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DARK MONOPOLES IN GRAND UNIFIED THEORIES

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Esta Dissertação foi julgada adequada para obtenção do Título de MESTRE EM FÍSICA na área de concentração Física Matemática e Teoria de Campos e aprovada em sua forma final pelo Programa de Pós-graduação em Física.

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To the people I consider as family.

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RESUMO

Neste trabalho nós revisamos a construção de soluções de monopolos magnéticos em teorias de Yang-Mills-Higgs com um campo escalar na representação adjunta e com um grupo de gauge G simples e simplesmente conexo. Nós revisitamos a solução do monopolo de 't Hooft-Polyakov e revemos como obter monopolos $SU(2)$ embutidos em teorias com grupos de gauge arbitrários. Nós analisamos alguns aspectos de monopolos não-abelianos, tais como os padrões de quebra de simetria e o chamado problema de "Global Color". Na sequência, nós usamos as nossas ferramentas anteriores para construir uma solução de monopolo com grupo de gauge $G = SU(n)$ quebrado em $G_0 = [SU(p) \times SU(n-p) \times U(1)]/Z$ por um campo de Higgs na adjunta. Nós obtemos soluções de monopolo cujo campo magnético não está na subálgebra de Cartan. E, como esse campo magnético é nulo na direção do gerador do grupo eletromagnético $U(1)_{em}$, nós chamamos estes monopolos de Monopolos Escuros. Estes Monopolos Escuros devem existir em algumas Teorias de Grande Unificação (GUTs) sem a necessidade de introduzirmos um setor escuro. Nós calculamos a hamiltoniana e equações de movimento (EoMs) para estes monopolos, assim como obtemos as soluções aproximadas nos limites $r \to 0$ and $r \to \infty$. Nós também mostramos que sua massa clássica é dada por $M = 4\pi v\, \widetilde{E}(\lambda/e^2)/e$, onde $\widetilde{E}(\lambda/e^2)$ é uma função monotonicamente crescente de λ/e^2. Para o caso particular de $G = SU(5)$, nós resolvemos as equações radiais numericamente e obtemos os limites inferior e superior da massa, dados por $\widetilde{E}(0) = 1.294$ e $\widetilde{E}(\infty) = 3.262$. Por fim, nós damos uma interpretação geométrica para a carga magnética não-abeliana destes monopolos e discutimos alguns problemas em aberto.

Palavras-chave: Teorias de Gauge, Monopolos Magnéticos, Quebra Espontânea de Simetria, Carga Magnética.

RESUMO EXPANDIDO

Introdução

Neste trabalho nós revisamos a construção de soluções de monopolos magnéticos em teorias de Yang-Mills-Higgs (YMH) com um campo de Higgs ϕ real na representação adjunta e com um grupo de gauge G simples e simplesmente conexo. Também discutimos alguns aspectos de monopolos não-abelianos, tais como os padrões de quebra de simetria e o problema de *Global Color*. Na sequência, nós construímos soluções de monopolos com carga magnética na direção abeliana nula. Para isso, consideramos uma teoria de Yang-Mills-Higgs com grupo de gauge $G = SU(n)$, $n \geq 3$, quebrado em

$$G_0 = [SU(p) \times SU(n-p) \times U(1)]/Z \, ,$$

por um campo escalar na representação adjunta. Pelo fato de não possuírem uma carga magnética na direção abeliana, nossos monopolos não interagem com o campo eletromagnético $U(1)_{em}$ e, por tal motivo, serão chamados de *Monopolos Escuros*. Por fim, discutimos uma interpretação geométrica para a carga magnética não-abeliana dos nossos monopolos e analisamos alguns problemas em aberto.

Objetivos

Revisar a construção de monopolos magnéticos em teorias de Yang-Mills-Higgs com um campo escalar ϕ na representação adjunta e com um grupo de gauge G simples e simplesmente conexo. Além disso, pretendemos investigar alguns aspectos das soluções de monopolos em teorias cujo grupo "não-quebrado" G_0 é não-abeliano. Contudo, nosso principal objetivo é usar estas ferramentas para construir uma solução de monopolo não-abeliano para o grupo de gauge $G = SU(n)$ com um campo magnético que não esteja na direção da subálgebra de Cartan $\mathcal{H}$, mas sim na direção de operadores escada E_α.

Metodologia

Durante a realização deste trabalho utilizamos de ferramentas matemáticas fornecidas pela teoria de grupos e álgebras de Lie. Também fizemos uso do software `MATLAB`® para resolver as equações radiais numericamente no caso particular dos Monopolos Escuros com $G = SU(5)$.

Resultados e Discussões

Durante nossa revisão sobre monopolos em teorias de Yang-Mills-Higgs, mostramos que a configuração assintótica das soluções de monopolo pode ser construída ao escolhermos um $\phi_0 = u \cdot H$ e uma subálgebra $su(2)$ com geradores M_i, $i = 1, 2, 3$, tal que $M_3 \in L(G_0)$ e $M_1, M_2 \notin L(G_0)$, onde $L(G_0)$ denota a álgebra de Lie do grupo não-quebrado G_0. A configuração assintótica é obtida a partir de uma configuração de vácuo $(\phi_0, W_i^{(0)})$, por uma transformação de gauge associada ao elemento $g \in G$ dado por

$$g(\theta, \varphi) = \exp(-i\varphi M_3) \exp(-i\theta M_2) \exp(+i\varphi M_3) \ .$$

Adicionalmente, construímos um ansatz bastante geral que é usado ao longo de toda a dissertação. Na sequência, discutimos o monopolo de 't Hooft-Polyakov e analisamos aspectos de soluções de monopolos não-abelianos.

Todavia, nossos principais resultados são relacionados aos Monopolos Escuros construídos para o grupo de gauge $G = SU(n)$. Nós mostramos que estes monopolos podem ser construídos com os geradores M_i definidos como combinação linear de operadores escada E_α. Ademais, provamos que estes monopolos não possuem uma carga magnética na direção $U(1)_{em}$. Nós calculamos a hamiltoniana e equações de movimento para estes monopolos, assim como obtivemos as soluções aproximadas nos limites $r \to 0$ e $r \to \infty$. Mostramos que sua massa clássica é dada por

$$M = \frac{4\pi v}{e} \tilde{E}(\lambda/e^2) \, ,$$

onde $\tilde{E}(\lambda/e^2)$ é uma função monotonicamente crescente de λ/e^2. Para o caso particular dos monopolos escuros com $G = SU(5)$, encontramos os limites inferior e superior de massa dados por $\tilde{E}(0) = 1.294$ e $\tilde{E}(\infty) = 3.262$. Por fim, nós damos uma explicação geométrica para a carga magnética não-abeliana destes monopolos, mostramos que ela é quantizada em unidades de $8\pi/e$ e está associada à uma simetria assintótica das configurações dos campos de gauge e Higgs.

Considerações Finais

Esta solução de Monopolos Escuros deve existir em algumas Teorias de Grande Unificação (GUTs), onde o campo de Higgs está na adjunta. Além disso, como nosso monopolo não possui uma carga magnética na direção $U(1)_{em}$, ele pode ter uma pequena contribuição à Matéria Escura no Universo. Contudo, ainda que tenhamos construído as soluções de Monopolos Escuros, existem algumas questões em aberto. Elas são

referentes à estabilidade dessas soluções e às implicações cosmológicas destes monopolos.

Palavras-chave: Teorias de Gauge, Monopolos Magnéticos, Quebra Espontânea de Simetria, Carga Magnética.

ABSTRACT

In this work we review the construction of magnetic monopole solutions in Yang-Mills-Higgs theories with an adjoint scalar field and a simple and simply-connected gauge group G. We revisit the 't Hooft-Polyakov monopole solution and review how one can construct $SU(2)$-embedded monopoles in theories with larger gauge groups. We analyze some aspects of non-abelian monopoles, such as the symmetry breaking pattern and the so-called "Problem of Global Color". Then, we use our previous tools to construct a monopole solution with gauge group $G = SU(n)$ broken to $G_0 = [SU(p) \times SU(n-p) \times U(1)]/Z$ by a Higgs field in the adjoint representation. We obtain monopole solutions whose magnetic field does not lie in the Cartan subalgebra. And, since their magnetic field vanish in the direction of the generator of the $U(1)_{em}$ electromagnetic group, we call them Dark Monopoles. These Dark Monopoles must exist in some Grand Unified Theories (GUTs) without the need to introduce a Dark sector. We calculate the hamiltonian and equations of motion (EoMs) for these Dark Monopoles. We obtain approximate solutions when $r \to 0$ and $r \to \infty$. We also show that their classical mass is $M = 4\pi v\, \widetilde{E}(\lambda/e^2)/e$, where $\widetilde{E}(\lambda/e^2)$ is a monotonically increasing function of λ/e^2. For the particular case of $G = SU(5)$, we numerically solve the radial EoMs and obtain the lower and upper bounds for the mass, given by $\widetilde{E}(0) = 1.294$ and $\widetilde{E}(\infty) = 3.262$. Finally, we give a geometrical interpretation to their non-abelian magnetic charge and discuss some open problems.

Keywords: Gauge theories, Magnetic Monopoles, Spontaneous Symmetry Breaking, Magnetic Charge.

CONTENTS

1. INTRODUCTION

In 1931, Dirac analyzed whether it was possible to have a magnetic monopole in ordinary electrodynamics [1]. This elusive particle was believed to be banned from electromagnetism by the sourceless Maxwell's equations $\partial_\mu {}^*F^{\mu\nu} = 0$, which, in principle, forbid the existence of a magnetic charge. However, because of some arguments of symmetry, which we shall discuss in section 2.1, Dirac insisted that this "outlaw" particle was compatible with Maxwell theory. He showed how one could construct the vector potential of a monopole and how this would lead to a spherically symmetric magnetic field of a point-like magnetic source. In fact, this vector potential had a singularity, the so-called *Dirac string*, which was a theoretical artifact corresponding to an infinitely long and thin solenoid. This was not a problem to classical electrodynamics, since only the field strengths **E** and **B** are measurable. However, in order for this monopole to be consistent with Quantum Mechanics, i.e., for the Dirac string to be undetectable in the Aharonov-Bohm experiment, Dirac showed that the electric and magnetic charges in nature should satisfy a *quantization condition*. This was an impressive result, since it implied that even if there is only one magnetic monopole in the whole universe, all electric charges should be quantized. Actually, Dirac gave the first theoretical explanation to the experimentally observed quantization of electric charge. The only problem in this story is that no monopole was ever detected[1].

Nevertheless, more than four decades later 't Hooft and Polyakov [3, 4] have independently found a magnetic monopole solution in the $SO(3)$ Georgi-Glashow model. But this time, the monopole was indeed a finite-energy solution to the equations of motion (EoMs) and its field configuration was regular at the origin. The monopole classical mass had upper and lower bounds and the magnetic charge g was conserved for topological reasons. This was the starting point for a new era in monopole theory, where monopoles were no longer *ad hoc* constructions, but topological solutions in Yang-Mills-Higgs theories.

From that point on, there have been many generalizations for these monopoles, for theories with larger gauge groups G. In many of these theories, there is a Higgs field in the adjoint representation, which can produce a symmetry breaking of the form [5–7] $G \to G_0 =$ "$K \times U(1)$", with a compact $U(1)$, which allows for the existence of topological

[1] Even though there were claims from Cabrera [2] of a possible monopole detection in 1982, which was later regarded as inconclusive.

monopoles. In general, these monopoles have magnetic charge in the abelian subalgebra of the unbroken group G_0, which can give rise to a non-vanishing magnetic charge for the electromagnetic $U(1)_{em}$ gauge group. It is interesting that these non-abelian monopoles also satisfy a generalized quantization condition, so that they also provide an explanation to the quantization of electric charge.

Now, there is another relevant point in this story. In 1974, the same year of the seminal papers of 't Hooft and Polyakov, Georgi and Glashow [8] proposed a Yang-Mills-Higgs theory where they conjectured that the strong, weak and electromagnetic forces have arisen from a single fundamental interaction based on the gauge group $G = SU(5)$. This theory was one of the first Grand Unified Theories (GUTs) to be proposed. These GUTs are based in the fact that the Standard Model, whose gauge group is $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$, can be conveniently embedded in a theory with a larger simple gauge group. Moreover, the fact that G is simple both implies that there is only a single gauge coupling constant and that it explains the quantization of electric charge [9]. It also turns out that the current observed particles could be accommodated in a relatively simpler multiplet structure. Some other examples of GUTs are the Pati-Salam model [10], the Trinification model [11] and also the E_6 and $SO(10)$ models.

The interesting point is that we expect that this larger gauge symmetry of nature was broken in the early universe after one (or some) stages of spontaneous symmetry breaking (SSB). This agrees with the fact that there are indications that the running gauge coupling constants for the weak, electromagnetic and strong interactions tend to the same value when evaluated at sufficiently high unification temperature[2] $T_c \sim 10^{16}$ GeV [9, 12].

Now, one can ask what is the role of magnetic monopoles in face of GUT theories. Well, one of the consequences of these GUTs is that they have topological magnetic monopoles. One example is the $SU(5)$ monopole of Dokos and Tomaras [13], associated to a SSB by a Higgs field in the adjoint representation. Thus, magnetic monopoles and grand unification are connected. The observation of such monopoles may be one of the few possibilities to obtain experimental support to the unification hypothesis [9].

In this work, we shall review the general construction of non-abelian magnetic monopoles from first principles. For simplicity, we shall work with theories with an adjoint Higgs field ϕ and with a simple and simply-connected gauge group G. We shall construct a quite general

[2]Note that we are working with units where the Boltzmann constant $K_B = 1$.

ansatz, that will be used throughout this dissertation. Then, we will revisit the 't Hooft-Polyakov monopole with a new perspective.

After that, we shall discuss some aspects of these non-abelian monopoles. This will involve an analysis of the symmetry breaking patterns and on the way one can embed 't Hooft-Polyakov monopoles in theories with larger gauge groups. We shall also investigate the generalized quantization condition and the "Problem of Global Color".

In the sequence, we shall tackle our main goal, which is to construct monopole solutions with vanishing abelian magnetic charge [14]. This implies that our monopoles do not interact with the $U(1)_{em}$ electromagnetic field, so that we shall call them Dark Monopoles. Moreover, it is well-known that the nature of Dark Matter is one of the biggest open problems in physics. In the last decades, many candidates have been proposed (see, for instance, [15, 16] and references therein) in a variety of distinct theories. Magnetic monopoles happen to be one of these candidates [17–23], usually associated to a dark (or hidden) sector coupled to the Standard Model. But, since our Dark Monopoles do not have a $U(1)_{em}$ electromagnetic field, we need not introduce a dark sector. This is an interesting feature, since we can have these monopoles contributing to dark matter in the standard Grand Unified Theories. And, even if future analysis shows that they do not have a relevant contribution to Dark Matter (due to inflation), they are still an interesting solution since they are a new type of GUT monopoles.

We emphasize that monopoles with a magnetic flux in a non-abelian direction have been constructed for a Yang-Mills-Higgs theory with $G = SU(3)$ broken to $U(2)$ [24] (see also [6, 25, 26]). They were associated to the $su(2)$ subalgebra generated by the Gell-Mann matrices λ_2, λ_5 and λ_7 and an ansatz was constructed using some general arguments of symmetry. On the other hand, in the present work we shall consider a Yang-Mills-Higgs theory with an arbitrary gauge group $SU(n)$, $n \geq 3$, broken to[3]

$$G_0 = [SU(p) \times SU(n-p) \times U(1)]/Z \,,$$

by a scalar field in the adjoint representation. We shall use our tools from chapter 2, which are based on the general procedure of [27, 28], in order to construct the ansatz for our Dark Monopoles. Its asymptotic configuration will be associated to some $su(2)$ subalgebras with generators M_a, which we choose to be given by linear combinations of some step operators. Then, we shall see that the asymptotic form of the gauge and magnetic fields are linear combinations of the generators

[3] In the case $p = 1$ or $p = n$, G_0 is given by $G_0 = \frac{SU(n-1)\times U(1)}{Z}$.

M_a, while the asymptotic form of the scalar field is a linear combination of generators S and Q_a, $a = 0, \pm 1, \pm 2$, which form, respectively, a singlet and a quintuplet under the $su(2)$ subalgebra. As a consequence, the Dark Monopole cannot satisfy the BPS condition $B_i = D_i\phi$.

From these asymptotic configurations, we construct an ansatz for the whole space and calculate the Hamiltonian. Then, we obtain the second order differential equations for the profile functions. Additionally, we obtain the numerical solution for these equations in the case $G = SU(5)$, for some particular coupling constant values.

Furthermore, we show that the classical mass of a Dark Monopole is a monotonically increasing function of λ/e^2, given by

$$M = \frac{4\pi v}{e}\,\widetilde{E}(\lambda/e^2)$$

where in the particular case of the $SU(5)$ Dark Monopoles, $\widetilde{E}(0) = 1.294$ and $\widetilde{E}(\infty) = 3.262$.

We shall also construct a Killing vector ζ associated to an asymptotic symmetry of the Dark Monopole and show that these monopoles have a conserved current in a non-abelian direction. The associated magnetic charge Q_M is quantized in multiples of $8\pi/e$ and we give a geometrical interpretation to this charge. And, although Dark Monopoles are associated to the trivial sector of $\Pi_1(G_0)$, the conservation of Q_M could prevent them to decay. Finally, we expect that our construction can be generalized to other gauge groups.

This dissertation is organized as follows. In chapter 2, we present the aforementioned construction of non-abelian monopoles in Yang-Mills-Higgs theories, while we also revisit the 't Hooft-Polyakov monopole. In chapter 3, we discuss symmetry breaking patterns, $SU(2)$-embedded monopoles and the generalized quantization condition. In the same chapter, we investigate some particularities of monopoles with a non-abelian unbroken symmetry. Finally, in chapter 4 we present our original construction of the Dark Monopoles in $SU(n)$. We conclude with a summary of the results and with a discussion on the possible cosmological implications of Dark Monopoles.

2. MAGNETIC MONOPOLES FROM FIRST PRINCIPLES

In this chapter we shall approach the construction of non-abelian magnetic monopoles from first principles. However, in order to do so we need to introduce the Dirac Monopole first, since we will need to borrow some concepts from ordinary electrodynamics. Then, we shall present our Lie algebra conventions, which will be used throughout this dissertation. In the sequence, we shall start the actual construction making use of some physical constraints, such as finite energy and spherical symmetry. After the final asymptotic configuration is obtained, we use our results in order to revisit the 't Hooft-Polyakov monopole under a new perspective.

2.1. THE DIRAC MONOPOLE

It is well-known that in the absence of sources Maxwell's equations can be written as complex equations of the form

$$\partial_\mu \left(F^{\mu\nu} + i\,^*F^{\mu\nu}\right) = 0\,, \tag{2.1}$$

which are invariant under the duality transformation

$$F^{\mu\nu} + i\,^*F^{\mu\nu} \to e^{i\alpha}\left(F^{\mu\nu} + i\,^*F^{\mu\nu}\right)\,, \tag{2.2}$$

where α is a constant phase. Now, since $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k$, we can also write this transformation in terms of the electric and magnetic fields

$$\mathbf{E} + i\mathbf{B} \to e^{i\alpha}(\mathbf{E} + i\mathbf{B})\,.$$

However, when sources are present the equations turn out to be

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial^*_\mu F^{\mu\nu} = 0 \tag{2.3}$$

and the symmetry under (2.2) is lost. This happens because only electric charges and currents appear. In order to restore the symmetry of (2.2) we must introduce a magnetic current $(K^\mu) = (\sigma, \mathbf{k})$ so that equations (2.3) take the form

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial^*_\mu F^{\mu\nu} = K^\nu\,. \tag{2.4}$$

Now, it is trivial to see that these equations can be written as

$$\partial_\mu \left(F^{\mu\nu} + i\,^*F^{\mu\nu}\right) = \left(J^\nu + iK^\nu\right)\,,$$

and that they are invariant under (2.2) as long as the currents transform as

$$J^\nu + iK^\nu \to e^{i\alpha} \left(J^\nu + iK^\nu \right) \, .$$

The addition of this magnetic current naturally leads to the idea of a magnetically charged particle, i.e., a magnetic monopole. This object would be the source of a Coulomb magnetic field of the form

$$\mathbf{B}_m = \frac{g}{r^3} \, \mathbf{r} \, , \tag{2.5}$$

where g is the magnetic charge. Note that from eq. (2.5)

$$\nabla \cdot \mathbf{B}_m = 4\pi g \, \delta^3(\mathbf{r}) \, ,$$

where $\delta^3(\mathbf{r})$ is the usual Dirac delta function. Thus, we should be careful, since the introduction of K^μ in standard electromagnetism implies that $\mathbf{B}$ is no longer divergenceless. This poses problems to the global definition of the magnetic field as $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. Nonetheless, this is not a surprise, since the addition of K^μ spoils the very own definition of the field strength $F_{\mu\nu}$ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A^μ is the 4-potential.

However, Dirac realized that the vector potential $\mathbf{A}$ need not be globally well-defined [29]. If magnetic charges were point-like, or at least confined to a finite volume, one could try to write $\mathbf{B}$ as a curl in the regions were magnetic charges were absent.

Let us suppose we have a point-like monopole at the origin with magnetic charge g. Let us define a vector potential $\mathbf{A}$ such that

$$\mathbf{A} \cdot d\mathbf{r} = g \left(\cos\theta - 1 \right) d\varphi \, . \tag{2.6}$$

This potential has a singularity, the so called *Dirac string*, along the negative z-axis. But one should note that, away from the singularity, $\nabla \times \mathbf{A} = \mathbf{B}_m$. Besides that, the location of the string is arbitrary, since we could have chosen another vector potential yielding the same $\mathbf{B}_m$, for instance,

$$\mathbf{A}' \cdot d\mathbf{r} = g \left(\cos\theta + 1 \right) d\varphi \, . \tag{2.7}$$

which has a Dirac string in the positive z-axis. The potentials $\mathbf{A}$ and $\mathbf{A}'$ are related by a singular gauge transformation

$$\mathbf{A}' - \mathbf{A} = \nabla\alpha \, ,$$

with $\alpha = 2g\varphi$. Also note that $\nabla\alpha$ is single-valued, while α is not. In fact, by an appropriate gauge transformation, the Dirac string can be made to coincide with an arbitrary curve starting at the origin and running out to spatial infinity [30]. But, it cannot be gauged away in a $U(1)$ gauge theory.

Now, we have to consider what are the implications of a singular vector potential. First, let us recall that in classical electrodynamics only the field strengths are measurable. Thus, we do not expect a singularity in $\mathbf{A}$ to have any consequences.

On the other hand the situation in the quantum theory is the opposite, since the vector potential can affect the phase of a wavefunction. Let us suppose that a particle with electric charge q travels around a closed circle C that enclose a region of non-zero magnetic flux. It is known from Quantum Mechanics that after a full turn, the particle's wavefunction is multiplied by the Aharonov-Bohm phase [30]

$$U[C] = \exp\left(iq \int_C d\mathbf{l} \cdot \mathbf{A} \right) . \tag{2.8}$$

If we choose a convenient loop, with θ fixed, in the case of a monopole whose Dirac string lies along the south pole, then

$$q \int_C d\mathbf{l} \cdot \mathbf{A} = q \int d\varphi A_\varphi = 2\pi\, qg\ (\cos\theta - 1) . \tag{2.9}$$

If C is contracted so that it turns into an infinitesimal curve around the string, then $\theta \to \pi$ and the line integral in eq. (2.9) becomes $-4\pi\, qg$. Now if we want the string to be undetectable, i.e. $U[C] = 1$, it follows from eq. (2.8) that[1]

$$qg = \frac{n}{2}, \quad n \in \mathbb{Z} .$$

This result is the famous Dirac quantization condition and it has deep consequences. It states that even if there is only one magnetic monopole with magnetic charge g_0 in the whole universe, all electrically charged particles possess a quantized electric charge such that $q = n\left(\frac{1}{2g_0}\right)$. This is a remarkable fact, since it was the first possible explanation to the observed quantization of electric charges.

Finally, it is necessary to say that there is another approach to the Dirac monopole that we have avoided here. It is based in the Wu-Yang procedure and it divides $\mathbb{R}^3 - \{0\}$ in two regions that can be chosen to be the North and South patches. We can use the potentials $\mathbf{A}$ and $\mathbf{A}'$, in eqs. (2.6) and (2.7) to define $\mathbf{A}^N$ and $\mathbf{A}^S$, respectively. The single-valuedness of the gauge transformation that relates both

[1] Considering $\hbar = 1$.

potentials, where the two regions intersect, leads (not surprisingly) to the same quantization condition. A nice explanation of this procedure is available at [31].

From now on, we will discuss more sophisticated theories, but we will have to keep in mind the concepts of the Dirac string singularities and that of the quantization condition.

2.2. LIE ALGEBRAS CONVENTIONS

In this section, we shall fix some conventions that will be used throughout this dissertation. Here, we do not intend to review the topic of Lie groups and Lie algebras, which can be found in [32–34].

Let G be a Lie group of rank r and $L(G)$ its d-dimensional Lie algebra. In the orthogonal basis of $L(G)$, the d generators T_a satisfy

$$\mathrm{Tr}\,(T_a T_b) = y\,\delta_{ab}\,,$$

where $\psi^2 y$ is the Dynkin index of the representation and ψ is the highest root of $L(G)$.

We shall also use the *Cartan-Weyl basis* with Cartan elements H_i, which form a basis for the Cartan subalgebra $\mathcal{H}$, and step operators E_α, satisfying the commutation relations

$$\begin{aligned}
[H_i, H_j] &= 0\,,\\
[H_i, E_\alpha] &= \alpha^{(i)} E_\alpha\,,\\
[E_\alpha, E_\beta] &= \begin{cases} N_{\alpha,\beta}\,E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root,}\\ \frac{2\alpha\cdot H}{\alpha^2} & \text{if } \alpha = -\beta,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned} \tag{2.10}$$

Moreover,

$$\begin{aligned}
\mathrm{Tr}\,(H_i H_j) &= y\,\delta_{ij}\,,\\
\mathrm{Tr}\,(E_\alpha E_\beta) &= y\,\frac{2}{\alpha^2}\delta_{\alpha,-\beta}\,,\\
\mathrm{Tr}\,(H_i E_\alpha) &= 0\,.
\end{aligned} \tag{2.11}$$

For an arbitrary root α we define the generators

$$\begin{aligned}
T_1^\alpha &\equiv \frac{E_\alpha + E_{-\alpha}}{2}\,,\\
T_2^\alpha &\equiv \frac{E_\alpha - E_{-\alpha}}{2i}\,,\\
T_3^\alpha &\equiv \frac{\alpha\cdot H}{\alpha^2}\,,
\end{aligned} \tag{2.12}$$

which form an $su(2)$ subalgebra. The weight states are such that

$$H_i \, |\mu\rangle = \mu_i \, |\mu\rangle \ ., \tag{2.13}$$

where μ_i are the components of a r-dimensional vector μ, which is an arbitrary weight. We will denote by α_i, $i = 1, 2, \ldots, r$, the simple roots, and by λ_i, $i = 1, 2, \ldots, r$ the fundamental weights of $L(G)$, which satisfy the relation

$$\frac{2\alpha_i \cdot \lambda_j}{\alpha_i^2} = \delta_{ij} \ . \tag{2.14}$$

The Cartan matrix and its inverse are defined to be

$$K_{ij} = \alpha_i \cdot \alpha_j^\vee \qquad \text{and} \qquad \left(K^{-1}\right)_{ij} = \lambda_i \cdot \lambda_j^\vee \ , \tag{2.15}$$

where $\alpha_i^\vee = 2\alpha_i/\alpha_i^2$ and $\lambda_i^\vee = 2\lambda_i/\alpha_i^2$ are the coroots and coweights of $L(G)$, respectively. It follows from the definition (2.15) and the orthogonality relation (2.14) that

$$\begin{aligned} \alpha_l^\vee &= \lambda_i^\vee \, K_{il} \ , \\ \lambda_l^\vee &= \alpha_i^\vee \left(K^{-1}\right)_{il} \ . \end{aligned} \tag{2.16}$$

Now, since any root can be written as a linear integer combination of the simple roots α_i, so that

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i \ , \quad n_i \in \mathbb{Z} \ , \tag{2.17}$$

the simple roots span the root lattice. The coefficients n_i will be all positive or all negative depending whether the root is said to be positive or negative. Similarly, the weights μ can be written as

$$\mu = \sum_{i=1}^{r} m_i \lambda_i \ , \quad m_i \in \mathbb{Z} \ , \tag{2.18}$$

so that the fundamental weights span the weight lattice. Furthermore, the lattices spanned by $\alpha_i^\vee$ and $\lambda_i^\vee$ are called coroot and coweight lattices, respectively.

2.3. MAGNETIC MONOPOLES IN NON-ABELIAN GAUGE THEORIES

Let us consider a Yang-Mills-Higgs theory in $3 + 1$ dimensions with a gauge group G, of rank r, which is compact and simple. Without loss of generality we can take G to be simply-connected. For simplicity, we shall consider a real scalar field $\phi = \phi_a T_a$ in the adjoint representation. The vacuum configuration of the Higgs field breaks G down to the subgroup G_0, the unbroken gauge group.

The action S is given by

$$S = \int d^4x \left\{ -\frac{1}{4y}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) + \frac{1}{2y}\mathrm{Tr}[(D_\mu\phi)(D^\mu\phi)] - V(\phi) \right\} , \tag{2.19}$$

where $V(\phi)$ is the potential term, with $V(\phi) \geq 0$. Besides that,

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu] , \tag{2.20}$$

$$D_\mu\phi = \partial_\mu\phi + ie[W_\mu, \phi] , \tag{2.21}$$

where $G_{\mu\nu}$, W_μ and D_μ are the field strength, the gauge field and the gauge covariant derivative, respectively, while e denotes the gauge coupling constant for the theory.

The equations of motion (EoMs) are given by

$$\begin{aligned} D_\mu G^{\mu\nu} &= -ie[\phi, D^\nu\phi] , \\ \left(D_\mu\left(D^\mu\phi\right)\right)_a &= -\frac{\partial V}{\partial\phi_a} . \end{aligned} \tag{2.22}$$

Furthermore, we have the Bianchi identities

$$D_\mu {}^*G^{\mu\nu} = 0 . \tag{2.23}$$

Now, since we want to construct a static magnetic monopole solution, we can work in a reference frame where the particle, with a mass M, is at rest. This is always possible, as long as $M \neq 0$.

The static solution will be characterized by the lack of time dependence in the fields, i.e., the gauge and Higgs fields will be of the form $W_\mu = W_\mu(\mathbf{r})$ and $\phi = \phi(\mathbf{r})$. In addition, we can use the gauge freedom to choose $W_0 = 0$. This implies that

$$D_0\phi = \partial_0\phi + ie[W_0, \phi] = 0 .$$

Recalling that $[D_\mu, D_\nu]\,\phi = ie\,G_{\mu\nu}\,\phi$, we will have that $G_{0i} = 0$. This means that, in the monopole rest frame, there will only be non-abelian magnetic fields. In fact, we can use the action in eq. (2.19) to obtain the hamiltonian for the monopole, which is

$$E = \int_{\mathbb{R}^3} d^3x \left\{ \frac{1}{2y}\,\mathrm{Tr}\left(B_iB_i\right) + \frac{1}{2y}\,\mathrm{Tr}\left(D_i\phi D_i\phi\right) + V(\phi) \right\} . \tag{2.24}$$

And, in the case we want to obtain a moving monopole, we only need to Lorentz transform the fields W_μ and ϕ. Moreover, the finite energy constraint will give us much more details on how to construct monopole solutions.

Nevertheless, before we end this section it is interesting to present a "trick", due to Bogomolny [35], so that we can find the global minimum of the hamiltonian. First, note that eq. (2.24) can be written as

$$E = \int_{\mathbb{R}^3} d^3x \left\{ \frac{1}{2y} \, \mathrm{Tr}\left(B_i \mp D_i\phi\right)^2 \pm \frac{1}{y} \, \mathrm{Tr}\left(B_i D_i \phi\right) + V(\phi) \right\} . \tag{2.25}$$

Since the first and third terms in eq. (2.25) are necessarily positive, we see that

$$E \geq \frac{1}{y} \int_{\mathbb{R}^3} d^3x \, \mathrm{Tr}\left(B_i D_i \phi\right) . \tag{2.26}$$

Now, making use of

$$D_i\left(B_i\phi\right) = \left(D_i B_i\right)\phi + B_i\left(D_i\phi\right) , \tag{2.27}$$

and the Bianchi identity $D_i B_i = 0$, we see that $B_i\left(D_i\phi\right) = D_i\left(B_i\phi\right)$. Moreover, from the cyclicity of the trace, it follows that $\mathrm{Tr}\left(D_i(B_i\phi)\right) = \partial_i \mathrm{Tr}\left(B_i\phi\right)$. Thus, eq. (2.26) can be conveniently written as

$$E \geq \frac{1}{y} \int_{\mathbb{R}^3} d^3x \, \partial_i \mathrm{Tr}\left(B_i\phi\right) . \tag{2.28}$$

Then, from the divergence theorem, it follows that the energy of a monopole solution in its rest frame, which is the classical mass of the particle, is always greater than

$$E \geq \frac{1}{y} \int_{S^2_\infty} d^2S_i \, \mathrm{Tr}\left(B_i\phi\right) . \tag{2.29}$$

We shall see in section 2.4.1 how this integral is associated to the abelian magnetic charge of a monopole. On the other hand, note that the equality holds if

$$V(\phi) = 0 \, , \tag{2.30a}$$

$$B_i = \pm D_i\phi \, . \tag{2.30b}$$

The condition (2.30a) is the *Prasad-Sommerfield limit* [36], while the eqs. (2.30b) are the Bogomolny equations [35]. The set of eqs. (2.30a) and (2.30b) are called *BPS equations*. Note that the condition $V(\phi) = 0$

must be understood as a limit, so that the boundary conditions for ϕ when $r \to \infty$ remain the same. This is essential in order for the spontaneous breaking of gauge symmetry to happen. Besides that, one can show that eqs. (2.30a) and (2.30b) are compatible with the second order equations (2.22). Monopole solutions satisfying eqs. (2.30a) and (2.30b) are called *BPS monopoles*.

Although they are quite important, in this work we shall be concerned with monopole solutions that satisfy the second order equations of motion, but not necessarily the first order ones. This is because the second order field equations (for static fields) are the condition for a stationary point of the energy functional, whereas the first order Bogomolny equation is the condition of the global minimum of the energy [29]. Then, we will focus on the more general solutions rather than the BPS ones. Additionally, we will see later on that our Dark Monopoles construction does not satisfy the BPS equations.

2.3.1 Finite Energy Solutions

In order for the static solution to have finite energy, we need the integrand in eq. (2.24) to have an appropriate behavior, since the integral is evaluated in a region where both $r = 0$ and $r \to \infty$ are included. That is, we seek for solutions which are regular at the origin and that fall sufficiently fast when $r \to \infty$. Thus, in this limit,

$$B_i \to 0\,, \tag{2.31}$$

$$D_i\phi \to 0\,, \tag{2.32}$$

$$V(\phi) \to 0\,. \tag{2.33}$$

One solution that satisfies all these constraints is the trivial field configuration, i.e., $\phi = \phi_0 =$ constant so that $V(\phi_0) = 0$ and $W_i = 0$. However, this is not physically interesting. But, an important concept in what follows is that of the *Higgs vacuum*. We shall say that the fields in a certain region are in the Higgs vacuum if equations (2.32) and (2.33), but not necessarily (2.31), are satisfied.

Now, it is relevant to recall that the values of ϕ which minimize the Higgs potential $V(\phi)$ lie on a manifold $\mathcal{M}$. Then, we can consider a nontrivial vacuum configuration $\phi_0 \in \mathcal{M}$, which produces a spontaneous symmetry breaking (SSB) of the form $G \to G_0$. From the invariance of $V(\phi)$ under the action of G, i.e. $V(g\,\phi\,g^{-1}) = V(\phi)$ for any $g \in G$, it follows that

$$\phi = g\,\phi_0\,g^{-1}\,, \tag{2.34}$$

is also a vacuum configuration lying in $\mathcal{M}$. To be more specific, this relation characterizes the so-called *G-orbit of* ϕ_0, which is defined to be [32]

$$\mathcal{O}(\phi_0) = \{\phi \in \mathcal{M} \,|\, \phi = g\,\phi_0\,g^{-1},\ \text{for some } g\ \in G\}\,.$$

In the special case $\mathcal{M}$ consists of a single orbit, then the action of G in $\mathcal{M}$ is said to be transitive [32]. Moreover, to any such ϕ_0 of $\mathcal{M}$, one associates a subgroup of G, the *isotropy subgroup* or *stabilizer* G_0, defined to be

$$G_0 = \{g \in G \,|\, g\,\phi_0\,g^{-1} = \phi_0\}\,,$$

which is the remaining gauge symmetry after the SSB, i.e., G_0 is the unbroken gauge group. Then, one can show that the generators T_a of the Lie algebra of G_0, which we shall denote by $L(G_0)$, satisfy

$$L(G_0) = \{T_a \in L(G) | [T_a, \phi_0] = 0\}\,. \tag{2.35}$$

Besides that, different points in the orbit $\mathcal{O}(\phi_0)$ will have stability groups which are isomorphic, but different, subgroups of G. If we move from one point of the orbit to another by means of eq. (2.34), we can get the generators of the isomorphic stability groups by conjugation. In the mathematical literature [32] is said that stabilizers of elements in the same orbit are conjugate subgroups. This means that the isomorphic unbroken subgroups will be given by $g\,G_0\,g^{-1}$.

Therefore, from the discussion above, we see that two elements g and g' will give the same ϕ if

$$g' = gg_0, \tag{2.36}$$

where g_0 denotes an element of G_0. Thus, because of eqs. (2.34) and (2.36) the set of all vacuum configurations lying in $\mathcal{M}$ which produce a SSB of the form $G \to G_0$, may be identified with the quotient space G/G_0. And, from now on, we shall refer to G/G_0 as the *vacuum manifold* associated with the desired symmetry breaking $G \to G_0$.

Therefore, from the analysis above, for the energy to be finite, the Higgs field must take values in G/G_0 when $r \to \infty$. This means that

$$\phi(\theta, \varphi) \equiv \lim_{r\to\infty} \phi(r, \theta, \varphi)\,, \tag{2.37}$$

gives a smooth map from the two-sphere at spatial infinity (S^2_∞) to the vacuum manifold G/G_0. Without loss of generality we can use the available gauge freedom to impose that $\phi(\theta = 0, \varphi) = \phi_0$. In mathematical language, this is our choice of a base-point.

If two configurations $\phi(\theta,\varphi)$ and $\phi'(\theta,\varphi)$ can be deformed into each other while keeping the value at $\theta=0$ fixed, then they are said to be in the same homotopy class. That is, if there exists a continuous function $\mathcal{T}(\theta,\varphi,t)$ such that

$$\begin{aligned}\mathcal{T}(\theta,\varphi,0) &= \phi(\theta,\varphi)\,,\\ \mathcal{T}(\theta,\varphi,1) &= \phi'(\theta,\varphi)\,,\\ \mathcal{T}(\theta=0,\varphi,t) &= \phi_0\,.\end{aligned}$$

Moreover, the set of homotopy classes of maps from the n-sphere to a topological space $\mathcal{X}$, with a group structure, defines the n-th homotopy group $\pi_n(\mathcal{X})$ [37]. Then, from what we have discussed earlier, if we want topological monopole solutions to arise in a certain theory, we need $\pi_2(G/G_0)$ to be non-trivial. Of course, this is a necessary, but not sufficient, condition for finding topologically stable monopoles. Furthermore, we should emphasize that in the physically-oriented literature, homotopy classes are usually called *topological sectors*. And this is a term we might use from now on.

From the above discussion, in order for the configuration to have finite energy, one can see that the asymptotic configuration of the Higgs field $\phi(\theta,\varphi)$ can be conveniently constructed as

$$\phi(\theta,\varphi) = g(\theta,\varphi)\,\phi_0\,g^{-1}(\theta,\varphi), \tag{2.38}$$

with $g(\theta=0,\varphi)=1$. However, note that for a given $\phi(\theta,\varphi)$, the choice of $g(\theta,\varphi)$ is not unique. Besides that, even though $\phi(\theta,\varphi)$ must be single-valued, $g(\theta,\varphi)$ can be multiple-valued, as long as the ambiguity corresponds to right multiplication by an element of G_0. One can show that [28] $g(\theta,\varphi)$ can always be chosen so that it is multiple-valued only when $\theta=\pi$, which means that

$$g(\pi,\varphi) = g(\pi,0)g_0(\varphi)\,, \tag{2.39}$$

where

$$g_0(\varphi+2\pi) = g_0(\varphi)\,. \tag{2.40}$$

The function $g_0(\varphi)$ maps the circle into G_0 and thus corresponds to an element of the first homotopy group $\pi_1(G_0)$. Two configurations $\phi(\theta,\varphi)$ and $\phi'(\theta,\varphi)$ belong to the same homotopy class if and only if the corresponding $g_0(\varphi)$ and $g_0'(\varphi)$ correspond to the same element of $\pi_1(G_0)$ [28]. In other words, this is the well-known result that $\pi_2(G/G_0)\cong\pi_1(G_0)$, where $\cong$ denotes an isomorphism, when G is compact and simply-connected. Therefore, monopoles with nontrivial topological charges might arise only if $\pi_1(G_0)$ is not trivial. We reinforce that this is a necessary, but not sufficient, condition. Throughout this work, it will be clear that this constraint implies that only monopoles with a non-trivial abelian magnetic charge, i.e., a charge in the abelian direction of ϕ_0, can be topologically stable.

Similarly, we can think of what must happen to the gauge field $W_i(\mathbf{r})$. Due to the finite energy constraint, we also need to impose that

$$W_i(\theta,\varphi) = \lim_{r\to\infty} W_i(r,\theta,\varphi) \tag{2.41}$$

and that the asymptotic configuration $W_i(\theta,\varphi)$ is a gauge transformation of some $W_i^{(0)}$, so that

$$W_i(\theta,\varphi) = g(\theta,\varphi)\, W_i^{(0)}\, g^{-1}(\theta,\varphi) + \frac{i}{e}\left(\partial_i g(\theta,\varphi)\right) g^{-1}(\theta,\varphi)\,. \tag{2.42}$$

In fact, $W_i^{(0)}$ must be a solution to the equations of motion in the absence of sources, i.e., $D_i G^{ij} = 0$. But, we shall discuss this issue in more details in section 2.3.3.

Finally, we recall that the finite energy constraint requires $D_i\phi \to 0$ when $r \to \infty$, i.e., in this limit the Higgs and gauge fields must be such that

$$\partial_i\phi + ie[W_i,\, \phi] \to 0\,.$$

Then, we should keep in mind for further analysis that any chosen field configuration must satisfy this constraint.

2.3.2 Spherical Symmetry

Firstly, the considerations we have done so far assure us that the monopole solution will be static and have finite energy. However, we can also demand it to have spherical symmetry, in order to be a single monopole instead of a bound state. Moreover, we expect more symmetric solutions to have lower energy.

Let

$$J_i = -i\epsilon_{ijk} r_j \partial_k + M_i \tag{2.43}$$

be the generator of generalized angular momentum, where M_i are the generators of a $su(2)$ subalgebra of G. A configuration involving $\phi(\mathbf{r})$ and $W_i(\mathbf{r})$ is spherically symmetric when [6, 38]

$$[J_i, \phi] = 0\,, \tag{2.44}$$

$$[J_i, W_j] = i\epsilon_{ijk} W_k\,. \tag{2.45}$$

The first condition implies that ϕ is a scalar under generalized rotations, while the second means that the gauge field transform as a vector under J_i. Even though one can be interested in monopoles with no spherical symmetry, in this work we shall only discuss spherically symmetric solutions.

2.3.3 The String Gauge Construction

Up to this point we have discussed how to construct the asymptotic configuration of the scalar and gauge fields. Our main result was that the finite energy constraint led the field configurations over S^2_∞ to be a gauge transformation of an original set up, which must lie in the Higgs vacuum. And even though we have postponed the choice of such configurations, we shall analyze it in this section. We will adopt an approach similar to [27], which is based on [39] and frequently called the *Dynamical (or GNO) Classification.*

Nevertheless, before this we have to analyze what happens to the scalar field ϕ. In fact, our choice of ϕ_0 determines the SSB pattern (see, for instance, [7, 40]). Since we are working with a real adjoint Higgs field, we can write that

$$\phi = \phi_a T_a \, , \tag{2.46}$$

where T_a are hermitian generators of $L(G)$. However, any normal generator of a compact Lie group can be rotated into the Cartan subalgebra $\mathcal{H}$, i.e., for any ϕ there is a $g \in G$ such that [40]

$$\phi' = g\,\phi\,g^{-1} = \sum_{i=1}^{r} \phi_i H_i \, , \tag{2.47}$$

where r denotes the rank of $L(G)$ and the H_i are the Cartan generators. The components of ϕ_i are not unique and, in fact, the ambiguity is given by the Weyl group. However, the discussion of the Weyl chambers is out of the scope of this work. For more details, please, see [40].

We shall write ϕ_0 in a more convenient form

$$\phi_0 = u \cdot H \, , \tag{2.48}$$

where u denotes a constant vector. Based on this choice and making use of the commutation relations between Cartan generators H_i and the step operators $E_{\pm\alpha}$, one can determine the gauge symmetry breaking pattern, just with the information of the Dynkin diagram of $L(G)$ [40]. However, we shall discuss this topic in more details in section 3.1. In summary, we know that ϕ_0 must be constant while it has to minimize the potential $V(\phi)$, such that $V(\phi_0) = 0$. We also need $D_i\phi_0 = 0$, which implies that any chosen configuration for $W_i^{(0)}$ will satisfy $[W_i^{(0)}, \phi_0] = 0$.

Now, we shall concentrate on the gauge field configurations. Far from the monopole "core", in a gauge where $\phi = \phi_0 =$ constant [30], we only need to deal with the sourceless field equations $D_\mu G^{\mu\nu} = 0$. In other words we shall look for monopole-like solutions in a specific gauge, called the *string gauge*. The reason for this name will be clear soon. It is also important to emphasize that we will not attempt to construct a complete series of solutions to these equations, but we will just try to find the non-abelian generalization of the magnetic monopole field.

As we have discussed previously, the fact that our solution is static, together with our choice of $W_0 = 0$, implies that $G_{0i} = 0$. However, we have the freedom to make time-independent gauge transformations.

Our first step is to make $W_r = 0$. The way we can do this is quite simple. Since the gauge fields transform as

$$W_i' = U\,W_i\,U^{-1} + \frac{i}{e}\,(\partial_i U)\,U^{-1}\,, \tag{2.49}$$

we can find a transformation U (with $U^\dagger U = \mathbb{I}$) such that the i-th component $W_i' = 0$, i.e.

$$\partial_i U = ie\,U\,W_i. \tag{2.50}$$

The formal solution to this problem is given by the Dyson series [41]. Then, in order to set $W_r = 0$, the necessary transformation is of the form

$$U(r,\theta,\varphi) = P\exp\left[-ie\int_R^r dr' W_r'(r',\theta,\varphi)\right]\,, \tag{2.51}$$

where P indicates that the terms in the expansion of the exponential should be ordered so that the smallest argument r appears on the left. Note that we have excluded the region $r < R$ from the integration, where R stands for some convenient distance from the core. We shall neglect a possible singularity at the origin (where radial lines intersect), since we have focused on the asymptotic behavior only.

The next step is to make $W_\theta = 0$ by means of a gauge transformation, independent of t and r, in a way similar to eq. (2.51). In this transformation we integrate along lines of fixed θ, meridians, starting from the north pole. This choice of gauge can lead to an artificial singularity at the place where all meridians intersect, the south pole. In particular, it can lead to a non-zero (and φ dependent) $W_\varphi(\pi,\varphi)$, i.e., to a Dirac string singularity. We will see later on that despite the abelian case, this singularity can be removed in non-abelian theories.

Then, we can finally use the field equations. The only non-vanishing component of the field strength tensor is

$$G_{\theta\varphi} = \partial_\theta W_\varphi \, . \tag{2.52}$$

In curvilinear coordinates, the sourceless Yang-Mills equations take the form

$$\partial_\mu \left(\sqrt{-g}\, G^{\mu\nu}\right) + ie \left[W_\mu, \sqrt{-g}\, G^{\mu\nu}\right] = 0 \, . \tag{2.53}$$

In our specific case,

$$\sqrt{-g}\, G^{\theta\varphi} = \frac{1}{r^2 \sin\theta}\, \partial_\theta W_\varphi \, , \tag{2.54}$$

where $[g_{\mu\nu}] = diag\left(1, -1, -r^2, -r^2 \sin^2\theta\right)$ and, as a consequence, $\sqrt{-g} = r^2 \sin\theta$.

Thus, there are two non-trivial field equations. One of them happens when $\nu = \varphi$, and it is given by

$$\partial_\theta \left(\frac{1}{\sin\theta}\, \partial_\theta W_\varphi\right) = 0 \, , \tag{2.55}$$

where we have, once more, considered that $r \neq 0$. We can look for a general solution under the boundary condition $W_\varphi(0, \varphi) = 0$, in agreement with our construction where the north pole has no singularities. The general solution is of the form

$$W_\varphi(\theta, \varphi) = (1 - \cos\theta)\, Q(\varphi) \, , \tag{2.56}$$

where $Q(\varphi)$ is a matrix-valued function. On the other hand, when $\nu = \theta$ the field equation is

$$\partial_\varphi \left(\partial_\theta W_\varphi\right) + ie\cancelto{0}{[W_\varphi, (\partial_\theta W_\varphi)]} = 0 \, , \tag{2.57}$$

which implies that

$$\partial_\varphi Q(\varphi) = 0 \, . \tag{2.58}$$

Therefore, Q must be a constant. However, we should also remember that, in this string gauge, for $\phi = \phi_0$ to be covariantly constant we need Q to take values in the Lie algebra of the unbroken gauge group G_0.

Thus, a magnetic monopole configuration in the *string gauge* is given by

$$\phi = \phi_0 \,, \tag{2.59}$$
$$W_\varphi^{(0)} = (1 - \cos\theta)\, Q \,, \tag{2.60}$$

with $Q \in L(G_0)$. Note that the name of this gauge is due to the existence of a string singularity in eq. (2.60) along the negative z-axis, which is similar to the abelian gauge field of eq. (2.6), in the case of a Dirac monopole. But, in the non-abelian case, we will show that under a convenient gauge transformation we can obtain a smooth asymptotic configuration for both $\phi(\theta, \varphi)$ and $W_i(\theta, \varphi)$.

2.3.4 The final asymptotic configuration

Since we have already found a general form for the field configurations in the string gauge, we are now able to use eqs. (2.38) and (2.42), together with eqs. (2.59) and (2.60), to construct a smooth asymptotic configuration for a magnetic monopole and then, propose an ansatz for the whole space. We want our non-abelian monopole to be regular everywhere, which implies that we shall be able to remove the Dirac string singularity. We will present a method for this construction here.

Let the generator Q be

$$Q = \frac{1}{e}\, M_3 \,, \tag{2.61}$$

with $M_3 \in L(G_0)$. Now, let us consider that there exists two other generators $M_1, M_2 \in L(G)$, but not in $L(G_0)$, such that

$$[M_i, M_j] = i\, \epsilon_{ijk}\, M_k \,. \tag{2.62}$$

We will call the M_i, $i = 1, 2, 3$, as the monopole generators. Then, with the commutation relations of eq. (2.62) together with the Baker-Campbell-Hausdorff (BCH) formula [32], we obtain that

$$e^{i\beta T_j} T_i e^{-i\beta T_j} = (\cos\beta)\, T_i + (\sin\beta)\, \epsilon_{ijk}\, T_k \,, \tag{2.63}$$

for $i \neq j$ and where β is an arbitrary parameter.

Moreover, let us define a group element $g(\theta, \varphi) \in G$, given by

$$g(\theta, \varphi) = \exp\left(-i\varphi M_3\right) \exp\left(-i\theta M_2\right) \exp\left(+i\varphi M_3\right) \,. \tag{2.64}$$

Note that the element in eq. (2.64) is single-valued, whenever $\theta \neq \pi$. When $\theta = \pi$, we recall the definition of eq. (2.39) to obtain that

$$g_0(\varphi) = g(\pi, \varphi) g^{-1}(\pi, 0) = \exp(-2i\varphi M_3)\,. \tag{2.65}$$

The condition (2.40) for this loop to be single-valued implies that

$$\exp\left(4\pi i M_3\right) = 1\,, \tag{2.66}$$

which is the generalization of the Dirac quantization condition for non-abelian monopoles. Provided that M_3 is a generator of a $su(2)$ subalgebra, its eigenvalues are integers or half-integers. This means that eq. (2.66) is always satisfied. Moreover, whether this loop is contractible or not will inform us when the monopole solutions belong to a non-trivial topological sector.

Besides the quantization condition, it is interesting to note that eq. (2.63) implies that

$$g(\theta, \varphi)\, M_3\, g^{-1}(\theta, \varphi) = n^a M_a\,, \tag{2.67}$$

with $n^a = x^a/r$. This is usually called the *hedgehog* configuration. Furthermore, using eq. (2.63), it is easy to show that under a gauge transformation as in eq. (2.42) by a gauge group element given by eq. (2.64), the Dirac string is removed. In appendix A we show this calculation in details. In fact, after the gauge transformation, the cartesian components of the gauge field can be written as

$$W_i(r \to \infty) = -\epsilon_{ijk}\, \frac{n^j}{er}\, M_k\,. \tag{2.68}$$

The gauge field configuration gives rise to the asymptotic magnetic monopole field

$$B_i(r \to \infty) = -\frac{n^i}{er^2}\, n^a M_a = -\frac{x^i}{er^3}\, g M_3 g^{-1}\,. \tag{2.69}$$

With regard to the scalar field ϕ, there is little we can in a general situation. Actually, the form of its asymptotic configuration, given by eq. (2.38), will strongly depends on ϕ_0 and also on the choice of the $su(2)$ subalgebra of the monopole generators. That is, we cannot calculate a completely general situation, since it depends on the commutation relations between the generators of $L(G)$. Instead, we present here a quite general situation, which will be useful for all the monopole solutions we analyze in this work.

Let us suppose that $\phi_0 = u \cdot H$ can be written as[2]

$$\phi_0 = v\,(S + \omega\, Q_0) \tag{2.70}$$

where v is the vacuum expectation value (VEV) of the Higgs field, while ω is some convenient constant in the decomposition and $[M_3, S] = 0 = [M_3, Q_0]$. Now, let us define

$$M_\pm = M_1 \pm i M_2\,. \tag{2.71}$$

In our construction the generator $S \in L(G)$ is such that $[S, M_\pm] = 0$. This means that S is a singlet under the $su(2)$ that generates the monopole since $[S, M_i] = 0, \forall\, i = 1, 2, 3$. On the other hand, let us suppose that Q_0 belongs to a set of generators Q_m, with $m = 0, \pm 1, \ldots, \pm l$, which satisfy the commutation relations

$$[M_3, Q_m] = m\, Q_m\,, \tag{2.72}$$

$$[M_\pm, Q_m] = c^{\pm}_{l,m}\, Q_{m\pm 1}\,, \tag{2.73}$$

where $c^{\pm}_{l,m} = \sqrt{l(l+1) - m(m\pm 1)}$. In fact, Q_m are the so-called *standard components of irreducible tensor operators* [32]. They form a (2l+1)-plet of the $su(2)$ subalgebra. Thus, in the situation we defined above we can find the asymptotic form of the Higgs field

$$\phi(\theta, \varphi) = v\, g(\theta, \varphi)\, (S + \omega\, Q_0)\, g^{-1}(\theta, \varphi)\,,$$

where $g(\theta, \varphi)$ is given by eq. (2.64).

First, let us recall that since S is a singlet, it is trivial to see that

$$g(\theta, \varphi)\, S\, g^{-1}(\theta, \varphi) = S\,. \tag{2.74}$$

Then, we only need to perform the following transformation

$$g(\theta, \varphi)\, Q_0\, g^{-1}(\theta, \varphi)\,,$$

which we shall compute making use of spherical harmonics and properties of irreducible representations of $su(2)$ algebras.

Let us recall that in a $(2j+1)$ irreducible representation of a $su(2)$ algebra with generators J_i, $i = 1, 2, 3$, and with eigenstates $|j, m\rangle$, the spherical harmonics can be written as [42],

$$Y^*_{jm}(\theta, \varphi) = \sqrt{\frac{2j+1}{4\pi}}\, D^j_{m0}(\phi, \theta, 0),$$

[2] We shall discuss a more general situation later on.

where

$$D^j_{m0}(\phi,\theta,0) = \langle j,m|\exp(-i\varphi J_3)\exp(-i\theta J_2)\,|j,0\rangle = e^{-i\varphi m}\, d^j_{m0}(\theta)$$

and

$$\begin{aligned} d^j_{m0}(\theta) &= \langle j,m|\exp(-i\theta J_2)\,|j,0\rangle \\ &= \delta_{m0} + \sum_{n=1}^{\infty} \frac{(-i\theta)^n}{n!} \left[\left(D^j(J_2)\right)^n\right]_{m0}\,, \end{aligned}$$

with $D^j(J_i)_{m'm} = \langle j,m'|\,J_i\,|j,m\rangle$.

The commutation relations (2.72) and (2.73) can be written as

$$[M_i, Q_m] = D^l(M_i)_{m'm} Q_{m'}\,, \tag{2.75}$$

where $D^l(M_i)_{m'm}$ is the $(2l+1)-$dimensional representation of the $su(2)$ generator M_i in the basis of the Q_m's. Then,

$$\begin{aligned} \exp(-i\theta M_2)\; Q_0\; \exp(i\theta M_2) &= Q_0 + \sum_{n=1}^{\infty} \frac{(-i\theta)^n}{n!} \underbrace{[M_2,[M_2,\ldots,[M_2,Q_0]]]}_{n} \\ &= \sum_m \left\{\delta_{m0} + \sum_{n=1}^{\infty} \frac{(-i\theta)^n}{n!} \left[\left(D^l(M_2)\right)^n\right]_{m0}\right\} Q_m \\ &= \sum_m d^l_{m0}(\theta)\, Q_m\,. \end{aligned}$$

Hence,

$$\begin{aligned} g(\theta,\varphi)\, Q_0\, g(\theta,\varphi)^{-1} &= \sum_m e^{-i\varphi m}\, d^l_{m0}(\theta)\, Q_m \\ &= \left(\frac{4\pi}{2l+1}\right)^{1/2} \sum_m Y^*_{lm}(\theta,\varphi)\, Q_m\,. \end{aligned} \tag{2.76}$$

Therefore, it follows from eqs. (2.74) and (2.76) that

$$\phi(\theta,\varphi) = v\,S + \alpha \sum_{m=-l}^{+l} Y^*_{lm}(\theta,\varphi)\, Q_m\,, \tag{2.77}$$

with

$$\alpha = v\omega\sqrt{\frac{4\pi}{2l+1}}\,. \tag{2.78}$$

In summary, when ϕ_0 can be decomposed in a singlet and a $(2l+1)$-plet, we obtain an asymptotic configuration of the form of eq. (2.77).

From the results of eqs. (2.68), (2.69) and (2.77) we can propose an ansatz for the whole space. From the asymptotic gauge field configuration (2.68), one can propose the ansatz

$$W_i = -\frac{[1-u(r)]}{er}\epsilon_{ijk}n^j M_k \,, \tag{2.79}$$

with the radial function $u(r)$ satisfying the conditions, $u(r=0)=1$ and $u(r\to\infty)=0$. From this gauge field we obtain the magnetic field[3]

$$B_i = \left(\frac{u'}{er}\,P_T^{ik} + \frac{u^2-1}{er^2}\,P_L^{ik}\right) M_k \,, \tag{2.80}$$

where $P_T^{ik} = \delta^{ik} - n^i n^k$, $P_L^{ik} = n^i n^k$ and $u'(r)$ stands for du/dr.

Finally, we recall eq. (2.77) to propose an ansatz for the scalar field, who takes the form

$$\phi(\mathbf{r}) = \phi_S + \phi_q \,, \tag{2.81}$$

where

$$\phi_s = v\, S, \tag{2.82}$$

$$\phi_q(r,\theta,\varphi) = \alpha\, f(r) \sum_m Y^*_{lm}(\theta,\varphi)\, Q_m \,. \tag{2.83}$$

where the radial function $f(r)$ satisfies $f(r=0)=0$ and $f(r\to\infty)=1$. Note that the boundary conditions for $u(r)$ and $f(r)$ are chosen such that the monopole is regular at the origin. This means that when $r\to 0$ the function $u(r)$ must behave as $u(0)-1 \propto -r^2$, while $f(r)$ must increase with r^l. On the other hand, when $r\to\infty$ both the gauge and Higgs fields must approach its asymptotic values, which justifies why $u(r\to\infty)=0$ and $f(r\to\infty)=1$.

Additionally, note that our construction indeed gives rise to a spherically symmetric monopole. Using the result

$$i\epsilon_{kab}\, x^a\, \partial_b Y^*_{lm} = D^l(M_k)_{m'm} Y^*_{lm'} \,, \tag{2.84}$$

it is straightforward to show that our scalar and gauge fields satisfy the conditions (2.44) and (2.45).

Before we proceed to the next section, let us add a relevant commentary. When we constructed our ansatz we have considered that the

[3] The detailed calculation is available at appendix A.

scalar field ϕ could be written as the sum of two terms, one that is a singlet under $su(2)$ and another one which transforms as a $(2l+1)$-plet. Although this is not the most general case, it is sufficient to construct all the monopoles we shall discuss. However, this construction can be extended to a more general class of vacuum configurations ϕ_0.

Let us suppose that ϕ_0 can be decomposed as

$$\phi_0 = v \sum_l c_l \, Q_0^l \tag{2.85}$$

where c_l are coefficients that may be necessary in the decomposition. The generators Q_0^l, with $l = 1, 2, \ldots, n$, are just a generalization of our original Q_0, i.e., each one of them belongs to a set of $2l+1$ generators Q_m^l, where $m = 0, \pm 1, \ldots, \pm l$, satisfying the commutation relations of eqs. (2.72) and (2.73). Thus, we can extend the result of eq. (2.76) to each one of the terms in the expansion (2.85). This implies that

$$\phi(\theta, \varphi) = v \sum_{l=0}^{n} \sum_{m=-l}^{+l} \alpha_l \, Y_{lm}^{*} Q_m^l \,, \tag{2.86}$$

where

$$\alpha_l = c_l \sqrt{\frac{4\pi}{2l+1}} \,. \tag{2.87}$$

Note how this is close to a multipole expansion in ordinary electrodynamics [43]. Now, with regard to an ansatz for the whole space, we must be careful. If, by any chance, one needs to construct magnetic monopoles in agreement with eq. (2.86), some caution must be taken with the profile functions to be introduced and the boundary conditions at the origin.

Finally, using the same arguments as before, the field configuration (2.86), together with eq. (2.79), is spherically symmetric with respect to the generalized angular momentum defined in eq. (2.43).

2.4. THE 'T HOOFT-POLYAKOV MONOPOLE REVISITED

In the previous sections we have constructed a quite general ansatz for non-abelian magnetic monopoles, for the physically interesting case of a scalar field ϕ in the adjoint representation. Now we shall use our general results in order to construct and discuss the simplest monopole solution, found in the $SO(3)$ Georgi-Glashow model by 't Hooft and Polyakov [3, 4], independently, in 1974.

Let us consider a theory with the gauge group $G = SU(2)$, whose generators M_a, $a = 1, 2, 3$, satisfy $Tr(M_a M_b) = y\,\delta_{ab}$. We also consider a Higgs field ϕ in the 3-dimensional representation. The action is given by eq. (2.19), while the potential term $V(\varphi)$ is taken to be the usual *mexican hat* potential, given by

$$V(\phi) = \frac{\lambda}{4}\left(\frac{\mathrm{Tr}(\phi\phi)}{y} - v^2\right)^2 , \tag{2.88}$$

where $\lambda \in \mathbb{R}^+$. The ground state ϕ_0 is chosen to be

$$\phi_0 = v\, M_3 , \tag{2.89}$$

which breaks $SU(2)$ down to $G_0 = U(1)$. Since the only choice for the monopole generators of eq. (2.62) is the full set of generators in $L(G)$, we have already named them as M_a. Then, from eq. (2.60) the gauge field configuration in the string gauge is

$$W_\varphi^{(0)} = \frac{1-\cos\theta}{e}\, M_3 .$$

Thus, the gauge field takes the exact form of eq. (2.68).

With regard to the Higgs field, note that there is a subtlety in the definition of the tensor operators Q_m. Since $L(G)$ contains only the generators of a $su(2)$ algebra, the correct commutation relations of eqs. (2.72) and (2.73), with $l = 1$, can be obtained with

$$\begin{aligned} Q_0 &= M_3 , \\ Q_{\pm 1} &= \mp \frac{M_\pm}{\sqrt{2}} . \end{aligned}$$

From the definition of $M_\pm$ in eq. (2.71), it follows directly that $[M_\pm, M_\mp] = \pm 2 M_3$ and $[M_\pm, M_3] = \mp M_\pm$. Then, it is trivial to check that $[M_\pm, Q_0] = \sqrt{2}\, Q_{\pm 1}$ and $[M_\pm, Q_{\mp 1}] = \sqrt{2} Q_0$. Likewise, it also follows that $[M_3, Q_m] = m\, Q_m$, with $m = -1,\, 0,\, +1$. Also note that there is no singlet term S, since from eq. (2.89) we see that $\phi_0 = v\, Q_0$.

Thus, we can use our general result of eq. (2.77), in the particular case of $l = 1$, in order to obtain that the Higgs field asymptotic configuration is given by

$$\phi(\theta, \varphi) = v \sqrt{\frac{4\pi}{3}} \sum_{m=-1}^{+1} Y_{1m}^* Q_m = v\, n^a M_a . \tag{2.90}$$

We could also be more compact and say that this is the result of eq. (2.86) when only $l = 1$ gives rise to non-trivial terms. Moreover, it is relevant to add that there is an easier way to obtain this asymptotic configuration, since one could recall the fact that both ϕ_0 and $W_\varphi^{(0)}$ are in the same direction and lie only in the $su(2)$ algebra of the monopole generators. Then, from eq. (2.67), we directly obtain that $\phi(\theta,\varphi) = v\, n^a M_a$. Nevertheless, our general procedure provides us with techniques to deal with more complicated situations. In chapter 4, for instance, we shall work with a configuration which is much more difficult to be computed by the standard methods.

Therefore, in full agreement with our previous results of eqs. (2.68) and (2.77), we propose an ansatz of the form

$$\begin{aligned} \phi &= v\, f(r)\, n^a M_a\,, \\ W_i &= -\frac{[1-u(r)]}{er}\epsilon_{ijk} n^j M_k\,, \end{aligned} \tag{2.91}$$

with the aforementioned boundary conditions. Note that, again, the magnetic field is given by eq. (2.80), i.e.,

$$B_i = \left(\frac{u'}{er}\, P_T^{ik} + \frac{u^2-1}{er^2}\, P_L^{ik}\right) M_k\,.$$

Then, we can plug the ansatz of eq. (2.91) into eq. (2.24) in order to obtain the well-known hamiltonian E of the 't Hooft-Polyakov monopole. For the sake of clarity, we shall show some steps of the calculation here. However, in section 4.2 we will show how one can calculate a general hamiltonian for a Higgs field of the form of eq. (2.77), i.e., written in terms of the spherical harmonics and the tensor operators Q_m. So, in order to avoid repetition of our calculations through the chapters, we will proceed with our present calculation, but making use of the form $\phi(\theta,\varphi) = v f(r)\, n^a M_a$ only.

First of all, let us calculate the term $\mathrm{Tr}(D_i\phi D_i\phi)$ in the hamiltonian. From the ansatz of eq. (2.91), we see that

$$D_i\phi = v\left[(\partial_i f(r))\, n^a M_a + f(r)\, D_i\left(n^a M_a\right)\right]\,.$$

Using that $\partial_i f(r) = n^i f'(r)$, where $f'(r)$ stands for $\frac{df(r)}{dr}$, and the ansatz of the gauge field, which is necessary in the gauge covariant derivative expression (2.21), we obtain that

$$D_i\phi = vf'(r)P_L^{ia}\, M_a + v\,\frac{f(r)}{r}\left[P_T^{ia} M_a + (1-u)\epsilon_{ijk}\epsilon_{abk} n^j n^a M_b\right]\,,$$

where we also used that $\partial_i\, n^a = (1/r)P_T^{ia}$. Then, from the properties of the Levi-Civita symbol, we get that

$$D_i\phi = v\left(f'\,P_L^{ia} + \frac{fu}{r}\,P_T^{ia}\right)M_a\,. \tag{2.92}$$

Thus, recalling that the projectors $P_L^{ia} = n^i n^a$ and $P_T^{ia} = (\delta^{ia} - n^i n^a)$ are such that

$$\begin{aligned} P_L^{ia}P_L^{ib} &= P_L^{ab}\,,\\ P_T^{ia}P_T^{ib} &= P_T^{ab}\,,\\ P_L^{ia}P_T^{ib} &= 0\,, \end{aligned}$$

where we used the implicit sum convention, we obtain that $P_L^{ia}P_L^{ia} = 1$ and $P_T^{ia}P_T^{ia} = 2$. Therefore, the kinetic term for the Higgs field in the hamiltonian will be given by

$$\frac{1}{2y}\,\mathrm{Tr}\,(D_i\phi D_i\phi) = \frac{1}{2}\,(f')^2 + \frac{(fu)^2}{r^2}\,. \tag{2.93}$$

Note that we used the fact that $\mathrm{Tr}(M_a M_b) = y\delta_{ab}$.

Now, using the same arguments of the trace between generators M_a and the properties of the above projectors, we obtain that

$$\frac{1}{2y}\,\mathrm{Tr}\,(B_iB_i) = \frac{(u')^2}{e^2r^2} + \frac{1}{2}\,\frac{(u^2-1)^2}{e^2r^4}\,. \tag{2.94}$$

Finally, the contribution from the mexican hat potential is the simplest one. From the fact that,

$$\frac{1}{y}\,\mathrm{Tr}\,(\phi\phi) = v^2 f^2\,,$$

it follows that

$$V(\phi) = \frac{\lambda}{4}\,v^2\left(f^2-1\right)^2\,. \tag{2.95}$$

Furthermore, we must join all the contributions from eqs. (2.93) to (2.95) and use the spherical symmetry of the integrand in order to obtain the 't Hooft-Polyakov hamiltonian. After a change of variables where $\xi = evr$, we obtain that E is given by

$$\begin{aligned} E = \frac{4\pi v}{e}\int_0^\infty d\xi\,\Bigg\{(u')^2 + \frac{1}{2}\,\frac{(u^2-1)^2}{\xi^2} + \frac{1}{2}\,\xi^2(f')^2 \\ + u^2f^2 + \frac{\lambda}{4e^2}\,\xi^2(f^2-1)^2\Bigg\}\,, \end{aligned} \tag{2.96}$$

where $u'(\xi)$ and $f'(\xi)$ denote derivatives with respect to ξ.

The conditions for E to be stationary with respect to $f(\xi)$ and $u(\xi)$ provide the equations of motion for the ansatz of the 't Hooft-Polyakov monopole

$$u'' = f^2 u + \frac{u(u^2-1)}{\xi^2}\,, \tag{2.97a}$$

$$f'' = -\frac{2}{\xi} f' + \frac{2}{\xi^2}\, f u^2 + \frac{\lambda}{e^2}\, f(f^2-1)\,. \tag{2.97b}$$

The appropriate boundary conditions (with respect to ξ) are

$$f(0) = 0, \qquad u(0) = 1 \tag{2.98}$$

$$f(\xi \to \infty) = 1, \qquad u(\xi \to \infty) = 0. \tag{2.99}$$

This set of second order ordinary differential equations possess an analytical solution in the limit of $\lambda \to 0$ (with λ/e^2 finite), i.e., in the Prasad-Sommerfield limit [36]. In this case, they reduce to first order equations, which could also have been obtained by plugging the ansatz (2.91) into the Bogomolny equations (2.30b).

Now, since it is not possible to find an analytical solution for a general value of λ to eqs. (2.97a) and (2.97b), we can analyze what happens in the limit $\xi \to 0$ and $\xi \to \infty$ and then implement a numerical method.

When $\xi \to 0$ one can see that $u(\xi) - 1 \propto -\xi^2$, while $f(\xi) \propto \xi$ in order for the equations of motion to be satisfied in this region. On the other hand, when $\xi \to \infty$ one can check that

$$u(\xi) = O\left[\exp\left(-\xi\right)\right]\,, \tag{2.100}$$

$$f(\xi) - 1 = O\left[\frac{\exp\left(-\sqrt{\frac{2\lambda}{e^2}}\,\xi\right)}{\xi}\right]. \tag{2.101}$$

Note that the behavior for $\xi \gg 1$ can be conveniently written in terms of r as

$$u(r) = O\left[\exp\left(-Mr/\hbar\right)\right] \quad \text{and} \quad f(r) - 1 = O\left[\frac{\exp\left(-\mu r/\hbar\right)}{evr}\right],$$

where $M = ev\hbar$ and $\mu = \sqrt{2\lambda}$ are the masses of the massive particles in the theory. The approach to the asymptotic form is thus given by the

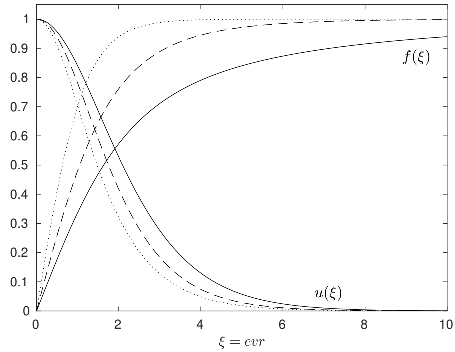

Figure 2.1: The monopole profile functions $u(\xi)$ and $f(\xi)$ for $\lambda/e^2 = 0$ (solid curves), $\lambda/e^2 = 0.1$ (dashed curves) and $\lambda/e^2 = 1$ (dotted curves).

Compton wavelength of the particle associated to the field in question. From these results, we see that we can think of the 't Hooft-Polyakov monopole as having a definite size, inside which the massive fields play a role in providing a smooth structure and outside which they rapidly vanish, leaving a field configuration exactly like the one in the Dirac Monopole [6].

With regard to the numerical solution we should recall that this problem has already been solved in details and with advanced numerical techniques. See, for instance, [44] and [45]. Nevertheless, we present in fig. 2.1 a numerical solution which was done with `MATLAB`® program `bvp4c`, for some particular coupling constant values. Besides that, in section 4.2.1, we will show how one can compute the approximate solutions for a broader case while in section 4.2.2, we shall also give more details on how the numerical solutions can be implemented.

At the moment we have the numerical solution, we can use it in order to obtain the classical mass of the monopole, given by eq. (2.96). First, let us define the rescaled mass $\tilde{E}_{tHP}(\lambda/e^2)$ [44], such that the hamiltonian (2.96) is written by

$$E = M_0 \tilde{E}_{tHP}(\lambda/e^2), \quad \text{where } M_0 = \frac{4\pi v}{e}\,.$$

Note that $\tilde{E}_{tHP}(\lambda/e^2)$ is a monotonically increasing function of λ/e^2, since

$$\frac{d\tilde{E}(\lambda/e^2)}{d(\lambda/e^2)} = \frac{1}{4}\int_0^\infty d\xi\,\xi^2(f^2-1)^2 > 0\,. \tag{2.102}$$

We obtain the lower bound when $\lambda = 0$ and numerical integration shows that [44] $\tilde{E}(0) = 1.000$, in agreement with the analytical solution found by Prasad and Sommerfield [36]. On the other hand, the upper bound can be obtained in the limit $\lambda \to \infty$, where the monopole mass stays finite. This is due to the fact that, in this limit, $f(\xi) = 1$, $\forall\,\xi > 0$ but $f(0) = 0$. Then, the mass is given by

$$E = \frac{4\pi v}{e}\int_0^\infty d\xi\left[(u'_\infty)^2 + \frac{(u_\infty^2-1)^2}{2\xi^2} + u_\infty^2\right]. \tag{2.103}$$

Thus, the only equation of motion is

$$u''_\infty = u_\infty + \frac{u_\infty(u_\infty^2-1)}{\xi^2}\,. \tag{2.104}$$

Performing the numerical solution of eq. (2.104) and making the integration of eq. (2.103) we obtain that $\tilde{E}(\infty) = 1.787$ [29, 44, 46] is the upper bound of the monopole mass.

2.4.1 The Magnetic Charge

In the last section, we have once more presented the form of the magnetic field (2.80) of a monopole constructed from the gauge field ansatz (2.79). But, note that far from the monopole core, B_i reduces to eq. (2.69), that is

$$B_i = -\frac{n^i}{er^2}\,g(\theta,\varphi)\,M_3\,g^{-1}(\theta,\varphi)\,,$$

where the magnetic field is spherically symmetric and it takes values in the local $U(1)$ symmetry, given by $g(\theta,\varphi)\,M_3\,g^{-1}(\theta,\varphi)$. This means that the 't Hooft-Polyakov monopole contributes to an abelian magnetic flux over S^2_∞. The magnetic charge g will be the natural measure of this flux.

Let us define

$$F^{\mu\nu} = \frac{1}{vy}\,\mathrm{Tr}\left(G^{\mu\nu}\phi\right)\,. \tag{2.105}$$

One can show that, at spatial infinity, this tensor satisfies Maxwell's equations in the Higgs vacuum [6, 47]. In order to prove this, we only need to use the condition $D_\mu\phi = 0$, the scalar field equations of motion and the Bianchi identity. Then, we shall call $F^{\mu\nu}$ the electromagnetic field strength. From this definition, the abelian magnetic charge in the $U(1)$ direction is defined as the surface integral

$$g = \oint_{S^2_\infty} dS_i \, {}^*F^{0i} = \frac{1}{vy} \oint_{S^2_\infty} dS_i \, \mathrm{Tr} \left({}^*G^{0i} \phi\right) \, . \tag{2.106}$$

Moreover, we should emphasize that this definition is valid for all the monopole solutions with an adjoint Higgs field, i.e., it is not restricted to the 't Hooft-Polyakov monopole.

Using the fact that

$$\frac{1}{vy} \, \mathrm{Tr} \left(B_i \phi\right) = -\frac{n^i}{er^2}$$

it follows that the magnetic charge of the 't Hooft-Polyakov monopole is

$$g = -\frac{4\pi}{e} \, . \tag{2.107}$$

On the other hand, we could have chosen an alternative approach where we define a magnetic current for the whole space, not only the Higgs vacuum. And even though there are two well-known possibilities for this, one suggested by 't Hooft [3] and the other one by Bogomolny [35] and Fadeev [48], we shall present the latter approach only, because the suggestion from 't Hooft is singular at the center of the monopole. Besides that, as pointed out by [49] there is no unambiguous way to measure the charge density of a monopole. Only the total charge makes sense.

Let us now extend the definition of eq. (2.105) to the whole space. The corresponding magnetic current is [6]

$$k^\mu = \frac{1}{vy} \, \partial_\nu \, \mathrm{Tr} \left({}^*G^{\mu\nu} \phi\right) \, . \tag{2.108}$$

The conservation of the magnetic current follows from its definition as the divergence of an antisymmetric and twice differentiable tensor. We will see later on how k^μ is indeed a topological current.

Now, since we are working in a frame where there are no electric fields, it follows that the only non-trivial component of k^μ is k^0 and it is given by

$$k^0 = \frac{1}{vy} \partial_i \mathrm{Tr} \left(B^i \phi\right) = \partial_i \left(\frac{u^2 - 1}{er^2} \, f(r) \, n^i \right) \, . \tag{2.109}$$

The integral

$$\int_{\mathbb{R}^3} d^3x\, k^0 = \frac{4\pi}{e} \int_0^\infty dr\, \frac{d}{dr} \left[f(u^2 - 1) \right] = -\frac{4\pi}{e}\,, \tag{2.110}$$

gives the magnetic charge g, as expected. Note that we have used the boundary conditions at the origin and at infinity for the radial functions u and f. Therefore, we know that the magnetic charge of a 't Hooft-Polyakov monopole is conserved. However, we can still use an alternative approach in order to prove it is also quantized.

2.4.2 The topological approach

In this section we shall follow the construction of [30]. In order to do so, let us first define some conventions.

Since ϕ, W_i and $D_i\phi$ take values in a $su(2)$ algebra, we can expand them as

$$\begin{aligned} \phi &= \phi_a M_a\,, \\ W_i &= W_{ia} M_a\,, \\ D_i\phi &= \left(D_i\phi\right)_a M_a\,, \end{aligned} \tag{2.111}$$

where

$$\left(D_i\phi\right)_a = \partial_a \phi_a - e\epsilon_{abc} W_{ib} \phi_c\,. \tag{2.112}$$

Besides that, let us define

$$\hat{\phi} = \frac{\phi}{|\phi|}, \quad \text{where } |\phi| = \sqrt{\phi_a \phi_a}\,. \tag{2.113}$$

We know from the finite energy constraint that, at spatial infinity, $|\phi| \to v$ and that $D_i\phi$ falls faster than $r^{-3/2}$. Then,

$$\frac{1}{8\pi y} \epsilon^{ijk} \oint_{S^2_\infty} dS_i\, \hat{\phi} \cdot \left(D_j \hat{\phi} \times D_k \hat{\phi} \right) = 0\,. \tag{2.114}$$

If we expand the covariant derivatives, we obtain that

$$\begin{aligned} 0 = \frac{1}{8\pi y}\, \epsilon_{ijk} \oint_{S^2_\infty} dS^i \Big\{ & \hat{\phi} \cdot \left(\partial_j \hat{\phi} \times \partial_k \hat{\phi} \right) \\ & + e\hat{\phi} \cdot \left[\partial_j \hat{\phi} \times \left(\vec{W}_k \times \hat{\phi} \right) + \left(\vec{W}_j \times \hat{\phi} \right) \times \partial_k \hat{\phi} \right] \\ & + e^2 \hat{\phi} \cdot \left(\vec{W}_j \times \hat{\phi} \right) \times \left(\vec{W}_k \times \hat{\phi} \right) \Big\}\,, \end{aligned}$$

which can be simplified to

$$0 = \frac{1}{8\pi y}\,\epsilon_{ijk} \oint_{S^2_\infty} dS^i \left\{ \hat{\phi} \cdot \left(\partial_j \hat{\phi} \times \partial_k \hat{\phi} \right) \right.$$
$$\left. + e\,\partial_j \hat{\phi} \cdot \vec{W}_k + e\,\partial_k \hat{\phi} \cdot \vec{W}_j + e^2 \left(\vec{W}_j \times \vec{W}_k \right) \cdot \hat{\phi} \right\} .$$

After an integration by parts, we get that

$$\frac{1}{8\pi y}\,\epsilon_{ijk} \oint_{S^2_\infty} dS^i \left\{ \hat{\phi} \cdot \left(\partial_j \hat{\phi} \times \partial_k \hat{\phi} \right) + e\,\hat{\phi} \cdot \vec{G}_{jk} \right\} = 0\,. \tag{2.115}$$

Now, from the definition

$$\vec{B}_i = -\frac{1}{2}\epsilon_{ijk}\vec{G}_{jk}$$

we can write eq. (2.115) as

$$N_\phi = \frac{e}{4\pi} \oint_{S^2_\infty} dS_i \hat{\phi} \cdot \vec{B}_i\,, \tag{2.116}$$

where N_ϕ is given by

$$N_\phi = \frac{1}{8\pi}\,\epsilon_{ijk} \oint_{S^2_\infty} dS_i\,\hat{\phi} \cdot \left(\partial_j \hat{\phi} \times \partial_k \hat{\phi} \right) . \tag{2.117}$$

Then, using the expression for the abelian magnetic charge (2.106), which can be written as

$$g = \oint_{S^2_\infty} dS_i\,\hat{\phi} \cdot \vec{B}_i\,, \tag{2.118}$$

we obtain that

$$g = \frac{4\pi}{e}\,N_\phi\,. \tag{2.119}$$

The integral N_ϕ is a topological quantity and it is an integer [50]. In fact, it is a winding number for a map between two 2-spheres, i.e., it measures the number of times $\hat{\phi}$ covers a 2-sphere Σ in internal space while $\hat{r}$ covers S^2_∞ once. In our specific case, $N_\phi = 1$, because we are working with only one 't Hooft-Polyakov monopole. However, the situation could be different in the case of a multi-monopole solution, for example.

3. ASPECTS OF NON-ABELIAN MONOPOLES

In this chapter we shall analyze some characteristics of non-abelian magnetic monopoles. In order to do so, we need to review the symmetry breaking patterns so that we can classify monopoles according to the unbroken gauge group. Besides that, we shall present a brief review of GUTs, in order to justify why it is interesting to study SSB patterns. After that, we shall discuss a way to embed the 't Hooft-Polyakov monopole in more general theories. Finally, we shall also review the so-called "Problem of Global Color".

3.1. SYMMETRY BREAKING PATTERNS

We have already discussed in section 2.3.3 under what circumstances a vacuum configuration such as $\phi_0 = \phi_{0a} T_a$ can be rotated to the Cartan subalgebra $\mathcal{H}$, so that

$$\phi_0 = u \cdot H \,,$$

where u is a constant vector. We know from eq. (2.35) that G_0, which is the unbroken gauge group with respect to the point where $\phi(\theta, \varphi) = \phi_0$, has generators such that $[\phi_0, T_a] = 0$. Since ϕ_0 commutes with itself and with all the other generators in $L(G_0)$, then, it immediately follows that ϕ_0 generates an invariant $U(1)$ subgroup of G_0. Thus, G_0 will be of the form [40]

$$G_0 = \text{“}K \times U(1)\text{”} \,, \tag{3.1}$$

where the quotation marks refer to the local structure of the unbroken gauge group, only.

Hence, let us look at the other generators of G commuting with ϕ_0. Clearly, all the Cartan generators H_i do, so the rank of the exact symmetry group G_0 is equal to the rank of G. Next, note that according to eq. (2.10)

$$[\phi_0, E_\alpha] = (u \cdot \alpha)\, E_\alpha \,.$$

Thus, the step operator E_α belongs to $L(G_0)$ if and only if $(u \cdot \alpha) = 0$. The same applies to $E_{-\alpha}$.

Now, let us recall the result (2.17) that any root α can be expanded in terms of the r simple roots α_a as

$$\alpha = \sum_{a=1}^{r} n_a \alpha_a \,,$$

where n_a are all positive or all negative integers, depending whether α is a positive or negative root. Then,

$$u \cdot \alpha = \sum_{a=1}^{r} n_a \left(u \cdot \alpha_a \right) \ . \tag{3.2}$$

Thus, we can enumerate the possibilities.

First, let us suppose that $u \cdot \alpha_a \neq 0$, $\forall\, a = 1, \ldots, r$. This implies that $L(G_0)$ has no step operators E_α and is, therefore, generated only by the Cartan generators H_i. That is, the unbroken gauge group will be of the form

$$G_0 = (U(1))^r = \underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{r \text{ times}} .$$

We shall denote this case by Maximal Symmetry Breaking (MSB).

Next, let us suppose that only $l < r$ products in the sum (3.2) are equal to zero. Let us denote by $\alpha_a^{(B)}$ the $r - l$ simple roots satisfying $u \cdot \alpha_a^{(B)} \neq 0$ and by $\alpha_a^{(U)}$ the l simple roots which satisfy $u \cdot \alpha_a^{(U)} = 0$. Then, note that from eq. (2.35) the step operators related to $\alpha_a^{(U)}$ will belong to $L(G_0)$, while those related to the simple roots $\alpha_a^{(B)}$ will be *broken generators*. Then, it follows that the unbroken gauge group will be of the form

$$G_0 = \text{``}H \times (U(1))^{r-l}\text{''} , \tag{3.3}$$

where H has rank l. Now, let us recall that each dot in the Dynkin diagram of a Lie algebra represents a simple root α_a. When we say that the step operators associated with $\alpha_a^{(B)}$ do not belong to $L(G_0)$, this means that, in order to obtain $L(H)$, we must eliminate the respective circles in the original diagram of $L(G)$. That is, $L(H)$ can be obtained by deleting the dots associated to the $r - l$ roots $\alpha_a^{(B)}$ from the Dynkin diagram of $L(G)$. For more details, please see [40]. We shall call this case a Non-Abelian Unbroken Symmetry (NUS).

Finally, let us define a special case of NUS, called Minimal Symmetry Breaking. In this case, $l = r - 1$ and so G_0 will be of the form of (3.1). Moreover, it can be shown that, in order for this $U(1)$ to be compact so that magnetic charge is quantized, the vector u must be proportional to a fundamental weight of G [7], let us say to λ_p. Then, in this case, G_0 will have the general form [5, 7]

$$G_0 = \frac{K \times U(1)}{Z} , \tag{3.4}$$

where K is a semisimple group, Z is a discrete subgroup of the center of K, $Z(K)$, which belongs to $U(1)$ and K, i.e., $Z = U(1) \cap K$. Moreover, from what we have discussed above, we can obtain the Dynkin diagram of $L(K)$ by deleting the root α_p in the original diagram of $L(G)$.

Then, let us give some simple examples of this minimal symmetry breaking patterns. When $G = SU(5)$, there are two possibilities for the SSB. They are:

a) $\alpha_p = \alpha_1$ or α_4, then $K = SU(4)$,

b) $\alpha_p = \alpha_2$ or α_3, then $K = SU(3) \times SU(2)$.

The latter example is the so-called minimal $SU(5)$ Grand Unified Theory broken by an adjoint Higgs field to $SU(3)_C \times SU(2)_L \times U(1)_Y$.

On the other hand, if $G = Spin(10)$[1], we can have the following possibilities:

a) $\alpha_p = \alpha_1$, then $K = Spin(8)$,

b) $\alpha_p = \alpha_2$, then $K = SU(2) \times SU(4)$,

c) $\alpha_p = \alpha_3$, then $K = SU(3) \times SU(2) \times SU(2)$,

d) $\alpha_p = \alpha_4$ or α_5, then $K = SU(5)$.

Note that, by the moment, we are only interested in the symmetry breaking patterns. However in the next subsection, we shall give a brief review of GUTs in order to explain why we have a physical motivation to analyze these SSB patterns.

Moreover, let us emphasize that in chapter 4 we shall use the minimal symmetry breaking given by eq. (3.4) in order to construct the Dark Monopole solutions.

We end this section by emphasizing that, even though we have determined the possible symmetry breaking patterns for an adjoint Higgs field, this problem is much more complicated for ϕ in other representations.

3.1.1 Brief Overview of GUTs

The Standard Model (SM) of the strong, weak and electromagnetic interactions is a gauge theory with symmetry group $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$, where $SU(3)_C$ is associated to the strong interaction, $SU(2)_L$ to the weak interaction and $U(1)_Y$ to *hypercharge*.

[1] $Spin(n)$ is the universal covering group of $SO(n)$.

The SM fits the observed data rather well, but it has a number of features which are somewhat unsatisfactory. It involves three independent gauge coupling constants, in addition to the large number of parameters needed to specify the Higgs potential and the fermion mass matrix. Moreover, the multiplet structure of the observed particles seems to be random. Finally, there is no explanation to the fact that electric charge is quantized.

One way to deal with these difficulties is to embed the Standard Model in a Grand Unified Theory, based on a simple gauge group $G \supset G_{SM}$. The fact that G is simple implies that there is only a single gauge coupling constant as well as explains the quantization of weak hypercharge (and hence of electric charge). It also turns out to be possible to construct GUTs in which the observed fermions fit into a relatively simple multiplet structure. For instance, for $G = SU(5)$ the set of all observed particles can be accommodated in a $\bar{5}$ representation and in a 10, with no new particles being needed. On the other hand, in the $SO(10)$ case the same particles can be put into a single 16-dimensional representation, with only one new necessary particle, which would correspond to the right-handed neutrino.

The symmetry breaking in GUTs can be viewed as occurring in stages [9]

$$G \to G_0^{(1)} \to G_0^{(2)} \to \cdots \to SU(3)_C \times SU(2)_L \times U(1)_Y \,.$$

Note that at each stage there exists a Higgs field responsible for the symmetry breaking.

Let us recall that the simplest GUT is [8]

$$SU(5) \xrightarrow{10^{16}\ \mathrm{GeV}} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{10^{2}\ \mathrm{GeV}} SU(3)_C \times U(1)_{em} \,,$$

where the first step of the SSB has been already introduced in section 3.1. The second stage is the symmetry breaking in the *Weinberg-Salam* model, where the Higgs field is in a 2-dimensional representation [51, 52].

Remember that more complex sequences are possible with larger gauge groups such as $SO(10)$ or E_6 [40]. If we restrict ourselves to the case of real adjoint Higgs fields, which we have thoroughly discussed in this dissertation, we can use the tools defined in the last section in order to determine all the possibilities of symmetry breaking patterns. However, we again emphasize that when ϕ is in another representation, then, the problem is much more complicated.

Finally, let us recall that there are predictions which are common to all GUTs [9]:

a) The sum of the electric charges of all particle species in a multiplet vanishes.

b) There are indications that the running gauge coupling constants (properly normalized) for the weak, electromagnetic and strong interactions tend to the same value when evaluated at sufficiently high unification temperature $T_c \sim 10^{16}$ GeV, where we take the Boltzmann constant to be $K_B = 1$.

c) Magnetic monopoles should exist. And the observation of such monopoles may be one of the few possibilities to obtain experimental support for the *unification hypothesis*.

Before we continue, let us explain the last item in more details. Let us recall that $\pi_2(G/G_{SM})$ is isomorphic to $\pi_1(G_{SM})$ as long as we take the unification gauge group G to be a universal covering group[2]. Then, it follows from $\pi_1(G_{SM}) = \mathbb{Z}$ that these theories possess topological magnetic monopoles. Besides that, it should be clear that in this argument we considered just the initial symmetry G and the surviving gauge symmetry G_{SM} of the the Standard Model. However, since these GUTs might involve many steps of symmetry breaking we may have some magnetic monopoles forming in the intermediate steps, but they may or may not survive the consecutive SSB steps down to G_{SM}. In addition, we remember that in theories with subsequent steps of symmetry breaking composite objects can be produced, such as monopoles attached to strings (see, for instance [53, 54] and references therein).

Therefore, there are good reasons to expect that a Grand Unification of the electroweak and strong interactions takes place at energies about 10^{16} GeV. Then, topological defects such as cosmic strings and monopoles, that occur naturally in GUT theories, might play some role in the early universe [55]. According to Kibble [56], these topological defects would be produced during the cosmological phase transitions by which the universe has been through.

Also note that, from the fact that the masses of GUT monopoles are proportional to the energy scale of grand unification, it implies that they are far beyond the reach of particle physics experiments. Then, the most plausible source of monopoles is as relics surviving from the early universe [30].

[2]Note that there is no loss of generality in this choice, since the universal covering group admits all the possible representations of $L(G)$. Moreover, a universal covering group G is such that $\pi_1(G) = 0 = \pi_0(G)$, i.e., it is *connected* and *simply-connected* [32].

In this section we shall use our results of chapter 2 in order to show how one can construct an embedded 't Hooft-Polyakov monopole in a more general theory. We shall present the construction due to Bais [57], which was proposed for monopoles in theories with MSB. However, we shall also discuss what happens to the NUS case.

Once more, let us consider a theory with an adjoint Higgs field ϕ with the vacuum configuration $\phi_0 = u \cdot H$, which breaks the compact and simply-connected gauge group G to G_0. Now, let us take the monopole generators M_i to be

$$M_i = T_i^\alpha \,, \tag{3.5}$$

where T_i^α, $i = 1, 2, 3$, are given by (2.12) and are the generators of a $su(2)$ subalgebra associated to the root α, such that $\alpha \cdot u \neq 0$, which implies that $M_3 \in L(G_0)$ and M_1, $M_2 \notin L(G_0)$. We can decompose $\phi_0 = u \cdot H$ as [57]

$$\phi_0 = \left(u - \frac{u \cdot \alpha}{\alpha^2}\,\alpha\right) \cdot H + (u \cdot \alpha)\; T_3^\alpha \,. \tag{3.6}$$

The first term is a singlet under the $su(2)$ generated by eq. (3.5), since

$$\left[\left(u - \frac{u \cdot \alpha}{\alpha^2}\,\alpha\right) \cdot H,\, T_i^\alpha\right] = 0 \,, \quad \forall\, i = 1,\, 2,\, 3 \,. \tag{3.7}$$

Therefore, we can use the result of eq. (2.77) with

$$S = \left(u - \frac{u \cdot \alpha}{\alpha^2}\,\alpha\right) \cdot H \,,$$

and with the generators $Q_0 = M_3$, $Q_{\pm 1} = \mp M_\pm/\sqrt{2}$, similar to what we did for the 't Hooft-Polyakov monopole. Then, from eqs. (2.79) and (2.81) to (2.83), these monopoles will have the form

$$\begin{aligned} \phi &= S + (u \cdot \alpha)\, f(r)\, n^a M_a, \\ W_i &= -\frac{[1 - u(r)]}{er} \epsilon_{ijk} n^j M_k \,, \end{aligned}$$

with the same boundary conditions as the 't Hooft-Polyakov monopole. Note that, except for the decomposition related to the singlet term S, this is similar to the ansatz of eq. (2.91). Moreover, from the fact that $D_i S = 0$, it follows from the general hamiltonian (2.24) that the kinetic terms of the Bais monopole will be of the same form of the 't Hooft-Polyakov case, up to some normalization.

On the other hand, the potential $V(\phi)$ does not need to be necessarily the same. Nevertheless, in principle, we can use the mexican hat potential (2.88). When this happens the second order equations of motion for the profile functions will be of the form of those in the 't Hooft-Polyakov monopole.

But, in fact, Bais [57] have only discussed monopole solutions in the Prasad-Sommerfield limit of vanishing potential, i.e., the specific form of $V(\phi)$ need not be determined. In this limit, the first order equations of motion will also be the same as those from the BPS limit of the 't Hooft-Polyakov case, regardless of the choice of $V(\phi)$.

It is relevant to note that, except for the possible magnetic charges, all the derivations we have done for the 't Hooft-Polyakov monopole can be repeated in the same way for this embedded solution. For this reason, we shall not discuss the details regarding the behavior of the function $u(r)$ and $f(r)$ nor the lower and upper bounds for the mass, for instance.

Now, let us recall that in the MSB case ϕ_0 is such that that it breaks G to $G_0 = (U(1))^r$. This implies that

$$\pi_2 \left(G/G_0 \right) \cong \pi_1 \left(G_0 \right) \cong \mathbb{Z}^r \, . \tag{3.8}$$

In this special case, the monopole will possess r topologically conserved charges.

However, note that this construction is equally valid for the NUS case of eq. (3.3). Of particular interest to Grand Unified Theories is the minimal symmetry breaking of eq. (3.4), where

$$\pi_2 \left(G/G_0 \right) \cong \pi_1 \left(G_0 \right) \cong \mathbb{Z} \, . \tag{3.9}$$

This means that the only magnetic charge to be topologically conserved, in the standard sense, is the one in the direction of the $U(1)$ gauge symmetry. In the next section, we shall analyze in more details the generalized quantization condition and the non-abelian magnetic charges.

Therefore, the important information we should keep is that by choosing the monopole generators to be given by (3.5) we can embed 't Hooft-Polyakov monopoles in more general theories.

3.3. GENERALIZED QUANTIZATION CONDITION

In section 2.1 we have discussed that, in order for the Dirac string to be undetectable, the electric and magnetic charges should satisfy a quantization condition, given by section 2.1. Moreover, when we constructed smooth non-abelian monopoles in section 2.3.4 we have also encountered a quantization condition, given by eq. (2.66). Then, in this section we shall analyze the generalized quantization condition in more details.

First, let us recall that since at $r \to \infty$, $\phi = \phi(\theta\,\varphi)$, the exact symmetry group G_0 is, in fact, position-dependent and given by $G_0(\hat{\mathbf{r}})$. For this reason, we shall focus our analysis in a point along the positive $z-$axis in the limit $r \to \infty$, i.e., a point to which the unbroken gauge group is indeed G_0. This is due to the fact that at the north pole $\phi = \phi_0$. Then, in order to make the notation clear, let us define $G_0 \equiv G_0(\hat{\mathbf{r}} = \hat{\mathbf{z}})$. From eq. (2.69) we know that, in this limit, the magnetic field at the north pole is given by

$$B_i = -\frac{x^i}{er^3}\, M_3 \,.$$

However, for further convenience, we shall write it as

$$B_i = -\frac{\mathcal{G}}{4\pi}\,\frac{x^i}{r^3} \,. \tag{3.10}$$

Now, note that the generalized quantization condition of eq. (2.66) can be rewritten as

$$\exp\left(ie\mathcal{G}\right) = 1 \,. \tag{3.11}$$

And it is easy to see that at any other point over S^2_∞, $B_i = -\frac{\mathcal{G}(\hat{\mathbf{r}})}{4\pi}\frac{x^i}{r^3}$ and the quantization condition turns out to be $\exp\left(ie\mathcal{G}\left(\hat{\mathbf{r}}\right)\right) = 1$. Also note that, again, in order to make the notation compact we use that $\mathcal{G} \equiv \mathcal{G}(\hat{\mathbf{r}} = \hat{\mathbf{z}})$. Now, even though there are more general ways of obtaining this quantization condition, see for instance [6, 39], the result is the same as we have obtained here. This shows that all the information about the topological quantum number is contained in the "generalized magnetic charge", $\mathcal{G}$, defined by the asymptotic generalized magnetic field.

Let us recall that in order to make ϕ_0 to lie in the Cartan subalgebra $\mathcal{H}$ we have used a gauge transformation $g \in G$. But, since $\mathcal{G} \in L(G_0)$, we still have the freedom to make a gauge transformation, with an element of the unbroken group G_0, in order to rotate $\mathcal{G}$ to the same Cartan subalgebra as ϕ_0. In the mathematical literature this is called *framing* [58, 59] and, unfortunately, it is quite common stated as the only way to solve the generalized quantization condition (3.11). As we shall see, this is indeed true for the case of MSB. However, in the case of minimal symmetry breaking it is not natural to require $\mathcal{G}$ to lie in $\mathcal{H}$ [58, 59]. When this condition is imposed, we might loose interesting non-equivalent monopole solutions.

For the sake of completeness, let us analyze what happens when we impose the framing condition. In this case, $\mathcal{G}$ can be written as

$$\mathcal{G} = \vec{g} \cdot \vec{H} \,, \tag{3.12}$$

where we needed to use an explicit vector notation in order to avoid confusion with any gauge group element $g \in G$. The r components of the vector $\vec{g}$ are called the *magnetic weights* of the monopole [6, 30] and one should remember that this "magnetic charge" is not gauge-invariant. Note that from (3.12), eq. (3.11) can be written as

$$\exp\left(ie\vec{g} \cdot \vec{H}\right) = 1 \,. \tag{3.13}$$

We can now determine the possible magnetic weights of $\vec{g}$ if we act with the quantization condition on an arbitrary weight state $|\mu\rangle$. Using the result (2.13) we obtain that

$$\exp\left(ie\vec{g} \cdot \vec{H}\right) |\mu\rangle = \exp\left(ie\vec{g} \cdot \vec{\mu}\right) |\mu\rangle \ .$$

Then, for the quantization condition (3.13) to be satisfied it follows that

$$\frac{e\,\vec{g} \cdot \vec{\mu}}{2\pi} = n, \quad \text{with } n \in \mathbb{Z} \,. \tag{3.14}$$

Let us recall that, from the fact that both roots and weights can be written as integer combinations of the simple roots α_i and the fundamental weights λ_i, respectively, as defined in eqs. (2.17) and (2.18), it follows from the orthogonality condition (2.14) that for any root α and any weight μ

$$\frac{2\alpha \cdot \mu}{\alpha^2} = N \quad \text{with } N \in \mathbb{Z} \,.$$

Therefore, the solution to eq. (3.13) is given by

$$\frac{e\,\vec{g} \cdot \vec{H}}{2\pi} = \sum_{a=1}^{r} n_a \, \alpha_a^\vee \cdot H \,, \tag{3.15}$$

where $n_\alpha \in \mathbb{Z}$ and $\alpha^\vee \equiv 2\alpha/|\alpha|^2$ is a coroot. This means that any monopole solution under the framing condition will have its vector magnetic charge $\vec{g}$ lying in the coroot lattice of $L(G)$.

Note that, in the case of MSB, the fact that $\mathcal{G} \in L(G_0)$ immediately implies that $\mathcal{G} \in \mathcal{H}$. In this special case all the magnetic weights are conserved topological charges. However, in the case of NUS theories, this is not true for all of them. In fact, this happens because the magnetic weights can be redefined up to a transformation by an element of the Weyl group [30, 39, 60]. Nevertheless, we shall not discuss this topic here since we will not need it in the subsequent chapters. The important information to keep is that these magnetic weights may have relevant physical consequences, as showed by Bais and Schroers [58,59] for the case of BPS solutions in the $SU(3) \to U(2)$ symmetry breaking.

There is one more point regarding the framed magnetic charge (3.12) we would like to emphasize. Note that $\mathcal{G}$ can always be transformed by a general element of the unbroken gauge group G_0. Also note that each one of the possible final configurations will satisfy the quantization condition (3.13), but in general they will not belong to the Cartan subalgebra anymore. In the case discussed by Bais and Schroers [58,59], the action of a $U(2)$ element on $\mathcal{G}$ generates 2-spheres of quantized radius and "height" in the coroot lattice of $su(3)$. Of course, a general case is much more complicated, but it is interesting to note that these new configurations will be associated to non-equivalent monopole solutions.

Now, let us come back to the general case of eq. (3.11) in order to make a final remark. Let us recall that this generalized condition came from our analysis (2.66) of closed loops in G_0. This means that the quantization condition is sensitive to the global structure of the exact symmetry group G_0, not only to its Lie algebra. This is why we have stressed in section 3.1 the cases where we were discussing the local structure, only.

3.4. THE PROBLEM OF GLOBAL COLOR

In section 2.3.1, we have showed that the unbroken gauge group G_0 is position-dependent, i.e, $G_0(\hat{\mathbf{r}})$. This is due to the fact that the monopole asymptotic configuration takes values in $L(G)$ according to eq. (2.34). Then, in order for the generators $T_a \in L(G_0)$ to belong to $L(G_0(\hat{\mathbf{r}}))$ they must transform as

$$T_a(\theta, \varphi) = g(\theta, \varphi)\, T_a\, g^{-1}(\theta, \varphi)\,, \qquad (3.16)$$

where $g(\theta, \varphi) \in G$ is the gauge group element used to construct the monopole, in our case it is given by eq. (2.64). This transformation is sometimes called a *parallel transport* of the generators [30].

Now, the interesting point here is that, in the presence of NUS monopoles, only the generators T_a which commute with the generalized magnetic charge $\mathcal{G}$ are globally-well defined. Furthermore, when there are generators in $L(G_0)$ which do not commute with $\mathcal{G} \propto M_3$, then, we say that there is no globally well-defined "rigid" copy of the unbroken gauge group. This topological obstruction in the definition of the unbroken gauge group $G_0(\hat{\mathbf{r}})$ is known as *The Problem of Global Color* [61–69]. This imposes difficulties to the construction of *chromodyons* and to the quantization of non-abelian monopoles [30,58,59].

But, in order to make clear what kind of problem we are talking about, let us give an example. Let us consider a YMH theory with $G = SU(3)$. We shall follow the conventions of section 2.2, where α_1, α_2 denote the simple roots of $su(3)$, while $\psi = \alpha_1 + \alpha_2$ is the remaining positive root. Next, let

$$\phi_0 = v\,\frac{\lambda_2 \cdot H}{|\lambda_2|}\,.$$

It follows from eq. (2.35) that $L(G_0)$ will be given by

$$L(G_0) = \left\{ \frac{\lambda_2 \cdot H}{|\lambda_2|},\, T_1^{\alpha_1},\, T_2^{\alpha_1},\, T_3^{\alpha_1} \right\}\,, \tag{3.17}$$

where $T_i^{\alpha_1}$, $i = 1,\,2,\,3$, follow the conventions of eq. (2.12). Then, it is trivial to see that, $L(G_0)$ is of the form $su(2) \oplus u(1)$, where $\lambda_2 \cdot H$ gives the $U(1)$ direction.

Now, since $\alpha_1^2 = 2 = \alpha_2^2$ and $\alpha_1 \cdot \alpha_2 = -1$, it follows that, for $su(3)$, $\alpha^\vee = \alpha$ and $\lambda^\vee = \lambda$. Besides that, the inverse Cartan Matrix is given by

$$K^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\,.$$

Then, from the eq. (2.16) it follows that

$$\frac{\lambda_2^\vee}{|\lambda_2|^2} = \frac{1}{2}\,\alpha_1^\vee + \alpha_2^\vee\,.$$

Thus, this implies that

$$\exp\left[2\pi i \left(\frac{\lambda_2^\vee}{|\lambda_2|^2} - \frac{1}{2}\,\alpha_1^\vee\right)\cdot H\right] = \exp\left(2\pi i\,\alpha_2^\vee \cdot H\right) = 1\,.$$

The last step follows from the fact that $\alpha_2^\vee \cdot H$ acting on any weight state $|n_1\lambda_1 + n_2\lambda_2\rangle$, with $n_1,\, n_2\, \in \mathbb{Z}$, gives

$$\alpha_2^\vee \cdot H\,|n_1\lambda_1 + n_2\lambda_2\rangle = n_2\,|n_1\lambda_1 + n_2\lambda_2\rangle\,.$$

Therefore,

$$\exp\left(2\pi i\,\frac{\lambda_2^\vee}{|\lambda_2|^2}\cdot H\right) = \exp\left(2\pi i\,\frac{1}{2}\,\alpha_1^\vee \cdot H\right)\,. \tag{3.18}$$

Then, we can see that the $U(1)$ subgroup of G, generated by $\frac{\lambda_2^\vee\cdot H}{|\lambda_2|^2}$, has an element in common with the unbroken $SU(2)$ subgroup, other than the identity. This is the nontrivial element of the center $\mathbb{Z}_2$ of $SU(2)$. Thus, there are two elements of $SU(2)\times U(1)$ corresponding to each element of G_0. Therefore, it follows from the result (3.18) that there is a $\mathbb{Z}_2$ identification between the elements of the $SU(2)$ and $U(1)$ factors, so that

$$G_0 = \frac{SU(2)\times U(1)}{\mathbb{Z}_2} \cong U(2)\,. \tag{3.19}$$

Since $\pi_1\left(U(2)\right) = \mathbb{Z}$, this theory supports magnetic monopoles with integer abelian magnetic charges. However, their generalized magnetic charge $\mathcal{G}$ may also contribute to a non-abelian magnetic flux.

Let us use the results of section 3.2 and take the monopole generators to be $M_i = T_i^{\alpha_2}$, $i = 1, 2, 3$. Furthermore, from eqs. (2.69) and (3.10) it follows that

$$\mathcal{G} = \frac{4\pi}{e}\, T_3^{\alpha_2}\,.$$

Now, we can explicitly check whether the generators of the local unbroken gauge group are well-defined over S^2_∞. We shall calculate $\lambda_2\cdot H(\theta,\varphi)$ and $T_3^{\alpha_1}(\theta,\varphi)$ first, because the calculation is straightforward. Since they are both in the Cartan subalgebra $\mathcal{H}$, we can use the "trick" of eq. (3.6) with $\alpha=\alpha_2$, the definition (2.64) of our element $g(\theta,\varphi)\in G$, with $M_i = T_i^{\alpha_2}$, as well as the result (2.63) provided by the BCH formula to obtain that

$$\begin{aligned} \lambda_2\cdot H(\theta,\varphi) &= \left(\lambda_2\cdot H - T_3^{\alpha_2}\right) + n^a T_a^{\alpha_2}\,,\\ T_3^{\alpha_1}(\theta,\varphi) &= \frac{1}{2}\left[\left(\alpha_1\cdot H + T_3^{\alpha_2}\right) - n^a T_a^{\alpha_2}\right]\,, \end{aligned} \tag{3.20}$$

with

$$n^a T_a^{\alpha_2} = \left(\sin\theta\cos\varphi\right)\,T_1^{\alpha_2} + \left(\sin\theta\sin\varphi\right)\,T_2^{\alpha_2} + \left(\cos\theta\right)\,T_3^{\alpha_2}\,. \tag{3.21}$$

Next, in order to obtain the remaining two generators $T_1^{\alpha_1}(\theta,\varphi)$ and $T_2^{\alpha_1}(\theta,\varphi)$ we will need to obtain some new results in advance. Firstly, let us recall that the BCH formula implies that

$$e^{i\theta T}\,A\,e^{-i\theta T} = \cos\left(\theta\sqrt{ab}\right)A - i\frac{b}{\sqrt{ab}}\,\sin\left(\theta\sqrt{ab}\right)B\,, \tag{3.22}$$

when

$$\begin{aligned} [A,T] &= b\,B\,,\\ [B,T] &= a\,A\,. \end{aligned} \tag{3.23}$$

We shall prove this result in appendix B. But, the important point here is to use this result in order to obtain that

$$
\begin{aligned}
e^{i\varphi T_3^{\alpha 2}}\, T_1^{\alpha_1}\, e^{-i\varphi T_3^{\alpha 2}} &= \cos(\varphi/2)\, T_1^{\alpha_1} + \sin(\varphi/2)\, T_2^{\alpha_1}\,, \\
e^{i\varphi T_3^{\alpha 2}}\, T_2^{\alpha_1}\, e^{-i\varphi T_3^{\alpha 2}} &= \cos(\varphi/2)\, T_2^{\alpha_1} - \sin(\varphi/2)\, T_1^{\alpha_1}\,, \\
e^{-i\varphi T_3^{\alpha 2}}\, T_1^{\psi}\, e^{i\varphi T_3^{\alpha 2}} &= \cos(\varphi/2)\, T_1^{\psi} + \sin(\varphi/2)\, T_2^{\psi}\,, \\
e^{-i\varphi T_3^{\alpha 2}}\, T_2^{\psi}\, e^{i\varphi T_3^{\alpha 2}} &= \cos(\varphi/2)\, T_2^{\psi} - \sin(\varphi/2)\, T_1^{\psi}\,,
\end{aligned}
\tag{3.24}
$$

which we will be need in our calculations. Note that, in order to obtain these results, we have used the fact that $[H_i, E_\alpha] = \alpha^{(i)} E_\alpha$, provided by eq. (2.10).

In addition, for future convenience, we identify two $su(2)$ subalgebras of $su(3)$. The first one is given by the set of generators

$$
\left\{ 2T_2^{\alpha_2}, 2T_1^{\alpha_1}, 2T_1^{\psi} \right\}\,,
$$

such that $[2T_2^{\alpha_2}, 2T_1^{\alpha_1}] = i\, 2T_1^{\psi}$. The second one is given by the set

$$
\left\{ 2T_2^{\alpha_2}, 2T_2^{\alpha_1}, 2T_2^{\psi} \right\}\,,
$$

such that $[2T_2^{\alpha_2}, 2T_2^{\alpha_1}] = i\, 2T_2^{\psi}$. One can obtain these results making use of the commutators between the step operators in $su(3)$. In the next chapter we shall present a review of the Lie algebra of $SU(n)$, where it will be clear how one can easily derive these commutation relations. Then, with the above results we can use eq. (2.63) in order to show that

$$
\begin{aligned}
e^{-i\theta T_2^{\alpha 2}}\, T_1^{\alpha_1}\, e^{i\theta T_2^{\alpha 2}} &= \cos(\theta/2)\, T_1^{\alpha_1} + \sin(\theta/2)\, T_1^{\psi}\,, \\
e^{-i\theta T_2^{\alpha 2}}\, T_2^{\alpha_1}\, e^{i\theta T_2^{\alpha 2}} &= \cos(\theta/2)\, T_2^{\alpha_1} + \sin(\theta/2)\, T_2^{\psi}\,.
\end{aligned}
\tag{3.25}
$$

Note that with the results of eqs. (3.24) and (3.25) we can easily compute

$$
T_I^{\alpha_1}(\theta,\varphi) = e^{-i\varphi T_3^{\alpha 2}}\, e^{-i\theta T_2^{\alpha 2}}\, e^{i\varphi T_3^{\alpha 2}}\, T_I^{\alpha_1}\, e^{-i\varphi T_3^{\alpha 2}}\, e^{i\theta T_2^{\alpha 2}}\, e^{i\varphi T_3^{\alpha 2}}\,,
$$

with $I = 1$, 2. After some trivial, but lengthy, steps we obtain that

$$
T_1^{\alpha_1}(\theta,\varphi) = \cos(\theta/2)\, T_1^{\alpha_1} + \sin(\theta/2)\left[\cos(\varphi)\, T_1^{\psi} + \sin(\varphi)\, T_2^{\psi}\right]\,, \tag{3.26}
$$

$$
T_2^{\alpha_1}(\theta,\varphi) = \cos(\theta/2)\, T_2^{\alpha_1} + \sin(\theta/2)\left[\cos(\varphi)\, T_2^{\psi} - \sin(\varphi)\, T_1^{\psi}\right]\,. \tag{3.27}
$$

Note that the generators $\lambda_2 \cdot H$ and $T_3^{\alpha_1}$ that commute with the generalized magnetic charge can be defined globally. However, the two that do not commute with $\mathcal{G}$, namely $T_1^{\alpha_1}$ and $T_2^{\alpha_1}$ fail to be well-defined at $\theta = \pi$, where they keep an azimuthal dependence. According to [30] this can be understood as follows. One way to define a global gauge rotation is to choose a Lie algebra element Ω at one point P on a sphere at large r and then use parallel transport to obtain Ω at any other point P' on the sphere. This only works if the result of the parallel transport is independent of the path from P to P'. This in turn requires that the surface integral of $[B_i, \Omega]$ over the area between any two such paths vanishes. In the limit $r \to \infty$ only the $1/r^2$ part of B_i, i.e., the generalized magnetic charge, contributes to this integral. Hence, only the generators that commute with $\mathcal{G}$ are well-defined.

Therefore, even in this simple model we can check that the presence of a monopole in a theory with a non-abelian unbroken symmetry may indeed obstruct the global definition of the unbroken gauge group G_0.

On the other hand, the situation is pretty different for monopoles in the MSB case. There, the unbroken gauge group $G_0 = (U(1))^r$ is globally well-defined since all the generators in $L(G_0)$, including $\mathcal{G}$, belong to the Cartan subalgebra $\mathcal{H}$. Therefore, they all commute with $\mathcal{G}$.

The simplest example of a MSB monopole is, in fact, the 't Hooft-Polyakov monopole. Thus, recalling the results of section 2.4, we know that the only unbroken generator in this case is M_3, which is the direction of both ϕ_0 and $\mathcal{G}$. And from the fact that $g(\theta,\varphi)\, M_3\, g^{-1}(\theta,\varphi) = n^a M_a$, we know that the unbroken $U(1)$ symmetry is well-defined over S^2_∞.

Finally, let us emphasize that the problem of obstructing the global definition of G_0 implies that the Noether charges cannot be globally extended, which means that some conservation laws may be spoiled and the particles in the theory experience exotic behaviors [64, 65]. This problem may also happen to non-abelian strings, but, in this case, the strings are called *Alice Strings* [53,70–75]. And even in the simplest model of *Alice Electrodynamics*, electric charge is not conserved in the ordinary sense.

4. DARK MONOPOLES IN GRAND UNIFIED THEORIES

In the previous chapters, we have analyzed the construction and characteristics of non-abelian monopoles. All the given examples involved a generalized magnetic charge $\mathcal{G} \propto M_3$ lying in the Cartan subalgebra $\mathcal{H}$. And this was on purpose, because in the known monopole solutions, it is usually considered that $[M_3, \phi_0] = 0$ and $M_3^\dagger = M_3$ imply that M_3 belongs to the same Cartan subalgebra as ϕ_0. However, as we mentioned in **??** and section 3.3, this is not necessary when G_0 is a non-abelian gauge group.

Thus, in this chapter we shall construct monopole solutions whose asymptotic magnetic field does not lie in the Cartan subalgebra $\mathcal{H}$, i.e., $M_3 \notin \mathcal{H}$. We will call them Dark Monopoles, since their magnetic field vanishes in the direction of the generator of the electromagnetic group $U(1)_{em}$, which we consider to be in $\mathcal{H}$. For the sake of clarity, we would like to emphasize that the content of this chapter is based on the publication [14]. Furthermore, it is important to mention that in the case of $\mathbb{Z}_2$ monopoles, for theories where ϕ is not in the adjoint representation, there already are monopole solutions with M_3 in the direction of some step operators [28, 76, 77]. Also note that string-vortex solutions with magnetic fields as combinations of step operators have been constructed for Yang-Mills-Higgs theories for various gauge groups [53, 74, 78–82].

4.1. THE DARK MONOPOLES CONSTRUCTION

4.1.1 Brief Review of $su(n)$

In this section we shall analyze the construction of our Dark Monopoles solutions in theories with gauge group $G = SU(n)$. But, in order to do so, we need to review some properties of $su(n)$ first.

In the *fundamental* representation, the group $SU(n)$ is realized through the set of $n \times n$ unitary matrices of determinant 1. It has dimension $n^2 - 1$, rank $n - 1$ and $n(n-1)$ roots. The elements of the Cartan subalgebra can be represented by a diagonal matrix of the form $A = diag(a_1, \ldots, a_n)$ with $\sum_i a_i = 0$, i.e., $(A)_{ij} = a_i \delta_{ij}$. This constraint arises since the elements of the Lie algebra have to be traceless.

Moreover, a convenient way to define the $n(n-1)$ step operators is to introduce $n \times n$ matrices E_{ij} for $i \neq j$, where the labels i, j are related to the name of the matrix instead of its components, defined by

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}\,, \tag{4.1}$$

where i, j, k, $l = 1, \ldots, n$. Now, note that

$$[A, E_{ij}]_{km} = (a_k - a_m)\ (E_{ij})_{km}\ ,$$

shows that the operators E_{ij} satisfy the properties of the step operators. Furthermore, they comprise *all* the step operators of $su(n)$, since there are $n(n-1)$ of them. Note that E_{ij} and E_{ji} are the step operators for two equal and opposite roots. Moreover,

$$[E_{ij}, E_{ji}]_{km} = \delta_{ik}\delta_{im} - \delta_{jk}\delta_{jm}\ ,$$

where there is no sum over the indices, unless otherwise stated. This shows that $[E_{ij}, E_{ji}]$ is a diagonal matrix with the i-th diagonal entry 1 and with the j-th diagonal entry -1, the rest being all zero. Such a matrix belongs to the Cartan subalgebra. Denoting the matrices h_i as $(h_i)_{kl} = \delta_{ik}\delta_{kl}$ we have that $[E_{ij}, E_{ji}] = h_i - h_j$. This verifies that indeed E_{ij} and E_{ji} are the step operators for two equal and opposite roots. By comparing with the forth commutation relation in eq. (2.10) we find that E_{ij} is associated with a root vector

$$\alpha = e_i - e_j\ , \tag{4.2}$$

where $(e_i)_k = \delta_{ik}$ are the unit vectors in the n-dimensional vector space. Thus, note that all the roots have equal length square, namely 2.

Now we may proceed to find the simple roots. Let us define the following vectors

$$\alpha_i = e_i - e_{i+1}\ , \tag{4.3}$$

with $i = 1, 2, \ldots, n-1$. Then, notice that any root $\alpha = (e_i - e_j)$ can be written as

$$\pm\,(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-2} + \alpha_{j-1}) = \pm\,(e_i - e_j)\,,\ 1 \leq i < j \leq n\,, \tag{4.4}$$

This shows that the set $(\alpha_1, \ldots, \alpha_{n-1})$ form a simple root basis, since the coefficients in the expansion of any roots are either all positive or all negative. A root $(e_i - e_j)$ is positive if $i < j$, is negative if $i > j$ and is a simple root if $j = i + 1$. Moreover, the definition (4.3) agrees with the fact that the simple roots α_i are the positive roots which cannot be written as a sum of any other positive roots.

Besides that, there is one more information that we need to extract from the fundamental representation of $SU(n)$, which is the commutators between the step operators. From the definition (4.1) we see that

$$(E_{ij}E_{pq})_{km} = \delta_{jp}\,(E_{iq})_{km}\ , \tag{4.5}$$

which implies that the commutator

$$[E_{ij}, E_{pq}] = \delta_{jp} E_{iq} - \delta_{iq} E_{pj} \,. \tag{4.6}$$

Note that the information we have extracted from the Lie algebra $su(n)$, such as the roots and commutation relations, is independent of the representation. This means that we have used the fundamental representation just for convenience.

After this brief review of the properties of $su(n)$ we still need to introduce another tool, which is the *Cartan involution*.

4.1.2 The Cartan Involution

In this subsection we shall give an introduction to the Cartan involution, since it will be necessary for the contruction of our monopole solution. For more details, please see [32] and [40]. Also note that in this section we will use a different notation for the Lie algebras, in order for the notation to be shorter.

An automorphism ω of a Lie algebra $\mathfrak{g}$ is by definition a map from $\mathfrak{g}$ to itself which satisfies two requirements. First, it preserves the structure of the Lie algebra, which means it is linear and compatible with the Lie bracket

$$\omega\left([x,y]\right) = \left[\omega\left(x\right), \omega\left(y\right)\right] \quad \forall\, x, y \in \mathfrak{g}\,.$$

And second, it is a bijection, i.e., a map which is one-to-one and onto.

The set of all automorphisms of $\mathfrak{g}$ is a group, denoted by $Aut(\mathfrak{g})$. This follows from the fact that the composition of two automorphisms ω and ω' is still an automorphism, while there is an identity element $\mathtt{id}$ given by the trivial automorphism. Besides that the composition of maps is associative and each automorphism ω has an inverse ω^{-1} satisfying $\omega \circ \omega^{-1} = \mathtt{id}$.

An automorphism is said to be of order N if there exists $N \in \mathbb{N}$ such that

$$\omega^N \equiv \underbrace{\omega \circ \omega \circ \cdots \circ \omega}_{N \text{ times}} = \mathtt{id}\,.$$

In the case such a number does not exist, ω is said to be of infinite order. Furthermore, if ω is of finite order N, then $\mathfrak{g}$ splits into the direct sum

$$\mathfrak{g} = \bigoplus_{j=0}^{N-1} \mathfrak{g}^{(j)}$$

of eigenspaces of ω, with

$$\mathfrak{g}^{(j)} = \{x \in \mathfrak{g} | \omega(x) = \exp(2\pi i \cdot j/N)\, x\} \ .$$

The automorphism property of ω implies that, upon taking Lie brackets, the subspaces $\mathfrak{g}^{(j)}$ behave as

$$[\mathfrak{g}^{(j)}, \mathfrak{g}^{(k)}] \subseteq \mathfrak{g}^{(j+k \, mod \, N)} \, . \tag{4.7}$$

Besides that, we will also need to use the fact that Killing form[1] κ is invariant under any automorphism. In practice, this means that the trace of generators of $\mathfrak{g}$ is invariant, i.e., $\mathrm{Tr}\,(\omega(x)\,\omega(y)) = \mathrm{Tr}(x\,y)$ for any $x, y \in \mathfrak{g}$.

In this work, however, we will only need to use the Cartan involution, which is an automorphism of order 2. It acts on the elements of an arbitrary semisimple Lie algebra $\mathfrak{g}$ as [83],

$$\sigma(H_i) = -H_i,$$
$$\sigma(E_\alpha) = -E_{-\alpha}\,,$$

so that $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}\,,$$

where

$$\begin{aligned} \mathfrak{g}^{(0)} &= \{-i\,(E_\alpha - E_{-\alpha})\,,\ \text{for}\ \alpha > 0\}\ , \qquad (4.8)\\ \mathfrak{g}^{(1)} &= \{H_i,\, i = 1,\, 2,\, \ldots,\, r;\ (E_\alpha + E_{-\alpha})\,,\ \ \text{for}\ \alpha > 0\}\ . \end{aligned}$$

Then, $\mathfrak{g}^{(0)}$ forms a subalgebra of $\mathfrak{g}$ and the generators of $\mathfrak{g}^{(1)}$ form a representation of $\mathfrak{g}^{(0)}$. In the particular case of $su(n)$, we can use the results of eqs. (4.1) and (4.4) in order to see that the generators of $\mathfrak{g}^{(0)}$ will be of the form

$$L_{ij} = -i\,(E_{ij} - E_{ji})\ , \quad i, j = 1, \ldots, n\,, \tag{4.9}$$

which are generators of a $so(n)$ Lie algebra contained in $su(n)$ [40]. As an example, let us take $\mathfrak{g} = su(3)$. There are three generators in $\mathfrak{g}^{(0)}$ which form a $su(2)$ subalgebra and there are five generators in $\mathfrak{g}^{(1)}$ which form a quintuplet of this $su(2)$ subalgebra.

[1] The Killing form is an inner product on a finite dimensional Lie algebra $\mathfrak{g}$ defined by $\kappa(x, y) \equiv \mathrm{Tr}_{adj}\,(x\,y)$, where the trace is taken in the adjoint representation.

4.1.3 The Monopole Generators and Ansatz

Now, since we have already reviewed the necessary tools for our calculations, we are finally able to construct the Dark Monopole solutions. In order to do so, we shall use the results of sections 2.3.4, 4.1.1 and 4.1.2. For simplicity, let us consider that the gauge group is $G = SU(n)$ and that

$$\phi_0 = v\,\frac{\lambda_p \cdot H}{|\lambda_p|}\,, \tag{4.10}$$

where λ_p is an arbitrary fundamental weight of $su(n)$. This vacuum, spontaneously breaks $SU(n)$ to[2] [7]

$$G_0 = [SU(p) \times SU(n-p) \times U(1)]/Z\,.$$

We shall consider that the monopole generators M_i, which form a $su(2)$ subalgebra, belong to $\mathfrak{g}^{(0)}$. Then, ϕ_0, which is in $\mathfrak{g}^{(1)}$ (because it belongs to the Cartan subalgebra), will be in a representation of this $su(2)$, in agreement with our results of eq. (2.70).

Using the definition of eq.(2.12), we will consider that

$$M_3 = 2T_2^{\alpha}\,, \quad M_1 = 2T_2^{\beta}\,, \quad M_2 = 2T_2^{\gamma}\,,$$

where α, β, γ are roots of $su(n)$. Now, let us consider these roots written according to eq. (4.2). Then, from eq. (4.6) we see that in order for the generators M_i to satisfy the commutation relations of a $su(2)$ algebra the roots are such that $\alpha + \beta + \gamma = 0$.

Moreover, from the commutation relations of eq. (2.10) we see that $[M_3, \phi_0]$ is proportional to

$$[2T_2^{\alpha}, \lambda_p \cdot H] = i\,(\alpha \cdot \lambda_p)\,2T_1^{\alpha}. \tag{4.11}$$

But, since we want $M_3 \in L(G_0)$, then $[M_3, \phi_0] = 0$, which implies that $\alpha \cdot \lambda_p = 0$. From the results (2.14) and (2.17) this means that α does not have the simple root α_p in its expansion in the simple root basis. Thus, for $\alpha = e_i - e_j$, if $i < j$, either $i > p$ or $j \leq p$, and if $i > j$, either $i \leq p$ or $j > p$. On the other hand, recalling that M_1 and M_2 do not belong to $L(G_0)$, we can proceed with an argument similar to eq. (4.11) to conclude that $\beta \cdot \lambda_p \neq 0$ and $\gamma \cdot \lambda_p \neq 0$, which implies that β and γ have the simple root α_p in their expansion in the simple root basis.

[2] In order to avoid a misunderstanding, we once more emphasize that for the case $p = 1$ or $p = n$, G_0 is given by $G_0 = \frac{SU(n-1)\times U(1)}{Z}$, in agreement with our discussion of section 3.1.

Then, denoting by T_a^{ij}, $a = 1, 2, 3$, the generators defined in eq. (2.12) for $\alpha = e_i - e_j$, we conclude that the possible monopole generators, for α positive are

$$\begin{aligned} M_3 &= 2T_2^{ij} \,, \\ M_1 &= 2T_2^{jk} \,, \\ M_2 &= 2T_2^{ki} \,, \end{aligned} \tag{4.12}$$

where there are two possibilities: a) $1 \leq i < j \leq p$ and $j < k$, with $p < k \leq n$; b) $p < i < j \leq n$ and $k < j$ with $1 < k \leq p$. Note that each of these $su(2)$ subalgebras can be labeled by these three numbers i, j, k. On the other hand, when α is a negative root, $i > j$, which can be seen as an exchange between $i \leftrightarrow j$ in the cases above.

Furthermore, it is easy to check that the generators (4.12) indeed satisfy an $su(2)$ algebra. Note that this can be done by means of the result (4.6), which implies that

$$[2T_2^{jk}, 2T_2^{ki}] = -[E_{jk} - E_{kj}, E_{ki} - E_{ik}] = i\, 2T_2^{ij} \,.$$

Also note that the remaining two commutation relations can be obtained with cyclic permutation of the labels i, j, k.

At this point we should also remark that there may be other $su(2)$ subalgebras, with M_3 being a combination of step operators, from which we could construct other Dark Monopole solutions. However, for simplicity, in this work we will only consider the $su(2)$ subalgebras related to positive roots, given by eq.(4.12).

In addition, it is relevant to note that from the result of eq. (4.9) each set of M_i, $i = 1, 2, 3$, generates an $SO(3)$ subgroup of $SU(n)$[3]. However, the associated closed loop $h(\varphi)$, $0 \leq \varphi \leq 2\pi$, given by eq. (2.65), is contractible. In a pictorial way, this can be seen as follows. In the minimal symmetry breaking scheme of eq. (3.4) we see that $\pi_1(G_0) = \mathbb{Z}$ because of the $U(1)$ factor in G_0. Then, the topologically nontrivial loops in G_0 consist of loops winding around the $U(1)$ subgroup of G_0, and also of all the loops which travel through the $U(1)$ subgroup passing by identified elements and completing the loop with a "walk" through K. But, note that since $\mathrm{Tr}(M_3\phi_0) = 0$, our loop does not prescribe any path in the $U(1)$ direction. Therefore, these monopoles are associated to the trivial topological sector of $\pi_1(G_0)$.

For each $su(2)$ subalgebra, we can construct a monopole solution. And in order to obtain the asymptotic configuration of the scalar field

[3]For $G = SU(3)$, in the three dimensional representation, these generators correspond to the Gell-Mann matrices λ_7, $-\lambda_5$, λ_2.

(2.38) for each of them, it is convenient to decompose ϕ_0 as

$$\phi_0 = v\left(S + \frac{2Q_0}{\sqrt{6}\,|\lambda_p|}\right), \tag{4.13}$$

where

$$Q_0 = \frac{2}{\sqrt{6}}\left(T_3^{ik} + T_3^{jk}\right),$$
$$S = \frac{\lambda_p \cdot H}{|\lambda_p|} - \frac{2Q_0}{\sqrt{6}\,|\lambda_p|},$$

with $\mathrm{Tr}(Q_0Q_0) = y$ and

$$[M_3, Q_0] = 0 = [M_3, S] \,.$$

Moreover, $[M_\pm, S] = 0$, where $M_\pm = M_1 \pm iM_2$. Therefore, S is a singlet. On the other hand, one can check that Q_0 belongs to a quintuplet together with the generators

$$Q_{\pm1} = \pm\left(T_1^{ik} \pm i\,T_1^{jk}\right),$$
$$Q_{\pm2} = -\left(T_3^{ij} \pm i\,T_1^{ij}\right),$$

satisfying the commutation relations (2.72) with $l = 2$. Note that this is exactly the case we have discussed in eq. (2.70) of section 2.3.4 with $\omega = \frac{2}{\sqrt{6}|\lambda_p|}$.

Although for any $su(2)$ subalgebra M_i, the generators Q_m always form a quintuplet and therefore $l = 2$, we will continue to write l to keep track of this constant. It can also be useful for possible generalizations of the Dark Monopole construction with different l for other gauge groups.

Since $M_i \in \mathfrak{g}^{(0)}$ and $Q_m \in \mathfrak{g}^{(1)}$, then, it follows from the invariance of the trace under the Cartan involution that

$$\mathrm{Tr}\left(M_iQ_m\right) = 0 \,.$$

Moreover, since

$$\mathrm{Tr}(Q_m[Q_p, M_3]) = \mathrm{Tr}(Q_p[M_3, Q_m]) \,,$$

it results from eq. (2.72) that

$$-p\,\mathrm{Tr}(Q_mQ_p) = m\,\mathrm{Tr}(Q_mQ_p) \,,$$

which means that $\mathrm{Tr}(Q_m Q_p) = 0$ if $p \neq -m$. Similarly, from

$$\mathrm{Tr}\left(Q_m\left[Q_{-(m+1)}, M_\pm\right]\right) ,$$

it results that

$$\mathrm{Tr}\left(Q_m Q_{-m}\right) = -\mathrm{Tr}\left(Q_{m+1} Q_{-(m+1)}\right) .$$

Note that this is a simple recurrence relation, which can be solved by m' subsequent substitutions of $m \to m-1$ until $m - m' = 0$. Therefore, we can conclude that

$$\mathrm{Tr}\left(Q_m Q_p\right) = (-1)^m \, y \, \delta_{m,-p} \, . \tag{4.14}$$

Finally, from the definition of the generators M_i in (4.12) and our convention for the trace between step operators (2.11), it follows that

$$\mathrm{Tr}(M_i M_j) = 2\, y\, \delta_{ij} \, , \tag{4.15a}$$

$$\mathrm{Tr}(M_+ M_-) = 4\, y \, . \tag{4.15b}$$

Now, since $Q_m \in \mathfrak{g}^{(1)}$, then it follows from (4.7) that $[Q_m, Q_p] \in \mathfrak{g}^{(0)}$. Thus,

$$[Q_m, Q_p] = A_{mp}\, M_3 + B^+_{mp}\, M_+ + B^-_{mp}\, M_- + \sum_\delta D^\delta_{mp}\, T^\delta_2 \, ,$$

where A_{mp}, $B^\pm_{mp}$, D^δ_{mp} are constants and T^δ_2 are other possible generators of $\mathfrak{g}^{(0)}$. Then, taking the trace of this commutator with M_3, $M_\pm$ and $T^{-\delta}_2$, and using the previous results, we can conclude that

$$[Q_m, Q_p] = (-1)^m \left(\frac{m}{2}\, M_3\, \delta_{m,-p} - \frac{1}{4}\, c^-_{l,p}\, M_+\, \delta_{m,-p+1} \right.$$
$$\left. - \frac{1}{4}\, c^+_{l,p}\, M_-\, \delta_{m,-(p+1)} \right) ,$$

where we recall that $c^\pm_{l,p} = \sqrt{l(l+1) - p(p \pm 1)}$ with $l = 2$. This set of generators $\{M_i,\, Q_m\}$ form a $su(3)$ subalgebra of $su(n)$, since they are linear combinations of the generators T^{ij}_a, T^{ik}_a, T^{jk}_a, $a = 1, 2, 3$, where only eight of these generators will be linearly independent.

Finally, we can use the results of the general ansatz of section 2.3.4, given by eqs. (2.79) and (2.81) to (2.83) to propose the ansatz

$$\begin{aligned} W_i(\mathbf{r}) &= -\frac{[1-u(r)]}{er} \epsilon_{ijk} n^j M_k , \\ \phi(\mathbf{r}) &= v\, S + \alpha f(r) \sum_{m=-2}^{2} Y^*_{lm}(\theta, \varphi)\, Q_m \, , \end{aligned} \tag{4.16}$$

where again the boundary conditions are given by $u(0) = 1$, $u(r \to \infty) = 0$, $f(0) = 0$ and $f(r \to \infty) = 1$, which are exactly the same as those in section 2.3.4. Now, note that the ansatz of the Dark Monopoles is very similar to the one of eqs. (2.79) and (2.81) to (2.83), with the constraint of $l = 2$ as the only difference. Then, from now on we shall use the ansatz given by eq. (4.16) in order to obtain a general hamiltonian and equations of motion for our Dark Monopoles and for some possible future generalizations. In order to do this, we shall keep the dependence in l in our calculations.

4.2. HAMILTONIAN AND EQUATIONS OF MOTION

In this section, we shall obtain the Hamiltonian for our Dark Monopole, as well as the equations of motion (EoMs) for the profile functions. It is important to note that the “traditional” BPS bound for this monopole is zero, since $\mathrm{Tr}(B_i\phi) = 0$ and therefore the magnetic charge g associated to the $U(1)$ group vanishes. However, since B_i is a linear combination of M_a and $D_i\phi$ is a linear combination of Q_m, then the Bogomolny equation [35] $B_i = D_i\phi$ does not have a non-trivial solution. Hence, there is no solution associated to this vanishing bound.

Let us now calculate the exact form of the hamiltonian (2.24) under the ansatz of eq. (4.16). Let us start with the kinetic term of the scalar field. Since the component $\phi_s = vS$ is such that $\partial_i(\phi_s) = 0$ and $[\phi_s, M_i] = 0$, it implies that $D_i\phi_s = 0$. Then, from eq. (2.75) one can obtain that

$$D_i\phi = \alpha\left[(\partial_i f)Y^*_{lm} + f(\partial_i Y^*_{lm}) - i\frac{f(1-u)}{r^2}\epsilon_{ijk}x^j D^l\left(M_k\right)_{m'm} Y^*_{lm'}\right] Q_m\,. \tag{4.17}$$

Making use of eq.(2.84), eq.(4.17) can be written as

$$D_i\phi = \alpha\left[\frac{f'}{r}(x^i\, Y^*_{lm}) + f u\,(\partial_i\, Y^*_{lm})\right] Q_m\,. \tag{4.18}$$

From eq.(4.14) and the fact that $Y_{lm} = (-1)^m\, Y^*_{lm}$, one can obtain that

$$\frac{1}{y}\,\mathrm{Tr}\left(D_i\phi D_i\phi\right) = \sum_{m=-l}^{l} \alpha^2\left[(f')^2 Y_{lm}Y^*_{lm} + f^2u^2\,\nabla Y_{lm}\cdot\nabla Y^*_{lm}\right]\,. \tag{4.19}$$

Moreover, we can use the following properties of Vector Spherical Harmonics (VSH) [84]

$$\int_{S^2} \mathbf{Y}_{lm} \cdot \mathbf{Y}^*_{lm}\, d\Omega = \delta_{ll'}\delta_{mm'}\,,$$
$$\int_{S^2} \mathbf{\Psi}_{lm} \cdot \mathbf{\Psi}^*_{lm}\, d\Omega = l(l+1)\,\delta_{ll'}\delta_{mm'}\,,$$

where $\mathbf{Y}_{lm} \equiv Y_{lm}\hat{\mathbf{r}}$, $\mathbf{\Psi}_{lm} \equiv r\nabla Y_{lm}$ and $d\Omega = \sin\theta\, d\theta d\varphi$. Then, we obtain that

$$\frac{1}{2y}\int d^3x\, \mathrm{Tr}\,(D_i\phi D_i\phi) = 4\pi\,\frac{2v^2}{3|\lambda_p|^2}\int_0^\infty dr\left[\frac{1}{2}r^2(f')^2 + \frac{l(l+1)}{2}f^2u^2\right]. \tag{4.20}$$

From the magnetic field (2.80) and the trace between generators M_i given by (4.15a) it follows that

$$\frac{1}{2y}\int d^3x\, \mathrm{Tr}\,(B_iB_i) = 4\pi\int_0^\infty dr\,\frac{1}{e^2r^2}\left[2r^2(u')^2 + (1-u^2)^2\right]. \tag{4.21}$$

Finally, we use eqs. (2.83) and (4.16), the fact that

$$\mathrm{Tr}\,(SS) = \left(1 - \frac{2}{3\,|\lambda_p|}\right)y\,,$$
$$\mathrm{Tr}\,(\phi_q\phi_q) = \frac{2v^2f^2}{3\,|\lambda_p|^2}\mathrm{Tr}\,\left(g\,Q_0\,g^{-1}g\,Q_0\,g^{-1}\right) = \frac{2v^2f^2y}{3\,|\lambda_p|^2}\,,$$

and $\mathrm{Tr}(SQ_0) = 0$ in order to obtain that

$$V(\phi) = \frac{\lambda v^4}{9\,|\lambda_p|^4}\,(f^2-1)^2. \tag{4.22}$$

Joining all the contributions and making the change of variables $\xi = evr$ the Hamiltonian (2.24) for the Dark Monopole will be

$$\begin{aligned} E = \frac{4\pi v}{e}\int_0^\infty d\xi\,\Bigg\{&\left[2(u')^2 + \frac{(1-u^2)^2}{\xi^2}\right] \\ &+\frac{2}{3|\lambda_p|^2}\left[\frac{1}{2}\xi^2(f')^2 + \frac{l(l+1)}{2}f^2u^2\right] \\ &+\frac{\lambda}{9e^2|\lambda_p|^4}\,\xi^2(f^2-1)^2\Bigg\}\,, \end{aligned} \tag{4.23}$$

where $u'(\xi)$, $f'(\xi)$ denote derivatives with respect to ξ.

The conditions for E to be stationary with respect to $f(\xi)$ and $u(\xi)$ provide the equations of motion for the ansatz of the Dark Monopole:

$$u'' = \frac{l(l+1)}{6|\lambda_p|^2}\, f^2 u + \frac{u(u^2-1)}{\xi^2}\,, \tag{4.24a}$$

$$f'' = -\frac{2}{\xi} f' + l(l+1)\,\frac{f u^2}{\xi^2} + \frac{2\lambda}{3e^2|\lambda_p|^2}\, f(f^2-1)\,. \tag{4.24b}$$

The appropriate boundary conditions for a non-singular finite-energy solution are

$$f(0) = 0, \qquad u(0) = 1 \tag{4.25}$$

$$f(\xi \to \infty) = 1, \qquad u(\xi \to \infty) = 0, \tag{4.26}$$

which are exactly the same as those from the 't Hooft-Polyakov monopole. Also note that the radial EoMs could have been obtained by substitution of our ansatz into the general Yang-Mills-Higgs equations (2.22). However, the calculations would be more complicated and lengthy.

Before looking for numerical solutions to eqs.(4.24a) and (4.24b), we shall analyze the behavior of the profile functions when $\xi \approx 0$ and also when $\xi \to \infty$.

4.2.1 Approximate Solutions

When $\xi \ll 1$, eq.(4.24a) remains non-linear, since the dominant contribution is of the form $u'' = u(u^2-1)/\xi^2$. However, since we are looking for approximate solutions, it is reasonable to series expand (4.24a) about $\xi = 0$ to order ξ^2. Since the series expansion is pretty long and elementary we shall omit it here. Besides that, it is more appropriate to use softwares such as `Mathematica`® and `wxMaxima`® to obtain all the required terms. Then, it is a trivial task to see that

$$u(\xi) = 1 - c_1\,\xi^2\,, \tag{4.27}$$

with $c_1 \in \mathbb{R}$, gives the behavior of $u(\xi)$, subject to the boundary conditions (4.25), near the origin. We do not bother to fix the constant c_1, since we are only interested in the behavior of the solution.

With regard to eq.(4.24b) one can see that the dominant contribution is of the form

$$\xi^2 f'' + 2\xi f' - l(l+1) f = 0\,,$$

where we used the approximation $u^2(\xi \to 0) \approx 1$. This equation is in the form of the Euler-Cauchy equation. Then, the solution which satisfies the boundary condition (4.25) is

$$f(\xi) = c_2\, \xi^l \,, \tag{4.28}$$

where $c_2 \in \mathbb{R}$ is also an arbitrary constant. It is important to stress that solutions (4.27) and (4.28) agree with the fact that we are looking for non-singular monopole solutions. In appendix C we explicitly check that the expression of ϕ, W_i and B_i are regular at the origin.

At this point, we can make an important comparison between the 't Hooft-Polyakov monopole and our Dark Monopoles. While the behavior of the profile function in the gauge field ansatz ($u(\xi)$) is the same for both, in the case of the Higgs field ($f(\xi)$) we see a distinct behavior. In the 't Hooft-Polyakov case, $f(\xi) \sim \xi$, although in our construction $f(\xi) \sim \xi^2$.

Finally we analyze how the asymptotic values (4.26) are approached. In order to do so, it is convenient to substitute $f = (h/\xi) + 1$ in the eqs.(4.24a) and (4.24b) and take $\xi \to \infty$, which results in

$$u'' = \frac{l(l+1)}{6|\lambda_p|^2}\, u \,, \tag{4.29a}$$

$$h'' = \frac{4\lambda}{3e^2|\lambda_p|^2}\, h \,. \tag{4.29b}$$

Thus, the solutions behave as

$$u(\xi) = O\left[\exp\left(-\sqrt{\frac{l(l+1)}{6|\lambda_p|^2}}\,\xi\right)\right] , \tag{4.30}$$

$$f(\xi) - 1 = O\left[\frac{\exp\left(-\sqrt{\frac{4\lambda}{3e^2\lambda_p|^2}}\,\xi\right)}{\xi}\right] . \tag{4.31}$$

Therefore, for distances larger than the monopole core

$$R_{\text{core}} = \frac{1}{ev}\sqrt{\frac{6\,|\lambda_p|^2}{l(l+1)}} \,,$$

the gauge field configuration in (4.16) reduces to the asymptotic form (2.68) and the magnetic field (2.80) takes the form of a hedgehog as in eq.(2.69).

4.2.2 Numerical Solution

From the fact that we cannot find an analytical solution to the set of equations (4.24a) and (4.24b), it is reasonable to look for numerical solutions. We numerically solved the problem making use of the `MATLAB`® program `bvp4c`, which implements the solution of boundary value problems (BVPs). The procedure to obtain the numerical solution is very similar to the case of the 't Hooft-Polyakov monopole, but here we shall give more details of the calculation.

First, the system of equations (4.24a) and (4.24b) were recast as a system of first order equations of the form

$$u' = v\,, \tag{4.32a}$$

$$v' = \frac{l(l+1)}{6|\lambda_p|^2}\, f^2 u + \frac{u(u^2-1)}{\xi^2}\,, \tag{4.32b}$$

$$f' = w\,, \tag{4.32c}$$

$$w' = -\frac{2}{\xi} w + l(l+1)\,\frac{fu^2}{\xi^2} + \frac{2\lambda}{3e^2|\lambda_p|^2}\, f(f^2-1)\,, \tag{4.32d}$$

where u, v, f and w are considered to be independent. Once more, we stress that in the case of our Dark Monopoles $l = 2$, and one can obtain several distinct solutions by choosing different SSB patterns through the choice of λ_p in the Lie algebra of G. These solutions must satisfy the constraints in the behavior imposed by the approximate solutions (4.27) and (4.28). Figure 4.1 shows the solution for the case of the SU(5) Dark Monopole, where the symmetry breaking is of the form $SU(5) \to$ "$SU(3) \times SU(2) \times U(1)$", where again the quotation marks refer to the local structure of the unbroken gauge group, only. In the $SU(5)$ case we can take the fundamental weight λ_p to be λ_2 or λ_3, since both of them generate the desired SSB. Then, it follows from the inverse Cartan matrix of $SU(5)$ [85] that $|\lambda_p|^2 = 6/5$. One can see that this solution agrees with the expected behavior, since $u - 1 \sim -\xi^2$ and $f \sim \xi^2$ near zero, while they both reach the asymptotic values rather fast.

The total energy of the solution, which is interpreted as the classical mass, is given by eq.(4.23) and to simplify the analysis we use the rescaled mass, $\tilde{E}$,

$$E = M_0\, \tilde{E}(\lambda/e^2), \quad \text{where } M_0 = \frac{4\pi v}{e}\,.$$

Performing an analysis similar to [44], we can obtain the mass range for the Dark Monopoles. Note first that $\tilde{E}$ is a monotonically increasing

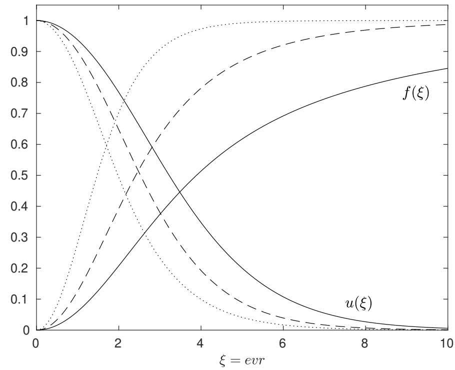

Figure 4.1: The monopole profile functions $u(\xi)$ and $f(\xi)$ for $\lambda/e^2 = 0$ (solid curves), $\lambda/e^2 = 0.1$ (dashed curves) and $\lambda/e^2 = 1$ (dotted curves).

function of λ/e^2, since

$$\frac{d\tilde{E}(\lambda/e^2)}{d(\lambda/e^2)} = \frac{1}{9|\lambda_p|^4} \int_0^\infty d\xi\, \xi^2 (f^2 - 1)^2 > 0\,.$$

The lower bound for the mass happens when $\lambda = 0$, and numerical integration shows that for the $SU(5)$ monopole $\tilde{E}(0) = 1.294$.

Similar to the case of the 't Hooft-Polyakov monopole [45], in the limit $\lambda \to \infty$ the mass of the monopole stays finite and it is given by

$$E = \frac{4\pi v}{e} \int_0^\infty d\xi \left[2(u'_\infty)^2 + \frac{(1-u_\infty^2)^2}{\xi^2} + \frac{l(l+1)}{3|\lambda_p|^2}\, u_\infty^2 \right], \qquad (4.33)$$

since $f(\xi) \equiv 1$, $\forall\, \xi > 0$ but $f(0) = 0$. Then, the only radial equation of motion is

$$u''_\infty = \frac{l(l+1)}{6|\lambda_p|^2}\, u_\infty + \frac{u_\infty (u_\infty^2 - 1)}{\xi^2}\,. \qquad (4.34)$$

Solving eq.(4.34) and performing the integration in (4.33) gives us the upper bound for the monopole mass. In the $SU(5)$ case, the upper bound is $\tilde{E}(\lambda \to \infty) = 3.262$. For comparison, for the 't Hooft-Polyakov monopole in the $SU(2)$ case, $\tilde{E}_{tHP}(\lambda = 0) = 1$ [36] and $\tilde{E}_{tHP}(\lambda \to \infty) = 1.787$ [45], as we have seen in section 2.4.

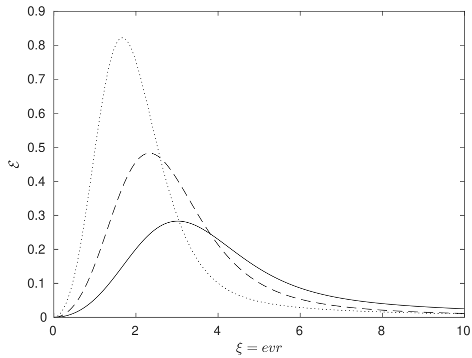

Figure 4.2: $\xi^2\mathcal{H}$ for the $SU(5)$ Dark Monopoles. The solid curves are from the solution with $\lambda/e^2 = 0$, while the dashed and dotted curves are from the $\lambda/e^2 = 0.1$ and $\lambda/e^2 = 1$, respectively.

In addition we can use the full numerical solution of eqs. (4.24a) and (4.24b) in order to plot the integrand of the rescaled mass $\tilde{E}(\lambda/e^2)$, which we will denote by $\mathcal{E}$. The result is presented in fig. 4.2, where we can see how well-localized is the size and shape of our $SU(5)$ Dark Monopole solution. This plot clearly indicates that the energy of the solution is finite.

Finally, note that for a given SSB, where λ_p is fixed, the value of monopole mass is the same for all the Dark Monopole solutions associated to the the $su(2)$ subalgebras (4.12). This follows directly from the fact that the hamiltonian is independent of the indices i, j, k that label those $su(2)$ subalgebras. Moreover, these are classical results. In order to determine the properties of the Dark Monopoles at the quantum level, one could use for example semi-classical quantization.

4.3. NON-ABELIAN MAGNETIC CHARGE

One of the main properties of the Dark Monopole solution is that its magnetic field is in a direction outside the Cartan subalgebra $\mathcal{H}$. Thus, as we mentioned before, this monopole has a vanishing abelian magnetic charge g, given by eq. (2.106), since $\mathrm{Tr}(B_i\phi) = 0$. However, from eq. (2.68) we see that far from the monopole core it has a non-abelian magnetic flux in the direction $g(\theta,\varphi)\, M_3\, g^{-1}(\theta,\varphi)$, with M_3 given by eq.(4.12). We shall define

$$\zeta(\vec{r}) = a(r)\, g(\theta,\varphi)\, M_3\, g^{-1}(\theta,\varphi) = a(r)\, n^a M_a\,, \qquad (4.35)$$

which is in the direction of the monopole non-abelian magnetic flux, where $a(r) \in \mathbb{R}$ is a radial function such that ζ is regular everywhere. This implies that when $r \to 0$, $a(r) \sim r$. On the other hand, when $r \gg R_{core}$, we consider that $a(r) = 1$. Then, using the fact that in this asymptotic region the gauge and the scalar fields assume the form (2.68) and (2.38), respectively, it is easy to verify that asymptotically ζ satisfies the conditions

$$D_\mu \zeta = 0 \, , \tag{4.36}$$

$$[\phi, \zeta] = 0 \, . \tag{4.37}$$

Recalling that under an infinitesimal gauge transformation of the form $1 + ic_a T_a$ with $c_a \in \mathbb{R}$ and $T_a \in L(G)$, the fields W_μ and ϕ transform as

$$\delta W_\mu = W'_\mu - W_\mu = \frac{1}{e} D_\mu (c_a T_a) \, ,$$
$$\delta \phi = \phi' - \phi = i \, [c_a T_a, \phi] \, ,$$

we can conclude from eqs. (4.36) and (4.37) that the asymptotic configuration of the monopole is invariant under a gauge transformation of the form $\exp(i\zeta)$. Therefore, ζ is a Killing vector which is associated to a symmetry of the asymptotic fields of the monopole. According to [86] and [87], from the existence of a Killing vector ζ for an asymptotic symmetry one can associate a conserved charge. It is interesting to note that ζ satisfies the same asymptotic conditions as the scalar field ϕ for the 't Hooft-Polyakov monopole, outside the monopole core. Therefore, in this special case ϕ can be identified with the Killing vector ζ. Note that if we perform an arbitrary gauge transformation U on the monopole's fields then, from eqs. (4.36) and (4.37), we obtain that ζ must transform as

$$\zeta \to \zeta' = U \, \zeta \, U^{-1} \, ,$$

in order for ζ' to be a Killing vector of the transformed fields.

Moreover, since ζ and W_i take values in the $su(2)$ subalgebra formed by M_a, we can expand them as

$$\begin{aligned} W_i &= W_{ia} M_a \, , \\ \zeta &= \zeta_a M_a \, , \\ D_i \zeta &= \left(D_i \zeta\right)_a M_a \, , \end{aligned} \tag{4.38}$$

where $\left(D_i \zeta\right)_a = \partial_i \zeta_a - e\epsilon_{abc} W_{ib} \zeta_c$. We shall also introduce the notation $\overline{\zeta}$ for the asymptotic configuration of ζ. Then, it follows from eq.(4.35) that $\bar{\zeta}_a = n^a$ is a unitary vector. Note that $\bar{\zeta}_a^2 = 1$ defines a 2-sphere, which we will denote by Σ.

Now, let us define a gauge-invariant magnetic current by taking a projection of ${}^*G^{\mu\nu}$ in the direction of the Killing vector ζ as

$$J_M^\mu \equiv \frac{1}{|\bar{\zeta}|y}\,\partial_\nu \mathrm{Tr}\left({}^*G^{\mu\nu}\zeta\right) , \tag{4.39}$$

where $|\bar{\zeta}| \equiv \sqrt{\bar{\zeta}_a\bar{\zeta}_a} = 1$. Besides that, ${}^*G^{0i} = B^i$ and ${}^*G^{ij} = -\epsilon_{ijk}E^k$. The conservation of the current J_M^μ follows from its definition as a divergence of an antisymmetric tensor and from the fact that $\mathrm{Tr}\left(B^i\zeta\right)$ is twice differentiable.

Thus, the conserved non-abelian magnetic charge is

$$Q_M = \int_{\mathbb{R}^3} d^3x\, J_M^0 = \frac{1}{|\bar{\zeta}|y}\oint_{S^2_\infty} dS_i\, \mathrm{Tr}\left(B^i\zeta\right) = -\frac{8\pi}{e}\,. \tag{4.40}$$

Note that eq.(4.40) is just a gauge-invariant measure of the non-abelian flux in the normalized $\bar{\zeta}(\theta,\varphi)$ direction. Furthermore, we must emphasize that the introduction of the radial function $a(r)$ has no contribution to the magnetic charge. This artifact was introduced so that we could define a regular magnetic current for the Dark Monopole. Besides that, as pointed out by [49] there is no unambiguous way to measure the charge density of a monopole. Only the total charge makes sense.

Let us now analyze the geometric meaning of the non-abelian magnetic charge Q_M. For this purpose, we shall use some arguments similar to those in section 2.4.1. First, it follows from the asymptotic condition (4.36) that

$$\frac{1}{8\pi y}\epsilon^{ijk}\oint_{S^2_\infty} dS_i \mathrm{Tr}\left\{\zeta\left[D_j\zeta, D_k\zeta\right]\right\} = 0\,. \tag{4.41}$$

Then, from eq.(4.38) and using vector notation, as well as the fact that $|\zeta| = 1$ when $r \to \infty$, eq.(4.41) can be written as

$$\frac{1}{8\pi}\epsilon^{ijk}\oint_{S^2_\infty} dS_i\left\{\hat{\zeta}\cdot\left(\partial_j\hat{\zeta}\times\partial_k\hat{\zeta}\right) - e\hat{\zeta}\cdot\vec{G}_{jk}\right\} = 0\,. \tag{4.42}$$

Now, using eq.(4.15a), the expression of the non-abelian magnetic charge (4.40) can be written as

$$Q_M = 2\oint_{S^2_\infty} dS_i \vec{B}^i\cdot\hat{\zeta}\,,$$

and from eq.(4.42) we conclude that

$$Q_M = -\frac{4\pi}{e}\,2N_\zeta\,,$$

where

$$N_\zeta = \frac{1}{8\pi}\epsilon^{ijk}\oint_{S^2_\infty} dS_i\left\{\hat{\zeta}\cdot\left(\partial_j\hat{\zeta}\times\partial_k\hat{\zeta}\right)\right\}\,.$$

As it is well-known, this integral is a topological quantity which is an integer and has the geometrical interpretation [50] which is to measure the number of times $\hat{\zeta}$ covers Σ as $\hat{r}$ covers S^2_∞ once. For our particular Dark Monopole construction, where $\bar{\zeta}^a = n^a$, $N_\zeta = 1$. However, in principle, one could obtain higher magnetic charges, generalizing our construction, considering for example a gauge transformation

$$g(\theta,\varphi) = \exp\left(-i\varphi k M_3\right)\exp\left(-i\theta M_2\right)\exp\left(i\varphi k M_3\right)\,,\quad k\in\mathbb{Z}\,,$$

which would be associated to $\hat{\zeta}$ covering Σ k times as $\hat{r}$ covers S^2_∞ once.

It is important to remark that for the Dark Monopole, the magnetic charge is not the usual one (in the abelian direction), associated to the homotopy classes of the scalar field, like in the 't Hooft-Polyakov case. In fact, Q_M is related to a gauge-invariant magnetic flux in the ζ direction. Moreover, we would like to stress that, even though non-abelian magnetic fields are not gauge-invariant, our magnetic current J^μ_M, the magnetic flux of eq. (4.40) and our magnetic charge Q_M are indeed gauge-invariant.

Therefore, from the results above we can conclude that the non-abelian magnetic charge of the Dark Monopole is conserved and quantized in multiples of $8\pi/e$. And even though they are associated to the trivial sector of $\Pi_1(G_0)$, the conservation of Q_M could prevent them to decay, at least classically. However, it is necessary to analyze in more detail the stability of the Dark Monopole.

5. CONCLUSIONS AND DISCUSSIONS

In this work we have reviewed the construction of non-abelian magnetic monopoles in Yang-Mills-Higgs theories with an adjoint Higgs field ϕ and a simple and simply-connected gauge group G. We revisited the 't Hooft-Polyakov monopole under a somewhat different perspective, while we have also analyzed its abelian magnetic charge in details.

Moreover, we have discussed symmetry breaking patterns and how to embed $SU(2)$ monopoles in theories with larger gauge groups, as well as some aspects of NUS monopoles. In particular, we presented a pedagogical review of the "Problem of Global Color".

Nevertheless, the most relevant point is that we have obtained a general procedure to construct monopole solutions whose magnetic field does not lie in the Cartan subalgebra $\mathcal{H}$ and, thus, vanishes in the direction of the generator of the $U(1)_{em}$ electromagnetic field. In order to do that, we considered theories with gauge group $SU(n)$ and a scalar field in the adjoint representation. But, we expect that this construction can be generalized to other gauge groups. These Dark Monopoles must exist in some Grand Unified Theories and we analyzed some of their properties for the $SU(5)$ case. In particular, we obtained their mass range.

We also have shown that our monopole solution has a conserved magnetic current J_M^μ in the direction of the Killing vector ζ. The associated charge is quantized and it measures the number of times $\hat{\zeta}$ covers Σ as $\hat{r}$ covers S^2_∞ once. In principle, the conservation of this non-abelian magnetic charge could prevent the Dark Monopoles to decay. However, the stability should be analyzed in more details in the future.

Another point that still needs to be analyzed is related to the cosmological implications of the Dark Monopoles. Nonetheless, in order to discuss some cosmological aspects, let us assume for a moment that our solution is indeed stable or has a reasonable lifetime.

We expect that the Dark Monopoles were created in a phase transition in the early universe by the Kibble mechanism [56] at a temperature of the order of the unification scale, along with the standard GUT monopoles. Under some general assumptions [55] one can show that their initial abundance $n_M(t_i)$ has evolved in time according to [88]

$$\dot{n}_M + 3Hn_M = -Dn_M^2\,,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, while $a(t)$ is the scale factor in the Robertson-Walker metric. The last term is associated to the annihilation mechanism and comes from the collision term in the Boltzmann equation [55]. This implies that the time evolution of the monopole density strongly depends on how the monopoles interact between themselves and also on how they interact with the plasma of particles in the universe. Their motion can be described as [79] a Brownian motion of heavy dust particles in a gas or liquid with a slight bias in their random walks caused by the interaction between monopoles and antimonopoles. But one should note that since our monopoles have a vanishing $U(1)$ magnetic charge, there may be some differences in the annihilation mechanism, such as a different mean free path l and capture radius r_c (high-temperature regime) as well as the cross-section for radiative capture (low-temperature regime). As a consequence, we expect the so called *monopole-to-entropy ratio* [30, 79] to be different. However, a future detailed analysis on how the monopoles interact is necessary in order to make estimates of this ratio.

Another relevant point is that when some possible ordinary monopoles interact with the magnetic field of our galaxy, they are accelerated. And it depends on the mass of these monopoles whether they will be ejected or slightly deflected [30]. In any case, the acceleration of these monopoles will drain energy from the galactic field. Now, note that since our Dark Monopoles do not interact with galactic magnetic fields, they will not be accelerated and, in principle, this means that they can cluster with the galaxy. The same reasoning can be applied to magnetic fields in galactic clusters.

Now, with regard to the Dark Matter problem we expect that Dark Monopoles might contribute to part of the mass usually attributed to Dark Matter. However, in face of the inflationary scenario [89], we expect that this contribution might be small. One way out of this is to investigate whether it is possible that any amount of Dark Monopoles were created during the *reheating* phase after inflation through energy density fluctuations. Although even if they do not have a relevant contribution to Dark Matter, they are still an interesting solution since they are a new type of GUT monopoles.

Finally, we recall that as the case of standard GUT monopoles the mass of our Dark Monopoles is set by the GUT scale, which is beyond the energy scale of particle physics experiments, and currently direct detection is unlikely. However, as we mentioned before, further analysis is needed on how our monopoles interact and this may give some hints on the way we can look for them.

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A. Calculations of the Gauge and Magnetic Fields

In this appendix, we have two separate goals. The first one is to prove that the Dirac string singularity can be removed in a non-abelian gauge theory, while the second one is to give the detailed calculations of the magnetic field $B_i(\mathbf{r})$ for non-abelian monopoles.

A.1. GAUGE FIELDS WITH NO DIRAC STRINGS

In this section we intend to show how the Dirac string singularity in eq. (2.60) can be removed with a gauge transformation with $g(\theta, \varphi) \in G$ given by eq. (2.64). First, let us recall that in the string gauge

$$W_r^{(0)} = 0 = W_\theta^{(0)} \,, \tag{A.1}$$

$$W_\phi^{(0)} = \frac{1-\cos\theta}{e} \, M_3 \,. \tag{A.2}$$

Furthermore, we know that under a gauge transformation, the gauge field W_μ transforms as eq. (2.42). Then, it is easy to calculate the components of the asymptotic gauge field in spherical coordinates. First, since $g \, W_r^{(0)} \, g^{-1} = 0$ and $\partial_r g(\theta, \varphi) = 0$, the transformed radial component is trivial, i.e.,

$$W_r(\theta, \varphi) = 0 \,. \tag{A.3}$$

Secondly, the component W_θ is also simple, since $g \, W_\theta^{(0)} \, g^{-1} = 0$, which implies that

$$W_\theta = \frac{i}{e} \, (\partial_\theta g) g^{-1} = \frac{1}{e} \, e^{-i\varphi M_3} \, M_2 \, e^{+i\varphi M_3} \,.$$

Thus, we can use the result of eq. (2.63) in order to obtain that

$$W_\theta(\theta, \varphi) = -\frac{1}{e} \left[(\sin\varphi) \, M_1 - (\cos\varphi) \, M_2 \right] \,. \tag{A.4}$$

Thirdly, we can compute the azimuthal component W_φ. From eq. (A.2) we obtain that $g \, W_\varphi^{(0)} \, g^{-1} = (1 - \cos\theta) \, g M_3 g^{-1} / e$. Besides that, it immediately follows from eq. (2.64) that

$$\frac{i}{e} \, (\partial_\varphi g) g^{-1} = \frac{1}{e} \left[M_3 - g \, M_3 g^{-1} \right] \,.$$

Thus,

$$W_\varphi(\theta,\varphi) = -\frac{1}{e}\,\left[(\cos\theta)\, g\, M_3\, g^{-1} - M_3\right]$$
$$= -\frac{1}{e}\,\sin\theta\,\left[\cos\theta\,(\cos\varphi\, M_1 + \sin\varphi\, M_2) - \sin\theta\, M_3\right]\,, \quad \text{(A.5)}$$

where we have used that $g\, M_3\, g^{-1}$ is given by

$$g\, M_3\, g^{-1} = [(\sin\theta\cos\varphi)\, M_1 + (\sin\theta\sin\varphi)\, M_2 + (\cos\theta)\, M_3]\ .$$

Note that, after this gauge transformation, the asymptotic configuration of the gauge field is free of singularities. However, if one wants to check that these spherical components, given by eqs. (A.3) to (A.5), are equivalent to the asymptotic gauge field configuration written in cartesian coordinates by eq. (2.68), one should proceed with a change of variables. We shall not present this calculation here because it is too simple and not worth it.

A.2. MAGNETIC FIELD IN CARTESIAN COORDINATES

Let us now show how one can obtain the magnetic field B_i, given by eq. (2.80), in details. First, let us recall that the ansatz for the gauge field is of the form of eq. (2.79), i.e.,

$$W_i = -\frac{[1-u(r)]}{er}\,\epsilon_{ijk} n^j M_k\ .$$

Moreover, we know that $2B_i = -\epsilon_{ijk}G_{jk}$, with G_{ij} given by eq. (2.20). Then,

$$2B_i = -\epsilon_{ijk}\left(2\partial_j W_k + ie[W_j, W_k]\right)\ . \quad \text{(A.6)}$$

For the sake of clarity, let us analyze both terms separately. The first one is given by

$$-\epsilon_{ijk}\,(2\,\partial_j W_k) = 2\,\epsilon_{ijk}\epsilon_{abk}\partial_j\left[\left(\frac{1-u}{er}\right) n^a\right] M_b$$
$$= \frac{2}{e}\,(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})\left[\frac{d}{dr}\left(\frac{1-u}{r}\right) P_L^{ja} + \frac{(1-u)}{r^2} P_T^{ja}\right] M_b\,,$$

where $P_L^{ja} = n^j n^a$ and $P_T^{ja} = \delta^{ja} - n^j n^a$. Also note that we have used the result $\partial_j n^a = P_T^{ja}/r$. After the contractions with the Kronecker delta, we obtain that

$$-\epsilon_{ijk}\,(2\,\partial_j W_k) = \frac{2}{e}\left[\frac{d}{dr}\left(\frac{1-u}{r}\right)\left(P_L^{ib} - \delta^{ib}\right) + \frac{(1-u)}{r^2}\left(P_T^{ib} - 2\delta^{ib}\right)\right] M_b\,.$$

But, we should note that $\left(P_L^{ib} - \delta^{ib}\right) = -P_T^{ib}$ and that $\left(P_T^{ib} - 2\delta^{ib}\right) = -\left(2P_L^{ib} + P_T^{ib}\right)$. Moreover, since $\frac{d}{dr}\left(\frac{1-u}{r}\right) + \left(\frac{1-u}{r^2}\right) = -\frac{u'}{r}$ the above result can be simplified to

$$-\epsilon_{ijk}\left(2\,\partial_j W_k\right) = \frac{2}{e}\left[\frac{u'}{r}\,P_T^{ib} + 2\left(\frac{1-u}{r^2}\right)P_L^{ib}\right]M_b\,. \tag{A.7}$$

Now, we are able to compute the second term in eq. (A.6). First, note that

$$ie[W_j, W_k] = -\frac{(1-u)^2}{er^2}\epsilon_{abc}\,\epsilon_{jmb}\,\epsilon_{knc}\,P_L^{mn}M_a\,.$$

Then, after the contraction with ϵ_{ijk} we get that

$$-ie\epsilon_{ijk}[W_j, W_k] = \frac{(1-u)^2}{er^2}\left(\delta_{mk}\delta_{ib} - \delta_{mi}\delta_{kb}\right)\epsilon_{abc}\epsilon_{knc}\,P_L^{mn}M_a\,.$$

Moreover, from the antisymmetry of the Levi-Civita symbol and the fact that P_L^{mn} is symmetric under $m \leftrightarrow n$, it is trivial to see that $\left(\delta_{mk}\delta_{ib} - \delta_{mi}\delta_{kb}\right)\epsilon_{abc}\epsilon_{knc}P_L^{mn}M_a = \epsilon_{akc}\epsilon_{nkc}\,P_L^{in}M_a = 2P_L^{ia}M_a$. Therefore,

$$-\,ie\epsilon_{ijk}[W_j, W_k] = \frac{2}{e}\,\frac{(1-u)^2}{r^2}P_L^{ia}M_a\,. \tag{A.8}$$

Then, joining the contributions from eqs. (A.7) and (A.8) we finally obtain that, in cartesian coordinates, the magnetic field B_i is of the form

$$B_i = \left(\frac{u'}{er}\,P_T^{ik} + \frac{u^2-1}{er^2}\,P_L^{ik}\right)M_k\,, \tag{A.9}$$

which is exactly the result of eq. (2.80).

B. Proof to Equation (3.22)

Let us now prove the result of eq. (3.22). First, let us recall that A, B and T are generators that satisfy (3.23). Then, we use the BCH formula to show that

$$\begin{aligned}
e^{i\theta T}\,A\,e^{-i\theta T} &= A + (-i\theta)[A,T] + \frac{(-i\theta)^2}{2!}[[A,T],\,T] \\
&+ \frac{(-i\theta)^3}{3!}[[[A,T],\,T],\,T] + \dots \\
&= A + (-i\theta)\,b\,B + \frac{(-i\theta)^2}{2!}\,ab\,A \\
&+ \frac{(-i\theta)^3}{3!}\,ab^2\,B + \frac{(-i\theta)^4}{4!}\,a^2b^2\,A + \dots \\
&= \left(1 - \frac{(\theta\sqrt{ab})^2}{2!} + \frac{(\theta\sqrt{ab})^4}{4!} + \dots\right) A \\
&- ib\left(\theta - \frac{\theta^3 ab}{3!} + \frac{\theta^5 (ab)^2}{5!} + \dots\right) B\,.
\end{aligned}$$

Now, recalling that

$$\theta^{2n+1}\,(ab)^n = \frac{1}{\sqrt{ab}}\,\theta^{2n+1}\,(ab)^{n+1/2} = \frac{1}{\sqrt{ab}}\,\left(\theta\sqrt{ab}\right)^{2n+1},$$

we obtain that

$$\begin{aligned}
e^{i\theta T}\,A\,e^{-i\theta T} &= \left(\sum_{n=0}^{+\infty} \frac{(-1)^n(\theta\sqrt{ab})^{2n}}{(2n)!}\right) A \\
&- i\frac{b}{\sqrt{ab}}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n(\theta\sqrt{ab})^{2n+1}}{(2n+1)!}\right) B \\
&= \cos\left(\theta\sqrt{ab}\right) A - i\frac{b}{\sqrt{ab}}\,\sin\left(\theta\sqrt{ab}\right) B\,,
\end{aligned}$$

which is exactly what we wanted to demonstrate.

C. Regularity at the Origin

Let us show that the fields $W_i(\mathbf{r})$, $B_i(\mathbf{r})$ and $\phi(\mathbf{r})$ are regular at the origin, as mentioned in section 4.2.1.

First, let us recall that the gauge field has the form of eq. (4.16), i.e.,

$$W_i = -\frac{[1-u(r)]}{er}\,\epsilon_{ijk}n^j M_k\,,$$

where again $n^j = x^j/r$. Then, we want to calculate

$$\lim_{\mathbf{r}\to 0} W_i = \lim_{\mathbf{r}\to 0} -\frac{[1-u(r)]}{er^2}\,\epsilon_{ijk}x^j M_k\,.$$

It is true that, formally, we should specify the paths by which we are taking the limit, for instance, setting $x = 0 = y$ and taking $z \to 0$ or any other possible way to approach the origin. However, it will soon be clear that for any of the possible paths the limit exists and it is zero. Now, we must remember that, even though the profile function $u(r)$ does not have an analytic expression, in the limit $r \to 0$ we can use the approximate solution (4.27) in order to analyze the regularity at the origin. This implies that

$$\begin{aligned}\lim_{\mathbf{r}\to 0} W_i &= \lim_{\mathbf{r}\to 0} -\frac{[1-(1-c_1 r^2)]}{er^2}\,\epsilon_{ijk}x^j M_k \\ &= \lim_{\mathbf{r}\to 0} \frac{c_1}{e}\,\epsilon_{ijk}x^j M_k \\ &= 0. \end{aligned} \tag{C.1}$$

Note that the last step is independent of the path we may choose, since for any of them either the component $x^j = 0$ or $x^j \to 0$ in the implicit sum. Therefore, the gauge field is indeed regular at the origin. Furthermore, let us emphasize that this result is also true for the cases of the 't Hooft-Polyakov and the $SU(2)$-embedded monopoles, since they all have the same expression for $W_i(\mathbf{r})$ and also an approximate solution of the form $(1-u) \propto -r^2$ when $r \to 0$.

With regard to the magnetic field B_i, given by eq. (2.80), the situation is similar. We know that B_i is of the form

$$B_i = \left(\frac{u'}{er}\,P_T^{ik} + \frac{u^2-1}{er^2}\,P_L^{ik}\right) M_k\,,$$

which can be conveniently rewritten as

$$B_i = \frac{u'}{er}M_i + \left(\frac{u^2-1-ru'}{er^4}\right) x^i x^k M_k\,,$$

using that $P_T^{ik} = \delta^{ik} - n^i n^k$ and $P_L^{ik} = n^i n^k$. Now, note that

$$\lim_{r\to 0} \frac{u'}{er} M_i = \lim_{r\to 0} -\frac{2c_1 r}{er} M_i = -\frac{2c_1}{e}$$

while

$$\lim_{\mathbf{r}\to 0} \left(\frac{u^2 - 1 - ru'}{er^4}\right) x^i x^k M_k = \lim_{\mathbf{r}\to 0} \frac{(1 - c_1 r^2)^2 - 1 + 2c_1 r^2}{er^4} x^i x^k M_k \,,$$

which can be simplified to

$$\lim_{\mathbf{r}\to 0} \frac{c_1^2}{e} x^i x^k M_k = 0 \,,$$

where again the limit is independent of the path by which we approach the origin. Hence,

$$\lim_{\mathbf{r}\to 0} B_i = -\frac{2c_1}{e} M_i \,, \tag{C.2}$$

which implies that the magnetic field is also non-singular.

Finally, we can analyze the behavior of the Higgs field in eq. (4.16). Since we do not have to worry about the constant term $\phi_s = vS$, we only need to consider the behavior of our general expression (2.83), i.e.,

$$\phi_q(r, \theta, \varphi) = \alpha\, f(r) \sum_m Y^*_{lm}(\theta, \varphi)\, Q_m \,.$$

Remember that, in the limit $r \to 0$, $f(r)$ is given by the approximate solution (4.28), which means that $f(r \to 0) \approx c_2 r^l$. Then,

$$\lim_{r\to 0} f(r) \sum_m Y^*_{lm}(\theta, \varphi)\, Q_m = \lim_{r\to 0} c_2 r^l \sum_m Y^*_{lm}(\theta, \varphi)\, Q_m = 0 \,,$$

which follows from the fact that when we write the spherical harmonics $Y_{lm}(\theta, \varphi)$ and their complex conjugates in terms of the cartesian coordinates we always get a product of r^{-l}, whose singularity is canceled by the r^l factor coming from $f(r)$, and a combination[1] of the coordinates x, y and z, which take the limit to zero. Thus,

$$\lim_{\mathbf{r}\to 0} \phi = vS \,, \tag{C.3}$$

which implies that it is regular at the origin. Also note that this result is valid in the special case when $l = 1$, for the $SU(2)$-embedded monopoles, and also to the more restrictive 't Hooft-Polyakov case, when $l = 1$ and $S = 0$.

Therefore, eqs. (C.1) to (C.3) prove that the Dark Monopole field configuration is well-defined at the origin, as well as the other monopoles we have discussed in this work.

[1] In general, it is a non-linear combination of the coordinates x, y and z.