$$x_{aa} = \begin{pmatrix} u_{aa} & v_{aa} \\ v_{aa} & -u_{aa} \end{pmatrix},$$

for some $u_{aa}, v_{aa} \in \mathbb{R}$. Coupled with the previous condition we get, for an angle parameter $\chi_a \in [0, 2\pi]$,

$$x_{aa} = \begin{pmatrix} \cos\chi_a & \sin\chi_a \\ \sin\chi_a & -\cos\chi_a \end{pmatrix}.$$

The remaining generator is given by

$$T_2 = -i[T_3, T_1] = -\frac{1}{2}\sum_a \beta_a J x_{aa}.$$

Writing these generators in the symmetric representation [8]

$$T_3 = \frac{1}{2i}\sum_a \beta_a(\mathbb{E}_{2a-1,2a} - \mathbb{E}_{2a,2a-1}),$$
$$T_1 = \frac{1}{2}\sum_a \cos\chi_a(\mathbb{E}_{2a-1,2a-1} - \mathbb{E}_{2a,2a}) + \sin\chi_a(\mathbb{E}_{2a-1,2a} + \mathbb{E}_{2a,2a-1}),$$
$$T_2 = \frac{1}{2}\sum_a \beta_a \sin\chi_a(\mathbb{E}_{2a-1,2a-1} + \mathbb{E}_{2a,2a}) - \beta_a\cos\chi_a(\mathbb{E}_{2a-1,2a} - \mathbb{E}_{2a,2a-1}),$$

Taking all $\chi_a = 0$ recovers the results of (KNEIPP; LIEBGOTT, 2010), for the cases where $n_p \neq 0$. The cases where some $n_p = 0$ can be treated by reducing the size of T_3 to its rank and applying the same method we employed, setting off–diagonal elements of T_1 to zero. This way we recover all cases previously considered.

Notice, however that χ_a are not the only internal degrees of freedom available. They are the only ones appearing here because we confined the solution to the diagonal blocks by setting $x_{ab} = 0$ for $a \neq b$. In order to find these remaining parameters one may follow the procedure of the last section of calculating the the orbits of T_1 under the stability group $G(T_3)$.

4.9 ASYMPTOTIC HIGGS FIELD

Because our gauge group G acts transitively over the vacuum manifold we may start from a particularly simple vacuum state

$$|\phi\rangle_0 = \frac{v}{\sqrt{2}}\sum_{k=1}^{4}|kk\rangle\,. \tag{4.9.1}$$

By choosing an $E \cong su(2)$ embedding, and a winding number q, we construct the corresponding asymptotic solution to this subgroup by applying the unitary transformation

$$U(\theta,\varphi) = \exp(-iq\varphi T_3)\exp(-i\theta T_2)\exp(iq\varphi T_3). \tag{4.9.2}$$

To the constant state (4.9.1) in the asymptotic sphere. In the tensor product representation a generic symmetric scalar field

$$|\phi\rangle = \sum_{ij}\phi_{ij}\,|ij\rangle\,,$$

transforms as

$$\begin{aligned}
D\otimes D(g)\,|\phi\rangle &= \sum_{ij}\phi_{ij}\,U\otimes U\,|ij\rangle\,,\\
&= \sum_{ijkl}\phi_{ij}\,|kl\rangle\,\langle kl|\,U\otimes U\,|ij\rangle\,,\\
&= \sum_{ijkl}\phi_{ij}\,|kl\rangle\,U_{ki}U_{lj},\\
&= \sum_{kl}(U\phi U^T)_{kl}\,|kl\rangle\,,
\end{aligned}$$

8 We adopt the notation $\mathbb{E}_{ij} = D\otimes D(E_{ij}) = E_{ij}\otimes I + I\otimes E_{ij}$.

in particular the coefficients for (4.9.1) are

$$\phi_{ij} = \frac{v}{\sqrt{2}}\delta_{ij},$$

and their transformation by (4.9.2) yields

$$\phi'_{ij} = \frac{v}{\sqrt{2}}(UU^T)_{ij}.$$

Notice that, as long as we choose $\psi = 0$, then for each of the families of embeddings considered, T_1 and T_2 are symmetric whereas T_3 is always antisymmetric. Therefore

$$U^T(\theta, \varphi) = \exp(-iq\varphi T_3)\exp(-i\theta T_2)\exp(iq\varphi T_3).$$

The matrix of coefficients for the asymptotic field becomes

$$\phi = \frac{v}{\sqrt{2}}\exp(-iq\varphi T_3)\exp(-2i\theta T_2)\exp(iq\varphi T_3). \tag{4.9.3}$$

The solution originating from the fundamental embedding $\beta = (1, 0)$

$$T_1 = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & 0\end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix}\sigma_1 & 0\\ 0 & 0\end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix}\sigma_2 & 0\\ 0 & 0\end{pmatrix}.$$

Yields, upon expansion,

$$\begin{aligned}\left|\phi\right\rangle_q = \frac{v}{\sqrt{2}}\{&(\cos\theta + i\sin\theta\sin q\varphi)\left|11\right\rangle + i\sin\theta\cos q\varphi(\left|12\right\rangle + \left|21\right\rangle)+\\ &+(\cos\theta - i\sin\theta\cos q\varphi)\left|22\right\rangle + \left|33\right\rangle + \left|44\right\rangle\}.\end{aligned}$$

This readily satisfies $\left|\phi\right\rangle(\theta, \varphi + 2\pi) = \left|\phi\right\rangle(\theta, \varphi)$ and is also well defined at $\theta = 0$ and $\theta = \pi$, that is, it does not depend on φ at the poles. Consequently it defines a continuous map from the 2–sphere to the vacuum manifold defining an element of $\mathrm{loop}_2(\mathcal{V})$.

For the fundamental embedding $\beta = (0, 1)$ one exchanges the pairs $\{1, 2\} \leftrightarrow \{3, 4\}$ in the above expression. For the isoclinic embedding $\beta = (1, 1)$ the solution yields

$$\begin{aligned}\left|\phi\right\rangle_q = \frac{v}{\sqrt{2}}\{&(\cos\theta + i\sin\theta\sin q\varphi)\left|11\right\rangle + i\sin\theta\cos q\varphi(\left|12\right\rangle + \left|21\right\rangle)+\\ &+(\cos\theta - i\sin\theta\cos q\varphi)\left|22\right\rangle + (\cos\theta + i\sin\theta\sin q\varphi)\left|33\right\rangle +\\ &+i\sin\theta\cos q\varphi(\left|34\right\rangle + \left|43\right\rangle) + (\cos\theta - i\sin\theta\cos q\varphi)\left|44\right\rangle\}\end{aligned}$$

4.9.1 Homotopies between Configurations

By the non–abelian nature of the unbroken gauge group one may deform different field configurations among themselves via further transformations. Take for instance the rotation of ψ radians about the $\{ij\}$ plane

$$R(\psi) = \exp\left(i\psi M_{ij}\right) \in H.$$

Because $R^T = R^\dagger$ we have

$$\begin{aligned}
\phi' &= R\phi R^T, \\
&= \frac{v}{\sqrt{2}} R\exp(-iq\varphi T_3)R^\dagger R\exp(-2i\theta T_2)R^\dagger R\exp(iq\varphi T_3)R^\dagger, \\
&= \frac{v}{\sqrt{2}} \exp(-iq\varphi T_3')\exp(-2i\theta T_2')\exp(iq\varphi T_3'),
\end{aligned}$$

where $T_i' = RT_iR^\dagger$ are new generators in the rotated frame. For definiteness let us take, again, the fundamental embedding $\beta = (1,0)$. By applying a rotation R about the plane $\{13\}$ we get

$$\begin{aligned}
T_3' &= \frac{1}{2}\exp(i\psi M_{13})M_{12}\exp(-i\psi M_{13}), \\
&= \frac{1}{2}\cos\psi M_{12} + \frac{1}{2}\sin\psi M_{23},
\end{aligned}$$

and

$$\begin{aligned}
T_2' &= \frac{1}{2}\exp(i\psi M_{13})(E_{12}+E_{21})\exp(-i\psi M_{13}), \\
&= \frac{1}{2}\cos\psi(E_{12}+E_{21}) + \frac{1}{2}\sin\psi(E_{32}+E_{23}).
\end{aligned}$$

Meaning that, as ψ varies continuously from zero to $\pi/2$, a solution confined to the block $\{12\}$ is deformed into a solution confined to the block $\{32\}$. The homotopy between the two is the continuous map $H : [0,\pi/2] \to \mathrm{loop}_2(\mathcal{V})$ Given by this global (homogeneous) rotation

$$H(\psi) = R(\psi)\left|\phi\right\rangle(\theta,\varphi).$$

Similarly one can rotate a solution from the block $\{32\}$ to the block $\{34\}$ by means of $R = \exp(i\psi M_{24})$. Incidentally this yields the other fundamental solution $\beta = (0,1)$. Therefore solutions $\beta = (1,0)$ and $\beta = (0,1)$ are homotopic to one another. By extension of this argument fundamental solutions within the blocks $\{ij\}$, namely

$$T_1 = \frac{1}{2}(E_{ii} - E_{jj}), \quad T_2 = \frac{1}{2}(E_{ij} + E_{ji}), \quad T_3 = \frac{1}{2}M_{ij},$$

for each i and j, $i \neq j$, are topologically equivalent; they lie in the same sector of $\pi_2(\mathcal{V})$. Next we show that there are only two distinct homotopy classes, as expected from $\pi_2(\mathcal{V}) = \mathbb{Z}_2$. To do this we need to find that whenever the winding number q in (4.9.7) is even, $q = 2k$, there is a smooth transformation from that configuration to the trivial one. This would identify all even q configurations to the same element of $\pi_2(\mathcal{V})$. Consequently, all odd q configurations will be identified to a single one where $q = 1$. Applying a transformation similar to the one considered in (WEINBERG; LONDON; ROSNER, 1984)

$$V = \exp(i\theta M_{13})\exp(-2ki\varphi T_3)\exp(-i\theta M_{13})\exp(i\theta T_2)\exp(2ki\varphi T_3). \tag{4.9.4}$$

Transposing yields

$$V^T = \exp(-2ki\varphi T_3)\exp(i\theta T_2)\exp(i\theta M_{13})\exp(2ki\varphi T_3)\exp(-i\theta M_{13}).$$

This transformation readily cancels all terms in (4.9.7):

$$\phi' = V\phi V^T = \frac{v}{\sqrt{2}} I.$$

And we are back to the trivial configuration. It remains to show that this transformation is continuous which is the purpose of the conjugation by M_{13}. We automatically have continuity at $V(\theta, \varphi + 2\pi) = V(\theta, \varphi)$ and $V(0, \varphi) = I$. At the south pole, $\theta = \pi$, we may write

$$V(\theta, \varphi) = V_A(\theta, \varphi) V_B(\theta, \varphi) \exp\left(i\theta \frac{1}{2}(E_{12} + E_{21})\right),$$

where

$$V_A(\theta, \varphi) = \exp(i\theta M_{13}) \exp(-ik\varphi M_{12}) \exp(-i\theta M_{13}) \tag{4.9.5}$$

$$V_B(\theta, \varphi) = \exp\left(i\theta \frac{1}{2}(E_{12} + E_{21})\right) \exp(+ik\varphi M_{12}) \exp\left(-i\theta \frac{1}{2}(E_{12} + E_{21})\right), \tag{4.9.6}$$

expanding

$$\exp(\pm ik\varphi M_{12}) = \cos\frac{q\varphi}{2}(E_{11} + E_{22}) \pm i \sin\frac{q\varphi}{2} M_{12} + (E_{33} + E_{44}).$$

Evaluating (4.9.5) and (4.9.6) at $\theta = \pi$, only the sign multiplying M_{12} on the above expression changes, thus

$$V_A(\pi, \varphi) = \exp(ik\varphi M_{12}), \qquad V_B(\pi, \varphi) = \exp(-ik\varphi M_{12}).$$

We find $V(\pi, \varphi)$ to be single–valued and conclude that $V(\theta, \varphi)$ is indeed continuous in the whole sphere and therefore a homotopy between $|\phi\rangle_{2k}$ and $|\phi\rangle_0$.

A similar, albeit longer, gauge transformation identifies $|\phi\rangle_{2k+1}$ to $|\phi\rangle_1$. Factoring it as $W = W_A W_B W_C$, it reads

$$\begin{aligned}
W_A &= \exp(-i\varphi T_3) \exp(-i\theta T_2), \\
W_B &= \exp(i\frac{\varphi}{2} M_{34}) \exp(i\theta M_{13}) \exp(-2ki\varphi T_3) \exp(-i\theta M_{13}) \exp(-i\frac{\varphi}{2} M_{34}), \\
W_C &= \exp(i\theta T_2) \exp((2k+1) i\varphi T_3).
\end{aligned}$$

First, notice that W_C is responsible for eliminating the original terms in ϕ_{2k+1} while W_B is canceled by its own transpose and W_A restores the expression to ϕ_1. The second exponential in W_B keeps W well defined at $\theta = 0$ while its conjugation by M_{13}, like in the previous case, guarantees that W is also well defined at $\theta = \pi$.

The remaining requirement is periodicity of $W(\theta, \varphi)$ in φ, which is ensured by the M_{34} conjugation. To see this compute $W(\theta, \varphi + 2\pi)$; The outermost conjugation by $-i\pi T_3$, which wasn't present in the $q = 2k$ case, has the effect of changing a sign in M_{13} which gets corrected by the $i\pi M_{34}$ conjugation in W_B. All other factors being either periodic or unaffected by these conjugations leads us to our result.

A straightforward generalization of (4.9.4) gives us a homotopy between the larger embeddings $\beta = (1, \pm 1)$, $\beta = (1, \pm 3)$ and the trivial ϕ_0 configuration for every q. Take $\beta = (1, 1)$,

$$T_1 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad T_3 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

so

$$\phi_q = \frac{v}{\sqrt{2}}\exp(-iq\varphi\frac{\varphi}{2}(M_{12}+M_{34}))\exp(-i\theta(E_{12}+E_{21}+E_{34}+E_{43}))\exp(iq\frac{\varphi}{2}(M_{12}+M_{34})), \tag{4.9.7}$$

and the transformation

$$\begin{aligned} V = &\exp(i\frac{\theta}{2}(M_{13}-M_{24}))\exp(-iq\frac{\varphi}{2}(M_{12}+M_{34}))\exp(-i\frac{\theta}{2}(M_{13}-M_{24}))\cdot \\ &\cdot\exp(i\frac{\theta}{2}(E_{12}+E_{21}+E_{34}+E_{43}))\exp(iq\frac{\varphi}{2}(M_{12}+M_{34})), \end{aligned}$$

which takes it to ϕ_0 is now periodic in φ for every q, even or odd. The reason for this is that now the sign arising from the $\pm i\pi T_3$ conjugation is global. Similarly monopoles constructed from $\beta = (1, \pm 3)$ embeddings are all also trivial; Apply a W transformation to the second block then repeat the previous V with appropriate signs.

4.10 FURTHER RESEARCH

The method developed for embedding $su(2)$ subalgebras lead the main equations (4.4.19) and (4.4.20). These are general enough so as to classify all $su(2)$ subalgebras of $su(2m)$ bearing one generator of $so(2m)$. Out of those some are straightforward generalizations, like the fundamental embeddings, $\beta = \pm e_i$, $i = 1, \dots, m$, and the isoclinic embeddings $\beta = \sum_i n_i e_i$, $n_i = -1, 0, 1$, where two or more n_i may be nonzero.

Besides those the three–to–one embedding generalizes to $\beta = (\pm 1, \pm 3, \dots, \pm(2m-1))$, the first generator yielding

$$\begin{aligned} T_1 = m e_{11} \otimes &\begin{pmatrix} \cos\chi & \sin\chi \\ \sin\chi & -\cos\chi \end{pmatrix} + \\ + \sum_{a=1}^{m-1} \sqrt{m^2-a^2}(e^{i\psi_a}e_{a,a+1} + e^{-i\psi_a}e_{a+1,a}) \otimes &\begin{pmatrix} \cos\zeta_a & \sin\zeta_a \\ k_{aa+1}\sin\zeta_a & -k_{aa+1}\cos\zeta_a, \end{pmatrix}. \end{aligned}$$

Another possibility is to consider different vacuum states $|\phi\rangle$ yielding different symmetry breaking patterns. One idea is that a result analogous to the one for the adjoint representation may be valid, along the lines of

$$SU(n) \to S(O(n_1) \times \cdots \times O(n_p)),$$

where n_i represent the multiplicities of the eigenvalues of the coefficient matrix ϕ.

5 CONCLUSION

In this thesis we studied how to obtain new $\mathbb{Z}_2$ monopole solutions in Yang–Mills–Higgs theories. For simplicity sake we focused on the case $su(4) \to so(4)$ symmetry break with the scalar field in the symmetric part of the 4×4 representation. A general method was devised to find subalgebras $su(2)$ from which one writes down the explicit form for the asymptotic fields of these $\mathbb{Z}_2$ monopoles.

We found that for the magnetic weights $\beta = (1, 0)$ and $\beta = (0, 1)$ our solutions lie in a one dimensional family just like the original 't Hooft–Polyakov monopole. When we considered the isoclinic embeddings $\beta = (1, \pm 1)$ we found four–dimensional families of solutions. Meanwhile the three–to–one embeddings $\beta = (1, \pm 3)$ generated a three–parameter family of monopoles.

Furthermore explicit homotopies between these asymptotic solutions were characterized. Upon inspection we find that there are indeed only two classes of monopoles, i.e. the trivial and the fundamental, in line with the fact that the second homotopy group of the vacuum manifold is $\mathbb{Z}_2$.

For future projects one may propose to generalize this idea to different symmetry breaking patterns. Also shedding a light on how to count the number of collective coordinates of these solutions. Perhaps even figuring out whether background gauge conditions could be introduced so as to allow for dynamic solutions like the dyon for different gauge symmetries. The possibilities seem enticing, hopefully this thesis has conveyed some excitement towards this subject.

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APPENDIX A – APPENDIX

A.1 SURFACE INTEGRAL AS TOPOLOGICAL DEGREE

Starting from the flux

$$g = \int \mathrm{d}^3x\, k_0 = \frac{1}{2ev^3}\int \mathrm{d}^3x\, \varepsilon_{abc}\varepsilon_{mnk}\partial^m(\Phi^a\partial^n\Phi^b\partial^k\Phi^c). \tag{A.1.1}$$

$$= \frac{1}{2ev^3}\int \mathrm{d}^2S^m\, \varepsilon_{abc}\varepsilon_{mnk}\Phi^a\partial^n\Phi^b\partial^k\Phi^c. \tag{A.1.2}$$

and parameterizing x^i by spherical coordinates ξ_α, $\alpha = 1, 2$:

$$\partial^j\Phi^b = \frac{\partial\xi_\gamma}{\partial x_j}\partial^{\alpha'}\Phi^b \tag{A.1.3}$$

$$\mathrm{d}^2S^i = \frac{1}{2}\varepsilon^{imn}\varepsilon^{\alpha\beta}\frac{\partial x_m}{\partial\xi_\alpha}\frac{\partial x_n}{\partial\xi_\beta}\mathrm{d}^2\xi \tag{A.1.4}$$

$$g = \frac{1}{2ev^3}\int \mathrm{d}^2\xi\, \varepsilon_{abc}\varepsilon_{\alpha\beta}\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c. \tag{A.1.5}$$

One may find that the square of the integrand is a determinant

$$\Big(\varepsilon_{abc}\varepsilon_{\alpha\beta}\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c\Big)^2 = \varepsilon_{abc}\varepsilon_{\alpha\beta}\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c\varepsilon^{a'b'c'}\varepsilon^{\alpha'\beta'}\Phi_{a'}\partial_{\alpha'}\Phi_{b'}\partial_{\beta'}\Phi_{c'} \tag{A.1.6}$$

$$= \delta^{a'b'c'}_{abc}\delta^{\alpha'\beta'}_{\alpha\beta}\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c\Phi_{a'}\partial_{\alpha'}\Phi_{b'}\partial_{\beta'}\Phi_{c'}, \tag{A.1.7}$$

where $\delta^{a'b'c'}_{abc}$ denotes the generalized Kronecker delta defined by the degree of the permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$, $\sigma(a) = a'$, $\sigma(b) = b'$, $\sigma(c) = c'$. Explicitly in three dimensions this can be written explicitly as

$$\delta^{a'b'c'}_{abc} = \delta^{a'}_a\delta^{b'}_b\delta^{c'}_c + \delta^{b'}_a\delta^{c'}_b\delta^{a'}_c + \delta^{c'}_a\delta^{a'}_b\delta^{b'}_c - \delta^{b'}_a\delta^{a'}_b\delta^{c'}_c - \delta^{a'}_a\delta^{c'}_b\delta^{b'}_c - \delta^{c'}_a\delta^{b'}_b\delta^{a'}_c, \tag{A.1.8}$$

And in two dimensions,

$$\delta^{\alpha'\beta'}_{\alpha\beta} = \delta^{\alpha'}_\alpha\delta^{\beta'}_\beta - \delta^{\beta'}_\alpha\delta^{\alpha'}_\beta. \tag{A.1.9}$$

Because $\Phi^a\Phi^a = v^2$ and $\Phi^a\partial_\alpha\Phi^a = \frac{1}{2}\partial_\alpha(\Phi^a\Phi^a) = 0$, all factors containing $\delta^{b'}_a, \delta^{c'}_a, \delta^{a'}_b, \delta^{a'}_c$ vanish

$$\Big(\varepsilon_{abc}\varepsilon_{\alpha\beta}\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c\Big)^2 = (\delta^{\alpha'}_\alpha\delta^{\beta'}_\beta - \delta^{\beta'}_\alpha\delta^{\alpha'}_\beta)\delta^{a'}_a(\delta^{c'}_b\delta^{c'}_c - \delta^{c'}_b\delta^{b'}_c)\Phi^a\partial^\alpha\Phi^b\partial^\beta\Phi^c\Phi_{a'}\partial_{\alpha'}\Phi_{b'}\partial_{\beta'}\Phi_{c'}, \tag{A.1.10}$$

$$= 2v^2(\partial_\alpha\Phi^b\partial^\alpha\Phi_c\partial_\beta\Phi^b\partial^\beta\Phi_c - \partial_\alpha\Phi^b\partial^\beta\Phi_b\partial_\alpha\Phi^c\partial^\beta\Phi_c), \tag{A.1.11}$$

$$= 2v^2((\mathrm{tr}\,\partial_\alpha\Phi^\beta\partial^\alpha\Phi_b)^2 - \mathrm{tr}(\partial_\alpha\Phi^b\partial^\beta\Phi_b)^2), \tag{A.1.12}$$

$$= 4v^2\det\partial_\alpha\Phi^b\partial^\beta\Phi_b. \tag{A.1.13}$$

The last line follows from an identity for determinants in two dimensions

$$\det A^\beta_\alpha = \varepsilon^{\alpha\beta}A^1_\alpha A^2_\beta = \frac{1}{2!}\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}A^\gamma_\alpha A^\delta_\beta =, \tag{A.1.14}$$

$$= \frac{1}{2}(A^\alpha_\alpha A^\beta_\beta - A^\beta_\alpha A^\alpha_\beta) = \frac{1}{2}(\mathrm{tr}^2 A - \mathrm{tr}\,A^2). \tag{A.1.15}$$

Thus

$$\varepsilon_{abc}\varepsilon_{\alpha\beta}\Phi^a\partial_\alpha\Phi^b\partial_\beta\Phi^c = \pm 2v\sqrt{\det\partial_\alpha\Phi^a\partial_\beta\Phi^a}. \tag{A.1.16}$$

as we wanted to show. Therefore

$$g = \pm\frac{1}{e}\int \mathrm{d}^2\xi\sqrt{\det\left(\partial_\alpha\hat{\Phi}^a\partial_\beta\hat{\Phi}^a\right)} \tag{A.1.17}$$

Where $\hat{\Phi}^a = \Phi^a/\sqrt{\Phi^b\Phi^b} = \Phi^a/v$ denotes the normilized field.

A.2 THE PFAFFIAN

Given a $2m \times 2m$ skewsymmetric matrix A the following invariant

$$\mathrm{pf}\,A = \frac{1}{2^m m!}\varepsilon_{i_1 j_1 \ldots i_m j_m} A_{i_1 j_1}\cdots A_{i_m j_m}$$

is called the *Pfaffian* of A. For $m = 1$ we have $\mathrm{pf}\,A = A_{12}$ while for $m = 2$ this reads

$$\mathrm{pf}\,A = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$$

The Pfaffian satisfies the following important property. Given any complex $2m \times 2m$ matrix B, the conjugation BAB^T is again skewsymmetric and its Pfaffian is given by

$$\mathrm{pf}(BAB^T) = \det B\,\mathrm{pf}\,A$$

. In particular, when $B = R \in SO(n)$, $\det R = +1$:

$$\mathrm{pf}(RAR^T) = \mathrm{pf}\,A$$

. So an orthogonal change of basis preserves the pfaffian. The six generators of $SO(4)$ are all skewsymetric and can be written as

$$M_{ij} = -i(E_{ij} - E_{ji}), \qquad i < j$$

Simultaneous rotations about the planes $\{ij\}$ and $\{kl\}$ are written as

$$R = \exp\left(i\theta_{ij}M_{ij} + i\theta_{kl}M_{kl}\right), \tag{A.2.1}$$

Therefore isoclinic rotations are generated by

$$M_1^L = \frac{1}{2}(M_{12} + M_{34}) \qquad M_1^R = \frac{1}{2}(M_{12} - M_{34}) \tag{A.2.2}$$

$$M_2^L = \frac{1}{2}(M_{13} - M_{24}) \qquad M_2^R = \frac{1}{2}(M_{13} + M_{24}) \tag{A.2.3}$$

$$M_3^L = \frac{1}{2}(M_{14} + M_{23}) \qquad M_3^R = \frac{1}{2}(M_{14} - M_{23}), \tag{A.2.4}$$

Note that $\mathrm{pf}\, M_i^L = +1$ while $\mathrm{pf}\, M_i^R = -1$. Here the signs are chosen such that each column generates an $su(2)$ subalgebra, this can be seen by applying the commutation relations

$$[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{ik}M_{jl}), \tag{A.2.5}$$

One checks that

$$[M_i^L, M_j^L] = i\varepsilon_{ijk}M_k^L \tag{A.2.6}$$
$$[M_i^R, M_j^R] = i\varepsilon_{ijk}M_k^R \tag{A.2.7}$$
$$[M_i^L, M_j^R] = 0. \tag{A.2.8}$$

Therefore they each generate an $su(2)$ subalgebra as desired.