Julio Eduardo Cáceres Gonzales

Cuntz-Pimsner algebras associated to vector bundles

Florianópolis

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Cuntz-Pimsner algebras associated to vector bundles

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To all the cats in my life, specially Pipi to whom I missed the chance to say goodbye.

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Resumo

Neste trabalho desenvolvemos técnicas para calcular a K-teoria associada à álgebra de Cuntz-Pimnser $\mathcal{O}_E$ proveniente de um B-módulo E com frame de Parseval finito. Nós aplicamos isto aos módulos de Hilbert que surgem naturalmente das seções contínuas de um fibrado vetorial sobre um espaço compacto e Hausdorff X. Para isto, apresentamos uma ligeira generalização de um resultado por Exel, an Huef e Raeburn que nos dá uma sequência exata de seis termos com morfismos bastante concretos que permitem o cálculo da K-teoria das referidas álgebras de Cuntz-Pimsner.

Palavras-chave: K-teoria; álgebras de Cuntz-Pimsner; módulos de Hilbert;

Abstract

In this work we develop techniques to calculate the K-theory associated to the Cuntz-Pimsner algebra $\mathcal{O}_E$ of a Hilbert B-module E with finite Parseval frame. We apply this to the Hilbert modules naturally arising from the continuous sections of a vector bundle over a compact Hausdorff space X. In order to do this we present a slight generalization of a result by Exel, an Huef and Raeburn that gives us a six-term exact sequence with rather concrete morphisms that permit the calculation of the K-theory of said Cuntz-Pimsner algebras.

Key-words: K-theory; Cuntz-Pimsner algebras; Hilbert modules;

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Introduction

In 1997 Pimsner introduced the concept of Cuntz-Pimsner algebras as a way to generalize certain crossed products and Cuntz-Krieger algebras. In the same paper he introduced a six-term exact sequence for their KK-theory whenever a certain Hilbert module over A is countably generated with A separable. One can use this sequence to induce a six-term exact sequence for the K-theory of these Cuntz-Pimsner algebras, however this sequence will end up having some morphisms which are not easy to compute. In 2011 Exel, an Huef and Raeburn proposed a method to compute the K-theory of some Cuntz-Pimsner algebras arising from dilation matrices. Since these unital algebras came from Hilbert modules with a finite orthonormal Parseval frame, they recreated the sequence using a more concrete approach. Our aim in this work is to show that this approach is still valid when dropping the condition of orthonormality.

This work is divided in four chapters in addition to some appendices. It is desirable that the reader is familiar with basic knowledge in C*-algebras, Topology and some Abstract Algebra.

In the first chapter we introduce the notion of Hilbert module over a C*-algebra together with some basic properties. These will be the building blocks for Cuntz-Pimsner algebras, moreover they will serve to define Morita-Rieffel equivalences. The main reference for this chapter will be [14].

In the second chapter we will look at Cuntz-Pimsner algebras and how are they constructed. We will also look at the

Fock representation which will allow us to have a more concrete representation of these algebras. We finish the chapter by showing how some familiar algebras can be realised as Cuntz-Pimsner algebras. The treatment of this chapter is heavily inspired by [17] and [20].

The third chapter contains a quick survey of K-theory for C*-algebras. We begin with some properties of projections and unitaries, then move on to define the K_0-group and K_1-group and explore their functorial properties. Finally we talk about Bott-periodicity and the six-term exact sequence it induces. The contents in this chapter are based on [18], [19] and [2], listed in increasing order of difficulty.

Finally, the last chapter contains the main results of this work. We explore how the six-term sequence obtained in [9] can still be applied to our particular case. To do this we determine which steps of the proof for the sequence are still valid and present slight modifications to the steps that require it. We also provide a simple application to Cuntz-Pimsner algebras arising from vector bundles.

There are two appendices in this work, the first one covering the basics of functors and categories which will mainly be used during chapter 3. The second appendix covering the basics of topological vector bundles which will be used as an application in the last chapter.

1 Hilbert modules

Hilbert modules were first introduced by Kaplansky in 1953 (see [12]) as a tool to study AW*-algebras. However this new structure proved to be really important to study C*-algebras in general, furthermore they served as the correct tool to extend the concept of Morita equivalence for C*-algebras. As we will see later on, Hilbert modules serve in a way as a non-commutative generalization of vector bundles.

In this chapter we will introduce the concept of Hilbert module together with some of the main results and examples. This will be one of the main building blocks when constructing Cuntz-Pimsner algebras. For more details regarding Hilbert modules refer to [14].

1.1 Basic definitions

In this section we introduce the main definitions regarding Hilbert modules. We also present various examples and some basic constructions we can do with those modules.

Definition 1.1.1. Let A be a C*-algebra and E a right A-module. A function $\langle \cdot, \cdot \rangle : E \times E \to A$ is called semi-inner-product over A if for any $x, y, z \in E$, $\lambda \in \mathbb{C}$ and $a \in A$ we have:

1. $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$,

2. $\langle x, ya \rangle = \langle x, y \rangle a$,

3. $\langle x, y \rangle = \langle y, x \rangle^*$,

4. $\langle x, x\rangle \geq 0$ in A.

If $\langle \cdot, \cdot \rangle$ also follows satisfies

5. $\langle x, x\rangle = 0$ implies that $x = 0$,

we say that $\langle \cdot, \cdot \rangle$ is an inner-product over A.

Remark 1.1.2. It is rather straightforward to note that for any $x, y \in E$ and $a \in A$ then $\langle xa, y\rangle = a^* \langle x, y\rangle$.

Definition 1.1.3. A pre-Hilbert A-module is a right A-module E that has an inner-product over A.

Remark. Note that if $A = \mathbb{C}$ then a pre-Hilbert A-module is nothing more than pre-Hilbert space.

It is a pleasant surprise to know that one of the most notable results for ordinary inner-products carries over to semi-inner-products:

Proposition 1.1.4 (Cauchy-Schwarz)**.** *Let $\langle \cdot, \cdot \rangle$ be a semi-inner-product over A in E. Then, for any $x, y \in E$ we have*

$$\langle x, y\rangle^* \langle x, y\rangle \leq \| \langle x, x\rangle \| \langle y, y\rangle$$

Proof. If $x = 0$ the result easily follows. If $x \neq 0$ we can assume without loss of generality that $\| \langle x, x\rangle \| = 1$, else we consider $x' = x/\| \langle x, x\rangle \|^{1/2}$. Given $a \in A$ we have

$$\begin{aligned} 0 \leq \langle xa - y, xa - y\rangle &= \langle xa, xa\rangle + \langle xa, -y\rangle + \langle -y, xa\rangle + \langle -y, -y\rangle \\ &= a^* \langle x, x\rangle a - a^* \langle x, y\rangle - \langle y, x\rangle a + \langle y, y\rangle \end{aligned}$$

but since $\| \langle x,x \rangle \| = 1$, we have $a^* \langle x,x \rangle a \le a^* a$ and therefore

$$\le a^*a - a^* \langle x,y \rangle - \langle y,x \rangle a + \langle y,y \rangle$$

Now, taking $a = \langle x,y \rangle$ we have

$$\begin{aligned} 0 &\le \langle y,x \rangle \langle x,y \rangle - \langle y,x \rangle \langle x,y \rangle - \langle y,x \rangle \langle x,y \rangle + \langle y,y \rangle \\ &\le - \langle y,x \rangle \langle x,y \rangle + \langle y,y \rangle \end{aligned}$$

and hence $\langle x,y \rangle^* \langle x,y \rangle \le \langle y,y \rangle$. □

Remark 1.1.5. Note that if E has a semi-inner-product, we can take the submodule $N = \{x \in E : \langle x,x \rangle = 0\}$ and in that case define $\langle \cdot,\cdot \rangle' : E/N \times E/N \to A$ by

$$\langle x+N, y+N \rangle' := \langle x,y \rangle .$$

The proposition above allows us to show that $\langle \cdot,\cdot \rangle'$ is a well defined inner-product over A in E/N.

Now if E is an A-module with inner-product we have the following result:

Corollary 1.1.6. *Define* $\| \cdot \| : E \to \mathbb{R}$ *by* $\|x\| := \| \langle x,x \rangle \|^{1/2}$. *Then* $\| \cdot \|$ *defines a norm over* E *(semi-norm if* $\langle \cdot,\cdot \rangle$ *is a semi-inner-product).*

Definition 1.1.7. If E is a pre-Hilbert A-module, we say that E is a Hilbert A-module if it is complete with respect to the norm defined above.

Remark 1.1.8. Clearly if $A = \mathbb{C}$, a Hilbert A-module is nothing but a Hilbert space. In this regard, a Hilbert module extends the notion of a Hilbert space.

Remark 1.1.9. In case a right A-module E with inner-product is not complete, we can consider the completion of E with respect to the norm induced by the inner-product, denoted by $\overline{E}^{\langle\cdot,\cdot\rangle}$, which will be a Hilbert A-module containing E isometrically.

Example 1.1.10. Let A be a C*-algebra and $I \subset A$ a right ideal in A, clearly I has a right A-module structure. We can define

$$\begin{aligned} \langle\cdot,\cdot\rangle : I \times I &\to A \\ (a,b) &\mapsto a^*b \end{aligned}$$

which will be an inner-product over A in I. Moreover,

$$\|a\| = \|a^*a\|^{1/2} = \| \langle a,a\rangle \|^{1/2}$$

for all $a \in I$ and therefore I is closed with respect to the norm induced by the inner-product. We can conclude that I is A-module de Hilbert, in particular we can always see A as a Hilbert A-module.

Example 1.1.11. Let B be a C*-algebra and E a subspace of B such that $E^*E \subset A$ and $E \cdot A \subset E$ where A is a C*-subalgebra of B. In this case E has a right A-module structure induced by B. Restricting the inner-product from the previous example to E, we obtain a Hilbert A-module if and only if E is a closed subspace in B. Later on we will see that all Hilbert modules can be identified with one of these, so that these are the most general examples, up to isomorphism. Due to this, moving forward we will try to use this example to motivate most of the constructions we do with Hilbert modules.

Example 1.1.12. Let A be a C*-algebra and $E = A^n = A \oplus \cdots \oplus A$. With the most obvious right A-module structure on E

we can define the following inner-product over A on E:

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle := \sum_{i=1}^{n} a_i^* b_i$$

in this case we will have that for every $a = (a_1, \ldots, a_n)$

$$\begin{aligned} \langle a, a \rangle &= \sum_{i=1}^{n} a_i^* a_i \\ &\leq \sum_{i=1}^{n} \|a_i^* a_i\| = \sum_{i=1}^{n} \|a_i\|^2 \leq n \max_i \|a_i\|^2 \end{aligned}$$

and that $a_j^* a_j \leq \langle a, a \rangle$. Taking norm and square root in both inequalities

$$\max_i \|a_i\| \leq \|a\|_{\langle \cdot, \cdot \rangle} \leq \sqrt{n} \max_i \|a_i\|$$

and hence the norm induced by the inner-product is equivalent to the norm of the maximum on A^n. We can conclude that E is closed with respect to the norm induced by the inner-product and therefore it is a Hilbert A-module.

Proposition 1.1.13. *Let E be a Hilbert A-module and $(u_\lambda)_{\lambda \in \Lambda}$ an approximate unit in A. Then $x \cdot u_\lambda \to x$ for every $x \in E$.*

Proof.

$$\begin{aligned} \|xu_\lambda - x\|^2 &= \| \langle xu_\lambda - x, xu_\lambda - x \rangle \| \\ &= \| \langle xu_\lambda, xu_\lambda \rangle - \langle xu_\lambda, x \rangle - \langle x, xu_\lambda \rangle + \langle x, x \rangle \| \\ &= \|u_\lambda \langle x, x \rangle u_\lambda - u_\lambda \langle x, x \rangle - \langle x, x \rangle u_\lambda + \langle x, x \rangle \| \\ &\leq \|u_\lambda\| \| \langle x, x \rangle u_\lambda - \langle x, x \rangle \| + \| \langle x, x \rangle - \langle x, x \rangle u_\lambda \| \\ &\leq \| \langle x, x \rangle u_\lambda - \langle x, x \rangle \| + \| \langle x, x \rangle - \langle x, x \rangle u_\lambda \| \to 0 \end{aligned}$$

□

Corollary 1.1.14. *If A is a unital C^*-algebra and E is a Hilbert A-module, then $x \cdot 1 = x$ for all $x \in E$.*

Corollary 1.1.15. *If E is a Hilbert A-module then $E \cdot A$ is dense in E.*

Let $(E_i)_{i\in I}$ be a family of Hilbert A-modules. Consider

$$E = \bigoplus_{i\in I}^{\text{alg}} E_i = \{(x_i)_{i\in I}, \text{ exists } F \subset I \text{ finite such that } x_i = 0 \text{ for all } i \notin F\}$$

which has a natural vector space structure. We define the following operations on E:

- Right product: $(x_i)_{i\in I} \cdot a := (x_i a)_{i\in I}$
- Inner-product: $\langle (x_i)_{i\in I}, (y_i)_{i\in I}\rangle := \sum_{i\in I} \langle x_i, y_i\rangle$

The inner-product is well defined because the sum will always have finitely many non-zero terms. With these two operations E is a pre-Hilbert A-module.

Definition 1.1.16. Given a family $(E_i)_{i\in I}$ of Hilbert A-modules we define their direct sum by

$$\bigoplus_{i\in I} E_i := \overline{\bigoplus_{i\in I}^{\text{alg}} E_i}^{\langle\cdot,\cdot\rangle}$$

which is a Hilbert A-module.

Remark 1.1.17. Note that if I is finite then

$$\overline{\bigoplus_{i\in I}^{\text{alg}} E_i}^{\langle\cdot,\cdot\rangle} = \bigoplus_{i\in I}^{\text{alg}} E_i$$

and by taking $I = \{1, \dots, n\}$ we have

$$\bigoplus_{i \in I} E_i = E_1 \oplus \cdots \oplus E_n$$

where the direct sums are as A-modules.

Example 1.1.18. Consider E a Hilbert A-module. We can see $E^n := \bigoplus_{i=1}^n E$ also as a Hilbert $M_n(A)$-module, to do this we define the following right module structure:

$$(x_1, \dots, x_n) \cdot (a_{ij})_{i,j=1}^n = \left(\sum_{i=1}^n x_i a_{i1}, \dots, \sum_{i=1}^n x_i a_{in} \right)$$

and the following inner-product:

$$\langle x, y \rangle_{M_n(A)} = (\langle x_i, y_j \rangle)_{i,j=1}^n$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. It is easy to verify that this inner-product satisfies all the desired properties. Something remarkable is that the norm induced by this inner-product, even though different, is equivalent to the norm induced by the inner-product over A. To verify this we just need to recall some basic properties regarding the norm on $M_n(A)$ found in [15, page 95]. Take any $x \in E^n$, then

$$\| \langle x_i, x_j \rangle \| \leq \left\| \langle x, x \rangle_{M_n(A)} \right\| \leq \sum_{h,k} \| \langle x_h, x_k \rangle \|$$

for any $i, j \in \{1, \dots, n\}$. This means that $\|x\|_A \leq n \|x\|_{M_n(A)}$ and since

$$\begin{aligned}\sum_{h,k} \| \langle x_h, x_k \rangle \| &\leq \sum_{h,k} \|x_h\| \|x_k\| \\ &\leq \left(\sum_h \|x_h\| \right) \left(\sum_k \|x_k\| \right) \leq n^2 \|x\|_A^2\end{aligned}$$

we obtain $\|x\|_{M_n(A)} \leq n\|x\|_A$, meaning both norms are equivalent. Since E^n was complete with the A-inner-product so it will be with the $M_n(A)$-inner-product and therefore with this structure E^n is a Hilbert $M_n(A)$-module.

This way of expressing the direct sum of Hilbert modules is convenient from a categoric point of view (it is easier to obtain the universal property of direct sums this way), but when dealing with elements inside the direct sum we will need a more precise description. Consider $(E_i)_{i\in\mathbb{N}}$ a sequence of Hilbert A-modules and

$$F = \left\{ (x_i)_{i\in\mathbb{N}},\ \sum_{i=1}^{\infty} \langle x_i, x_i \rangle \text{ converges unconditionally}^1 \right\}.$$

Lemma 1.1.19. *With the coordinate-wise operations, F is a right A-module. Moreover,*

$$((x_i),(y_i)) \mapsto \sum_{i=1}^{\infty} \langle x_i, y_i \rangle$$

defines an inner-product over A on F. This means that F is a pre-Hilbert A-module.

Proof. We verify first that F is indeed a vector space. If $(x_i)_{i\in\mathbb{N}} \in F$ and $\lambda \in \mathbb{C}$ we have that

$$\sum_{i=1}^{\infty} \langle \lambda x_i, \lambda x_i \rangle = \sum_{i=1}^{\infty} |\lambda|^2 \langle x_i, y_i \rangle = |\lambda|^2 \sum_{i=1}^{\infty} \langle x_i, y_i \rangle$$

[1] We say that a series $\sum_{i=1}^{\infty} q_i$ in a Banach space converges unconditionally if for all $\varepsilon > 0$ there exists $J \subset \mathbb{N}$ finite such that for all $I \supseteq J$ finite then $\|\sum_{i\notin I} q_i\| < \varepsilon$.

and therefore $\sum_{i=1}^{\infty} \langle \lambda x_i, \lambda x_i \rangle$ converges unconditionally, this means that $\lambda(x_i) = (\lambda x_i) \in F$. In a similar fashion, since the product of elements in A is continuous we show that $(x_i)a \in F$ for all $a \in A$.

Let $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \in F$ and fix $\varepsilon > 0$. We have there exists a $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\left\| \sum_{i=n}^{m} \langle x_i, x_i \rangle \right\| < \varepsilon, \quad \left\| \sum_{i=n}^{m} \langle y_i, y_i \rangle \right\| < \varepsilon$$

Now, looking at $(x_i)_{i=n}^m$ and $(y_i)_{i=n}^m$ as elements from $\bigoplus_{i=n}^m E_i$ and using Cauchy-Schwarz (1.1.4) we obtain

$$\left\| \sum_{i=n}^{m} \langle x_i, y_i \rangle \right\| \leq \left\| \sum_{i=n}^{m} \langle x_i, x_i \rangle \right\|^{1/2} \left\| \sum_{i=n}^{m} \langle y_i, y_i \rangle \right\|^{1/2} \leq \varepsilon$$

and therefore $\sum_{i=1}^{\infty} \langle x_i, y_i \rangle$ converges. Now we verify that the convergence is unconditional, considering the same $\varepsilon > 0$, there exists $J \subset \mathbb{N}$ finite such that for all $I \supseteq J$ finite we have

$$\left\| \sum_{i\notin I} \langle x_i, x_i \rangle \right\| < \varepsilon, \quad \left\| \sum_{i\notin I} \langle y_i, y_i \rangle \right\| < \varepsilon$$

Fixing $I \supseteq J$ and considering $n \in \mathbb{N}$ such that $\left\| \sum_{i=n+1}^{\infty} \langle x_i, y_i \rangle \right\| < \varepsilon/2$ and $I \subset \{1, \ldots, n\} = I_n$ we obtain that

$$\begin{aligned} \left\| \sum_{i\notin I} \langle x_i, y_i \rangle \right\| &= \left\| \sum_{i=n+1}^{\infty} \langle x_i, y_i \rangle + \sum_{i\in I_n \setminus I} \langle x_i, y_i \rangle \right\| \\ &\leq \frac{\varepsilon}{2} + \left\| \sum_{i\in I_n \setminus I} \langle x_i, y_i \rangle \right\| \end{aligned}$$

but $I_n \backslash I$ is finite and therefore we can use Cauchy-Schwarz,

$$\leq \frac{\varepsilon}{2} + \left\| \sum_{i \in I_n \backslash I} \langle x_i, x_i \rangle \right\|^{1/2} \left\| \sum_{i \in I_n \backslash I} \langle y_i, y_i \rangle \right\|^{1/2}$$
$$\leq \frac{\varepsilon}{2} + \left\| \sum_{i \notin I} \langle x_i, x_i \rangle \right\|^{1/2} \left\| \sum_{i \notin I} \langle y_i, y_i \rangle \right\|^{1/2} < \varepsilon$$

We conclude that $\sum_{i=1}^{\infty} \langle x_i, y_i \rangle$ converges unconditionally. Finally, since

$$\sum \langle x_i + y_i, x_i + y_i \rangle = \sum \langle x_i, x_i \rangle + \sum \langle x_i, y_i \rangle + \sum \langle y_i, x_i \rangle + \sum \langle y_i, y_i \rangle$$

since every term on the right hand side converges unconditionally then $(x_i + y_i)_{i \in \mathbb{N}} \in F$. □

Proposition 1.1.20. *With the above notation, F is a Hilbert A-module.*

Proof. Due to the previous lemma we only need to prove that F is complete with respect to the norm induced by the inner-product. Let $x^n = (x_i^n)_{i \in \mathbb{N}} \in F$ be a Cauchy sequence, it is clear that the sequences $(x_i^n)_{n \in \mathbb{N}}$ are also Cauchy and therefore they converge for some $x_i \in E_i$. Consider $x := (x_i)_{i \in \mathbb{N}}$, we need to show that $x \in F$. Take $\varepsilon > 0$, since the sequence is Cauchy there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\left\| \sum_{i=1}^{\infty} \langle x_i^n - x_i^m, x_i^n - x_i^m \rangle \right\|^{1/2} = \|x^n - x^m\| < \sqrt{\varepsilon}/2$$

since each term on the sum is positive, then for any finite set $K \subset \mathbb{N}$

$$\left\| \sum_{i \in K} \langle x_i^n - x_i^m, x_i^n - x_i^m \rangle \right\|^{1/2} < \sqrt{\varepsilon}/2$$

and taking limit $m \to \infty$, then for all finite $K \subset \mathbb{N}$ and $n \geq N$:

$$\|(x_i^n - x_i)_{i \in K}\| = \left\| \sum_{i \in K} \langle x_i^n - x_i, x_i^n - x_i \rangle \right\|^{1/2} \leq \sqrt{\varepsilon}/2 \qquad (1.1)$$

where the norm on the left hand side can be seen as a norm on $\bigoplus_{i \in K} E_i$. Since $(x_i^N) \in F$, there exists a finite $J \subset \mathbb{N}$ such that for all finite $I \supset J$ then

$$\left\| \sum_{i \notin I} \langle x_i^N, x_i^N \rangle \right\|^{1/2} < \sqrt{\varepsilon}/2.$$

Fix a finite $I \supset J$ and take $M \geq \max I$, considering $I_M = \{1, \ldots, M\}$, by the triangular inequality in $\bigoplus_{I_M \setminus I} E_i$

$$\|(x_i)_{i \in I_M \setminus I}\| \leq \|(x_i^N)_{i \in I_M \setminus I}\| + \|(x_i^N - x_i)_{i \in I_M \setminus I}\|$$

and since $I_M \setminus I$ is finite, by (1.1) we have that

$$\begin{aligned} \|(x_i)_{i \in I_M \setminus I}\| &\leq \|(x_i^N)_{i \in I_M \setminus I}\| + \sqrt{\varepsilon}/2 \\ &= \left\| \sum_{i \in I_M \setminus I} \langle x_i^N, x_i^N \rangle \right\|^{1/2} + \sqrt{\varepsilon}/2 \\ &\leq \left\| \sum_{i \notin I} \langle x_i^N, x_i^N \rangle \right\|^{1/2} + \sqrt{\varepsilon}/2 < \sqrt{\varepsilon} \end{aligned}$$

and finally $\left\| \sum_{i \in I_M \setminus I} \langle x_i, x_i \rangle \right\| < \varepsilon$. Since we can take M arbitrarily large, this shows that the series converges (since it is Cauchy) but that it also converges unconditionally.

□

Proposition 1.1.21. *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of Hilbert A-modules. Then*

$$\bigoplus_{i\in\mathbb{N}} E_i \cong \left\{ (x_i)_{i\in\mathbb{N}},\ \sum_{i=1}^{\infty} \langle x_i, x_i\rangle \ \textit{converges unconditionally} \right\}$$

Proof. Clearly, there exists an isometric application φ' from ${\bigoplus_{i\in\mathbb{N}}}^{\mathrm{alg}} E_i$ to F and therefore it can be extended to an application φ from $\bigoplus_{i\in\mathbb{N}} E_i$ to F, it only remains to prove that image of φ is dense. Take $x = (x_i)_{i\in\mathbb{N}} \in F$ and consider $x^n = (x_1, \dots, x_n, 0, \dots) \in {\bigoplus_{i\in\mathbb{N}}}^{\mathrm{alg}} E_i$. Since $\varphi(x^n) = x^n$ we have to show that $x^n \to x$ in F. Fix $\varepsilon > 0$, since $x \in F$ there exists a finite $J \subset \mathbb{N}$ such that for all finite $I \supseteq J$ we have that

$$\left\| \sum_{i\notin I} \langle x_i, x_i\rangle \right\|^{1/2} < \varepsilon$$

Consider $N = \max J$, in this case for all $n \geq N$,

$$\left\| \sum_{i=n+1}^{\infty} \langle x_i, x_i\rangle \right\|^{1/2} < \varepsilon$$

since $I_n = \{1, \dots, n\} \supseteq J$. But

$$\begin{aligned} \|x - x^n\| &= \left\| \sum_{i=1}^{\infty} \langle x_i - x_i^n, x_i - x_i^n\rangle \right\|^{1/2} \\ &= \left\| \sum_{i=1}^{n} \langle x_i - x_i, x_i - x_i\rangle + \sum_{i=n+1}^{\infty} \langle x_i - 0, x_i - 0\rangle \right\|^{1/2} \\ &= \left\| \sum_{i=n+1}^{\infty} \langle x_i, x_i\rangle \right\|^{1/2} < \varepsilon \end{aligned}$$

and therefore $x^n \to x$. $\square$

The proposition above gives us a concrete description of the direct sum of a sequence of Hilbert A-modules. Note that we can not avoid the hypothesis of unconditional convergence since without it there's a chance that F will not be complete.

1.2 Adjointable transformations

As with any algebraic structure, we need to establish which morphisms will be of our interest when studying Hilbert modules. Since they're also modules in the classical algebraic sense, it is natural to try to extend the notion of module morphism so that it takes into account the extra structure Hilbert modules have. The most ingenuous form to do this would be to consider those morphisms that preserve the inner-product, the problem with this is that such morphisms are immediately injective (moreover, they're isometric) leaving us with few interesting morphisms to work with. It is because of this that we decide to consider adjointable morphisms.

Definition 1.2.1. Given E and F two Hilbert A-modules, we say that a function $T : E \to F$ is **adjointable** if there exists $S : F \to E$ such that

$$\langle Tx, y \rangle_F = \langle x, Sy \rangle_E \,, \; x \in E \, y \in F.$$

An S satisfying this, if it exists, is unique. We call S the adjoint of T and will be denoted by $T^* := S$.

Note that a priori, these operators don't preserve any additional structure. The next proposition shows us that these kind of morphisms will indeed preserve all the structure we desire and hence serve as a good extension of the notion of module morphism.

Proposition 1.2.2. *Let E and F two Hilbert A-modules. Consider $\mathcal{L}(E,F) = \{T : E \to F;\ T$ is adjointable$\}$ and $L_A(E,F) = \{T : E \to F;\ T$ is a continuous A-module morphism$\}$. Then $\mathcal{L}(E,F) \subset L_A(E,F)$ is a closed subspace and therefore a Banach space.*

Proof. Let $T : E \to F$ be an adjointable transformation and take $S = T^* : F \to E$. To verify that T is $\mathbb{C}$-linear we consider $x_1, x_2 \in E$ and $\lambda \in \mathbb{C}$, then for all $f \in F$ we have

$$\begin{aligned}
\langle T(x_1 + \lambda x_2), f\rangle &= \langle x_1 + \lambda x_2, Sf\rangle \\
&= \langle x_1, Sf\rangle + \overline{\lambda}\,\langle x_2, Sf\rangle \\
&= \langle Tx_1, f\rangle + \overline{\lambda}\,\langle Tx_2, f\rangle \\
&= \langle Tx_1 + \lambda Tx_2, f\rangle
\end{aligned}$$

and therefore we conclude that $T(x_1 + \lambda x_2) = Tx_1 + \lambda Tx_2$. Similarly to check A-linearity, for any $a \in A$ we have

$$\begin{aligned}
\langle T(x_1 a), f\rangle &= \langle x_1 a, Sf\rangle \\
&= a^*\,\langle x_1, Sf\rangle \\
&= a^*\,\langle Tx_1, f\rangle \\
&= \langle T(x_1)a, f\rangle
\end{aligned}$$

and we can conclude that $T(x_1 a) = T(x_1)a$. We proceed to verify continuity, in order to do this we will use the closed graph theorem. Consider $(x_n)_{n\in\mathbb{N}} \subset E$ such that $x_n \to x$ and $T(x_n) \to y$. Take any $z \in F$, we have that

$$\begin{aligned}\langle Tx, z\rangle &= \langle x, Sz\rangle \\ &= \lim \langle x_n, Sz\rangle \\ &= \lim \langle Tx_n, z\rangle \\ &= \langle y, z\rangle\end{aligned}$$

and therefore $Tx = y$, meaning the graph of T is closed and hence T is continuous. Finally we need to prove that $\mathcal{L}(E,F)$ is a closed subspace, consider $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(E,F)$ such that $T_n \to T \in L_A(E,F)$ in the operator norm. Note that since $\|Tx\| = \sup_{\|y\|\leq 1} \| \langle Tx, y\rangle \|$ for all $x \in E$ and that $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$ we have that

$$\|T\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \| \langle Tx, y\rangle \| = \sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} \| \langle Tx, y\rangle \|$$

but this means that for any $R \in \mathcal{L}(E,F)$ $\|R\| = \|R^*\|$ and since (T_n) is Cauchy we get that (T_n^*) is also Cauchy and therefore converges to some operator S. Now we verify that this operator works as an adjoint for T,

$$\langle Tx, y\rangle = \lim \langle T_n x, y\rangle = \lim \langle x, T_n^* y\rangle = \langle x, Sy\rangle$$

hence we can conclude that $T \in \mathcal{L}(E,F)$ and therefore $\mathcal{L}(E,F)$ closed. Since $L_A(E,F)$ is Banach so is $\mathcal{L}(E,F)$. □

Proposition 1.2.3. *Take E, F, G Hilbert A-modules, $T \in \mathcal{L}(E,F)$ and $S \in \mathcal{L}(F,G)$. Then $S \circ T \in \mathcal{L}(E,G)$ and $(S \circ T)^* = T^* \circ S^*$.*

Proof. Take any $x \in E$ and $z \in G$, then

$$\begin{aligned}\langle (S\circ T)x, z\rangle &= \langle S(T(x)), z\rangle \\ &= \langle Tx, S^*(z)\rangle \\ &= \langle x, T^* \circ S^*(z)\rangle\end{aligned}$$

and therefore $(S \circ T)^* = T^* \circ S^*$. □

It is not hard to verify that if $T \in \mathcal{L}(E, F)$ then $T^* \in \mathcal{L}(F, E)$ with $(T^*)^* = T$. We will denote $\mathcal{L}(E, E)$ by $\mathcal{L}(E)$.

Proposition 1.2.4. *If E is a Hilbert A-module, then $\mathcal{L}(E)$ is a C^*-algebra with the composition as product, adjoint as involution and the operator norm.*

Proof. Due to the previous propositions $\mathcal{L}(E)$ is a Banach space and also a sub-algebra of $L(E) := L(E, E)$ (which is a Banach algebra itself). This means that $\mathcal{L}(E)$ is a Banach *-algebra. To check the C*-identity, we just need to verify that $\|T\|^2 \leq \|T^*T\|$:

$$\begin{aligned}\|T\|^2 &= \sup_{\|x\| \leq 1} \|T(x)\|^2 \\ &= \sup_{\|x\| \leq 1} \| \langle T(x), T(x) \rangle \| \\ &= \sup_{\|x\| \leq 1} \| \langle x, T^*T(x) \rangle \| \\ &\leq \sup_{\|x\| \leq 1} \|T^*T(x)\| = \|T^*T\|.\end{aligned}$$

□

Example 1.2.5. Take $E = A$ as a Hilbert A-module and suppose A has a unit, we claim that $\mathcal{L}(A) \cong A$ as C*-algebras. To see this first consider $T \in L(A)$, then for every $a \in A$

$$T(a) = T(1 \cdot a) = T(1) \cdot a = \lambda_{T(1)}(a) \tag{1.2}$$

where $\lambda_a \in L(A)$ is given by $\lambda_a(b) = ab$. The equation above implies that $\lambda : A \to L(A)$ is surjective. Moreover if $\lambda_a = 0$ then $0 = \lambda_a(1) = a$ and therefore λ is injective. This means that λ is a bijective linear application.

Now take any $a, b, c \in A$,

$$\lambda_{ab}(c) = abc = a(bc) = \lambda_a \circ \lambda_b(c)$$

this means that $\lambda_{ab} = \lambda_a \circ \lambda_b$ and therefore λ is an isomorphism of algebras. Similarly we have

$$\langle \lambda_a(b), c\rangle = \langle ab, c\rangle = (ab)^* c = b^*(a^* c) = \langle b, \lambda_{a^*}(c)\rangle$$

this means that λ_a is adjointable with adjoint λ_{a^*}. This means that $\lambda_a \in \mathcal{L}(A)$ for all $a \in A$ and therefore $\lambda : A \to \mathcal{L}(A) \subset L(A)$ is a *-isomorphism, but λ is surjective and therefore $\mathcal{L}(A) = L(A)$. We conclude that

$$A \cong \mathcal{L}(A) = L(A)$$

In case A is not unital we get an injective *-homomorphism $\lambda : A \to \mathcal{L}(A)$ (non surjective), it can be shown that $M(A) \cong \mathcal{L}(A)$ and that λ actually coincides with the canonical inclusion $A \subset M(A)$.

Example 1.2.6. Let E be a Hilbert A-module and consider E^n as a Hilbert $M_n(A)$-module as in Example 1.1.18. Then $\mathcal{L}_{M_n(A)}(E^n)$ is isomorphic to $\mathcal{L}_A(E^n)$ as C*-algebras. To show this we define

$$\begin{aligned} \varphi : \mathcal{L}_{M_n(A)}(E) &\to \mathcal{L}_A(E^n) \\ T &\mapsto T. \end{aligned}$$

In order to verify that this is well defined we just need to show that $T \in \mathcal{L}_A(E)$ when $T \in \mathcal{L}_{M_n(A)}(E)$. Write $T = (T_1, \ldots, T_n) \in \mathcal{L}_{M_n(A)}(T)$ and $T^* = (S_1, \ldots, S_n)$ where $T_i, S_i \in E^n \to E$ are linear maps. Consider $x, y \in E^n$, we have that

$$\begin{aligned} (\langle T_i(x), y_j\rangle)_{i,j=1}^n &= \langle T(x), y\rangle_{M_n(A)} \\ &= \langle x, T^*(y)\rangle_{M_n(A)} = (\langle x_i, S_j(y)\rangle)_{i,j=1}^n \end{aligned}$$

and therefore $\langle T_i(x), y_j \rangle = \langle x_i, S_j(y) \rangle$ for all $1 \leq i, j \leq n$. Now we have that

$$\begin{aligned} \langle T(x), y \rangle_A &= \sum_{i=1}^{n} \langle T_i(x), y_i \rangle \\ &= \sum_{i=1}^{n} \langle x_i, S_i(y) \rangle = \langle x, T^*(y) \rangle_A \end{aligned}$$

this means that $T \in \mathcal{L}_A(E)$ and that its adjoint in $\mathcal{L}_A(E)$ coincides with the one in $\mathcal{L}_{M_n(A)}(E)$. We can conclude that φ is a *-isomorphism.

Example 1.2.7. Take A as a Hilbert A-module and E another Hilbert A-module. Now fix $x \in E$ and define

$$\begin{aligned} \lambda_x : A &\to E \\ a &\mapsto xa \end{aligned}$$

it is easily verifiable that $\lambda_x \in \mathcal{L}(A, E)$ with $(\lambda_x)^*(y) = \langle x, y \rangle$. We get an application $\lambda : E \to \mathcal{L}(A, E)$ and we will use the following notation:

$$|x\rangle := \lambda_x \in \mathcal{L}(A, E), \quad \langle x| := \lambda_x^* \in \mathcal{L}(E, A)$$

1.3 Compact operators

In the previous section we generalized the notion of the algebra $B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space to the case of Hilbert A-modules. A very important ideal in $B(\mathcal{H})$ is the ideal of compact operators $\mathcal{K}(\mathcal{H})$. In this section we will extend this notion to the setting of Hilbert A-modules and do a quick survey of the differences and similarities with the Hilbert space case. Throughout this section A will always be a C*-algebra and E, F and G will be Hilbert A-modules.

In functional analysis we have a couple of ways of characterizing compact operators, one is via a topological condition regarding its action on the unit sphere and another way is via operators of finite rank. We will choose the later to define our compact operators for Hilbert A-modules. Consider $x \in E$ and $y \in F$, we can define our standard "rank 1" operator:

$$\begin{aligned} \theta_{y,x} : E &\to F \\ z &\mapsto y \langle x, z \rangle \end{aligned}$$

clearly $\theta_{y,x} = |y\rangle\langle x|$ and therefore $\theta_{y,x} \in \mathcal{L}(E, F)$.

Definition 1.3.1. If E and F are two Hilbert A-modules we define

$$\mathcal{K}(E, F) := \overline{\text{span}}\{\theta_{y,x},\ x \in E,\ y \in F\}$$

which will be called the **space of compact operators from E to F.**

The first difference we encounter with the Hilbert space case is regarding the compactness of the identity. It is a known fact that the identity is a compact operator if and only if the Hilbert space is finite dimensional. However if we consider A an infinite dimensional unital C*-algebra and look at it as a Hilbert A-module then $\lambda : A \to \mathcal{L}(A)$ is surjective. Now, since $\lambda_a = |a\rangle\langle 1| \in \mathcal{K}(A)$, every $T \in \mathcal{L}(A)$ is compact and therefore the identity is compact. On the other hand when we take $A = \mathbb{C}$ then $\mathcal{K}(E, F)$ coincides with the classical space of compact operators.

The next proposition provides some general algebraic properties of compact operators.

Proposition 1.3.2. *For all $x \in E$ and $y \in F$ we have:*

1. $(|y\rangle\langle x|)^* = |x\rangle\langle y|$;
2. $T \cdot |y\rangle\langle x| = |T(y)\rangle\langle x|$ *and* $|y\rangle\langle x| \cdot S = |y\rangle\langle S^*(x)|$ *for all* $T \in \mathcal{L}(F,G)$ *and* $S \in \mathcal{L}(G,E)$;
3. $|ya\rangle\langle x| = |y\rangle\langle xa^*|$ *for all* $a \in A$;
4. $\||x\rangle\langle y|\| = \| \langle x,x\rangle^{1/2} \langle y,y\rangle^{1/2} \|$.

Proof. 1. Recall that $\langle x|^* = |x\rangle$, therefore it follows immediately.

2. Consider $a \in A$ and $e' \in E$, then

$$T \cdot |y\rangle(a) = T(ya) = T(y)a = |T(y)\rangle(a)$$
$$\langle x| \cdot S(x') = \left\langle x, S(x')\right\rangle = \left\langle S^*(x), x'\right\rangle = \langle S^*(x)|(x')$$

therefore $T \cdot |y\rangle = |T(y)\rangle$ and $\langle x| \cdot S = \langle S^*(x)|$.

3. Consider any $x' \in E$, then

$$|ya\rangle\langle x|(x') = ya\left\langle x, x'\right\rangle = y\left\langle xa^*, x'\right\rangle = |y\rangle\langle xa^*|(x').$$

4. First we show that $\||x\rangle\langle x|\| = \| \langle x,x\rangle \| = \|x\|^2$. Take $x' \in E$, then

$$\||x\rangle\langle x|(x')\| = \|x\left\langle x, x'\right\rangle\| \leq \|x\|\|\left\langle x, x'\right\rangle\| \leq \|x\|^2\|x'\|$$

meaning $\||x\rangle\langle x|\| \leq \|x\|^2$. But also $\||x\rangle\langle x|(x)\| = \|x\langle x,x\rangle\| = \|x\|^3$, hence $\||x\rangle\langle x|\| = \|x\|^2$.

Now $\| |x\rangle\langle y| \|^2 = \| (|x\rangle\langle y|)^* |x\rangle\langle y| \| = \| |y \langle x, x\rangle\rangle\langle y| \|$. Since $\langle x, x\rangle \geq 0$ we can write $\langle x, x\rangle = \langle x, x\rangle^{1/2} \langle x, x\rangle^{1/2}$ and therefore due to the previous item

$$\begin{aligned}
\| |x\rangle\langle y| \|^2 &= \| |y \langle x, x\rangle\rangle\langle y| \| \\
&= \left\| |y \langle x, x\rangle^{1/2}\rangle\langle y \langle x, x\rangle^{1/2} | \right\| \\
&= \left\| y \langle x, x\rangle^{1/2} \right\|^2 \\
&= \left\| \left\langle y \langle x, x\rangle^{1/2}, y \langle x, x\rangle^{1/2} \right\rangle \right\| \\
&= \left\| \langle x, x\rangle^{1/2} \langle y, y\rangle \langle x, x\rangle^{1/2} \right\| \\
&= \left\| \langle x, x\rangle^{1/2} \langle y, y\rangle^{1/2} \langle y, y\rangle^{1/2} \langle x, x\rangle^{1/2} \right\| \\
&= \left\| \langle x, x\rangle^{1/2} \langle y, y\rangle^{1/2} \right\|^2 .
\end{aligned}$$

□

We will write $\mathcal{K}(E)$ for $\mathcal{K}(E, E)$. The second item in the proposition above gives us a rather useful corollary about $\mathcal{K}(E)$,

Corollary 1.3.3. *$\mathcal{K}(E)$ is a closed two-sided ideal in $\mathcal{L}(E)$.*

The next result will allow us to prove something equivalent to Riesz's representation theorem.

Proposition 1.3.4. *For all $x \in E$ we have $|x\rangle \in \mathcal{K}(E)$. Moreover the map $|\cdot\rangle : E \to \mathcal{K}(A, E)$ is linear, bijective and isometric.*

Proof. First we will show that $\lambda_a \in \mathcal{K}(A)$ for any $a \in A$ when A is looked at as a Hilbert A-module. Note that since A is not necessarily unital then $\lambda : A \to \mathcal{L}(A)$ is just an injective *-homomorphism. Consider any $b \in A$ with $\|b\| \leq 1$, then

$$\|\lambda_a(b)\| = \|ab\| \leq \|a\|$$

meaning that $\|\lambda_a\| \leq \|a\|$, but since $\|\lambda_a(a^*)\| = \|a\|$ then $\|\lambda_a\| = \|a\|$. This means that λ is isometric and therefore continuous. Consider (u_μ) an approximate unit in A, since $a = \lim_\mu au_\mu$ then $\lambda_a = \lim_\mu \lambda_{au_\mu}$. Take any $b \in A$,

$$\lambda_{au_\mu}(b) = au_\mu b = a \left\langle u_\mu^*, b\right\rangle = |a\rangle\langle u_\mu^*|(b)$$

and therefore $\lambda_a = \lim_\mu |a\rangle\langle u_\mu^*| \in \mathcal{K}(A)$.

Initially we have the map $|\cdot\rangle : E \to \mathcal{L}(A, E)$, we will show that's isometric and later restrict is codomain. Consider any $x \in E$ and $a \in A$ with $\|a\| \leq 1$,

$$\||x\rangle(a)\| = \|xa\| \leq \|x\|\|a\| \leq \|x\|$$

and therefore $\||x\rangle\| \leq \|x\|$. Due to Proposition 1.1.13 we also get that for any approximate unit (u_λ) in A it follows that

$$\begin{aligned} \||x\rangle(u_\lambda)\| = \|xu_\lambda\| &\leq \||x\rangle\| \\ \lim_\lambda \|xu_\lambda\| &\leq \||x\rangle\| \\ \|x\| &\leq \||x\rangle\| \end{aligned}$$

and consequently $\||x\rangle\| = \|x\|$. Since $|\cdot\rangle$ is isometric it is also continuous, therefore for any $x \in E$ we have that

$$|x\rangle = \lim_\lambda |eu_\lambda\rangle$$

but $|xu_\lambda\rangle = |x\rangle\lambda_{u_\mu} \in \mathcal{K}(A, E)$ since $\lambda_{u_\mu} \in \mathcal{K}(A)$, therefore $|x\rangle \in \mathcal{K}(A, E)$. This means that we can restrict the codomain of $|\cdot\rangle$ to $\mathcal{K}(A, E)$, to finish the proof we need to prove that it is surjective with this new codomain. Take $x \in E$ and $a \in A$, for any $b \in A$ we have

$$|x\rangle\langle a|(b) = xa^*b = |xa^*\rangle(b)$$

and therefore $|x\rangle\langle a| \in |E\rangle$. Since $\mathcal{K}(A, E) = \overline{\text{span}}\{|x\rangle\langle a|,\ x \in E, a \in A\}$, $|\cdot\rangle$ is isometric and E is complete we conclude that $|\cdot\rangle$ is indeed surjective. □

Corollary 1.3.5. *Any $T \in \mathcal{K}(E, A)$ is of the form $T = \langle x|$ for a unique $x \in E$.*

Proof. Take $T \in \mathcal{K}(E, A)$, then $T^* \in \mathcal{K}(A, E)$ and due to the previous proposition, since $|\cdot\rangle$ is injective, $T^* = |x\rangle$ for some unique $x \in E$. Since $T = (T^*)^* = (|x\rangle)^* = \langle x|$ we can conclude the desired result. □

Corollary 1.3.6. *If A is a C^*-algebra, considering the canonical Hilbert A-module structure, then $\mathcal{K}(A) \cong A$.*

Proof. From the proof of the previous proposition we have

$$\lambda(A) \subset \mathcal{K}(A)$$

To see the surjectivity, consider $|a\rangle\langle b| \in \mathcal{K}(A)$, then for any $x \in A$

$$\lambda(ab^*)(x) = ab^*x = a\,\langle b, x\rangle = |a\rangle\langle b|(x)$$

meaning $\lambda(ab^*) = |a\rangle\langle b|$ and hence $\lambda : A \to \mathcal{K}(A)$ is a bijective linear map. To show it's a *-homomorphism consider $a, b \in A$ then

$$\lambda(ab)(x) = abx = a(bx) = \lambda(a)(bx) = \lambda(a)\lambda(b)(x)$$

for any $x \in A$, meaning $\lambda(ab) = \lambda(a)\lambda(b)$. Finally, since

$$\lambda(a)^*(x) = \langle a, x\rangle = a^*x = \lambda(a^*)(x)$$

we can conclude that λ is a *-isomorphism. □

Proposition 1.3.7. *For any Hilbert A-module E we have*

$$M_n(\mathcal{L}_A(E)) \cong \mathcal{L}_A(E^n), \text{ and } M_n(\mathcal{K}_A(E)) \cong \mathcal{K}_A(E^n)$$

Proof. Consider $P_i : E \to E^n$ defined by

$$P_i(x) = (0, \cdots, 0, \overbrace{x}^{\text{i-th entry}}, 0, \cdots, 0)$$

it's not hard to verify that $P_i \in \mathcal{L}_A(E, E^n)$ with

$$P_i^*(x_1, \ldots, x_n) = x_i.$$

Moreover we have $P_i^* P_j = \delta_{ij} \,\mathrm{id}_E$ and $\sum_{i=1}^n P_i P_i^* = \mathrm{id}_{E^n}$. Consider the following *-homomorphism

$$\begin{aligned} \phi : M_n(\mathcal{L}_A(E)) &\to \mathcal{L}_A(E^n) \\ T = (T_{ij}) &\mapsto \sum_{i,j=1}^n P_i T_{ij} P_j^* \end{aligned}$$

it is clearly well defined, we just need to verify that it is a *-homomorphism. Take $T, S \in M_n(\mathcal{L}_A(E))$, then

$$\begin{aligned} \phi(TS) &= \phi\left(\sum_{k=1}^n T_{ik} S_{kj}\right) \\ &= \sum_{i,j=1}^n P_i \sum_{k=1}^n T_{ik} S_{kj} P_j^* \\ &= \sum_{i,j,k=1}^n P_i T_{ik} P_k^* P_k S_{kj} P_j^* \\ &= \sum_{i,k=1}^n P_i T_{ik} P_k^* \sum_{j,k=1}^n P_k S_{kj} P_j^* \\ &= \phi(T)\phi(S) \end{aligned}$$

Similarly

$$\phi(T)^* = \left(\sum_{i,j=1}^{n} P_i T_{ij} P_j^* \right)^* = \sum_{i,j=1}^{n} P_j T_{ij}^* P_i^* = \phi(T^*).$$

We now verify that ϕ is surjective. Take $R \in \mathcal{L}_A(E^n)$ and define $T_{ij} : E \to E$ by $T_{ij} = P_i^* R P_j$ and consider $T = (T_{ij}) \in M_n(\mathcal{L}_A(E))$, we have

$$\begin{aligned}\phi(T) &= \sum_{i,j=1}^{n} P_i T_{ij} P_j^* \\ &= \sum_{i,j=1}^{n} P_i P_i^* R P_j P_j^* \\ &= \left(\sum_{i=1}^{n} P_i P_i^* \right) R \left(\sum_{j=1}^{n} P_j P_j^* \right) = R\end{aligned}$$

meaning ϕ is surjective. Finally to show that ϕ is injective take $T \in \mathrm{M}_n(\mathcal{L}_A(E))$ such that $\phi(T) = 0$, then

$$\sum_{i,j=1}^{n} P_i T_{ij} P_j^* = 0$$

but for any $h, k \in \{1, \ldots, n\}$ we have

$$\begin{aligned}0 &= P_h^* \left(\sum_{i,j=1}^{n} P_i T_{ij} P_j^* \right) P_k \\ &= \sum_{i,j=1}^{n} \delta_{hi} \delta_{jk} T_{ij} = T_{hk}\end{aligned}$$

and therefore $T = 0$. Note that when restricting ϕ to $M_n(\mathcal{K}_A(E))$, due to Proposition 1.3.2 we have that $\phi(M_n(\mathcal{K}_A(E))) \subset \mathcal{K}_A(E^n)$.

To show that in fact $\phi(M_n(\mathcal{K}_A(E))) = \mathcal{K}_A(E^n)$ we proceed in a similar fashion as we did to show the surjectivity of ϕ and using Proposition 1.3.2 we can show that the pre-image lies in $M_n(\mathcal{K}_A(E))$. $\square$

1.4 Tensor product

There are two different ways of defining a tensor product of two Hilbert modules, we will only focus on the "internal" tensor product. Details regarding the construction of the "exterior" tensor product can be found in [14, Chapter 4]. Throughout this section A and B will be two C*-algebras, E will be a Hilbert A-module and F a Hilbert B-module.

We will use Example 1.1.11 to motivate the construction we are about to present. Consider C a C*-algebra that contains A and B as C*-subalgebras and E, F as closed subspaces, such that $E^*E \subset A$, $E \cdot A \subset E$, $A \cdot F \subset F$, $F^*F \subset B$ and $F \cdot B \subset F$. We can consider the closed subspace $G = \overline{\text{span}} E \cdot F \subset C$, which is in fact a B-module:

$$\begin{aligned} G^*G &= \overline{\text{span}} F^*E^*EF \subset \overline{\text{span}} F^*AF \subset \overline{\text{span}} F^*F \subset B, \\ G \cdot B &= \overline{\text{span}} EFB \subset \overline{\text{span}} EF = G. \end{aligned}$$

We want our internal tensor product of E and F to coincide with G in this scenario.

Definition 1.4.1. An A-B correspondence is a Hilbert B-module F endowed with a *-homomorphism $\varphi : A \to \mathcal{L}_B(F)$. We will use the pair (F, φ) to denote the correspondence or simply F when it is clear which *-homomorphism we are using.

Our idea will be to construct a new Hilbert B-module arising from E and an A-B correspondence (F, φ). First we consider the algebraic tensor product of E and F (as vector spaces), which will be denoted by $E \otimes^{\text{alg}} F$. We can define the following right B-module structure on $E \otimes^{\text{alg}} F$,

$$(x \otimes y) \cdot b = x \otimes yb$$

for all elementary tensors $x \otimes y \in E \otimes^{\text{alg}} F$ and all $b \in B$. This product is extended by linearity to all of $E \otimes^{\text{alg}} F$. Throughout the rest of this section φ will be such that (F, φ) is an A-B correspondence.

Proposition 1.4.2. *The right B-module structure above is well defined.*

Proof. First we look at the mapping $(x, y) \in E \times F \mapsto x \otimes yb \in E \otimes^{\text{alg}} F$ which is bilinear, by the universal property of the algebraic tensor product we have a unique linear map that sends $x \otimes y$ to $x \otimes yb$. This means that our right B-module structure is compatible with the scalar multiplication and finite sums. Now, consider $b, c \in B$ and an elementary tensor $x \otimes y \in E \otimes^{\text{alg}} F$,

$$\begin{aligned}
(x \otimes y) \cdot (b + c) &= x \otimes y(b + c) \\
&= x \otimes (yb + yc) = x \otimes yb + x \otimes yc \\
&= (x \otimes y) \cdot b + (x \otimes y) \cdot c
\end{aligned}$$

Similarly, we can see that

$$\begin{aligned}
(x \otimes y) \cdot (bc) &= x \otimes ybc \\
&= x \otimes (yb)c \\
&= (x \otimes yb) \cdot c \\
&= ((x \otimes y) \cdot b) \cdot c
\end{aligned}$$

By linearity we obtain both results for any element in the tensor product. ☐

To turn this module into a Hilbert B-module we need a B-valued inner-product. We consider the following bilinear function defined on elementary tensors:

$$\begin{aligned}\langle\cdot,\cdot\rangle : (E \otimes^{\text{alg}} F) \times (E \otimes^{\text{alg}} F) \to B \\ (x_1 \otimes y_1, x_2 \otimes y_2) \mapsto \langle y_1, \varphi(\langle x_1, x_2\rangle) y_2\rangle .\end{aligned}$$

In order to prove that this will serve to obtain our desired inner-product we will need the following technical lemmas.

Lemma 1.4.3. *For any $T \in \mathcal{L}_B(F)$ the following are equivalent:*

1. *$T \geq 0$ in $\mathcal{L}_B(F)$,*
2. *$\langle x, Tx\rangle \geq 0$ in B for all $x \in F$.*

Proof. To show that 1. implies 2. we write $T = S^*S$ where $S \in \mathcal{L}_B(E)$. Then $\langle x, Tx\rangle = \langle Sx, Sx\rangle \geq 0$ for any $x \in F$. Now to show the converse we write $T = R + iS$ where both R and S are self-adjoint. Consider any $x \in F$, then

$$\begin{aligned}0 \leq \langle x, Tx\rangle &= \langle x, Rx + iSx\rangle \\ &= \langle x, Rx\rangle + i\,\langle x, Sx\rangle\end{aligned}$$

this means the right-hand side is also self adjoint and therefore

$$\langle x, Rx\rangle + i\,\langle x, Sx\rangle = (\langle x, Rx\rangle + i\,\langle x, Sx\rangle)^* = \langle x, Rx\rangle - i\,\langle x, Sx\rangle$$

meaning $\langle x, Sx\rangle = 0$ for any $x \in F$. Consider $x, y \in F$, then

$$\begin{aligned}0 = \langle S(x+y), x+y\rangle &= \langle Sx, x\rangle + \langle Sy, x\rangle + \langle Sx, y\rangle + \langle Sy, y\rangle \\ &= \langle Sy, x\rangle + \langle Sx, y\rangle \\ &= \langle y, Sx\rangle + \langle Sx, y\rangle\end{aligned}$$

but also

$$0 = \langle y, S(ix)\rangle + \langle S(ix), y\rangle = i\,\langle y, Sx\rangle - i\,\langle Sx, y\rangle$$

meaning $\langle Sx, y\rangle = -\langle Sx, y\rangle$ and therefore $\langle Sx, y\rangle = 0$ for all $x, y \in F$. Since $\|S\| = \sup_{\substack{\|x\|\leq 1\\ \|y\|\leq 1}} \|\langle Sx, y\rangle\|$ we get that $S = 0$.

We have $T = R$ with R self-adjoint and therefore can write $T = P - Q$ where $P, Q \geq 0$ and $PQ = 0$ in $\mathcal{L}_B(F)$. This means that for any $x \in F$

$$0 \leq \langle Qx, TQx\rangle = -\left\langle Qx, Q^2x\right\rangle = -\left\langle x, Q^3x\right\rangle$$

But since $Q \geq 0$ we also have $Q^3 \geq 0$ and therefore $\left\langle x, Q^3x\right\rangle \geq 0$. This means that $\left\langle x, Q^3x\right\rangle = 0$ for all $x \in F$, with a similar reasoning to the one above we conclude that $Q^3 = 0$. Since Q is self-adjoint then $\|Q^3\| = \|Q\|^3$ and therefore $Q = 0$. We can conclude that $T = P \geq 0$. □

Corollary 1.4.4. *For any $T \in \mathcal{L}_B(F)$ and $x \in E$ we have that*

$$\langle Tx, Tx\rangle \leq \|T\|^2\,\langle x, x\rangle\,.$$

Proof. Since $\mathcal{L}_B(F)$ is a unital C*-algebra then $\|T\|^2 1 - T^*T \geq 0$. Using the previous lemma, for any $x \in F$

$$\begin{aligned} 0 &\leq \left\langle x, (\|T\|^2 1 - T^*T)x\right\rangle \\ 0 &\leq \|T\|^2\,\langle x, x\rangle - \langle x, T^*Tx\rangle \\ \langle Tx, Tx\rangle &\leq \|T\|^2\,\langle x, x\rangle \end{aligned}$$

□

Corollary 1.4.5. *Let A be a C^*-algebra, consider the C^*-algebra $M_n(A)$ and $a = (a_{ij}) \in M_n(A)$. Then $a \geq 0$ if and only if*

$$\sum_{i,j=1}^{n} x_i^* a_{ij} x_j \geq 0$$

for all $x_1, \ldots, x_n \in A$.

Proof. It is enough to use the previous lemma together with the identification $M_n(A) = \mathcal{K}(A^n)$, then for any $x = (x_1, \ldots, x_n)$ we have

$$\langle x, ax \rangle = \sum_{i,j=1}^{n} x_i^* a_{ij} x_j$$

which will be positive if and only if $a \geq 0$. □

Lemma 1.4.6. *Let E be a Hilbert B-module and $x_1, \ldots, x_n \in E$. Then*

1. *The matrix $x = (\langle x_i, x_j \rangle) \in M_n(B)$ is positive.*

2. *If $T \in \mathcal{L}_B(E)$, the matrix $a = (\langle Tx_i, Tx_j \rangle) \in M_n(B)$ satisfies*

$$0 \leq a \leq \|T\|^2 x.$$

Proof. 1. We use the previous corollary, take any $a_1, \ldots, a_n \in B$ then

$$\sum_{i,j=1}^{n} a_i^* \langle x_i, x_j \rangle a_j = \sum_{i,j=1}^{n} \langle x_i a_i, x_j a_j \rangle$$
$$= \left\langle \sum_{i=1}^{n} x_i a_i, \sum_{j=1}^{n} x_j a_j \right\rangle \geq 0$$

2. Using the previous item and taking $x'_i = Tx_i$ we get that $a \geq 0$. We already know that $a \geq 0$. To see that $a \leq \|T\|^2 x$ we will use once again the previous corollary. Take $a_1, \ldots, a_n \in B$, then

$$\sum_{i,j=1}^{n} a_i^* (\|T\|^2 \langle x_i, x_j \rangle - \langle Tx_i, Tx_j \rangle) a_j = \sum_{i,j=1}^{n} a_i^* \left\langle x_i, (\|T\|^2 - T^*T) x_j \right\rangle) a_j \quad (1.3)$$

but since $\|T\|^2 - T^*T \geq 0$ in $\mathcal{L}_B(E)$, then $\|T\|^2 - T^*T = S^*S$ for some $S \in \mathcal{L}_B(E)$. Now

$$\left(\left\langle x_i, (\|T\|^2 - T^*T) x_j \right\rangle\right)_{i,j} = (\langle Sx_i, Sx_j \rangle)_{i,j} \geq 0$$

the last inequality due to the previous item. From this we obtain that (1.3) is positive for any $a_1, \ldots, a_n$ and due to the previous corollary we have that

$$\|T\|^2 x - a = (\|T\|^2 \langle x_i, x_j \rangle - \langle Tx_i, Tx_j \rangle)_{i,j} \geq 0.$$

□

Now we can proceed to show that our bilinear function defined previously is indeed a semi-inner-product.

Proposition 1.4.7. *The bilinear function on $E \otimes^{alg} F$ is a semi-inner-product on B. Moreover $\langle x, x \rangle \geq 0$ for any $x \in E \otimes^{alg} F$.*

Proof. Take $x, y \in (E \otimes^{\mathrm{alg}} F)$ and $b \in B$, then $x = \sum_{i=1}^{n} x_i \otimes y_i$

and $y = \sum_{j=1}^{m} x'_j \otimes y'_j$ and therefore

$$\begin{aligned}
\langle x, y \cdot b \rangle &= \left\langle \sum_{i=1}^{n} x_i \otimes y_i, \sum_{j=1}^{m} x'_j \otimes y'_j b \right\rangle \\
&= \sum_{i,j} \left\langle x_i \otimes y_i, x'_j \otimes y'_j b \right\rangle \\
&= \sum_{i,j} \left\langle y_i, \varphi(\left\langle x_i, x'_j \right\rangle) y'_j b \right\rangle \\
&= \sum_{i,j} \left\langle y_i, \varphi(\left\langle x_i, x'_j \right\rangle) y'_j \right\rangle b \\
&= \sum_{i,j} \left\langle x_i \otimes y_i, x'_j \otimes y'_j \right\rangle b = \langle x, y \rangle b.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle x, y \rangle^* &= \sum_{i,j} \left\langle y_i, \varphi(\left\langle x_i, x'_j \right\rangle) y'_j \right\rangle^* \\
&= \sum_{i,j} \left\langle \varphi(\left\langle x_i, x'_j \right\rangle) y'_j, y_i \right\rangle \\
&= \sum_{i,j} \left\langle y'_j, \varphi(\left\langle x_i, x'_j \right\rangle)^* y_i \right\rangle \\
&= \sum_{i,j} \left\langle y'_j, \varphi(\left\langle x'_j, x_i \right\rangle) y_i \right\rangle = \langle y, x \rangle .
\end{aligned}$$

We now prove that $\langle x, x \rangle \geq 0$ for any $x \in (E \otimes^{\mathrm{alg}} F)$, this is where our lemmas will play a key role. Once again, writing $x = \sum_{i=1}^{n} x_i \otimes y_i$, we have

$$\langle x, x \rangle = \sum_{i,j=1}^{n} \langle y_i, \varphi(\langle x_i, x_j \rangle) y_j \rangle .$$

Consider $X = (\langle x_i, x_j \rangle)_{i,j} \in M_n(A)$ then, due to the previous lemma, $X \geq 0$. Now, the *-homomorphism $\varphi : A \to \mathcal{L}_B(F)$

induces a *-homomorphism:

$$\begin{aligned}\varphi^{(n)} : M_n(A) &\to M_n(\mathcal{L}_B(F)) \cong \mathcal{L}_B(F^n)\\ (a_{ij}) &\mapsto (\varphi(a_{ij})).\end{aligned}$$

Since $\varphi^{(n)}$ is a *-homomorphism it is also a positive function and therefore $\varphi^{(n)}(X) \geq 0$. Finally, using Lemma 1.4.3,

$$0 \leq \left\langle y, \varphi^{(n)}(X)\, f \right\rangle = \sum_{i,j=1}^{n} \langle y_i, \varphi(\langle x_i, x_j\rangle) y_j\rangle = \langle x, x\rangle$$

for $f = (y_1, \ldots, y_n) \in F^n$. □

Remark 1.4.8. Note that the semi-inner-product we just defined will not usually be an inner-product. This means there will usually exists elements $x \in E \otimes^{\mathrm{alg}} F$ such that $x \neq 0$ and $\langle x, x\rangle = 0$. In fact, consider any $x \in E$, $y \in F$, $a \in A$ and take

$$z = xa \otimes y - x \otimes \varphi(a)y$$

then

$$\begin{aligned}\langle z, z\rangle &= \langle xa \otimes y - x \otimes \varphi(a)y, xa \otimes y - x \otimes \varphi(a)y\rangle\\ &= \langle xa \otimes y, xa \otimes y\rangle - \langle xa \otimes y, x \otimes \varphi(a)y\rangle\\ &\quad - \langle x \otimes \varphi(a)y, xa \otimes y\rangle + \langle x \otimes \varphi(a)y, x \otimes \varphi(a)y\rangle\\ &= \langle y, \varphi(\langle xa, xa\rangle)y\rangle - \langle y, \varphi(\langle xa, x\rangle)\varphi(a)y\rangle\\ &\quad - \langle \varphi(a)y, \varphi(\langle x, xa\rangle)y\rangle + \langle \varphi(a)y, \varphi(\langle x, x\rangle)\varphi(a)y\rangle \qquad (1.4)\end{aligned}$$

but since φ is a *-homomorphism and $\langle xa, xa\rangle = a^* \langle x, xa\rangle = \langle xa, x\rangle a = a^* \langle x, x\rangle a$ then the expression (1.4) is equal to zero.

Now we can define the interior inner-product of two Hilbert modules E and F, we start by considering $N = \{x \in E \otimes^{\mathrm{alg}} F;\ \langle x, x\rangle = 0\}$. We obtain the pre-Hilbert B-module

$$E \otimes^{\mathrm{alg}} F \big/ N$$

The **internal tensor product** of E and F will be the completion of the quotient above with respect to the norm induced by the inner-product and will be denoted either by $E \otimes_\varphi F$ or $E \otimes_A F$.

Remark 1.4.9. The fact that (1.4) is zero also means that our tensor product is A-balanced in the same way algebraic tensor products of modules are. It is worth noting that we could have began with an algebraic tensor product of modules, defined the same inner-product and the end result would have been isomorphic.

The following proposition shows that our construction coincides with the one presented at the beginning of the section as motivation.

Proposition 1.4.10. *Consider C a C^*-algebra such that A and B are C^*-subalgebras, E and F are closed subspaces of C, such that $E^*E \subset A$, $E \cdot A \subset E$, $A \cdot F \subset F$, $F^*F \subset B$ and $F \cdot B \subset F$. Then $G := \overline{\operatorname{span}}\, E \cdot F \cong E \otimes_\varphi F$ where $\varphi(a)y := ay$ for all $a \in A$ and $y \in F$.*

Proof. First of all we need to verify that G is a Hilbert B-module. By definition G is closed and hence complete with the induced norm. Now, since

$$\begin{aligned} G \cdot B &= \overline{\operatorname{span}}\, E \cdot F \cdot B \subset \overline{\operatorname{span}}\, E \cdot F = G \\ G^* \cdot G &= \overline{\operatorname{span}}\, F^* \cdot E^* \cdot E \cdot F \\ &\subset \overline{\operatorname{span}}\, F^* \cdot A \cdot F \subset \overline{\operatorname{span}}\, F^* \cdot F \subset B \end{aligned}$$

we can assert that G is indeed a Hilbert B-module.

Consider the bilinear application $(x, y) \in E \times F \mapsto xy \in G$, it induces a linear function $\psi : E \otimes^{\mathrm{alg}} F \to G$ such that $\psi(x \otimes y) = xy$. Moreover since $\psi(x \otimes y \cdot b) = \psi(x \otimes yb) = xyb = \psi(x \otimes y)b$, then ψ is B-linear. By definition of G it is clear that ψ has dense range. In order to show that ψ induces an isomorphism between $E \otimes_\varphi F$ and G we just need to verify that ψ is isometric. We will verify this for elementary tensors and extend by bilinearity, consider $x = x_1 \otimes y_1$ and $y = x_2 \otimes y_2$ with $x_i \in E$ and $y_i \in F$:

$$\begin{aligned}\langle \psi(x), \psi(y) \rangle &= \psi(x)^* \psi(y) \\ &= (x_1 y_1)^* (x_2 y_2) \\ &= y_1^* (x_1^* x_2) y_2 \\ &= y_1^* \varphi(x_1^* x_2) y_2 \\ &= \langle y_1, \varphi(x_1^* x_2) y_2 \rangle \\ &= \langle y_1, \varphi(\langle x_1, x_2 \rangle) y_2 \rangle = \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\end{aligned}$$

which means φ is isometric. We can now factor ψ to a morphism $E \otimes^{\mathrm{alg}} F /_N \to G$ which remains isometric. Finally, by continuity we extend our morphism to $E \otimes_A F$ and since it is isometric it will also be injective and hence an isomorphism between $E \otimes_A F$ and G. $\square$

Lemma 1.4.11. *Given $T \in \mathcal{L}_A(E)$, there exists $\hat{T} \in \mathcal{L}_B(E \otimes_A F)$ such that $\widehat{T}(x \otimes y) = T(x) \otimes y$. Moreover the mapping $T \mapsto \widehat{T}$ is a *-homomorphism. We will also denote such $\widehat{T}$ by $T \otimes_A \mathrm{id}_F$ or simply $T \otimes_A \mathrm{id}$ when the Hilbert module we are using is clear from the context.*

Proof. Consider the bilinear application $(x, y) \mapsto T(x) \otimes y$. It induces a unique linear function $s_0 : E \otimes^{\mathrm{alg}} F \to E \otimes_A F$ such that $s_0(x \otimes y) = T(x) \otimes y$. We claim that $\langle s_0(x), s_0(x) \rangle \leq \|T\|^2 \langle x, x \rangle$ for any $x \in E \otimes^{\mathrm{alg}} F$. For any $x_1, \ldots, x_n \in E$ and $T \in \mathcal{L}_A(E)$, due to Lemma 1.4.6, we have

$$a = (\langle x_i, x_j \rangle)_{i,j=1}^n \geq 0$$
$$b = (\langle Tx_i, Tx_j \rangle_{i,j=1}^n) \geq 0$$

and $b \leq \|T\|^2 a$. Now write $x = \sum_{i=1}^n x_i \otimes y_i$ and set $y = (y_1, \ldots, y_n) \in F^n$, we have

$$\begin{aligned} \langle s_0(x), s_0(x) \rangle &= \sum_{i,j=1}^n \langle T(x_i) \otimes y_i, T(x_j) \otimes y_j \rangle \\ &= \sum_{i,j=1}^n \langle y_i, \varphi(\langle T(x_i), T(x_j) \rangle) y_j \rangle \\ &= \left\langle y, \varphi^{(n)}(b) y \right\rangle \\ &\leq \|T\|^2 \left\langle y, \varphi^{(n)}(a) y \right\rangle = \|T\|^2 \langle x, x \rangle . \end{aligned}$$

This means that s_0 is bounded with norm less than $\|T\|$ and is 0 when $\langle x, x \rangle = 0$, therefore it factors through to a bounded linear function $s : {E \otimes^{alg} F}/{N} \to E \otimes_A F$. Since s is bounded we can further extend this to a bounded linear operator $\widehat{T} : E \otimes_A F \to E \otimes_A F$ with $\|\widehat{T}\| \leq \|T\|$ and $\widehat{T}(x \otimes y) = T(x) \otimes y$. In order to show that $\widehat{T}$ is in fact adjointable consider $x_1, x_2 \in E$, $y_1, y_2 \in F$ and set $x = x_1 \otimes y_1$ and $y = x_2 \otimes y_2$. Then

$$\begin{aligned} \left\langle \widehat{T}(x), y \right\rangle &= \langle T(x_1) \otimes y_1, x_2 \otimes y_2 \rangle \\ &= \langle y_1, \varphi(\langle T(x_1), x_2 \rangle) y_2 \rangle \\ &= \langle y_1, \varphi(\langle x_1, T^*(x_2) \rangle) y_2 \rangle \\ &= \langle x_1 \otimes y_1, T^*(x_2) \otimes y_2 \rangle = \left\langle x, \widehat{T^*}(y) \right\rangle \end{aligned}$$

hence we can extend this by bilinearity to show that $\widehat{T}$ is adjointable and $\widehat{T}^* = \widehat{T^*}$. It is pretty clear that the mapping $T \mapsto \widehat{T}$ is linear and since for any $T, S \in \mathcal{L}_A(E)$, $x \in E$ and $y \in F$ we have that

$$\widehat{TS}(x \otimes y) = TS(x) \otimes y = \widehat{T}(S(x) \otimes y) = \widehat{T}\widehat{S}(x \otimes y)$$

and by linearity $\widehat{TS} = \widehat{T}\widehat{S}$, we can conclude that the map is a *-homomorphism. □

Corollary 1.4.12. *We can view at $E \otimes_A F$ as a $\mathcal{L}_A(E)$-B correspondence. Moreover, if we consider C a C^*-algebra and $\psi : C \to \mathcal{L}_A(E)$ such that (E, ψ) is a C-A correspondence, then we can view $E \otimes_A F$ as a C-B correspondence.*

Proof. The first assertion is a direct consequence of the mapping $\mathcal{L}_A(E) \to \mathcal{L}_B(E \otimes F)$ from the previous lemma being a *-homomorphism. The second assertion follows from the fact that the mapping $\widehat{\psi} : C \to \mathcal{L}_B(E \otimes_A F)$, $c \mapsto \widehat{\psi(c)}$ will be a *-homomorphism. □

Proposition 1.4.13. *If (F, φ) is a proper A-B correspondence, in the sense that $\varphi(A) \subset \mathcal{K}_B(F)$, then $E \otimes_A F$ can be seen as a proper $\mathcal{K}_A(E)$-B correspondence. This means that if $T \in \mathcal{K}_A(E)$ then $\widehat{T} \in \mathcal{K}_B(E \otimes_A F)$.*

Proof. Consider $T = |x_1 a\rangle\langle x_2|$ where $x_1, x_2 \in E$ and $a \in A$. Define the **creation operator** $T_{x_i} : F \to E \otimes_A F$ by $T_{x_i}(y) = x_i \otimes y$, clearly T_{x_i} is a morphism of B-modules, moreover for any $y, z \in F$ and $x \in E$ we have that

$$\begin{aligned} \langle T_{x_i}(z), x \otimes y\rangle &= \langle x_i \otimes z, x \otimes y\rangle \\ &= \langle z, \varphi(\langle x_i, x\rangle)y\rangle \end{aligned}$$

and therefore T_{x_i} is adjointable with $T^*_{x_i}(x \otimes y) = \varphi(\langle x_i, x\rangle)y$. Now

$$\begin{aligned}\widehat{T}(x \otimes y) &= T(x) \otimes y \\ &= x_1 a \langle x_2, x\rangle \otimes y \\ &= x_1 \otimes \varphi(a \langle x_2, x\rangle)y \\ &= x_1 \otimes \varphi(a)\varphi(\langle x_2, x\rangle)y \\ &= T_{x_1}\varphi(a)T^*_{x_2}(x \otimes y)\end{aligned}$$

and therefore $\widehat{T} = T_{x_1}\varphi(a)T^*_{x_2}$, since $\varphi(a)$ is compact we conclude that $\widehat{T}$ is compact.

Now, due to Proposition 1.1.13, we have $\overline{EA} = E$ and consequently any rank 1 operator in $\mathcal{K}_A(E)$ can be expressed as a limit of operators of the form $|x_1 a\rangle\langle x_2|$. Hence, if T is a rank 1 operator then $\widehat{T}$ is compact. Since the linear span of rank 1 operators is dense in $\mathcal{K}_A(E)$ we conclude that $\widehat{T}$ is compact for all $T \in \mathcal{K}_A(E)$. □

1.5 Morita-Rieffel equivalence

Before concluding this chapter we introduce the notion of Morita-Rieffel equivalence. In abstract algebra we say two rings are Morita equivalent if their categories of left modules are equivalent. In 1973 Rieffel extended this notion for C*-algebras with the use of Hilbert bimodules. Throughout this section we will refer to pre-Hilbert modules as right pre-Hilbert modules, Hilbert modules as right-Hilbert modules and inner-products as right inner-products. Most of the results in this section can be found in [11, Section 4.5].

Definition 1.5.1. Let E be a left B-module. A function $\{\cdot,\cdot\}: E\times E \to A$ is called left inner-product over B if for any $x, y, z \in E$, $\lambda \in \mathbb{C}$ and $a \in A$ we have:

1. $\{x + \lambda y, z\} = \{x, z\} + \lambda \{y, z\}$,
2. $\{bx, y\} = b\{x, y\}$,
3. $\{x, y\} = \{y, x\}^*$,
4. $\{x, x\} > 0$ in B if $x \neq 0$.

Definition 1.5.2. A left pre-Hilbert B-module is a left B-module that has a left inner-product over B.

All of the basic results we have with right inner-products have an equivalent for left inner-products. This means that $\|x\| := \|\{x, x\}\|^{1/2}$ also defines a norm in E. We say E is a left Hilbert B-module if it is a left pre-Hilbert module which is complete with respect to this norm.

Definition 1.5.3. Let E be a B-A-bimodule. If E is a left pre-Hilbert B-module and also a right pre-Hilbert A-module such that:

$$x\langle y, z\rangle = \{x, y\} z$$

for all $x, y, z \in E$, we say E is a pre-Hilbert B-A-bimodule.

We say a pre-Hilbert B-A-bimodule E is a Hilbert B-A if it is complete with respect to the norms induced by both inner-products.

Remark 1.5.4. In [3, II.7.6.3] it is shown that if E is a pre-Hilbert B-A bimodule, the norm induced by the left-inner-product and the right-inner-product coincide. Meaning the completion of E with respect to both norms is the same.

Similarly to the way we constructed the internal tensor product for A-B correspondences we can define the tensor product for Hilbert bimodules. Take F a Hilbert A-B bimodule and G a Hilbert B-C bimodule, the tensor product $F \otimes_B G$ (as modules) has a pre-Hilbert A-C bimodule structure with the following product:

$$a \cdot (x \otimes y) = ax \otimes y,$$
$$(x \otimes y) \cdot c = x \otimes yc$$

for $x \in E$, $y \in F$, $a \in A$ and $c \in C$, and left and right inner-products:

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_C = \langle y_1, \langle x_1, x_2 \rangle_B \, y_2 \rangle_C \in C,$$
$$\{x_1 \otimes y_1, x_2 \otimes y_2\}_A = \{y_1 \{x_1, x_2\}_B \, , y_2\}_A \in A$$

for all $x_1, x_2 \in F$ and $y_1, y_2 \in G$. We can take quotient and completion in a similar manner as we did in the previous section to obtain a Hilbert A-C-bimodule which we will also denote by $F \otimes_B G$.

Definition 1.5.5. Two C*-algebras A and B are *Morita-Rieffel* equivalent if there exists a Hilbert A-B bimodule E and a Hilbert B-A bimodule F such that

$$E \otimes_B F \cong A \text{ and}$$
$$F \otimes_A E \cong B$$

as A- and B- bimodules respectively. We denote this by $A \overset{\mathrm{MR}}{\sim} B$.

Remark 1.5.6. Morita-Rieffel equivalence is an equivalence relation as shown in [11, Proposition 4.23]. It is also easy to verify that if $A \cong B$ then $A \overset{\text{MR}}{\sim} B$.

Example 1.5.7. Let A be a C*-algebra and $p \in A$ a full projection (that is $\overline{\text{span}}\, ApA = A$), then $pAp \overset{\text{MR}}{\sim} A$. To verify this we consider $E = Ap$ and $F = pA$, which have a natural A-pAp and pAp-A bimodule structure respectively. We define the following morphism

$$\begin{aligned} f : Ap \otimes_{pAp} pA &\to A \\ ap \otimes pb &\mapsto apb \end{aligned}$$

We begin by verifying it is well defined, to do this we check if it is bounded. Take $x = \sum_{i=1}^{n} x_i p \otimes p y_i \in Ap \otimes_{pAp} pA$, then:

$$\begin{aligned} \|f(x)\|^2 &= \|f(x)^* f(x)\| \\ &= \left\| f\left(\sum_{i=1}^{n} x_i p \otimes p y_i\right)^* f\left(\sum_{j=1}^{n} x_j p \otimes p y_j\right) \right\| \\ &= \left\| \left(\sum_{i=1}^{n} x_i p y_i\right)^* \left(\sum_{j=1}^{n} x_j p y_j\right) \right\| \\ &= \left\| \left(\sum_{i=1}^{n} y_i^* p x_i^*\right) \left(\sum_{j=1}^{n} x_j p y_j\right) \right\| \\ &= \left\| \sum_{i,j=1}^{n} y_i^* p x_i^* x_j p y_j \right\| \\ &= \left\| \sum_{i,j=1}^{n} \left\langle p y_i, \langle x_i p, x_j p \rangle_{pAp}\, p y_j \right\rangle_A \right\| \\ &= \left\| \sum_{i,j=1}^{n} \langle x_i p \otimes p y_i, x_j p \otimes p y_j \rangle \right\| \end{aligned}$$

$$= \left\| \left\langle \sum_{i=1}^{n} x_i p \otimes p y_i, \sum_{j=1}^{n} x_j p \otimes p y_j \right\rangle \right\|$$
$$= \|\langle x, x\rangle\| = \|x\|^2.$$

This shows that f is well defined and isometric, therefore injective. Since p is a full projection, f is surjective. Similarly we define

$$g : pA \otimes_A Ap \to pAp$$
$$pa \otimes bp \mapsto pabp.$$

In a similar manner we verify g is an isometry meaning it is injective. Clearly it also is surjective. We can conclude that $pAp \overset{\mathrm{MR}}{\sim} A$.

2 Cuntz-Pimsner algebras

In this chapter we introduce Cuntz-Pimsner algebras, which are a very vast class of C*-algebras. Initially introduced by Pimsner in [17] as a way to generalize Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, their structure (with some modifications) has manifested in many other classes of C*-algebras such as graph C*-algebras (as seen in [13]) and Exel's crossed products (as seen in [6]).

The first section of this chapter will be dedicated to the concrete construction of Toeplitz algebras. In the second section we will show the construction of Cuntz-Pimsner algebras and show some of the main examples.

2.1 Fock representation

If (E, φ) is an A-A correspondence we want to construct a C*-algebra that in a way contains copies of A and E such that all of their operations are well represented, this C*-algebra will be called the Toeplitz algebra and denoted $\mathcal{T}_E$.

The motivation for the construction of the Toeplitz algebra of a correspondence can be explained once again using Example 1.1.11. Let B be a C*-algebra and E a closed subspace such that $E^*E \subset A$, $E \cdot A \subset E$ and $A \cdot E \subset E$ where A is a C*-subalgebra of B. It's clear that E will have a Hilbert A-module structure. Defining $\varphi : A \to \mathcal{L}_A(E)$ by $\varphi(a)(x) = ax$, we can view E as an A-A correspondence. The Toeplitz algebra $\mathcal{T}_E$ will be the C*-

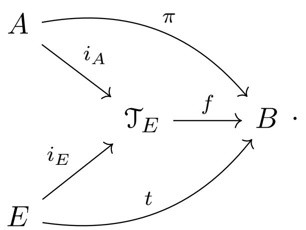
algebra generated by E and A in B. To deal with the general case we need to define a special kind of representation.

Definition 2.1.1. Let (E, φ) be an A-A correspondence. A **Toeplitz representation** of E in a C*-algebra B is a triple (π, t, B) such that $\pi : A \to B$ is a *-homomorphism and $t : E \to B$ is a linear map such that for all $x, y \in E$ and $a \in A$ we have

1. $t(x)\pi(a) = t(xa)$,
2. $t(x)^*t(y) = \pi(\langle x, y\rangle)$,
3. $\pi(a)t(x) = t(\varphi(a)x)$.

A Toeplitz representation is called injective if π is injective and will be surjective if the C*-algebra generated by $\pi(A)$ and $t(E)$, denoted by $C^*(\pi, t)$, is equal to B.

The Toeplitz algebra $\mathcal{T}_E$ will be the C*-algebra that is universal for these representations. By this we mean that there exists i_A, i_E such that $(i_A, i_E, \mathcal{T}_E)$ is a Toeplitz representation and if (π, t, B) is another representation, then there exists a unique $f : \mathcal{T}_E \to B$ *-homomorphism such that the following diagram commutes:



(2.1)

We can construct $\mathcal{T}_E$ as a universal C*-algebra. Consider the set of generators $G = \{[a],\ a \in A\} \cup \{[x],\ x \in E\}$ subject to the following relations:

1. $[a][x] = [\varphi(a)x]$, for all $a \in A$ and $x \in E$,

2. $[x]^*[y] = [\langle x, y\rangle]$, for all $x, y \in E$,

3. $[x][a] = [xa]$, for all $a \in A$ and $x \in E$,

4. all relations in A,

5. all relations in E.

We will define $\mathcal{T}_E$ to be the universal C*-algebra generated by G subject to the relations above. The following lemma shows that $\mathcal{T}_E$ is well defined.

Lemma 2.1.2. *The relations above are admissible and therefore induce a universal C*-algebra.*

Proof. We need to verify that for any $x \in \mathcal{G}$ there exists $C_x \geq 0$ such that $\|f(x)\| \leq C_x$ for all representations $f : \mathcal{G} \to B$. Here $\mathcal{G}$ is the universal *-algebra subject to the relations above and B is a C*-algebra. We fix a representation $f : \mathcal{G} \to B$ and note that any element $x \in \mathcal{G}$ will be the finite sum of products of elements of the form $[a], [x], [y]^*$, where $a \in A$ and $x, y \in E$. This means, due to the triangular inequality, that we just need to bound $\|f(x)\|$ when x is a product of elements of the form $[a], [x], [y]^*$. Due to the relations that spawned $\mathcal{G}$, we have all products must be either of the form $[a]$ for $a \in A$ or

$$[x_1] \cdots [x_n][y_1]^* \cdots [y_m]^*, \quad x_i, y_j \in E$$

This is because any term of the form $[x]^*[y]$ with $x, y \in E$ becomes $[\langle x, y\rangle]$. Since $g : A \to B$ given by $g(a) = f([a])$ is a *-homomorphism, we have $\|f([a])\| \leq \|a\|$. Now, if we take $x = [x_1] \cdots [x_n][y_1]^* \cdots [y_m]^*$ for $x_i, y_j \in E$ then

$$\begin{aligned} \|f(x)\| &= \|f([x_1]) \cdots f([x_n]) f([y_1])^* \cdots f([y_m])^*\| \\ &\leq \|f([x_1])\| \cdots \|f([x_n])\| \|f([y_1])\| \cdots \|f([y_m])\| \end{aligned}$$

but we also have

$$\|f([x_i])\|^2 = \|f([x_i]^*[x_i])\| = \|f([\langle x_i, x_i \rangle])\| \leq \| \langle x_i, x_i \rangle \| =: C_{x_i}^2$$

and therefore $\|f(x)\| \leq C_{x_1} \cdots C_{x_n} C_{y_1} \cdots C_{y_m} = C_x$. □

The lemma above shows that

$$\|x\| := \sup\{\|f(x)\|,\ f : \mathcal{G} \to B \text{ is a representation}\}$$

defines a seminorm in $\mathcal{G}$ and therefore $\mathcal{G}/\mathcal{N}$ can be viewed as a normed *-algebra where $\mathcal{N} = \{x,\ \|x\| = 0\}$. We have that $\mathcal{T}_E$ will the completion of $\mathcal{G}/\mathcal{N}$ with respect to this norm. Now we verify that $\mathcal{T}_E$ we just constructed satisfies our desired universal property.

Proposition 2.1.3. *$\mathcal{T}_E$ satisfies the universal property from diagram (2.1).*

Proof. We begin defining $i_A : A \to \mathcal{T}_E$ by $i_A(a) = q([a])$ and $i_E : E \to \mathcal{T}_E$ by $i_E(x) = q([x])$ where $q : \mathcal{G} \to \mathcal{T}_E$ is the quotient map. It is rather straight forward to show that $(i_A, i_E, \mathcal{T}_E)$ is a Toeplitz representation of E. If (π, t, B) is another Toeplitz representation of E, we would have that $\pi(A) \cup t(E) \subset B$ satisfies all the relations from above and therefore we have a unique *-homomorphism $g : \mathcal{G} \to B$ such that $g([x]) = t(x)$ and $g([a]) =$

$\pi(a)$ for all $x \in E$ and $a \in A$. We also have that for any $x \in \mathcal{G}$, by definition

$$\|g(x)\| \leq \|x\|$$

This means that $\mathcal{N} \subset \ker g$ and therefore g factors through to a *-homomorphism from $\mathcal{G}/\mathcal{N}$ to B. Since g is bounded we can extend this *-homomorphism to a *-homomorphism $f : \mathcal{T}_E \to B$. Now, take $a \in A$ and $x \in E$, by definition of f we have

$$f(i_A(a)) = f(q([a])) = g([a]) = \pi(a),$$
$$f(i_E(x)) = f(q([x])) = g([x]) = t(a)$$

and therefore f makes diagram (2.1) commute. The uniqueness of f comes from the fact that any other f' that makes the diagram commute would coincide with f on a set of generators of $\mathcal{T}_E$. $\square$

We can construct $\mathcal{T}_E$ in a more concrete way, to do this it is enough to find an injective Toeplitz representation, this will be done using the Fock representation.

Definition 2.1.4. If (E, φ) is an A-A correspondence then the associated **Fock space** is defined by

$$F_E := \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

where $E^{\otimes 0} = A$ when $n = 0$ and

$$E^{\otimes n} = E \otimes_A E^{\otimes n-1} \cong E^{\otimes n-1} \otimes_A E$$

when $n \geq 1$ (this is well defined due to Corollary 1.4.12).

Remark 2.1.5. Due to the definition of $\mathcal{T}_E$, i_A and i_E, $\mathcal{T}_E$ can be described as the C*-algebra generated by $i_A(A) \cup i_E(E)$.

Now, since $i_E(x)^* i_E(y) = i_A(\langle x, y\rangle)$, $i_E(x)i_A(a) = i_E(xa)$ and $i_A(a)i_E(x) = i_E(\varphi(a)x)$ for all $x, y \in E$ and $a \in A$ then an arbitrary element in the *-algebra generated by $i_A(A) \cup i_E(E)$ will be either the form $i_A(a)$ with $a \in A$ or of the form

$$i_E(x_1)\cdots i_E(x_n)i_E(y_1)^* i_E(y_m)^*,\ x_i, y_j \in E.$$

This means that

$$\mathcal{T}_E = \overline{\text{span}}\{(i_E)^{\otimes n}(x)(i_E)^{\otimes m}(y)^*,\ x \in E^{\otimes n},\ y \in E^{\otimes m},\ n, m \geq 0\}$$

where $(i_E)^{\otimes n} : E^{\otimes n} \to \mathcal{T}_E$ is given by $(i_E)^{\otimes n}(x_1 \otimes \cdots \otimes x_n) = i_E(x_1)\cdots i_E(x_n)$ for $n \geq 1$ and $(i_E)^{\otimes 0}(a) = i_A(a)$. The construction of the Fock representation will be made to reflect the structure above.

Note that since F_E is a Hilbert A-module it makes sense to talk about its adjointable operators $\mathcal{L}_A(F_E)$ which is a C*-algebra. We shall use $\mathcal{L}_A(F_E)$ to represent our correspondence (E, φ), this means our goal now is to construct $i_A : A \to \mathcal{L}_A(F_E)$ and $i_E : E \to \mathcal{L}_A(F_E)$. Before proceeding further we will need a tool to easily construct morphisms in direct sums.

Proposition 2.1.6. *Let F_n and G_n be two sequences of Hilbert A-modules. Consider a sequence $T_n \in \mathcal{L}_A(F_n, G_n)$ such that $\|T_n\| \leq K$ for all $n \in \mathbb{N}$ for some constant $K \geq 0$. Then $\bigoplus_{n=0}^{\infty} T_n := T : \bigoplus_{n=0}^{\infty} F_n \to \bigoplus_{n=0}^{\infty} G_n$, defined by $T(x_n) = (T_n(x_n))_{n\in\mathbb{N}}$ for all $(x_n) \in \bigoplus_{n\in\mathbb{N}}^{alg} F_n$, is a well defined adjointable operator.*

Proof. To check that T is well defined we just need to verify that it is bounded on $\bigoplus_{n\in\mathbb{N}}^{\text{alg}} F_n$. Consider $(x_n) \in \bigoplus_{n\in\mathbb{N}}^{\text{alg}} F_n$, then

$$\begin{aligned}\|T(x_n)\|^2 &= \| \langle T(x_n), T(x_n)\rangle \| \\ &= \left\| \sum_{n=0}^{\infty} \langle T_n(x_n), T_n(x_n)\rangle \right\|\end{aligned}$$

using Corollary 1.4.4 we have

$$\begin{aligned}&\leq \left\| \sum_{n=0}^{\infty} \|T_n\|^2 \langle x_n, x_n\rangle \right\| \\ &\leq K^2 \left\| \sum_{n=0}^{\infty} \langle x_n, x_n\rangle \right\| = K^2 \|(x_n)\|\end{aligned}$$

concluding that T is bounded and therefore well defined. To check that it is adjointable just consider $S := \bigoplus_{n=0}^{\infty} T_n^*$, a very quick calculation shows that it is indeed the adjoint of T. □

Consider $\varphi_\infty : A \to \mathcal{L}_A(F_E)$ defined by

$$\varphi_\infty(a) = \bigoplus_{n=0}^{\infty} \varphi_n(a)$$

where $\varphi_n(a) = \varphi(a) \otimes \text{id}_{E^{\otimes n-1}} \in \mathcal{L}_A(E^{\otimes n})$ (well defined since $\|\varphi(a) \otimes \text{id}_{E^{\otimes n-1}}\| \leq \|\varphi(a)\| \leq \|a\|$) for $n \geq 1$ and $\varphi_0(a)(b) = ab$ for all $b \in A$. Since each mapping $a \mapsto \varphi(a) \otimes \text{id}_{E^{\otimes n-1}} = \varphi_n(a)$ is a *-homomorphism , it is clear that so will be $a \mapsto \varphi_\infty(a)$.

Lemma 2.1.7. *$\varphi_\infty : A \to \mathcal{L}_A(F_E)$ is injective.*

Proof. If $\varphi_\infty(a) = 0$ then $\varphi_n(a) = 0$ for all $n \geq 0$. In particular $\varphi_0(a) = 0$ and therefore $ab = 0$ for all $b \in A$, which means $a = 0$. □

We now recall the creation operators introduced in the proof of Proposition 1.4.13. For any $x \in E$, $T_{x,n} : E^{\otimes n} \to E \otimes_A E^{\otimes n} = E^{\otimes n+1}$ defined by $T_{x,n}(y) = x \otimes y$ is an adjointable operator. Note that for $y \in E^{\otimes n}$ with $\|y\| \leq 1$ we have

$$\begin{aligned} \| \langle T_{x,n}(y), T_{x,n}(y) \rangle \| &= \| \langle x \otimes y, x \otimes y \rangle \| \\ &= \| \langle y, \varphi(\langle x, x \rangle) y \rangle \| \leq \|\varphi(\langle x, x \rangle)\| \end{aligned}$$

and therefore, using the previous proposition, it makes sense to consider $T(x) := \bigoplus_{n=0}^{\infty} T_{x,n} \in \mathcal{L}_A(F_E)$ (in this case $T_{x,0} = |x\rangle$ and we look at $\bigoplus_{n=0}^{\infty} E^{\otimes n+1} \subset F_E$). It is not hard to verify that T is indeed linear. It is also worth noting that $T(x)^*(b) = 0$ for all $x \in E$ and $b \in A$, since

$$\langle T(x)(y), b \rangle = 0$$

for all $y \in F_E$ and $b \in B$ as $T(x)(y)$ lies in $\bigoplus_{n=1}^{\infty} E^{\otimes n}$.

We move onto verifying that the triple $(\varphi_\infty, T, \mathcal{L}_A(F_E))$ we just defined is indeed a Toeplitz representation. Since F_E is generated by elements of the form $x_1 \otimes \cdots \otimes x_n$ with $n \in \mathbb{N}$, it is enough to verify each condition for those kind of elements. Take $x, y \in E$, $x' = x_1 \otimes \cdots \otimes x_n \in E^{\otimes n}$ and $a \in A$,

$$\begin{aligned} T(x)\varphi_\infty(a)(x') &= T(x)(\varphi(a)x_1 \otimes \cdots \otimes x_n) \\ &= x \otimes \varphi(a)x_1 \otimes \cdots \otimes x_n \end{aligned}$$

and using Remark 1.4.8,

$$= xa \otimes x_1 \otimes \cdots \otimes x_n = T(xa)(x').$$

This means that $T(x)\varphi_\infty(a) = T(xa)$. Similarly

$$\varphi_\infty(a)T(x)(x') = \varphi_\infty(a)(x \otimes x') = \varphi(a)x \otimes x' = T(\varphi(a)x)(x')$$

and therefore $\varphi_\infty(a)T(x) = T(\varphi(a)x)$. Finally, since

$$\begin{aligned} T(x)^*T(y)(x') &= T(x)^*(y \otimes x') \\ &= \varphi(\langle x, y\rangle)x' = \varphi_\infty(\langle x, y\rangle)(x') \end{aligned}$$

we conclude that $T(x)^*T(y) = \varphi_\infty(\langle x, y\rangle)$.

We have shown that for any A-A correspondence (E, φ) there exists an injective Toeplitz representation $(\varphi_\infty, T, \mathcal{L}_A(F_E))$. It can be shown that $C^*(\varphi_\infty, T)$ is isomorphic to $\mathcal{T}_E$. Moreover, the fact that $(\varphi_\infty, T, \mathcal{L}_A(F_E))$ is an injective representation implies that so is the universal Toeplitz representation $(i_A, i_E, \mathcal{T}_E)$:

Theorem 2.1.8. *The universal Toeplitz representation of E, $(i_A, i_E, \mathcal{T}_E)$, is injective.*

Proof. Consider the Toeplitz representation $(\varphi_\infty, T, \mathcal{L}_A(F_E))$, due to the universal property there exists $f : \mathcal{T}_E \to C^*(\varphi_\infty, T)$ such that $f \circ i_A = \varphi_\infty$. Since φ_∞ is injective we conclude that so is i_A, this means that $(i_A, i_E, \mathcal{T}_E)$ is an injective representation. □

Example 2.1.9. Consider $A = \mathbb{C}$ and $E = \mathbb{C}^n$ with the inner product $\langle x, y\rangle = \sum_{i=1}^n \overline{x_i} y_i$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Taking $\varphi : \mathbb{C} \to \mathcal{L}_\mathbb{C}(\mathbb{C}^n) \cong M_n(\mathbb{C})$ to be $\varphi(\lambda) = \lambda\,\mathrm{id}_n$ we can look at $\mathbb{C}^n$ as a $\mathbb{C}$-$\mathbb{C}$ correspondence. Note that $i_\mathbb{C}(1)i_{\mathbb{C}^n}(x) = i_{\mathbb{C}^n}(x)$ since $\varphi(1)(x) = x$. Therefore $i_\mathbb{C}(1)$ is a unit in $\mathcal{T}_{\mathbb{C}^n}$ and $i_\mathbb{C}(\lambda) = \lambda 1_{\mathcal{T}_{\mathbb{C}^n}}$.

Now, since $\mathbb{C}^n$ generated by the canonical base

$$e_i = (0, \ldots, 1, \ldots, 0),$$

consider $s_i = i_{\mathbb{C}^n}(e_i)$ then

$$s_i^* s_j = i_{\mathbb{C}^n}(e_i)^* i_{\mathbb{C}^n}(e_j) = i_A(\langle e_i, e_j\rangle) = i_A(\delta_{ij}1) = \delta_{ij}1_{\mathcal{T}_{\mathbb{C}^n}}.$$

Meaning $\mathcal{T}_{\mathbb{C}^n}$ is generated by n mutually orthogonal isometries, it can be shown that $\mathcal{T}_{\mathbb{C}^n} \cong \mathcal{T}\mathcal{O}_n$ the Toeplitz-Cuntz algebra, in particular when $n = 1$, by Coburn's theorem [7], we obtain the Toeplitz algebra.

2.2 Cuntz-Pimsner representations

Given (E, φ) an A-A correspondence, our goal with the Cuntz-Pimsner algebra $\mathcal{O}_E$ will be to generalize the Cuntz algebra in the same sense that the Toeplitz algebra of E generalizes the regular Toeplitz algebra, that is we want that $\mathcal{O}_{\mathbb{C}^n} \cong \mathcal{O}_n$. We proceed in a similar way as we did with the Toeplitz algebra, considering a special class of representations of (E, φ) and taking the C*-algebra which is universal for these representations.

We begin by fixing (E, φ) a A-A correspondence. If (π, t, B) is a representation of (E, φ) we can induce a *-homomorphism $(\pi, t)^{(1)} : \mathcal{K}(E) \to B$ by $(\pi, t)^{(1)}(|x\rangle\langle y|) = t(x)t(y)^*$.

Lemma 2.2.1. *Let C be a C*-algebra, F be a Hilbert C-module and $x_1, \ldots, x_n, y_1, \ldots, y_n \in F$. Then*

$$\left\| \sum_{i=1}^{n} |x_i\rangle\langle y_i| \right\| = \left\| \left((\langle x_i, x_j \rangle)_{i,j=1}^{n} \right)^{1/2} \left((\langle y_i, y_j \rangle)_{i,j=1}^{n} \right)^{1/2} \right\|$$

where the norm on the right hand side is the one on $\mathrm{M}_n(C)$.

Proof. First we fix $\gamma = \sum_{i=1}^{n} |x_i\rangle\langle y_i|$ then let $x = (x_i)$, $y = (y_i)$, we have $|x\rangle\langle y| \in \mathcal{L}_{\mathrm{M}_n(A)}(E^n)$ and due to Proposition 1.3.2 it follows

$$\begin{aligned} \||x\rangle\langle y|\| &= \| \langle x, x \rangle_{\mathrm{M}_n(A)}^{1/2} \langle y, y \rangle_{\mathrm{M}_n(A)}^{1/2} \| \\ &= \left\| \left((\langle x_i, x_j \rangle)_{i,j=1}^{n} \right)^{1/2} \left((\langle y_i, y_j \rangle)_{i,j=1}^{n} \right)^{1/2} \right\|. \end{aligned}$$

Our goal now is to prove that $\| |x\rangle\langle y| \| = \| \sum_{i=1}^n |x_i\rangle\langle y_i| \|$. In order to do this we will utilize the *-isomorphism between $\mathcal{L}_{\mathrm{M}_n(A)}(E^n)$ and $\mathcal{L}_A(E^n)$ shown in Example 1.2.6. It is not difficult to verify that $\mathcal{L}_A(E^n) \cong \mathrm{M}_n(\mathcal{L}_A(E))$ and therefore we have an *-isomorphism $\Omega : \mathcal{L}_{\mathrm{M}_n(A)}(E^n) \to \mathrm{M}_n(\mathcal{L}_A(E))$. This means that $\| |x\rangle\langle y| \| = \| \Omega(|x\rangle\langle y|) \|$. Doing some simple calculations we obtain

$$\begin{aligned} \Omega(|x\rangle\langle y|) &= \begin{pmatrix} \sum_{i=1}^n |x_i\rangle\langle y_i| & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^n |x_i\rangle\langle y_i| & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \sum_{i=1}^n |x_i\rangle\langle y_i| \end{pmatrix} \\ &= \mathrm{diag}_n\left(\sum_{i=1}^n |x_i\rangle\langle y_i| \right) \end{aligned}$$

where $\mathrm{diag}_n(T)$ is the matrix with n copies of T in its diagonal and 0 elsewhere. Since $\mathrm{diag}_n : \mathcal{L}_A(E) \to \mathrm{M}_n(\mathcal{L}_A(E))$ is a unital *-homomorphism we obtain

$$\left\| \sum_{i=1}^n |x_i\rangle\langle y_i| \right\| = \left\| \mathrm{diag}_n\left(\sum_{i=1}^n |x_i\rangle\langle y_i| \right) \right\| = \| \Omega(|x\rangle\langle y|) \| = \| |x\rangle\langle y| \|$$

and finally we conclude the desired result. □

Proposition 2.2.2. *$(\pi, t)^{(1)}$ is a well-defined *-homomorphism.*

Proof. If $\mathcal{F}(E) = \mathrm{span}\{|x\rangle\langle y|,\ x, y \in E\}$, then $\overline{\mathcal{F}(E)} = \mathcal{K}(E)$. Take $\gamma = \sum_{i=1}^n |x_i\rangle\langle y_i| \in \mathcal{F}(E)$ with $x_i, y_i \in E$. We need to verify that if $\gamma = 0$ then $(\pi, t)^{(1)}(\gamma) = 0$:

$$\begin{aligned} (\pi, t)^{(1)}(\gamma)(\pi, t)^{(1)}(\gamma)^* &= \left(\sum_{i=1}^n t(x_i)t(y_i)^* \right) \left(\sum_{j=1}^n t(x_j)t(y_j)^* \right)^* \\ &= \left(\sum_{i=1}^n t(x_i)t(y_i)^* \right) \left(\sum_{j=1}^n t(y_j)t(x_j)^* \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j=1}^{n} t(x_i)t(y_i)^* t(y_j)t(x_j) \\
&= \sum_{i,j=1}^{n} t(x_i)\pi(\langle y_i, y_j\rangle)t(x_j) \\
&= \sum_{i,j=1}^{n} t(x_i \langle y_i, y_j\rangle)t(x_j) \\
&= \sum_{j=1}^{n} t\left(\sum_{i=1}^{n} x_i \langle y_i, y_j\rangle\right) t(x_j) \\
&= \sum_{j=1}^{n} t\left(\gamma(y_j)\right) t(x_j) = 0.
\end{aligned}$$

This means that $(\pi, t)^{(1)}$ is well defined in $\mathcal{F}(E)$. Now we need to show that $(\pi, t)^{(1)}$ is bounded in $\mathcal{F}(E)$ for us to be able to extend $(\pi, t)^{(1)}$ to the compact operators . We have,

$$\begin{aligned}
\|(\pi, t)^{(1)}(\gamma)\| &= \left\|\sum_{i=1}^{n} t(x_i)t(y_i)^*\right\| \\
&= \left\|\sum_{i=1}^{n} |t(x_i)\rangle\langle t(y_i)|\right\|
\end{aligned}$$

by using the previous lemma we obtain

$$\begin{aligned}
\|(\pi, t)^{(1)}(\gamma)\| &= \left\|\left((\langle t(x_i), t(x_j)\rangle)_{i,j}\right)^{1/2} \left((\langle t(y_i), t(y_j)\rangle)_{i,j}\right)^{1/2}\right\| \\
&= \left\|\left((t(x_i)^* t(x_j))_{i,j}\right)^{1/2} \left((t(y_i)^* t(y_j))_{i,j}\right)^{1/2}\right\| \\
&= \left\|\left((\pi(\langle x_i, x_j\rangle))_{i,j}\right)^{1/2} \left((\pi(\langle y_i, y_j\rangle))_{i,j}\right)^{1/2}\right\| \\
&= \left\|\pi^{(n)}\left[\left((\langle x_i, x_j\rangle)_{i,j}\right)^{1/2} \left((\langle y_i, y_j\rangle)_{i,j}\right)^{1/2}\right]\right\|
\end{aligned}$$

but $\pi^{(n)}$ is a *-homomorphism,

$$\leq \left\| \left((\langle x_i, x_j \rangle)_{i,j} \right)^{1/2} \left((\langle y_i, y_j \rangle)_{i,j} \right)^{1/2} \right\|$$
$$= \left\| \sum_{i=1}^{n} |x_i\rangle\langle y_i| \right\| = \|\gamma\|$$

and therefore $(\pi, t)^{(1)}$ is bounded on $\mathcal{F}(E)$.

In order to prove that $(\pi, t)^{(1)}$ is indeed a *-homomorphism we will use the properties show in Proposition 1.3.2, take $x, x', y, y' \in E$ then

$$\begin{aligned}
(\pi, t)^{(1)}(|x\rangle\langle y| \cdot |x'\rangle\langle y'|) &= (\pi, t)^{(1)}(|\,|x\rangle\langle y|(x')\rangle\langle y'|) \\
&= (\pi, t)^{(1)}(|x \left\langle y, x' \right\rangle\rangle\langle y'|) \\
&= t(x \left\langle y, x' \right\rangle)t(y')^* \\
&= t(x)\pi(\left\langle y, x' \right\rangle)t(y')^* \\
&= t(x)t(y)^*t(x')t(y')^* \\
&= (\pi, t)^{(1)}(|x\rangle\langle y|)(\pi, t)^{(1)}(|x'\rangle\langle y'|)
\end{aligned}$$

and

$$\begin{aligned}
(\pi, t)^{(1)}(|x\rangle\langle y|)^* &= (t(x)t(y)^*)^* \\
&= t(y)t(x)^* \\
&= (\pi, t)^{(1)}(|y\rangle\langle x|) = (\pi, t)^{(1)}(|x\rangle\langle y|^*).
\end{aligned}$$

Since $(\pi, t)^{(1)}$ is clearly linear we can conclude that it is a *-homomorphism.

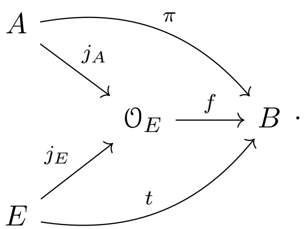
□

We will use this *-homomorphism to define our special kind of representations that will give rise to Cuntz-Pimsner algebras. Consider $J_E = \varphi^{-1}(\mathcal{K}(E)) \cap \ker(\varphi)^{\perp}$, where $I^{\perp} = \{a \in A,\ aI = \{0\}\}$ for $I \subset A$.

Definition 2.2.3. Let (π, t, B) be a Toeplitz representation of E. We say that this representation is **Cuntz-Pimsner covariant** if

$$(\pi, t)^{(1)}(\varphi(a)) = \pi(a), \, \forall a \in J_E.$$

The Cuntz-Pimsner algebra $\mathcal{O}_E$ is, by definition, the universal C*-algebra for Cuntz-Pimsner covariant representations. This means that there exists $j_A : A \to \mathcal{O}_E$ and $j_E : E \to \mathcal{O}_E$ such that $(j_A, j_E, \mathcal{O}_E)$ is a Cuntz-Pimsner covariant representation and if (π, t, B) is other Cuntz-Pimsner covariant representation then there exists a unique *-homomorphism $f : \mathcal{O}_E \to B$ such that the following diagram commutes:



(2.2)

We can explicitly construct this C*-algebra, to do this we define $\mathcal{T}(J_E)$ as the ideal generated by the set

$$\{(i_A, i_E)^{(1)}(\varphi(a)) - i_A(a), \, a \in J_E\}.$$

Theorem 2.2.4. *For any (E, φ) correspondence, we have*

$$\mathcal{T}_E / \mathcal{T}(J_E) \cong \mathcal{O}_E.$$

Proof. Consider $Q : \mathcal{T}_E \to \mathcal{T}_E / \mathcal{T}(J_E)$ the quotient map. Define $j_A := Q \circ i_A$ and $j_E = Q \circ i_E$, we now verify that $\left(j_A, j_E, \mathcal{T}_E / \mathcal{T}(J_E)\right)$ satisfies the universal property for Cuntz-Pimsner covariant representations. First we need to verify that $\left(j_A, j_E, \mathcal{T}_E / \mathcal{T}(J_E)\right)$ is a Cuntz-Pimsner covariant representation, the fact that it is a Toeplitz representation is pretty clear. Take $a \in J_E \subset \mathcal{K}(E)$ and $\gamma_n = \sum_{i=1}^{n} |x_i^n\rangle\langle y_i^n| \in \mathcal{K}(E)$ such that $\gamma_n \to \varphi(a)$,

$$\begin{aligned}(j_A, j_E)^{(1)}(\gamma_n) &= \sum_{i=1}^{n} j_E(x_i^n) j_E(y_i^n)^* \\ &= Q\left(\sum_{i=1}^{n} i_E(x_i^n) i_E(y_i^n)^*\right) \\ &= Q((i_A, i_E)^{(1)}(\gamma_n)).\end{aligned}$$

Due Proposition 2.2.2, both $(j_A, j_E)^{(1)}$ and $(i_A, i_E)^{(1)}$ are *-homomorphisms and therefore continuous. Taking limit when $n \to \infty$ above we have

$$(j_A, j_E)^{(1)}(\varphi(a)) = Q((i_A, i_E)^{(1)}(\varphi(a))) = Q(i_A(a)) = j_A(a)$$

and therefore $\left(j_A, j_E, {}^{\mathcal{T}_E}\!\big/_{\mathcal{T}(J_E)}\right)$ is Cuntz-Pimsner covariant.

To check the universal property let (π, t, B) be a Cuntz-Pimsner covariant representation of (E, φ), since (π, t, B) is also a Toeplitz representation there exists a unique *-homomorphism $f' : \mathcal{T}_E \to B$ such that $f' \circ i_A = \pi$ and $f' \circ i_E = t$. Take $a \in J_E$ and $\gamma_n = \sum_{i=1}^{n} |x_i^n\rangle\langle y_i^n| \in \mathcal{K}(E)$ such that $\gamma_n \to \varphi(a)$,

$$\begin{aligned}f'((i_A, i_E)^{(1)}(\gamma_n) - i_A(a)) &= f'\left(\sum_{i=1}^{n} i_E(x_i^n) i_E(y_i^n)^*\right) - \pi(a) \\ &= \sum_{i=1}^{n} t(x_i^n) t(y_i^n)^* - \pi(a) \\ &= (\pi, t)^{(1)}\left(\sum_{i=1}^{n} |x_i^n\rangle\langle y_i^n|\right) - \pi(a) \\ &= (\pi, t)^{(1)}(\gamma_n) - \pi(a)\end{aligned}$$

taking limit above when $n \to \infty$ we have

$$f'((i_A, i_E)^{(1)}(\varphi(a)) - i_A(a)) = (\pi, t)^{(1)}(\varphi(a)) - \pi(a) = 0$$

and therefore $\mathcal{T}(J_E) \subset \ker f'$. We can now factor f' to a *-homomorphism $f : {}^{\mathcal{T}_E}\!/\!_{\mathcal{T}(J_E)} \to \mathcal{O}_E$ that is such that $f(Q(s)) = f'(s)$ for all $s \in \mathcal{T}_E$. For $a \in A$ we have,

$$f(j_A(a)) = f(Q(i_A(a))) = f'(i_A(a)) = \pi(a)$$

and for $x \in E$,

$$f(j_E(x)) = f(Q(i_E(x))) = f'(i_E(x)) = t(x)$$

hence, f makes the desired diagram commute.

To show that f is unique, consider another *-homomorphism g such that the diagram commutes. We have that for $a \in A$

$$g(Q(i_A(a))) = g(j_A(a)) = \pi(a) = f(Q(i_A(a)))$$

and for $x \in E$,

$$g(Q(i_E(x))) = g(j_E(x)) = t(x) = f(Q(i_E(x)))$$

since $Q(i_A(A))$ and $Q(i_E(E))$ generate ${}^{\mathcal{T}_E}\!/\!_{\mathcal{T}(J_E)}$ we have $f = g$ and therefore f is unique. $\square$

Now that it is clear how we obtain Cuntz-Pimsner algebras from an A-A correspondence we move on to explain why do they carry this name. The following example shows how these algebras generalize Cuntz algebras.

Example 2.2.5. Consider $A = \mathbb{C}$ and $\mathbb{C}^n$ as a $\mathbb{C}$-$\mathbb{C}$ correspondence as in Example 2.1.9. Note that for $x \in \mathbb{C}^n$

$$x = \mathrm{id}_n(x) = \sum_{i=1}^{n} e_i \langle e_i, x \rangle = \sum_{i=1}^{n} |e_i\rangle\langle e_i|(x)$$

where $\{e_i\}$ is the canonical basis for $\mathbb{C}^n$. This means that

$$\sum_{i=1}^{n} |e_i\rangle\langle e_i| = \mathrm{id}_n = \varphi(1)$$

and that $\varphi(\mathbb{C}) \subset \mathcal{K}(\mathbb{C}^n)$, therefore $J_{\mathbb{C}^n} = \mathbb{C}$. Now,

$$\begin{aligned}(i_{\mathbb{C}}, i_{\mathbb{C}^n})^{(1)}(\varphi(\lambda)) &= \lambda \sum_{i=1}^{n} t(e_i)t(e_i)^* \\ &= \lambda \sum_{i=1}^{n} s_i s_i^*\end{aligned}$$

and $i_{\mathbb{C}}(\lambda) = \lambda 1_{\mathcal{T}_{\mathbb{C}^n}}$, meaning that $\mathcal{T}(J_{\mathbb{C}^n})$ is generated by

$$\lambda \sum_{i=1}^{n} s_i s_i^* - \lambda 1_{\mathcal{T}_{\mathbb{C}^n}},\ \lambda \in \mathbb{C}$$

meaning that

$$\sum_{i=1}^{n} Q(s_i)Q(s_i)^* = 1_{\mathcal{O}_{\mathbb{C}^n}}$$

in $\mathcal{O}_{\mathbb{C}^n}$. Since $\{s_i\}_{i=1}^n$ are isometries that generated $\mathcal{T}_{\mathbb{C}^n}$, then $\mathcal{O}_{\mathbb{C}^n}$ is generated by orthogonal isometries $\{v_i\}_{i=1}^n$ where $v_i = Q(s_i)$ such that

$$\sum_{i=1}^{n} v_i v_i^* = 1_{\mathcal{O}_{\mathbb{C}^n}}.$$

This means that $\mathcal{O}_{\mathbb{C}^n} \cong \mathcal{O}_n$, where $\mathcal{O}_n$ is the Cuntz algebra.

In the original paper where Pimsner [17] introduced this class of algebras, he made a remark that they not only generalized Cuntz algebras but also crossed products by $\mathbb{Z}$ and Cuntz-Krieger algebras. The next example shows how this is done for the case of crossed products.

Example 2.2.6. Let $\alpha : A \to A$ be an automorphism of a unital C*-algebra A, this implicitly defines a C*-dynamical system $(A, \mathbb{Z}, \alpha)$ where "$\alpha(n) = \alpha^n$" for $n \in \mathbb{Z}$. Now consider $E = A$ with the following Hilbert A-module structure,

$$x \cdot a := x\alpha(a),\ x \in E,\ a \in A$$

and $\langle x, y\rangle := \alpha^{-1}(x^*y)$ for $x, y \in E$. It's not hard to verify that this indeed turns E into a Hilbert A-module, furthermore we can define $\varphi : A \to \mathcal{L}_A(E)$ by $\varphi(a)(x) := ax$. This means we can view (E, φ) as an A-A correspondence. Consider (π_u, U) the universal covariant representation of $(A, \mathbb{Z}, \alpha)$, and define $t_U : E \to A \rtimes_\alpha \mathbb{Z}$ by

$$t_U(x) := \pi_u(x)U.$$

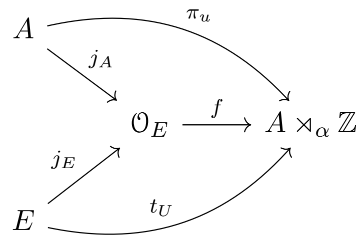
We have

$$\begin{aligned} t_U(x \cdot a) &= t_U(x\alpha(a)) = \pi_u(x\alpha(a))U \\ &= \pi_u(x)U\pi_u(a) = t_U(x)\pi_u(a), \\ t_U(x)^*t(y) &= (\pi_u(x)U)^*\pi_u(y)U = U^*\pi_u(x^*y)U \\ &= \pi_u(\alpha^{-1}(x^*y)) = \pi_u(\langle x, y\rangle), \\ t_U(\varphi(a)x) &= \pi_u(ax)U = \pi_u(a)t_U(x) \end{aligned}$$

so that $(\pi_u, t_U, A \rtimes_\alpha \mathbb{Z})$ is a Toeplitz covariant representation. Moreover, since $\varphi(A) \subset \mathcal{K}_A(E)$ and φ is injective then $J_E = A$. Now take $a \in A$

$$\begin{aligned} (\pi_u, t_U)^{(1)}(\varphi(a)) - \pi_u(a) &= (\pi_u, t_U)^{(1)}(|a\rangle\langle 1|) - \pi_u(a) \\ &= t_U(a)t_U(1)^* - \pi_u(a) \\ &= \pi_u(a)UU^*\pi_u(1) - \pi_u(a) \\ &= \pi_u(a) - \pi_u(a) = 0 \end{aligned}$$

hence $(\pi_u, t_U, A \rtimes_\alpha \mathbb{Z})$ is also Cuntz-Pimsner covariant and therefore there exists a unique *-homomorphism $f : \mathcal{O}_E \to A \rtimes_\alpha \mathbb{Z}$ such that the following diagram commutes



(2.3)

Now consider $(j_A, j_E(1))$, lets verify that this is a covariant representation of $(A, \alpha, \mathbb{Z})$. First we have

$$\begin{aligned} j_E(1)^* j_E(1) &= j_A(\langle 1, 1\rangle) = j_A(\varphi^{-1}(1)) = 1 \\ j_E(1) j_E(1)^* &= (j_A, j_E)^{(1)}(|1\rangle\langle 1|) \\ &= (j_A, j_E)^{(1)}(\varphi(1)) \\ &= j_A(1) = 1 \end{aligned}$$

meaining $V = j_E(1)$ is unitary. We also have that

$$\begin{aligned} V j_A(a) &= j_E(1) j_A(a) \\ &= j_E(1 \cdot a) \\ &= j_E(\varphi(a)) = j_A(\varphi(a)) j_E(1) = j_A(\varphi(a)) V \end{aligned}$$

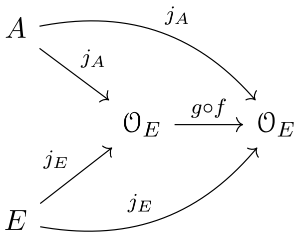
and therefore (j_A, V) is a covariant representation into $\mathcal{O}_E$. This means that there exists a *-homomorphism $g : A \rtimes_\alpha \mathbb{Z} \to \mathcal{O}_E$ such that $g \circ \pi_u = j_A$ and $g(U) = V$. We now verify that $f = g^{-1}$, since

$$g \circ f \circ j_A = g \circ \pi_u = j_A$$

for any $x \in E$ we have,

$$\begin{aligned} (g \circ f \circ j_E)(x) &= (g \circ t_U)(x) = g(\pi_u(x) U) \\ &= j_A(x) V = j_A(x) j_E(1) = j_E(\varphi(x) 1) = j_E(x) \end{aligned}$$

and therefore $(g \circ f) \circ j_E = j_E$. This means that $g \circ f$ makes the following diagram commute:



and due to the uniqueness of the middle *-homomorphism we have $g \circ f = \mathrm{id}$. To show that $f \circ g = \mathrm{id}$ we proceed in a similar way,

$$(f \circ g) \circ \pi_u = f \circ j_A = \pi_u,$$
$$(f \circ g)(U) = f(V) = f(j_E(1)) = t_U(1) = \pi_u(1)U = U$$

and due to the uniqueness of the morphism that satisfies those relations we have $f \circ g = \mathrm{id}$. We can conclude that

$$A \rtimes_\alpha \mathbb{Z} \cong \mathcal{O}_E$$

Remark 2.2.7. In [6] it is shown that we can also view Exel's crossed product as a relative Cuntz-Pimsner algebra (these are a slight generalizations of Cuntz-Pimsner algebras). This is done in a similar manner as we did in the previous example, we just replace all occurences of α^{-1} by the transfer operator L.

3 K-theory

K-theory has its origins as a tool to study vector bundles over topological spaces. The first notions of K-theory started with Grothendieck and his development of the Grothendieck-Riemann-Roch theorem and later became more widespread in the study of schemes and vector bundles over topological spaces.

In operator algebras, K-theory will present us with two functors that under certain conditions allow us to further study the structure of C*-algebras and in some cases classify C*-algebras. The topic of K-theory for C*-algebras is pretty vast and full of technical details with matrices, we will only cover enough for the results in Chapter 4 to make sense. The interested reader can go to [18] for an accessible introduction, to [19] for a more detailed treatment and to [2] to delve into the more advanced topics of K-theory for operator algebras.

Throughout this chapter A, B and C will always be C*-algebras. If A is a C*-algebra A^+ will denote its unitization obtained by adjointing a unit. Here A^+ will be equal as a vector space to $A \oplus \mathbb{C}$ but with the following product and involution:

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$
$$(a, \lambda)^* = (a^*, \overline{\lambda})$$

In case A is already unital we have a unital *-isomorphism between A^+ and $A \oplus \mathbb{C}$, this time the direct sum of C*-algebras, given by

$$f : A^+ \to A \oplus \mathbb{C}$$
$$(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$$

To avoid keeping track if the algebra we are handling is unital or not we will consider the following C*-algebra,

$$\tilde{A} := \begin{cases} A^+ & \text{, if } A \text{ is not unital} \\ A & \text{, if } A \text{ is unital.} \end{cases}$$

In a sense $\tilde{A}$ is the smallest unital C*-algebra containing A as an ideal. Some extra notation we will use throughout the chapter:

- $\mathrm{M}_\infty(A) := \cup_{n\in\mathbb{N}} \mathrm{M}_n(A)$,
- $\mathrm{GL}(A)$ to denote the group of invertibles in A,
- $\mathrm{GL}_n(A) := \mathrm{GL}(\mathrm{M}_n(A))$,
- $\mathrm{GL}_\infty(A) := \cup_{n\in\mathbb{N}} \mathrm{GL}_n(A)$,
- $\mathcal{U}(A)$ to denote the group of unitaries in A,
- $\mathcal{U}_n(A) := \mathcal{U}(\mathrm{M}_n(A))$,
- $\mathcal{U}_\infty(A) := \cup_{n\in\mathbb{N}}\mathcal{U}_n(A)$,
- $\mathcal{P}(A)$ to denote the set of projections in A,
- $\mathcal{P}_n(A) := \mathcal{P}(\mathrm{M}_n(A))$,
- $\mathcal{P}_\infty(A) := \cup_{n\in\mathbb{N}}\mathcal{P}_n(A)$.

All of the unions above are disjoint unions.

3.1 Projections and unitaries

Projections and unitaries play a key role in K-theory. Projections will be the noncommutative equivalent of complex vector bundles over topological spaces, this will be clear in the appendix. For this reason we are going to explore some properties and relations regarding projections before proceeding to the definition of the K_0 group. We will also revise some properties of unitaries as these will be used to define the K_1 group.

Definition 3.1.1. An element $p \in A$ is called a *projection* if it is self-adjoint and idempotent. Two projections p, q are called *orthogonal* whenever $pq = 0$; this will be denoted by $p \perp q$.

Remark 3.1.2. If p is a projection, then $1-p \in \tilde{A}$ is an orthogonal projection to p.

It is clear that any projection p is positive, furthermore due to the previous remark we also have $1 - p \geq 0$. This means that for any projection p we have $0 \leq p \leq 1$. Moreover, if $p \neq 0$ then

$$\|p\|^2 = \|p^*p\| = \|p^2\| = \|p\|$$

and therefore $\|p\| = 1$. Since $p - p^2 = 0$, we have

$$\{0\} = \sigma(0) = \sigma(p - p^2) = \{\lambda - \lambda^2,\ \lambda \in \sigma(p)\}$$

and consequently $\sigma(p) \subset \{0, 1\}$. If $p \neq 0, 1$, then $\sigma(p) = \{0, 1\}$. The converse is true as long as the element is normal:

Proposition 3.1.3. *If $a \in A$ is normal and $\sigma(a) = \{0, 1\}$ then a is a projection.*

Proof. Denote by $\varphi_a : C(\sigma(a)) \to A$ the functional calculus of a. Now, if $\sigma(a) \subset \{0, 1\}$, then $\mathrm{id}^2_{\sigma(a)} = \mathrm{id}_{\sigma(a)} = \mathrm{id}^*_{\sigma(a)}$. By functional calculus we have $\varphi_a(\mathrm{id}_{\sigma(a)}) = a$ and

$$a = \varphi_a(\mathrm{id}_{\sigma(a)}) = \varphi_a(\mathrm{id}^2_{\sigma(a)}) = \varphi_a(\mathrm{id}_{\sigma(a)})^2 = a^2$$

$$a = \varphi_a(\mathrm{id}_{\sigma(a)}) = \varphi_a(\mathrm{id}^*_{\sigma(a)}) = \varphi_a(\mathrm{id}_{\sigma(a)})^* = a^*$$

therefore a is a projection. □

Lemma 3.1.4. *The sum of two projections is a projection if and only if they are orthogonal.*

Proof. Let p, q be two projections. If p and q are orthogonal we have

$$(p+q)^2 = p^2 + pq + qp + q^2 = p^2 + q^2 = p + q$$

and therefore $p + q$ is a projection.

On the other hand, if $p+q$ is a projection then $pq+qp = 0$ and therefore

$$0 = q(pq + qp)q = 2qpq = 2qp^2q = 2(pq)^*(pq).$$

Finally, by the C*-identity we have

$$\|pq\|^2 = \|(pq)^*(pq)\| = 0$$

and we can conclude that p and q are orthogonal. □

Definition 3.1.5. We say that $v \in A$ is a *partial isometry* if v^*v is a projection. In case A is unital and $v^*v = 1$, we say that v is an *isometry*.

Lemma 3.1.6. *An element $v \in A$ is a partial isometry if and only if $v = vv^*v$.*

Proof. If v is a partial isometry, consider $x = v - vv^*v$ and the projection $p = v^*v$, then

$$\begin{aligned}\|x\|^2 = \|x^*x\| &= \|(v - vv^*v)^*(v - vv^*v)\| \\ &= \|v^*v - 2v^*vv^*v + v^*vv^*vv^*v\| \\ &= \|p - 2p^2 + p^3\| = 0.\end{aligned}$$

On the other hand, if $v = vv^*v$ then $p = v^*v$ is clearly self-adjoint and $p^2 = v^*vv^*v = v^*v = p$, hence p is a projection. □

Remark 3.1.7. A direct consequence of the previous lemma is that if v is a partial isometry, then so is v^* and therefore vv^* is also a projection.

To define the K_0 group we need certain equivalence relations. The following are three relations that later on we will see to be equivalent in the context of $\mathrm{M}_\infty(A)$.

Definition 3.1.8. Let p and q be two projections in a C*-algebra A. We say they are:

- *equivalent* ($p \sim_0 q$), if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$, in this case we say $p \sim_0 q$ via v;
- *unitarily equivalent* ($p \sim_u q$), if $p = u^*qu$, with $u \in \tilde{A}$ unitary;
- *homotopic* ($p \sim_h q$), when p and q are connected by a continuous path in $\mathcal{P}(A)$.

Remark 3.1.9. In case A is commutative, two equivalent projections are equal. Moreover, if $0 \sim_0 p$ then $p = 0$ and if $1 \sim_u q$ then $q = 1$.

Consider p a projection in the multiplier algebra $\mathcal{M}(A)$ then it can be proved that pAp is a hereditary subalgebra of A called a *corner*. A projection $p \in \mathcal{M}(A)$ is called *full* if $\overline{\mathrm{span}} ApA = A$, in this case we say that pAp is called a *full corner*.

Proposition 3.1.10. *Let $p, q \in \mathcal{P}(\mathcal{M}(A))$, if $p \sim_0 q$ then pAp is isomorphic to qAq.*

Proof. Consider $v \in \mathcal{M}(A)$ partial isometry such that $vv^* = p$ and $v^*v = q$. Then define the following *-homomorphism

$$\begin{aligned} \mathrm{Ad}(v) : qAq &\to pAp \\ x &\mapsto vxv^*. \end{aligned}$$

It is a well defined *-homomorphism since any $x \in qAq$ is of the form $x = qaq = v^*vav^*v$, then

$$vxv^* = vv^*(vav^*)vv^* = p(vav^*)p \in pAp.$$

To verify injectivity we consider $x \in qAq$ such that $vxv^* = 0$, then

$$0 = v^*vxv^*v = qxq = x.$$

Finally, for any $pap \in pAp$ we have

$$\mathrm{Ad}(v)(qv^*avq) = vqv^*avqv^* = p^2ap^2 = pap$$

therefore $\mathrm{Ad}(v)$ is surjective. It is not hard to verify that $\mathrm{Ad}(v)^{-1} = \mathrm{Ad}(v^*)$. □

Remark 3.1.11. The proposition above tells us that equivalent projections produce isomorphic corner algebras where the isomorphism is of the form $\mathrm{Ad}(v)$ for some partial isometry $v \in \mathcal{M}(A)$. In case v is an isometry in $\mathcal{M}(A)$ then $\mathrm{Ad}(v)$ is an isomorphism between A and pAp for $p = vv^*$.

Definition 3.1.12. We will call two invertibles $x, y \in \mathrm{GL}(\tilde{A})$ homotopic if there exists a continuous path in $\mathrm{GL}(\tilde{A})$ that connects them. In a similar manner, two unitaries $u, v \in \mathcal{U}(\tilde{A})$ are said to be homotopic if there exists a continuous path of unitaries connecting them.

The following lemmas will be useful to prove various properties of K_0 and K_1.

Lemma 3.1.13. *Let $p_1, p_2, q_1, q_2 \in \mathcal{P}(A)$ such that $p_1 \sim_0 q_1$ via the partial isometry v, $p_2 \sim_0 q_2$ via the partial isometry w, $p_1 \perp p_2$ and $q_1 \perp q_2$. Then $p_1 + p_2 \sim_0 q_1 + q_2$ via the partial isometry $v + w$.*

Proof. Let $v, w \in A$ be two partial isometries such that $p_1 = v^*v$, $q_1 = vv^*$, $p_2 = w^*w$ and $q_2 = ww^*$. By orthogonality we have $v^*vw^*w = 0$ and $vv^*ww^* = 0$. Due to lemma (3.1.6),

$$v^*w = v^*vv^*ww^*w = 0$$
$$vw^* = vv^*vw^*ww^* = 0$$

and by taking $s = v + w$ we have

$$s^*s = (v+w)^*(v+w) = v^*v + v^*w + w^*v + w^*w = p_1 + p_2$$
$$ss^* = (v+w)(v+w)^* = vv^* + vw^* + wv^* + ww^* = q_1 + q_2$$

hence $p_1 + p_2 \sim_0 q_1 + q_2$. □

Lemma 3.1.14. *If $p, q \in \mathcal{P}(A)$ and $z \in \tilde{A}$ is an invertible element such that $q = zpz^{-1}$, then $p \sim_u q$.*

Proof. We have $qz = zp$ and therefore $z^*zp = z^*qz = pz^*z$. This means that p commutes with $f(z^*z)$ for all $f \in C(\sigma(z^*z))$, in particular it commutes with $|z|^{-1} := (z^*z)^{-1/2}$. Taking $u := z|z|^{-1}$ we verify it is unitary,

$$u^*u = |z|^{-1}z^*z|z|^{-1} = |z|^{-2}z^*z = 1$$
$$uu^* = z|z|^{-2}z^* = z(|z|^{-2}z^*z)z^{-1} = zz^{-1} = 1.$$

Now, we have

$$upu^* = z|z|^{-1}p|z|^{-1}z^* = zp|z|^{-2}z^* = qz|z|^{-2}z^* = quu^* = q$$

hence $p \sim_u q$. □

Lemma 3.1.15. *Let $p, q \in \mathcal{P}(A)$, then $p \sim_u q$ if and only if $p \sim_0 q$ and $1 - p \sim_0 1 - q$.*

Proof. Consider $v, w \in \tilde{A}$ two partial isometries such that $v^*v = p$, $vv^* = q$, $w^*w = 1 - p$ and $ww^* = 1 - q$. Since $p \perp 1 - p$ and $q \perp 1-q$, by lemma (3.1.13) we have $p+1-p = 1 \sim_0 1 = q+1-q$ via the partial isometry $u = v + w$, this means $1 = u^*u = uu^*$. Then, since $vw^* = v^*w = 0$,

$$\begin{aligned} upu^* = (v + w)v^*v(v^* + w^*) &= (v + w)(v^*vv^* + v^*vw^*) \\ &= (v + w)(v^*vv^* + 0) \\ &= (v + w)v^* = vv^* + wv^* \\ &= vv^* = q. \end{aligned}$$

The converse is easily obtained from $p = u^*qu = u^*q^2u = u^*q^*qu$. Taking $v = qu$ as a partial isometry for $p \sim_0 q$ and $w = u - qu$ as partial isometry for $1 - p \sim_0 1 - q$. □

The following two results will give us simple criteria to determine when two projections or two unitaries are homotopic.

Proposition 3.1.16. *Let A be a unital C^*-algebra and $u \in \mathcal{U}(A)$ such that $\|u - 1\| < 2$. Then, there exists a continuous path in $\mathcal{U}(A)$ between u and 1. Moreover, if $u, v \in \mathcal{U}(A)$ with $\|u - v\| < 2$ then there exists a continuous path in $\mathcal{U}(A)$ connecting them.*

Proof. Let $u \in A$ be a unitary and hence normal. Since u is unitary, $\sigma(u) \subset \mathbb{T}$. We verify that $-1 \notin \sigma(u)$. If that was not the case we would have

$$-1 - 1 \in \sigma(u) - 1 = \sigma(u - 1)$$

but $r(u-1) = \|u - 1\| < 2$. Since $\sigma(u) \subsetneq \mathbb{T}$ then there exists a logarithm branch defined on $\sigma(u)$. We now consider $u_t = \varphi_u(f_t)$, where φ_u is the functional calculus on u and $f_t(z) = e^{t\log(z)}$. Moreover, since $\sigma(u) \subsetneq \mathbb{T}$ we have $\log(z) = i\theta$ with $\theta \in \mathbb{R}$ and hence $\log(u) := \varphi_u(\log) = ih$ with $h \in A_{sa}$ from which it follows that $u_t = e^{ith}$ is a unitary. Finally we have $u_0 = 1$ and $u_1 = e^{ih} = u$.

If u, v are unitaries such that $\|u - v\| < 2$ then $\|uv^* - 1\| < 2$ and the result follows. □

Theorem 3.1.17. *Let $p, q \in \mathcal{P}(A)$ such that $\|p - q\| < 1$, then they are homotopic in $\mathcal{P}(A)$. Moreover, there exists a continuous path of unitaries u_t such that $u_0 = 1$ and $q = u_1 p u_1^*$.*

Proof. In $\tilde{A}$ we define $v_p := 2p - 1$, $v_q := 2q - 1$. Since p and q are projections, v_p and v_q are unitaries

$$v_p^* v_p = v_p v_p = v_p v_p^* = (2p-1)(2p-1) = 4p - 4p + 1 = 1$$

and therefore $\|v_p\| = \|v_q\| = 1$. Now, if $z_q := v_q v_p + 1$, then

$$\begin{aligned}\|(1/2)z_q - 1\| &= \frac{\|v_q v_p - 1\|}{2} = \frac{\|v_q v_p - v_q v_q\|}{2} \\ &= \frac{\|v_q(v_p - v_q)\|}{2} \leq \frac{\|v_p - v_q\|}{2} = \frac{2\|p - q\|}{2} < 1\end{aligned}$$

and therefore z_q is invertible. Next, since

$$qz_q = q(2q-1)(2p-1) + q = 4qp - 2q - 2qp + q + q = 2qp$$

$$z_q p = (2q-1)(2p-1)p + p = 4qp - 2qp - 2p + p + p = 2qp$$

we have $q = z_q p z_q^{-1}$. By lemma (3.1.14) we have $q \sim_u p$, furthermore, we have $q = u_q p u_q^*$ where $u_q = z_q |z_q|^{-1}$.

Fixing $p \in A$ projection, we see that $q \mapsto z_q$ is a continuous map in A,

$$\|z_q - z_{q'}\| = \|v_q v_p - v_{q'} v_p\| \leq \|v_q - v_{q'}\| \|v_p\| = 2\|q - q'\|$$

and, since $z \mapsto z|z|^{-1}$ is also a continuous map, $q \mapsto u_q$ is continuous in $\{q \in A : \ z_q \in \mathrm{GL}(\tilde{A})\}$. Since $z_q \in \mathrm{GL}(\tilde{A})$ whenever q is a projection and $\|p - q\| < 1$, $q \mapsto u_q$ is continuous in $X = \{q \in A : \ q = q^2 = q^*, \ \|q - p\| < 1\}$.

Now we consider the path $t \mapsto z_{t,q} := t z_q + 2 - 2t$, $t \in [0, 1]$. Since $\|z_{t,q} - 2\| = t\|z_q - 2\| < 2$ as long as $\|p - q\| < 1$, it follows that $z_{t,q}$ is invertible for $t \in [0, 1]$ and $q \in X$. This means that $z_{t,q}$ is a continuous path of invertibles joining $z_{0,q} = 2$ and $z_{1,q} = z_q$ whenever $\|q - p\| < 1$ and q is a projection. If we consider $u_{t,q} = z_{t,q}|z_{t,q}|^{-1}$ we have a path of unitaries such that $u_{0,q} = 1$ and $u_{1,q} = u_q$. Finally, for every projection q with $\|q - p\| < 1$ we have

$$r_{t,q} = u_{t,q} p u_{t,q}^* \in A$$

is a path of projections in A such that $r_{0,q} = p$ and $r_{1,q} = q$.

□

Remark 3.1.18. Let $u, v \in \mathcal{U}(\tilde{A})$ such that they are homotopic in $\mathrm{GL}(\tilde{A})$. Using the techniques used in the proof above, we can adjust the continuous path in $\mathrm{GL}(\tilde{A})$ to be in $\mathcal{U}(\tilde{A})$. This means that using polar decomposition we can transform a path of invertibles between two unitaries into a path of unitaries without altering the beginning or the end of the path.

Corollary 3.1.19. *Let $p, q \in \mathcal{P}(A)$, then $p \sim_h q$ if and only if there exists a unitary u_1 homotopic to 1 such that $p = u_1 q u_1^*$.*

Proof. If $p \sim_h q$, let p_t be the continuous path such that $p_0 = p$ and $p_1 = q$. Suppose that $\|p - p_t\| < 1$ for all $t \in [0, 1]$. By the previous theorem there exists a path of unitaries u_t such that $u_0 = 1$ and $p = u_1 q u_1^*$.

In the general scenario, by compactness we can take $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|p_{t_{k+1}} - p_{t_k}\| < 1$. We now have paths of unitaries $u_{t,k}$ such that $p_{t_{k+1}} = u_{1,k} p_{t_k} u_{1,k}^*$ and $u_{0,k} = 1$. Finally we consider $u_t = u_{t,n-1} \cdots u_{t,1} u_{t,0}$, we have $u_0 = 1$ and

$$q = p_1 = u_1 p_0 u_1^* = u_1 p u_1^*.$$

The converse implication is immediate. □

We can now establish implications between the three equivalence relations defined at the beginning of this section.

Proposition 3.1.20. *Let $p, q \in \mathcal{P}(A)$, we have*

1. *if $p \sim_h q$ then $p \sim_u q$,*
2. *if $p \sim_u q$ then $p \sim_0 q$.*

Proof. By the previous corollary, $p \sim_h q$ implies $p \sim_u q$. Finally, if $p \sim_u q$ we have $q = u^* p u$ and by taking $v = pu$ it follows,

$$q = u^* p u = u^* p^2 u = u^* p^* p u = (pu)^* (pu) = v^* v$$
$$p = p^2 = p u u^* p = p u u^* p^* = (pu)(pu)^* = v v^*$$

□

The converse does not hold in general. However we can obtain a slightly weaker result if we allow ourselves to work in $\mathrm{M}_2(A)$. The following will be an important tool during our work with the K_0 group.

Definition 3.1.21. An *elementary operation* in $\mathrm{M}_n(A)$ refers to:

- Multiplying a matrix row or column by a non-zero scalar.
- Exchanging two rows or columns.
- Multiplying a row or column by an element $a \in A$ and adding up the result to another row or column, respectively

By basic linear algebra we know that we can obtain elementary operations by multiplying a matrix (from left or right) by one of the following matrices in $\mathbb{M}_n(\tilde{A})$:

$$T_{ij} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 0 & 1 & \\ & & \ddots & & \\ & & 1 & 0 & \\ & & & & \ddots \\ 0 & & & & 1 \end{pmatrix}, \quad D_i(\lambda) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ & & & & 1 \\ & & & & \ddots \\ 0 & & & & 1 \end{pmatrix},$$

$$L_{i,j}(a) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & a & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{pmatrix}.$$

Remark 3.1.22. Since $T_{ij}, D_i(\lambda) \in \mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ is connected, there exists a continuous path of invertibles that connects them to $1 \in \mathrm{GL}_n(\mathbb{C})$. For $L_{ij}(a)$ we can consider $h(t) = L_{ij}(ta)$, which begins at 1 and ends up in $L_{ij}(a)$. We also note that T_{ij}

is unitary, therefore if $q \in \mathrm{M}_n(A)$ is the result of simultaneously exchanging rows and columns of a matrix $p \in \mathrm{M}_n(A)$ (that is, conjugating by T_{ij}), we have $p \sim_u q$ and by proposition 3.1.20 we obtain $p \sim_0 q$.

To facilitate the notation, if $a_1, \dots, a_n \in A$ then

$$\operatorname{diag}(a_1, a_2, \dots, a_n) := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix} \in \mathrm{M}_n(A).$$

Similarly we define $x \oplus y \in \mathrm{M}_{m+n}(A)$ for $x \in \mathrm{M}_n(A)$ and $y \in \mathrm{M}_m(A)$. Take $x \in \mathrm{M}_n(A)$ and $y \in \mathrm{M}_m(A)$ such that

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mm} \end{pmatrix}$$

we define $x \oplus y$ as

$$x \oplus y := \begin{pmatrix} x_{11} & \cdots & x_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ x_{n1} & \cdots & x_{nn} & 0 & \cdots & 0 \\ 0 & 0 & 0 & y_{11} & \cdots & y_{1m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & y_{m1} & \cdots & y_{mm} \end{pmatrix}.$$

It is not hard to verify that the operation $\oplus$ is associative and therefore expressions like $x \oplus y \oplus z$ for $x \in \mathrm{M}_n(A)$, $y \in \mathrm{M}_m(A)$ and $z \in \mathrm{M}_p(A)$ make sense.

Lemma 3.1.23. *If $u, v \in \tilde{A}$ are unitaries (or invertibles), then there exist continuous paths of unitaries (or invertibles) that connect the following matrices:*

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \text{ and } \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

Proof. In the first place we have the following,

$$\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now, for $t \in [0, 1]$ consider:

$$r_t := \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix}$$

Which gives us a continuous path of unitaries in $\mathrm{M}_2(\mathbb{C})$ that joins $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Now, consider $w_t = r_t \operatorname{diag}(u, v) r_t$ and $z_t = \operatorname{diag}(u, 1) r_t \operatorname{diag}(v, 1) r_t$. By the previous equatilities we verify

$$w_0 = \operatorname{diag}(u, v) \qquad z_0 = \operatorname{diag}(uv, 1)$$
$$w_1 = \operatorname{diag}(v, u) \qquad z_1 = \operatorname{diag}(u, v).$$

The same proof works in the case of invertibles. □

Lemma 3.1.24. *For any $x \in \tilde{A}$ there exists a continuous path of invertibles that connects the following matrices to the identity matrix in $\mathrm{M}_2(\tilde{A})$:*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

Proof. Simply consider the continuous path given by

$$X_t = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}$$

The other matrix is treated in an analogous way. □

Proposition 3.1.25. *Let $u \in A$ be a unitary, then there exists a continuous path between u and 1 if and only if $u = e^{ih_1} \cdots e^{ih_n}$ with $h_i \in A_{sa}$.*

Proof. Let X be the path connected component of 1 in the unitaries of A, which by 3.1.16 is open in $\mathcal{U}(A)$. Take $G = \{e^{ih_1} \cdots e^{ih_n} : h_i \in A_{sa}\}$, it is clear that $G \subset X$. Now, by Proposition 3.1.16, G is open. It is straightforward to see that G is a subgroup of $\mathcal{U}(A)$, hence we can write $\mathcal{U}(A) = \cup_{x \in \Lambda} xG$ for some $\Lambda \subseteq G$ such that xG are disjoint cosets. Since xG is open, $\mathcal{U}(A) \backslash G$ is also open and therefore G is closed. By connectedness we conclude that $G = X$. □

Proposition 3.1.26. *Let $p, q \in \mathcal{P}(A)$, then*

$$p \sim_0 q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \quad p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

Proof. Assume $p \sim_0 q$ and take v with $p = v^*v$ and $q = vv^*$.

Consider $u := \left(\begin{smallmatrix} v & 1-q \\ 1-p & v^* \end{smallmatrix}\right) \in \mathbb{M}_2(\tilde{A})$. We verify that u is unitary,

$$\begin{aligned} u^*u &= \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \\ &= \begin{pmatrix} v^*v + 1 - p & v^* - v^*q + v^* - pv^* \\ v - qv + v - vp & 1 - q + vv^* \end{pmatrix} \\ &= \begin{pmatrix} p + 1 - p & v^* - v^*vv^* + v^* - v^*vv^* \\ v - vv^*v + v - vv^*v & 1 - q + q \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

in the same manner we verify $uu^* = 1$. Observe that

$$\begin{aligned} u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* = u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^* & 1-q \\ 1-p & v \end{pmatrix} \\ &= \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \begin{pmatrix} pv^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} vpv^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now, if $p \sim_u q$ with $q = upu^*$ and $u^* \in \tilde{A}$, due to the previous lemma there exists a path of unitaries w_t that joins $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} u^* & 0 \\ 0 & u \end{smallmatrix}\right)$. Taking the path of projections in $\mathrm{M}_2(A)$, $p_t = w_t^* \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) w_t$, we have $p_0 = \mathrm{diag}(p, 0)$ and

$$p_1 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} upu^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

□

Remark 3.1.27. The previous proposition shows us that if $p \sim_0 q$ in A, then in $\mathrm{M}_4(A)$ we have $\mathrm{diag}(p, 0, 0, 0) \sim_h \mathrm{diag}(q, 0, 0, 0)$.

3.2 The K_0 group

Now that we have all the tools necessary we can begin by defining the K_0 group for unital C*-algebras. When the algebra is not unital we must proceed more carefully as we will see later on.

Definition 3.2.1. Let A be a unital C*-algebra $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say that p and q are equivalent if there exists $v \in \mathrm{M}_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. We will denote this by $p \sim q$.

Remark 3.2.2. Here $\mathrm{M}_{m,n}(A)$ is the set of $m \times n$ matrices with entries on A. The adjoint of a matrix here is the matrix in $\mathrm{M}_{n,m}(A)$ obtained by taking transpose and adjoint in each entry.

It is clear that this equivalence relation extends the relation $\sim_0$ defined for projections in A. The next proposition shows us that it even retains most of the properties $\sim_0$ has.

Proposition 3.2.3. *Let $p, q, p', q' \in \mathcal{P}_\infty(A)$, then*

1. *$p \sim p \oplus 0_n$ for all $n \in \mathbb{N}$, where 0_n is the zero matrix in $\mathrm{M}_n(A)$,*

2. *if $p \sim p'$ and $q \sim q'$, then $p \oplus q \sim p' \oplus q'$,*

3. *$p \oplus q \sim q \oplus p$,*

4. *if $p, q \in \mathrm{M}_n(A)$ and $p \perp q$ then $p + q \sim p \oplus q$.*

Proof. 1. Take $p \in \mathcal{P}_m(A)$ and $n \in \mathbb{N}$. Consider

$$v = \begin{pmatrix} p \\ 0_{n,m} \end{pmatrix} \in \mathrm{M}_{m+n,m}(A)$$

where $0_{n,m}$ is the zero matrix in $\mathrm{M}_{n,m}(A)$. Then

$$v^*v = p$$
$$vv^* = \begin{pmatrix} p & 0_{m,n} \\ 0_{n,m} & 0_n \end{pmatrix} = p \oplus 0_n$$

and therefore $p \sim p \oplus 0_n$.

2. If $p \sim p'$ and $q \sim q'$, there exist $v \in \mathrm{M}_{m,n}(A)$, $w \in \mathrm{M}_{r,s}(A)$ such that

$$p = v^*v, \quad p' = vv^*, \quad q = w^*w, \quad q' = ww^*.$$

Now, we consider

$$u = \begin{pmatrix} v & 0_{m,s} \\ 0_{r,n} & w \end{pmatrix} \in \mathrm{M}_{m+r,n+s}(A)$$

A simple calculation shows that $u^*u = v^*v \oplus w^*w = p \oplus q$ and $uu^* = vv^* \oplus ww^* = p' \oplus q'$, so that $p \oplus q \sim p' \oplus q'$.

3. If $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ are two projections, then by setting

$$v = \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix}$$

we have $v^*v = p \oplus q$ and $vv^* = q \oplus p$, so that $p \oplus q \sim q \oplus p$.

4. If $p, q \in \mathcal{P}_n(A)$ and $p \perp q$, consider

$$v = \begin{pmatrix} p \\ q \end{pmatrix}$$

We have $v^*v = p + q$ and $vv^* = p \oplus q$, so that $p + q \sim p \oplus q$.

$\square$

Proposition 3.2.4. *Let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ such that $p \sim q$ and $n \leq m$. Then there exists $w \in \mathrm{M}_m(A)$ such that*

$$p \oplus 0_{m-n} = w^*w, \quad q = ww^*$$

Proof. If $p \sim q$ then there exists $v \in \mathrm{M}_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$, now take

$$w = \begin{pmatrix} v & 0_{m,m-n} \end{pmatrix} \in \mathrm{M}_m(A)$$

and we have $w^*w = p \oplus 0_{m-n}$ and $ww^* = q$. $\square$

The previous proposition means that we can always look at the relation $\sim$ as our previously defined relation $\sim_0$ by adding zeros to the matrices to have their sizes match.

Definition 3.2.5. We define the projection semigroup by

$$V(A) := \mathcal{P}_\infty(A)/\sim$$

where the sum is given by $[p] + [q] := [p \oplus q]$.

Remark 3.2.6. Note that $V(A)$ is a commutative semigroup by proposition 3.2.3 and has $[0]$ as its identity element.

Proposition 3.2.7. *Let A be a unital C^*-algebra and $[p], [q]$ in $V(A)$. Then*

1. *$[p] = [q]$ if and only if there exists $m \in \mathbb{N}$ and $p', q' \in \mathcal{P}_m(A)$ such that $p' \sim_u q'$ and $[p'] = [p] = [q] = [q']$,*

2. *$[p] = [q]$ if and only if there exists $m \in \mathbb{N}$ and $p'', q'' \in \mathcal{P}_m(A)$ such that $p'' \sim_h q''$ and $[p''] = [p] = [q] = [q'']$.*

Proof. 1. If $[p] = [q]$, by 3.2.4, we can assume $p \sim_0 q$ in $\mathrm{M}_n(A)$, now by Proposition 3.1.26 we have $p \oplus 0_n \sim_u q \oplus 0_n$. Taking $p' = p \oplus 0_n$ and $q' = q \oplus 0_n$ we have $[p'] = [p] = [q] = [q']$ and $p' \sim_u q'$.

   The converse is a direct consequence of Proposition 3.1.20, since $p' \sim_u q'$ implies $p' \sim_0 q'$ and hence $p \sim p' \sim q' \sim q$.

2. If $[p] = [q]$, using the previous item, we have $p', q' \in \mathrm{M}_m(A)$ such that $p' \sim_u q'$ with $[p'] = [p] = [q] = [q']$, by Proposition 3.1.26 we have $p' \oplus 0_m \sim_h q' \oplus 0_m$. Taking $p'' = p' \oplus 0_m$ and $q'' = q' \oplus 0_m$ we have $p'' \sim_h q''$ and

$$[p''] = [p'] = [p] = [q] = [q'] = [q'']$$

   The converse is a direct consequence of Proposition 3.1.20, since $p' \sim_h q'$ implies $p' \sim_0 q'$ and hence $p \sim p' \sim q' \sim q$.

□

This means that in $V(A)$ we can see $\sim$ as either $\sim_h$ or $\sim_u$, whichever is more convenient.

Proposition 3.2.8. *Let A and B be two unital C^*-algebras and $\alpha : A \to B$ a *-homomorphism. Then we can induce a semigroup homomorphism $\alpha_* : V(A) \to V(B)$ by*

$$\alpha_*([p]) = [\alpha(p)]$$

*where $\alpha(p)$ is understood as the entrywise induced *-homomorphism on the matrix algebras. Moreover, $A \mapsto V(A)$ and $\alpha \mapsto \alpha_*$ define a covariant functor from the category of unital C^*-algebras to the category of abelian semigroups.*

Proof. First, if $p \in \mathcal{P}_\infty(A)$ then

$$\alpha(p)^* = \alpha(p^*) = \alpha(p) = \alpha(p^2) = \alpha(p)^2$$

and therefore $\alpha(p)$ is also a projection. Now, if $[p] = [q]$ with $p \in \mathrm{M}_n(A)$, $q \in \mathrm{M}_m(A)$ and $m \geq n$ by proposition 3.2.4 there exists $v \in \mathrm{M}_\infty(A)$ with $p \oplus 0_{m-n} = v^*v$ and $q = v^*v$,

$$\alpha(p) \oplus 0_{m-n} = \alpha(p \oplus 0_{m-n}) = \alpha(v^*v) = \alpha(v^*)\alpha(v) = \alpha(v)^*\alpha(v)$$
$$\alpha(q) = \alpha(vv^*) = \alpha(v)\alpha(v^*) = \alpha(v)\alpha(v)^*$$

hence $[\alpha(p)] = [\alpha(p) \oplus 0_{m-n}] = [\alpha(q)]$ and α_* is well defined. To see that it is a semigroup homomorphism consider $[p], [q] \in V(A)$, it is pretty clear that $\alpha(p \oplus q) = \alpha(p) \oplus \alpha(q)$ and therefore

$$\begin{aligned} \alpha_*([p] + [q]) = \alpha_*([p \oplus q]) &= [\alpha(p \oplus q)] \\ &= [\alpha(p) \oplus \alpha(q)] \\ &= [\alpha(p)] + [\alpha(q)] \\ &= \alpha_*([p]) + \alpha_*([q]). \end{aligned}$$

Now, if $\alpha : A \to B$ and $\beta : B \to C$ are $*$-homomorphisms, we have

$$(\beta \circ \alpha)_*([p]) = [(\beta \circ \alpha)(p)] = [\beta(\alpha(p))] = \beta_*([\alpha(p)]) = \beta_* \circ \alpha_*([p])$$

and for $\mathrm{id}_* : V(A) \to V(A)$ we have $\mathrm{id}_*([p]) = [\mathrm{id}(p)] = [p]$. □

Example 3.2.9. If A is a unital C*-algebra, we have $V(A) \cong V(\mathrm{M}_n(A))$ for all $n \in \mathbb{N}$. To see this consider $\alpha : A \to \mathrm{M}_n(A)$ by $\alpha(a) = a \oplus 0_{n-1}$. We can verify that α_* is an isomorphism. We first observe that if $p \in \mathrm{M}_m(A)$ is a projection then

$$\alpha(p) = \begin{pmatrix} p_{11} \oplus 0_{n-1} & \cdots & p_{1m} \oplus 0_{n-1} \\ \vdots & \ddots & \vdots \\ p_{m1} \oplus 0_{n-1} & \cdots & p_{mm} \oplus 0_{n-1} \end{pmatrix} \in \mathrm{M}_m(\mathrm{M}_n(A))$$

but by using the identification $\mathrm{M}_m(\mathrm{M}_n(A)) \cong \mathrm{M}_{mn}(A)$ and doing some simultaneous exchange of rows and columns we obtain

$$\alpha(p) \sim_u p \oplus 0_{m(n-1)}$$

and therefore $\alpha_*([p]) = [\alpha(p)] = [p \oplus 0_{m(n-1)}] = [p]$. From this and the identification $\mathrm{M}_m(\mathrm{M}_n(A)) \cong \mathrm{M}_{mn}(A)$ we quickly conclude that α_* is bijective.

Definition 3.2.10. We say that $([p],[q]),([p'],[q']) \in V(A) \times V(A)$ are equivalent if there exists $[r] \in V(A)$ such that

$$[p] + [q'] + [r] = [p'] + [q] + [r].$$

This defines an equivalence relation on $V(A) \times V(A)$. The quotient space defined by this relation is denoted by $K_{00}(A)$. We will use the following notation for its elements:

$$[p] - [q] := [([p],[q])].$$

Furthermore, if we consider the operation

$$([p] - [q]) + ([p'] - [q']) := ([p] + [p']) - ([q] + [q'])$$

we observe that $K_{00}(A)$ has an abelian group structure with identity element given by $0 = [0] - [0]$ and $-([p] - [q]) = [q] - [p]$.

Remark 3.2.11. The construction done on the previous definition, also known as the Grothendieck group construction, allows us to obtain a group from any abelian semigroup, in fact it defines a functor from the category of abelian semigroups to the category of abelian groups. For example, one can use this construction to obtain $\mathbb{Z}$ from $\mathbb{N}$.

Proposition 3.2.12. *Given A unital C*-algebra, if $[p] - [q] = 0$ in $K_{00}(A)$, then there exists $m \in \mathbb{N}$ such that $p \oplus 1_m \sim q \oplus 1_m$.*

Proof. By definition if $[p] - [q] = 0$ then there exists $r \in \mathcal{P}_{m'}(A)$ such that

$$[p] + [0] + [r] = [q] + [0] + [r]$$

which means $[p] + [0 \oplus r] = [q] + [0 \oplus r]$, by taking $m = m' + 1$ and $r' = 0 \oplus r \in \mathrm{M}_m(A)$ we have by proposition 3.2.3

$$\begin{aligned}
[p \oplus 1_m] &= [p] + [1_m] \\
&= [p] + [r' + (1_m - r')] \\
&= [p] + [r'] + [1_m - r'] \\
&= [q] + [r'] + [1_m - r'] = [q] + [1_m] = [q \oplus 1_m]
\end{aligned}$$

therefore $p \oplus 1_m \sim q \oplus 1_m$. □

We can see $K_{00}(-)$ as a functor from the category of C*-algebras to the category of abelian groups. If $\alpha : A \to B$ is a *-homomorphism, then $K_{00}(\alpha) : K_{00}(A) \to K_{00}(B)$ will be given by

$$K_{00}(\alpha)([p] - [q]) = \alpha_*([p]) - \alpha_*([q]).$$

To avoid clutter we will use the notation $\alpha_* := K_{00}(\alpha)$. For any C*-algebra we can consider the morphism $i_A : V(A) \to K_{00}(A)$, given by $i_A([p]) = [p] - [0]$. If $\alpha : A \to B$ is a *-homomorphism we obtain the following commutative diagram:

$$\begin{array}{ccc}
V(A) & \xrightarrow{\alpha_*} & V(B) \\
\downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} \\
K_{00}(A) & \xrightarrow{\alpha_*} & K_{00}(B)
\end{array} \tag{3.1}$$

Example 3.2.13. If $A = \mathbb{C}$ then $K_{00}(\mathbb{C}) \cong \mathbb{Z}$, to see this we first verify $V(\mathbb{C}) \cong \mathbb{N} \cup \{0\}$. Take $x \in \mathrm{M}_n(\mathbb{C})$ a projection with rank k. By linear algebra there exists $u \in \mathcal{U}_n(\mathbb{C})$ unitary such that $I_k \oplus 0_{n-k} = u^*xu$, where I_k is the identity matrix in $\mathrm{M}_k(\mathbb{C})$. This means that for all projections $x \in \mathrm{M}_n(\mathbb{C})$, we have $[x] = [I_k]$ where k is its rank. Finally we can conclude that

$$V(\mathbb{C}) = \{[I_k] : \ k \in \mathbb{N} \cup \{0\}\} \cong \mathbb{N} \cup \{0\}$$

and therefore $K_{00}(\mathbb{C}) \cong \mathbb{Z}$.

Example 3.2.14. If $A = \mathrm{M}_k(\mathbb{C})$ then $K_{00}(A) \cong \mathbb{Z}$. This is because $V(\mathrm{M}_k(\mathbb{C})) \cong V(\mathbb{C})$.

For any C*-algebra A we have the following split exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} A^+ \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \tag{3.2}$$

where $\pi(a, z) = z$ and $\lambda(z) = (0, z)$, here 1 denotes the unit of A^+ disregarding whether A is unital or not. We define

$$K_0(A) := \ker(\pi_* : K_{00}(A^+) \to \mathbb{Z})$$

and we also define the *scalar map* as $s := \lambda \circ \pi$.

Proposition 3.2.15. *Let A_1, A_2 be two C*-algebras and consider $A_1 \oplus A_2$ with the canonical projections $\pi_k : A_1 \oplus A_2 \to A_k$ for $k = 1, 2$. These projections induce the following isomorphisms:*

$$\pi_{1*} \oplus \pi_{2*} : V(A_1 \oplus A_2) \to V(A_1) \oplus V(A_2),$$
$$\pi_{1*} \oplus \pi_{2*} : K_{00}(A_1 \oplus A_2) \to K_{00}(A_1) \oplus K_{00}(A_2),$$
$$\pi_{1*} \oplus \pi_{2*} : K_0(A_1 \oplus A_2) \to K_0(A_1) \oplus K_0(A_2)$$

Proof. Let A_1, A_2 be two C*-algebras and $p \in \mathrm{M}_n(A_1), q \in \mathrm{M}_n(A_2)$ projections. It is clear that $p\hat{\oplus}q \in \mathrm{M}_n(A_1 \oplus A_2)$ defined by

$$(p\hat{\oplus}q)_{ij} := (p_{ij}, q_{ij}) \in A_1 \oplus A_2$$

is a projection and that all projections in $M_n(A_1 \oplus A_2)$ are of this form. Now,

$$\begin{aligned}(\pi_{1*} \oplus \pi_{2*})([p\hat{\oplus}q]) &= \big(\pi_{1*}([p\hat{\oplus}q]), \pi_{2*}([p\hat{\oplus}q])\big) \\ &= \big([\pi_1(p\hat{\oplus}q)], [\pi_2(p\hat{\oplus}q)]\big) \\ &= ([p], [q]) \, ,\end{aligned}$$

which implies that $(\pi_{1*} \oplus \pi_{2*}) : V(A_1 \oplus A_2) \to V(A_1) \oplus V(A_2)$ is an isomorphism. By functoriality of the Grothendieck group construction, we have an isomorphism between the K_{00} groups. Moreover $\ker \pi_{1*} \cong K_{00}(A_2)$ and $\ker \pi_{2*} \cong K_{00}(A_1)$.

To verify the result for K_0, we just need to check that $K_0(-)$ is a functor (proposition 3.3.1) and that it is half-exact (Proposition 3.3.4). □

From diagram (3.2), by functoriality, we obtain the following split exact sequence:

$$0 \longrightarrow K_0(A) \xrightarrow{\iota} K_{00}(A^+) \underset{\lambda_*}{\overset{\pi_*}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0 \, . \tag{3.3}$$

We have π_* surjective since $\pi_* \circ \lambda_* = \mathrm{id}_{A^+ *}$ and the exactness comes from the definition of $K_0(A)$. Remember that if A is unital we have $A^+ \cong A \oplus \mathbb{C}$ as C*-algebras via the unital *-isomorphism $f : A^+ \to A \oplus \mathbb{C}$, $f(a, z) = (a + z1_A, z)$ which has inverse $g : A \oplus \mathbb{C} \to A^+$ given by $g(a, z) = (a - z1_A, z)$. By the previous proposition we have $K_{00}(A^+) \simeq K_{00}(A) \oplus \mathbb{Z}$ via $\pi_1 \oplus \pi_2$ with $\ker(\pi_{2*}) \cong K_{00}(A)$. This induces the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \xrightarrow{\iota} & K_{00}(A^+) & \xrightarrow{\pi_*} & \mathbb{Z} & \longrightarrow & 0 \\ & & & & g_* \uparrow\downarrow f_* & & \downarrow \mathrm{id}_* & & \\ 0 & \longrightarrow & K_{00}(A) & \underset{\pi_{1*}}{\overset{\iota}{\rightleftarrows}} & K_{00}(A \oplus \mathbb{C}) & \xrightarrow{\pi_{2*}} & \mathbb{Z} & \longrightarrow & 0 \end{array} \tag{3.4}$$

and since the square on the right commutes we can conclude that $K_{00}(A) \cong K_0(A)$ for A unital where the isomorphism is given by $F = \pi_{1*} \circ f_* \circ \iota$ with inverse $G = g_* \circ \iota$.

3.3 K_0 as a functor

To make $K_0(-)$ into a functor we first need to define $K_0(\alpha)$ for a *-homomorphism $\alpha : A \to B$. The following proposition shows the natural way to construct this morphism and shows it satisfies all of the functorial properties.

Proposition 3.3.1. *K_0 is a covariant functor from the category of C^*-algebras to the category of abelian groups. Moreover, given a *-homomorphism $\alpha : A \to B$ and $p, q \in \mathcal{P}_\infty(A^+)$ we have*

$$\alpha_*([p] - [q]) := K_0(\alpha)([p] - [q]) = [\alpha^+(p)] - [\alpha^+(q)]$$

*where $\alpha^+ : A^+ \to B^+$ is the unital *-homomorphism given by $\alpha^+(a, z) = (\alpha(a), z)$.*

Proof. By definition we have $\pi(\alpha^+(a, z)) = z = \pi(a, z)$, which means that the following diagram commutes

$$\begin{array}{ccc} A^+ & \xrightarrow{\pi} & \mathbb{C} \\ {\scriptstyle \alpha^+}\big\downarrow & \nearrow_{\pi} & \\ B^+ & & \end{array} \tag{3.5}$$

Now, from the functoriality of K_{00} we have the following commutative diagram

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\iota} & K_{00}(A^+) & \xrightarrow{\pi_*} & \mathbb{Z} \\ & & \big\downarrow{\scriptstyle \alpha_*^+} & & \big\downarrow{\scriptstyle =} \\ K_0(B) & \xrightarrow{\iota} & K_{00}(B^+) & \xrightarrow{\pi_*} & \mathbb{Z} \end{array} \tag{3.6}$$

Therefore $\alpha_* := (\alpha^+)_* \circ \iota : K_0(A) \to K_0(B)$ is well defined. Moreover, if we have $\alpha : A \to B$ and $\beta : B \to C$ then

$$\begin{aligned}(\beta \circ \alpha)_* &= [(\beta \circ \alpha)^+]_* \circ \iota \\ &= [\beta^+ \circ \alpha^+]_* \circ \iota \\ &= (\beta^+)_* \circ (\alpha^+)_* \circ \iota = (\beta^+)_* \circ \alpha_* = \beta_* \circ \alpha_*\end{aligned}$$

and $(\mathrm{id}_A)_* = (\mathrm{id}_A^+)_* \circ \iota = (\mathrm{id}_{A^+})_* \circ \iota = \mathrm{id}_{K_{00}(A)} \circ \iota = \mathrm{id}_{K_0(A)}$. We can conclude that K_0 is a covariant functor. □

Remark 3.3.2. Recall that for a unital C*-algebra $K_0(A)$ is isomorphic to $K_{00}(A)$. Something remarkable is that $K_0(\alpha)$ and $K_{00}(\alpha)$ will coincide for any *-homomorphism $\alpha : A \to B$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} K_{00}(A) & \xrightarrow{K_{00}(\alpha)} & K_{00}(B) \\ \downarrow G & & \downarrow G \\ K_0(A) & \xrightarrow{K_0(\alpha)} & K_0(B) \end{array}$$

where G is the isomorphism defined by the end of the previous section. To verify this we take a projection $p = (p_{ij}) \in \mathcal{P}_r(A)$, then

$$\begin{aligned}G(K_{00}(\alpha)([p])) &= G([(\alpha(p_{ij}))]) \\ &= g_* \circ \iota([(\alpha(p_{ij}))]) \\ &= g_*([(\alpha(p_{ij}), 0)]) \\ &= [(\alpha(p_{ij}), 0)] \\ K_0(\alpha)(G([p])) &= K_0(\alpha)(g_*([(p_{ij}, 0)])) \\ &= K_0(\alpha)([(p_{ij}, 0)]) \\ &= [(\alpha(p_{ij}), 0)]\end{aligned}$$

meaning the diagram commutes.

The following result will helps us get a clearer picture of the elements in $K_0(A)$. Before proceeding observe that for the scalar map $s : A^+ \to A^+$ we have $s(a, z) = (0, z)$.

Theorem 3.3.3 (Portrait of K_0)**.** *Let A a C^*-algebra. We can see the elements of $K_0(A)$ as differences of the form $[p] - [s(p)]$ with $p \in \mathcal{P}_k(A^+)$ for some k and $p - s(p) \in \mathrm{M}_k(A)$. Moreover, if $\alpha : A \to B$ is a *-homomorphism then*

$$\alpha^*([p] - [s(p)]) = [\alpha^+(p)] - [s(\alpha^+(p))]$$

Proof. Take $x = [p'] - [q'] \in K_0(A)$ with $p', q \in \mathcal{P}_k(A^+)$. Consider the following matrices in $\mathcal{P}_{2k}(A)$

$$p = p' \oplus (1_k - q'), \qquad q = 0_k \oplus 1_k$$

We have

$$\begin{aligned}
[p] - [q] &= ([p'] + [1_k - q']) - [1_k] + 0 \\
&= ([p'] + [1_k - q']) - [1_k] + [q'] - [q'] \\
&= ([p'] + [1_k]) - ([1_k] + [q']) = [p'] - [q']
\end{aligned}$$

Note that $s(q) = q$ and since $[p] - [q] \in K_0(A) = \ker(\pi_*)$ we have

$$\begin{aligned}
[s(p)] - [q] &= [s(p)] - [s(q)] \\
&= \lambda_* \circ \pi_*([p] - [q]) = 0
\end{aligned}$$

so that $[p] - [q] = [p] - [s(p)]$. Clearly $p - s(p) \in \mathrm{M}_k(A)$.

For the second part, consider $\alpha : A \to B$ a *-homomorphism. Writing s for the scalar map in both A and B, we note that $\alpha^+ \circ s(a, z) = (0, z) = s \circ \alpha^+(a, z)$ and therefore

$$\alpha_*([p] - [s(p)]) = [\alpha^+(p)] - [\alpha^+(s(p))] = [\alpha^+(p)] - [s(\alpha^+(p))].$$

□

So far we have not talked much about ways to calculate the K-theory of a given C*-algebra. Generally speaking this is done by using exact sequences relating our algebra with other algebras with known K-theory. To construct these sequences one first needs to show that K_0 is a half-exact functor.

Theorem 3.3.4 (Half-exactness of K_0). *The following exact sequence of C*-algebras*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{3.7}$$

induces a short exact sequence of K_0-groups,

$$K_0(A) \xrightarrow{\alpha_*} K_0(B) \xrightarrow{\beta_*} K_0(C) \tag{3.8}$$

Moreover, if the first sequence splits then α_* is injective and β_* is surjective.

The proof of this theorem requires some lifting properties for unitaries and some lemmas regarding the behavior of β_* when β is surjective. The details can be found in [18, page 66] and [19, page 119]. The following results are also mentioned without proofs and are included just for the sake of completeness. See [19, page 116] and [18, page 97] for a detailed treatment of them.

Proposition 3.3.5 (Continuity of K_0). *Let A be the inductive limit of a directed system $\{A_i, \Phi_{ij}\}_{\mathfrak{I}}$ of C*-algebras, then $\{K_0(A_i), \Phi_{ij*}\}_{\mathfrak{I}}$ is a directed system of abelian groups and*

$$K_0(A) = K_0(\varinjlim A_i) \simeq \varinjlim K_0(A_i).$$

Proposition 3.3.6 (Stability of K_0). *Let A be a C*-algebra, then $K_0(A) \simeq K_0(A \otimes \mathcal{K})$ where the isomorphism is induced by $a \mapsto a \otimes e_{11}$.*

Remark 3.3.7. Here e_{ij} is the finite rank operator $|e_i\rangle\langle e_j|$ where $\{e_i\}_{i\in\mathbb{N}}$ is the canonical base in our separable Hilbert space.

Example 3.3.8. Due to the previous proposition and since $\mathcal{K} \cong \mathbb{C} \otimes \mathcal{K}$ we have $K_0(\mathcal{K}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$.

The previous proposition tells us that two stably equivalent C*-algebras A and B, that means $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, have the same K_0 group. This means that K_0 group is preserved under relationships weaker than isomorphism, it would be interesting to find even weaker relationships that preserve K_0. In Section 1.5 we introduced the notion of Morita-Rieffel equivalence, it is shown in [11, Corollary 4.29] that Morita-Rieffel equivalence is weaker than stable equivalence and hence a candidate to preserve K_0. Exel proved in [8] that this was actually the case. A summarized proof of this theorem can be found in [11, page 165].

Theorem 3.3.9 (Exel, 1993)**.** *Let A and B be two Morita-Rieffel equivalent C*-algebras, then $K_0(A) \cong K_0(B)$.*

3.4 The K_1 group

The K_1 group, similarly to K_0, is built upon matrices over a C*-algebra A. However, its construction is slightly less complicated. Morever, we will see that we can construct K_1 from K_0 using suspension algebras.

Definition 3.4.1. Let A be a unital algebra. If $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say $u \sim_1 v$ if there exists a continuous path w_t in $\mathcal{U}_k(A)$ $(k \geq n, m)$ such that $w_0 = u \oplus 1_{k-n}$ and $w_1 = v \oplus 1_{k-m}$. It is straightforward to show that $\sim_1$ defines an equivalence relation in $\mathcal{U}_\infty(A)$. We will use $[u]_1$ to denote the equivalence class of u in $\mathcal{U}_\infty(A)$.

Remark 3.4.2. We can define a similar equivalence relation in $\mathrm{GL}_\infty(A)$. Given $x \in \mathrm{GL}_n(A)$ and $y \in \mathrm{GL}_m(A)$, we say $x \sim_1 y$ in $\mathrm{GL}_\infty(A)$ if there exists a continuous path of invertibles that connects them.

Remark 3.4.3. In case A is a non-unital C*-algebra, since $\mathcal{U}_n(\mathbb{C})$ is path connected for all n then for any $u \in \mathcal{U}_n(A^+)$ there exists $v \in \mathcal{U}_n(A^+)$ such that $u \sim_1 v$ and $\pi(v) = 1_n$. Consider $w = \pi(u)$, there exists a continuous path $w_t \in \mathcal{U}_n(\mathbb{C})$ such that $w_0 = 1_n$ and $w_1 = w^*$. Now take $v = uw_1 \in \mathcal{U}_n(A^+)$ and consider the path $u_t := uw_t \in \mathcal{U}_n(A^+)$. We have $u_0 = u$, $u_1 = v$ and $\pi(v) = \pi(uw_1) = \pi(u)w_1 = ww^* = 1_n$.

The following proposition allows us to handle $\mathcal{U}_\infty(A)/\sim_1$ as a group, in fact by using invertibles instead of unitaries we obtain the same result for $\mathrm{GL}_\infty(A)/\sim_1$.

Proposition 3.4.4. *Let A be a unital C*-algebra. We define $[u]_1[v]_1 := [u \oplus v]_1$ which makes $\mathcal{U}_\infty(A)/\sim_1$ into an abelian group. Moreover we have $[u]_1[1_n]_1 = [1_n]_1[u]_1 = [u]_1$ for all $u \in \mathcal{U}_\infty(A)$ and if $u, v \in \mathcal{U}_n(A)$ then $[u]_1[v]_1 = [uv]_1$.*

Proof. First of all, consider $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$ then by setting

$$z = \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix} \in \mathcal{U}_{n+m}(A)$$

we have

$$z(u \oplus v)z^* = \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1_n \\ 1_m & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$
$$= \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} = v \oplus u$$
$$z^* z(u \oplus v) = u \oplus v.$$

Since $z(u \oplus v), z^* \in \mathcal{U}_{n+m}$, we can apply lemma (3.1.23) and obtain $v \oplus u = z(u \oplus v)z^* \sim_1 z^* z(u \oplus v) = u \oplus v$. This means that if our product is well defined then it is commutative. Now we verify that the product is well defined. Let $u \in \mathcal{U}_{n_1}(A)$, $u' \in \mathcal{U}_{m_1}$, $v \in \mathcal{U}_{n_2}(A)$ and $v' \in \mathcal{U}_{m_2}(A)$ such that $u \sim_1 u'$ and $v \sim_1 v'$ then we have paths $u_t \in \mathcal{U}_{k_1}$ and $v_t \in \mathcal{U}_{k_2}(A)$ connecting each pair respectively. We consider the following path in $\mathcal{U}_{k_1+k_2}(A)$,

$$w_t = \begin{pmatrix} u_t & 0 \\ 0 & v_t \end{pmatrix}$$

we have

$$u \oplus 1_{k_1-n_1} \oplus v \oplus 1_{k_2-n_2} \sim_1 u' \oplus 1_{k_1-m_1} \oplus v' \oplus 1_{k_2-m_2}$$

$$w_0 = u \oplus 1_{k_1-n_1} \oplus v \oplus 1_{k_2-n_2}, \quad w_1 = u' \oplus 1_{k_1-m_1} \oplus v' \oplus 1_{k_2-m_2}$$

Due to the argument at the beginning, there are continuous paths between w_0 and $u \oplus v \oplus 1_{k_2+k_1-n_1-n_2}$ and between w_1 and $u' \oplus v' \oplus 1_{k_2+k_1-m_1-m_2}$. This means $u \oplus v \sim_1 u' \oplus v'$ and therefore the product is well defined.

It is clear that $[u]_1[1_n]_1 = [u]_1$, we just need to verify that if $u, v \in \mathcal{U}_n(A)$ then $[u]_1[v]_1 = [uv]_1$. Once again we obtain this from lemma (3.1.23), since there exists a path of unitaries between $\left[\begin{smallmatrix} uv & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right]$. □

Example 3.4.5. For any unital C*-algebra A and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ in the unit circle we have $[\mathrm{diag}(\lambda_1 \cdot 1, \ldots, \lambda_n \cdot 1)]_1 = [1_n]_1$. This is because

$$\mathrm{diag}(\lambda_1 \cdot 1, \ldots, \lambda_n \cdot 1) = \lambda_1 \cdot 1 \oplus \cdots \oplus \lambda_n \cdot 1$$

and since the unit circle is connected we have $[\lambda_i \cdot 1]_1 = [1]_1$ for all $i \in \{1, \ldots n\}$. Therefore

$$\begin{aligned}[\mathrm{diag}(\lambda_1 \cdot 1, \ldots, \lambda_n \cdot 1)]_1 &= [\lambda_1 \cdot 1]_1 \cdots [\lambda_n \cdot 1] \\ &= [1]_1 \cdots [1]_1 = [1_n]_1\end{aligned}$$

Using the previous proposition and noting that $[u]_1^{-1} = [u^*]_1$, we conclude that $\mathcal{U}_\infty(A)/\sim_1$ is an abelian group with $[1_n]_1$ as its identity element.

Proposition 3.4.6. *For any unital C*-algebra A, $\mathcal{U}_\infty(A)/\sim_1$ and $\mathrm{GL}_\infty(A)/\sim_1$ are isomorphic as groups.*

Proof. Consider $\phi : \mathrm{GL}_\infty(A) \to \mathcal{U}_\infty(A)$, given by $\phi(x) = x|x|^{-1}$. First we have $x \sim_1 \phi(x)$ in $\mathrm{GL}_\infty(A)$ since by taking

$$x_t = xe^{-t \log |x|}$$

we have $x_0 = x$ and $x_1 = \phi(x)$ (log is well defined for $|x|$ since it is positive, self-adjoint and invertible). From this, if $\phi(x) \sim_1 \phi(y)$ then, in $\mathrm{GL}_\infty(A)$, we have

$$x \sim_1 \phi(x) \sim_1 \phi(y) \sim_1 y. \tag{3.9}$$

This induces a function $\phi_1 : \mathrm{GL}_\infty(A)/\sim_1 \to \mathcal{U}_\infty(A)/\sim_1$, given by $\phi_1([x]_1) = [\phi(x)]_1$. To see that ϕ_1 is well defined take $x_t \in \mathrm{GL}_\infty(A)$ such that $x_0 = x$ and $x_1 = y$ then $x_t|x_t|^{-1}$ is a continuous path of unitaries connecting $\phi(x)$ and $\phi(y)$ ($|x_t|$ is continuous since $x_t^* x_t \geq 0$). Since for any unitary u we have $\phi_1(u) = u$ we have ϕ_1 surjective and due to (3.9) it is injective, moreover we have that ϕ_1^{-1} will coincide with the inclusion of $\mathcal{U}_\infty(A)/\sim_1$ into $\mathrm{GL}_\infty(A)/\sim_1$.

Now we verify that ϕ_1 is a group homomorphism. Take $x, y \in \mathrm{GL}_\infty(A)$ and consider the paths $v_t = x|x|^{-t}y|y|^{-t}$, with $|z|^{-t} := e^{-t\log|z|}$, and $u_t := v_t|v_t|^{-1}$. We have $u_t \in \mathcal{U}_\infty(A)$, $u_0 = xy|xy|^{-1} = \phi(xy)$ and $u_1 = x|x|^{-1}y|y|^{-1} = \phi(x)\phi(y)$, which means

$$\phi_1([xy]_1) = \phi_1([x]_1)\phi_1([y]_1).$$

□

Definition 3.4.7. If A is a C*-algebra, we define the group

$$K_1(A) := \mathcal{U}_\infty(A^+)/\sim_1 .$$

Proposition 3.4.8. *Whenever A is unital then*

$$K_1(A) \cong \mathcal{U}_\infty(A)/\sim_1 .$$

Proof. Recall that when A is unital then $A^+ \cong A \oplus \mathbb{C}$ as C*-algebras, we can take $A^+ = A \oplus \mathbb{C}f$ with $f = 1_{A^+} - 1_A$. Note that $A \cdot f = f \cdot A = \{0\}$, which means that if $x \in \mathrm{M}_n(A)$ and $y \in \mathrm{M}_n(\mathbb{C}f)$ then $xy = yx = 0$. From the unital *-homomorphism $\mu : A^+ \to A$ given by $\mu(a + \lambda f) = a$ we induce a morphism in the unitaries $\mu : \mathcal{U}_\infty(A^+) \to \mathcal{U}_\infty(A)$. To see that it induces a group homomorphism in the quotient space we need to verify that if $u \sim_1 v$ then $\mu(u) \sim_1 \mu(v)$. This is straightforward since μ can carry any path of unitaries in $\mathcal{U}_\infty(A^+)$ to a path of unitaries in $\mathcal{U}_\infty(A)$. Moreover, from the definition of μ it is clear that

$\mu(u \oplus v) = \mu(u) \oplus \mu(v)$ for all $u, v \in \mathcal{U}_\infty(A^+)$. Hence we have a well defined group homomorphism $\mu_* : \mathcal{U}_\infty(A^+)/\sim_1 \to \mathcal{U}_\infty(A)/\sim_1$.

The fact that μ_* is surjective is immediate as for any unitary matrix $u \in \mathcal{U}_n(A)$ we have $\mu(u + f \cdot 1_n) = u$ where $u + f \cdot 1_n \in \mathcal{U}_\infty(A^+)$. To verify it is injective take $\mu(u) \sim_1 \mu(v)$, since $u - \mu(u), v - \mu(v) \in \mathcal{U}_\infty(\mathbb{C}f)$ and the latter being path connected, we have $u - \mu(u) \sim_1 v - \mu(v)$ in $\mathcal{U}_\infty(\mathbb{C}f)$. Take a_t the path in $\mathcal{U}_\infty(A)$ that connects $\mu(u)$ with $\mu(v)$ and b_t the path in $\mathcal{U}_\infty(\mathbb{C})f$ that connects $u - \mu(u)$ with $v - \mu(v)$. Since $a_t \in \mathrm{M}_\infty(A)$ and $b_t \in \mathrm{M}_\infty(\mathbb{C}f)$ we have

$$\begin{aligned}
(a_t + b_t)^*(a_t + b_t) &= a_t^* a_t + a_t^* b_t + b_t^* a_t + b_t^* b_t \\
&= 1_{\mathrm{M}_n(A)} + 0 + 0 + 1_{\mathrm{M}_n(\mathbb{C}f)} \\
&= 1_{\mathrm{M}_n(A^+)}, \\
(a_t + b_t)(a_t + b_t)^* &= a_t a_t^* + a_t b_t^* + b_t a_t^* + b_t b_t^* \\
&= 1_{\mathrm{M}_n(A)} + 0 + 0 + 1_{\mathrm{M}_n(\mathbb{C}f)} \\
&= 1_{\mathrm{M}_n(A^+)}
\end{aligned}$$

and therefore $a_t + b_t$ is a path in $\mathcal{U}_\infty(A^+)$ connecting u with v, in other words $u \sim_1 v$. □

3.5 K_1 as a functor

Similarly to K_0, K_1 also defines a continuous half-exact covariant functor.

Proposition 3.5.1. *K_1 is a covariant functor from the category of C^*-algebras to the category of abelian groups. Where, given a *-homomorphism $\alpha : A \to B$ and $u \in \mathcal{U}_\infty(A^+)$ we have $\alpha_* : K_1(A) \to K_1(B)$ given by*

$$\alpha_*([u]_1) = K_1(\alpha)([u]_1) = [\alpha^+(u)]_1.$$

Proof. First, if $u \in \mathcal{U}_\infty(A^+)$ then

$$\alpha^+(u)\alpha^+(u^*) = \alpha^+(uu^*) = \alpha^+(1) = 1$$

and therefore $\alpha^+(u) \in \mathcal{U}_\infty(B^+)$. Now if u_t is the path in $\mathcal{U}_\infty(A^+)$ connecting u and u', then $\alpha^+(u_t)$ is a path between $\alpha^+(u)$ and $\alpha^+(u')$ in $\mathcal{U}_\infty(B^+)$. hence α_* is well defined.

Let $\alpha : A \to B$ and $\beta : B \to C$, we verify $(\beta\circ\alpha)_* = \beta_*\circ\alpha_*$:

$$\begin{aligned}(\beta \circ \alpha)_*([u]_1) &= [(\beta \circ \alpha)(u)]_1 \\ &= \beta_*([\alpha(u)]_1) = \beta_* \circ \alpha_*([u]_1),\end{aligned}$$

moreover $\mathrm{id}_A : A \to A$, $\mathrm{id}_{A*}([u]_1) = [u]_1$. □

Using the identification from Proposition 3.4.8 we have the following picture for induced morphism between unital C*-algebras:

Proposition 3.5.2. *Let A and B be unital C^*-algebras and $\alpha : A \to B$ a *-homomorphism, then for any $u \in \mathcal{U}_n(A)$ we have*

$$\mu_* \circ \alpha_* \circ (\mu_*)^{-1}([u]_1) = [\alpha(u - 1_n) + 1_n]_1 \tag{3.10}$$

where μ_* is the morphism from Proposition 3.4.8.

Proof. Take $u \in \mathcal{U}_n(A)$, then $(\mu_*)^{-1}([u]_1) = [u + f \cdot 1_n]_1$ hence

$$\begin{aligned}\alpha_* \circ (\mu_*)^{-1}([u]_1) &= [\alpha^+(u + f \cdot 1_n)]_1 \\ &= [\alpha(u - 1_n) + 1_{B^+} \cdot 1_n]\end{aligned}$$

composing with μ_* we obtain

$$\begin{aligned}\mu_* \circ \alpha_* \circ (\mu_*)^{-1}([u]_1) &= \mu_*([\alpha(u - 1_n) + 1_{B^+} \cdot 1_n]_1) \\ &= \mu_*([\alpha(u - 1_n) + 1_B \cdot 1_n + f \cdot 1_n]_1) \\ &= [\alpha(u - 1_n) + 1_B \cdot 1_n]_1 \\ &= [\alpha(u - 1_n) + 1_n]_1.\end{aligned}$$

□

Remark 3.5.3. When it is clear that we are using the identification from Proposition 3.4.8, we will omit μ_* in (3.10) when computing the induced morphism. That is, we will simply write $\alpha_*([u]_1) = [\alpha(u - 1_n) + 1_n]_1$.

Proposition 3.5.4. *If A and B are two unital C^*-algebras and $\alpha : A \to B$ is a *-homomorphism then, using the identification from Proposition 3.4.8, for any $u \in \mathcal{U}_n(A)$ we have*

$$\alpha_*([u]_1) = [\alpha(u - 1_n) + 1_n]_1 = [\alpha(u - z \cdot 1_n) + z \cdot 1_n]_1$$

for any $z \in \mathbb{T}$. In particular, taking $z = -1$ we have

$$\alpha_*([u]_1) = [\alpha(u - 1_n) + 1_n]_1 = [\alpha(u + 1_n) - 1_n]_1$$

Proof. First we show that $\alpha(u - z \cdot 1_n) + z \cdot 1_n$ is unitary for all $z \in \mathbb{T}$:

$$\begin{aligned}&(\alpha(u - z \cdot 1_n) + z \cdot 1_n)^*(\alpha(u - z \cdot 1_n) + z \cdot 1_n) = \\ &\quad = (\alpha(u^* - \overline{z} \cdot 1_n) + \overline{z} \cdot 1_n)(\alpha(u - z \cdot 1_n) + z \cdot 1_n) \\ &\quad = \alpha(u^*u - \overline{z}u - zu^* + 1_n) + z\alpha(u^* - \overline{z} \cdot 1_n) + \overline{z}\alpha(u - z \cdot 1_n) + 1_n \\ &\quad = \alpha(1_n - \overline{z}u - zu^* + 1_n + zu^* - 1_n + \overline{z}u - 1_n) + 1_n \\ &\quad = \alpha(0) + 1_n = 1_n.\end{aligned}$$

Similarly we verify that the other product is the identity and therefore $\alpha(u - z \cdot 1_n) + z \cdot 1_n$ is unitary. Now, since $\mathbb{T}$ is connected and $1 \in \mathbb{T}$ we have

$$[\alpha(u - 1_n) + 1_n]_1 = [\alpha(u - z \cdot 1_n) + z \cdot 1_n]_1$$

for all $z \in \mathbb{T}$. □

Proposition 3.5.5 (Half-exactness of K_1)**.** *If we are given the following exact sequence,*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{3.11}$$

then the following diagram is exact,

$$K_1(A) \xrightarrow{\alpha_*} K_1(B) \xrightarrow{\beta_*} K_1(C) \tag{3.12}$$

Moreover, if the first sequence splits then α_* is injective and β_* is surjective.

Proof. By functoriality we have $\alpha_*(K_1(A)) \subset \ker \beta_*$. Let $[u]_1 \in \ker \beta_*$ with $u \in \mathcal{U}_n(B^+)$ then $\beta^+(u \oplus 1_m) = \beta^+(u) \oplus 1_m \sim_1 1_{n+m}$ and by (3.1.25) we have $\beta^+(u \oplus 1_m) = e^{ih_1} \cdots e^{ih_n}$ with h_j self-adjoint. Since β^+ is surjective, $h_i = \beta^+(g_i)$ where we can choose $g_i \in \mathrm{M}_{n+m}(B^+_{sa})$. Now if $v = e^{ig_1} \cdots e^{ig_n}$ we have $\beta^+(v) = \beta^+(u \oplus 1_m)$ and $v \sim_1 1$, hence $\beta^+((u \oplus 1_m)v^*) = 1_{n+m}$. Since $\ker \beta = \alpha(A)$ and $\beta^+((u \oplus 1_m)v^*)$ has no non-zero entries on C, there exists $w \in A^+$ such that $\alpha^+(w) = (u \oplus 1_m)v^*$. Finally $\alpha_*([w]) = [(u \oplus 1_m)v^*]_1 = [(u \oplus 1_m)]_1[v^*]_1 = [u]_1$.

As with K_0, suppose the first sequence splits, this means that there exist *-homomorphisms $\phi : C \to B$ and $\psi : B \to A$ such that $\phi \circ \beta = \mathrm{id}_A$ and $\psi \circ \alpha = \mathrm{id}_C$. By functoriality, β_* is surjective and α_* injective. □

Example 3.5.6. Since $\mathcal{U}_n(\mathbb{C})$ is path connected for all $n \in \mathbb{N}$, we have $K_1(\mathbb{C}) = 0$. Moreover, from the short exact sequence that splits

$$0 \longrightarrow A \longrightarrow A^+ \longrightarrow \mathbb{C} \longrightarrow 0$$

due to the previous proposition we have $K_1(A) \cong K_1(A^+)$.

Once again we mention without proof the results for continuity and stability for K_1. The details can be found in [18, page 142] and [19, page 133].

Proposition 3.5.7 (Continuity of K_1)**.** *Let A be the direct limit of the directed system of C^*-algebras $\{A_i, \Phi_{ij}\}_{\mathfrak{I}}$, then $\{K_1(A_i), \Phi_{ij_*}\}_{\mathfrak{I}}$ is a directed system of abelian groups and*

$$K_1(A) = K_1(\varinjlim A_i) \simeq K_1(A_i)$$

Proposition 3.5.8 (Stability de K_0)**.** *Let A a C^*-algebra, then $K_1(A) \simeq K_1(\mathrm{M}_n(A))$ for all $n \in \mathbb{N}$ and $K_1(A) \simeq K_1(A \otimes \mathcal{K})$. The isomorphism are induced by $a \mapsto a \oplus 0_{n-1}$ and $a \mapsto a \otimes e_{11}$ respectively.*

We know from Proposition 3.1.10 that corner algebras generated by equivalent projections are isomorphic and therefore they have the same K-theory. However we have not studied how does the K-theory of a corner algebra relates with the K-theory of the algebra as a whole. If $p \in \mathcal{P}(M(A))$ is a *full* projection on a separable C*-algebra A, that is $\overline{\mathrm{span}} ApA = A$, then we can show that the inclusion $\iota : pAp \to A$ induces an isomorphism for the K_1 groups. To show this we first need a result by Brown found in [5, Lemma 2.5]:

Lemma 3.5.9 (Brown, 1977)**.** *If A is a C^*-algebra with an strictly positive element and $p \in \mathcal{P}(\mathcal{M}(A))$ a full projection on A then there is an isometry $v \in M(A \otimes \mathcal{K})$ such that $vv^* = p \otimes 1 \in M(\in A \otimes \mathcal{K})$.*

Remark 3.5.10. Here we are using the fact that $M(A) \otimes M(\mathcal{K}) = M(A) \otimes B(\mathcal{H})$ can be seen as a subalgebra of $M(A \otimes \mathcal{K})$ and $p \otimes 1 \in M(A \otimes \mathcal{K})$ is understood as the corresponding element in $M(A) \otimes B(\mathcal{H})$.

Note that all separable non-zero C*-algebras have a strictly positive element and therefore the lemma above can be applied to them. The following proposition can also be found in [16, 1.2]:

Proposition 3.5.11. *Let p be a full projection on a separable C^*-algebra A, then the inclusion induces an isomorphism between $K_1(pAp)$ and $K_1(A)$.*

Proof. By the previous lemma we have an isometry $v \in M(A \otimes \mathcal{K})$ such that $vv^* = p \otimes 1$. Now consider the following diagram:

$$\begin{array}{ccc} pAp & \xrightarrow{\quad\iota\quad} & A \\ \Big\downarrow{\scriptstyle \cdot\otimes e_{11}} & & \Big\downarrow{\scriptstyle \cdot\otimes e_{11}} \\ pAp \otimes \mathcal{K} & \xrightarrow{\iota\otimes\mathrm{id}} & A \otimes \mathcal{K} \end{array}.$$

By Proposition 3.5.8 we know the vertical morphisms induce isomorphisms for the K_1 groups. We will show that $\iota \otimes \mathrm{id}$ also induces an isomorphism for the K_1 groups and therefore $\iota_* : K_1(pAp) \to K_1(A)$ is an isomorphism. First we identify $pAp \otimes \mathcal{K}$ with $(p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1)$, under this identification $\iota \otimes \mathrm{id}$ looks as the inclusion. Due to Proposition 3.1.10 we have the isomorphism $\mathrm{Ad}(v^*) : (p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1) \to A \otimes \mathcal{K}$. Lets verify that $K_1(\mathrm{Ad}(v^*)) = K_1(\iota \otimes \mathrm{id})$, take $u \in \mathcal{U}_n(((p \otimes 1)(A \otimes \mathcal{K})(p \otimes$

$1))^+)$ such that $\pi(u) = 1_n$. This means we can write $u = w + 1_n$ where $w \in \mathrm{M}_n((p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1))$. Since

$$\begin{aligned} uu^* &= u^*u \\ (w + 1_n)(w^* + 1_n) &= (w^* + 1_n)(w + 1_n) \\ ww^* + w + w^* + 1_n &= w^*w + w + w^* + 1_n \end{aligned}$$

we have that w is normal and therefore we can use continuous functional calculus. This means we can factor $w = ab$ with $a, b \in C^*(w) \subset \mathrm{M}_n((p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1))$. Taking $x = b(v \cdot 1_n)$ and $y = (v^* \cdot 1_n)a$ we have $x, y \in \mathrm{M}_n(A \otimes \mathcal{K})$, since v is a multiplier of $A \otimes \mathcal{K}$. Moreover, since $C^*(w)$ is commutative we have

$$\begin{aligned} xy &= b(v \cdot 1_n)(v^* \cdot 1_n)a \\ &= b(vv^* \cdot 1_n)a \\ &= b((p \otimes 1) \cdot 1_n)a \end{aligned}$$

but $(p \otimes 1)$ is the unit in $(p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1)$, therefore

$$= ba = ab = w.$$

Clearly we also have $yx = (v^* \cdot 1_n)w(v \cdot 1_n) = \mathrm{Ad}(v^*)(w)$. Now we use the following decomposition for $(xy + 1_n) \oplus 1_n$:

$$\begin{pmatrix} xy + 1_n & 0_n \\ 0_n & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & -x \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} 1_n & x \\ -y & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ y & 1_n \end{pmatrix}. \tag{3.13}$$

By Lemma 3.1.24 this is homotopic in $\mathrm{GL}_{2n}(A \otimes \mathcal{K})$ to

$$\begin{pmatrix} 1_n & x \\ -y & 1_n \end{pmatrix}.$$

Similarly we obtain that $(yx + 1_n) \oplus 1_n$ is homotopic to

$$\begin{pmatrix} 1_n & -y \\ x & 1_n \end{pmatrix}$$

But these last two can easily be seen to be homotopic, therefore $(xy + 1_n) \oplus 1_n$ is homotopic to $(yx + 1_n) \oplus 1_n$ in $\mathrm{GL}_{2n}(A \otimes \mathcal{K})$. Due to Remark 3.1.18 this means that they are homotopic in $\mathcal{U}_{2n}(A \otimes \mathcal{K})$, therefore

$$\begin{aligned} [u]_1 = [w + 1_n]_1 = [xy + 1_n]_1 = [yx + 1_n]_1 \\ = [\mathrm{Ad}(v^*)(w) + 1_n]_1 = [\mathrm{Ad}(v^*)^+(u)]_1. \end{aligned}$$

We can conclude that $K_1(\mathrm{Ad}(v^*)) = K_1(\iota \otimes \mathrm{id})$ and therefore $K_1(\iota) = \iota_*$ is an isomorphism between $K_1(pAp)$ and $K_1(A)$. □

3.6 The six-term exact sequence

In this section we will provide one of the main tools used to compute K-theory. First we will introduce the index map, then the suspension algebras, look at their role in the Bott periodicity theorem and finally look into the six-term exact sequence they allow us to build. Many of the results in this section will be presented without a proof as most proofs are rather technical and will require many results regarding unitaries and projections which we omitted in the first section.

In essence, the six-term exact sequence gives us two homomorphism connecting the K_0 groups of an exact sequence with its K_1 groups. The first of these homomorphims is the index map. In a way the index map is a generalization of the Fredholm index defined for Fredholm operators on a Hilbert space.

Theorem 3.6.1 (The index map)**.** *Given an exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad (3.14)$$

there exists a group homomorphism $\delta_1 : K_1(C) \to K_0(A)$ *such the following diagram is exact:*

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{\alpha_*} & K_1(B) & \xrightarrow{\beta_*} & K_1(C) \\ & & & & \downarrow{\delta_1} \\ K_0(C) & \xleftarrow[\beta_*]{} & K_0(B) & \xleftarrow[\alpha_*]{} & K_0(A) \end{array} . \tag{3.15}$$

The complete proof of this theorem can be found throughout [18, Chapter 9]. We will only provide the definition of the index map.

Proof. Take $u \in \mathcal{U}_n(C^+)$ such that $\pi(u) = 1_n$, since $u \oplus u^* \sim_1 1_{2n}$ by 3.1.25 we have $u \oplus u^* = e^{ih_1} \cdots e^{ih_m}$ where $h_i \in \mathrm{M}_{2n}(C^+)_{sa}$. Since β is surjective (and consequently so is β^+) there exists $k_i \in \mathrm{M}_{2n}(B^+)_{sa}$ such that $\beta^+(k_i) = h_i$. Taking $w = e^{ik_1} \cdots e^{ik_m}$, it is straightforward to see $\beta^+(w) = u \oplus u^*$. For $p_n = 1_n \oplus 0_n$, we have $\beta^+(wp_nw^*) = p_n$ and therefore $\beta^+(wp_nw^* - p_n) = 0$. Since (3.14) is exact, there exists $q \in \mathrm{M}_{2n}(A)$ such that $\alpha(q) = wp_nw^* - p_n$. Taking $p = q + p_n \in \mathrm{M}_{2n}(A^+)$, we prove that p is a projection. First, given that α is injective and

$$\begin{aligned} \alpha(q^2) &= (wp_nw^* - p_n)(wp_nw^* - p_n) \\ &= wp_nw^* - p_nwp_nw^* - wp_nw^*p_n + p_n \\ &= wp_nw^* - p_n + p_n(p_n - wp_nw^*) + (p_n - wp_nw^*)p_n \\ &= \alpha(q) - \alpha(p_nq) - \alpha(qp_n) = \alpha(q - p_nq - qp_n) \end{aligned}$$

we have $q^2 = q - p_nq - qp_n$, but

$$\begin{aligned} p^2 &= (q + p_n)(q + p_n) \\ &= q^2 + p_nq + qp_n + p_n \\ &= q - p_nq - qp_n + p_nq + qp_n + p_n \end{aligned}$$

$$= q + p_n = p$$

hence p is idempotent. Similarly one proves it is self-adjoint and therefore a projection. Defining $\delta([u]_1) := [p] - [s(p)] \in K_0(A)$, it can be proved this is a well defined group homomorphism between $K_1(C)$ and $K_0(A)$ that makes (3.15) exact. □

Example 3.6.2. Let $\mathcal{H}$ be a Hilbert space and consider the well known exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B(\mathcal{H}) \longrightarrow B(\mathcal{H})/\mathcal{K} \longrightarrow 0 \ .$$

Recall that an operator $T \in B(\mathcal{H})$ is Fredholm if and only if it is invertible in the Calkin algebra $B(\mathcal{H})/\mathcal{K}$. Since all invertibles induce a unitary by taking $U = T|T|^{-1}$, we have a morphism from the Fredholm operators to $\mathcal{U}(B(\mathcal{H})/\mathcal{K})$. From Example 3.3.8 we have $K_0(\mathcal{K}) = \mathbb{Z}$, so that $\delta_1([U]_1) \in \mathbb{Z}$. In [18, page 166] it is shown that $\delta_1([U]_1)$ actually coincides with the Fredholm index of T. This means that our index map is indeed a generalization of the Fredholm index.

We will use the index map to construct a long exact sequence, to do this we introduce the higher K-groups. Given a C*-algebra A we can always construct the suspension algebra given by

$$SA = \{f : [0,1] \to A;\ f(0) = 0 = f(1)\}.$$

Some quick observations regarding the suspension are that $SA \cong C_0(\mathbb{R}, A) \cong A \otimes C_0(\mathbb{R})$ and $\mathrm{M}_n(SA) \cong S(\mathrm{M}_n(A))$ for any $n \in \mathbb{N}$. Moreover, the unitization $(SA)^+$ can be identified with the following algebra:

$$\{f : [0,1] \to A^+;\ f \text{ continuous},$$
$$f(1) = f(0) = (0,z),\ f(t) = (a_t, z)\,\forall t \in [0,1], \text{for some } z \in \mathbb{C}\}.$$

In fact, if $\alpha : A \to B$ is a *-homomorphism and we define $S(\alpha)(f) := \alpha \circ f$ then S defines a functor from the category of C*-algebras to itself.

Remark 3.6.3. It should be no surprise that the functor S preserves Morita-Rieffel equivalence. If $A \overset{\text{MR}}{\sim} B$ then there exists a Hilbert A-B bimodule E and a Hilbert B-A bimodule F such that $E \otimes_B F \cong A$ and $F \otimes_A E \cong B$. By taking $E' = E \otimes_{\mathbb{C}} C_0(\mathbb{R})$ with the natural A-B-bimodule structure it can be shown that $E' \otimes_B F \cong SA$ and $F \otimes E' \cong SB$, meaning $SA \overset{\text{MR}}{\sim} SB$.

Lemma 3.6.4. *Let A be a C*-algebra. The set*

$$\text{span}\{f \cdot a \in SA :\ f \in S\mathbb{C}, a \in A\}$$

is dense in SA, where $(f \cdot a)(t) := f(t)a$.

Proof. Let $g \in SA$, since $[0,1]$ is compact then g is uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|t-s\| < \delta$ then $\|g(t) - g(s)\| < \varepsilon$. Consider $N > 0$ such that $1/N < \delta/2$ and take $U_n = (n/N - \delta/2, n/N + \delta/2)$ for $n = 0, \dots, N$, $\{U_n\}_{n=0}^N$ is an open cover for $[0,1]$. Now we set $t_n = n/N \in U_n$. There exists functions $f_0, \dots, f_N \in C([0,1])$ such that $0 \le f_n \le 1$, $f_0 + \cdots + f_N = 1$ and each f_n is 0 outside U_n (essentially a partition of unity, although in our case it can be constructed manually). Since the diameter of each U_n is less than δ we have

$$\|g(t) - g(t_n)\| < \varepsilon \quad , t \in U_n$$

and therefore $\|f_n(t) \cdot g(t) - f_n(t) \cdot g(t_n)\| < \varepsilon f_n(t)$ for all $t \in [0,1]$. Adding all of the inequalities we obtain

$$\left\| g(t) - \sum_{n=1}^{N} f_n(t) \cdot g(t_n) \right\| < \varepsilon$$

Taking $a_n = g(t_n)$ we obtain $\|g - \sum_{n=1}^{N} f_n \cdot a_n\| < \varepsilon$. Since $a_0 = g(0) = 0$ and $a_N = g(1) = 0$ we have $\sum_{n=1}^{N} f_n \cdot a_n \in SA$. □

Proposition 3.6.5. *The functor S is exact.*

Proof. Consider the exact sequence at (3.14), we need to show that the sequence

$$0 \longrightarrow SA \xrightarrow{S(\alpha)} SB \xrightarrow{S(\beta)} SC \longrightarrow 0$$

is also exact. Clearly $S(\alpha)$ is injective. Moreover, since

$$S(\beta)S(\alpha)(f) = \beta(\alpha(f)) = 0,$$

we have $S(\beta) \circ S(\alpha) = 0$. Take $g \in SB$ with $S\beta(g) = 0$, this means $\beta(g(t)) = 0$ for all $t \in [0,1]$. Due to the exactness of (3.14), for all $t \in [0,1]$ there exists a unique $a_t \in A$ such that $\alpha(a_t) = g(t)$. Since α is injective, it is isometric and therefore

$$\|a_t - a_s\| = \|\alpha(a_t - a_s)\| = \|g(t) - g(s)\|$$

so that $t \mapsto a_t$ is continuous. Taking $f(t) = a_t$ we have $S(\alpha)(f) = g$. Finally, if $f \in S\mathbb{C}$ and $b \in B$ then $S(\beta)(f \cdot b) = f \cdot \beta(b)$. Since β is surjective we have $f \cdot c \in S(\beta)(SB)$ for all $f \in S\mathbb{C}$ and $c \in \mathbb{C}$, due to the previous lemma. we obtain $SC = S(\beta)(SB)$ □

Something remarkable about this new functor is the following:

Theorem 3.6.6. *The functors K_1 and $K_0 \circ S$ are naturally isomorphic. More precisely, for each C^*-algebra A there exists an isomorphism $\theta_A : K_1(A) \to K_0(SA)$ such that for any *-homomorphism $\alpha : A \to B$ the following diagram commutes:*

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\alpha_*} & K_1(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ K_0(SA) & \xrightarrow{S(\alpha)_*} & K_0(SB). \end{array} \tag{3.16}$$

The proof of this theorem is rather technical and can be found in [19, page 138], in [18, page 177] there is another approach using the index map and the fact that K_0 is homotopy invariant (a property we have not talked about).

Corollary 3.6.7. *If A and B are two Morita-Rieffel equivalent C^*-algebras then $K_1(A) \cong K_1(B)$.*

Proof. From Remark 3.6.3 we have $SA \overset{\text{MR}}{\sim} SB$ and by Theorem 3.3.9 we obtain $K_0(SA) \cong K_0(SB)$. Finally, due to the previous theorem:

$$K_1(A) \cong K_0(SA) \cong K_0(SB) \cong K_1(B).$$

□

The previous theorem allows us to define the higher K-groups by

$$K_n(A) := K_0(S^n A), \ n \in \mathbb{N}$$

where $S^n A$ is understood as the functor S applied n-times to A. Theorem 3.6.6 tells us that this new definition of K_1 coincides with the previous one. Also note that since S is exact and K_0 is half-exact then K_n is half-exact. Applying the functor S^n to the exact sequence (3.14) we obtain a new exact sequence:

$$0 \longrightarrow S^n A \xrightarrow{S^n(\alpha)} S^n B \xrightarrow{S^n(\beta)} S^n C \longrightarrow 0$$

which induces the exact sequence:

$$\begin{array}{ccccc} K_1(S^n A) & \xrightarrow{\alpha_*} & K_1(S^n B) & \xrightarrow{\beta_*} & K_1(S^n C) \\ & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(S^n C) & \xleftarrow[\beta_*]{} & K_0(S^n B) & \xleftarrow[\alpha_*]{} & K_0(S^n A) \end{array} \tag{3.17}$$

using the definition of higher K-groups and defining

$$\delta_{n+1} := \delta_1 : K_1(S^n C) \to K_0(S^n A)$$

we have

$$\begin{array}{ccccc} K_{n+1}(A) & \xrightarrow{\alpha_*} & K_{n+1}(B) & \xrightarrow{\beta_*} & K_{n+1}(C) \\ & & & & \downarrow{\scriptstyle \delta_{n+1}} \\ K_n(C) & \xleftarrow[\beta_*]{} & K_n(B) & \xleftarrow[\alpha_*]{} & K_n(A) \end{array} . \tag{3.18}$$

Joining the sequences 3.18 for all $n \in \mathbb{N}$ we have the following long exact sequence:

$$\begin{array}{l} \cdots \longrightarrow K_n(A) \longrightarrow K_n(B) \longrightarrow K_n(C) \longrightarrow \\ \longrightarrow K_{n-1}(A) \longrightarrow K_{n-1}(B) \longrightarrow K_{n-1}(C) \longrightarrow \\ \longrightarrow \cdots \qquad\qquad \cdots \longrightarrow \\ \longrightarrow K_0(A) \longrightarrow K_0(B) \longrightarrow K_0(C) \end{array} . \tag{3.19}$$

The only missing piece to turn the sequence above into a six-term exact sequence is Bott periodicity.

Theorem 3.6.8 (Bott periodicity)**.** *There exists a natural isomorphism between K_0 and K_1S. This means that for every C*-algebra A there exists a group isomorphism $\beta_A : K_0(A) \to K_1(SA)$ such that for any *-homomorphism $\alpha : A \to B$ the following diagram commutes*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\alpha_*} & K_0(B) \\ \downarrow{\scriptstyle \beta_A} & & \downarrow{\scriptstyle \beta_B} \\ K_1(SA) & \xrightarrow{S(\alpha)_*} & K_1(SB) \end{array} . \tag{3.20}$$

In the literature β_A is often called the Bott map.

There are many proofs of this theorem, the original one attributed to Atiyah. Atiyah's proof (originally meant for a commutative context) uses pretty elementary objects but is long and complicated at times. A thorough look at this proof can be found in [19, chapter 9] and [18, chapter 11]. Cuntz also provided another proof of the theorem using the machinery of Bott functors, this can be found in [19, page 190], this proof is more abstract and heavily relies on the non-commutative nature of the Toeplitz algebra.

Applying Bott periodicity to (3.19) and the identification $K_2 = K_0S^2$ we obtain the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\alpha_*} & K_0(B) & \xrightarrow{\beta_*} & K_0(C) \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(C) & \xleftarrow[\beta_*]{} & K_1(B) & \xleftarrow[\alpha_*]{} & K_1(A) \end{array} \tag{3.21}$$

where $\delta_0 := \delta_2 \circ \beta_C$. This is called the **induced six-term induced sequence**.

Lemma 3.6.9. *For any C^*-algebra A, $C([0,1],\mathcal{M}(A))$ can be identified with a subalgebra of $\mathcal{M}(SA)$.*

Proof. We only need to show that $C([0,1],\mathcal{M}(A))$ contains SA as an essential ideal. Clearly $SA \subset C([0,1],\mathcal{M}(A))$, now take $f \in C([0,1],\mathcal{M}(A))$ such that $fSA = 0$. Consider $g(t) = e^{-t^2}$ and $a \in A$ then $f(g \cdot a) = 0$, therefore for all $t \in [0,1]$ we have

$$f(g \cdot a)(t) = 0$$
$$f(t)g(t)a = 0$$

but $g(t) > 0$, therefore

$$f(t)a = 0$$

meaning $f(t)A = 0$ for all $t \in [0,1]$, since A is an essential ideal in $\mathcal{M}(A)$ we have $f = 0$. We conclude that SA is an essential ideal in $C([0,1],\mathcal{M}(A))$, by the universal property of the multiplier algebra it can be identified with a subalgebra of $\mathcal{M}(SA)$. □

Corollary 3.6.10. *If p is a full projection in a separable C^*-algebra A, then the inclusion induces an isomorphism between $K_0(pAp)$ and $K_0(A)$.*

Proof. Let p be a full projection in A and define $\tilde{p} \in C([0,1],\mathcal{M}(A))$ by $\tilde{p}(t) = p$ for all $t \in [0,1]$. Due to the previous lemma $\tilde{p}$ can be considered as a projection in $\mathcal{M}(SA)$. We proceed to show that $\tilde{p}$ is a full projection in SA and that $S(pAp) = \tilde{p}SA\tilde{p}$. Take $a \in A$ and $f \in S\mathbb{C}$, since p is full there exists $a_n, b_n \in A$ such that

$$a = \lim_{N\to\infty} \sum_{n=1}^{N} a_n p b_n.$$

Consider $h = |f|^{1/2}$ and $g = f/h$, both are well defined functions in $S\mathbb{C}$ that satisfy $f = gh$ (any discontinuity f/h might present can be removed by defining the function at that point as 0). We now have

$$\begin{aligned} f \cdot a &= f \cdot \lim_{N\to\infty} \sum_{n=1}^{N} a_n p b_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} f \cdot a_n p b_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} (g \cdot a_n)\tilde{p}(h \cdot b_n) \in \overline{\operatorname{span} SA\,\tilde{p}\,SA} \end{aligned}$$

Due to Lemma 3.6.4 this implies that $\tilde{p}$ is a full projection. The second statement is easily obtained from the fact that for any $f \in S\mathbb{C}$ and $a \in A$ we have

$$f \cdot (pap) = \tilde{p}(f \cdot a)\tilde{p}.$$

We can now apply Proposition 3.5.11 to SA and obtain the following isomorphism

$$\iota_* : K_1(S(pAp)) = K_1(\tilde{p}SA\tilde{p}) \to K_1(SA)$$

Now, consider the inclusion $\iota : pAp \to A$, for any $f \in S\mathbb{C}$ and $a \in A$ we have

$$\begin{aligned} S(\iota)(\tilde{p}(f \cdot a)\tilde{p}) &= \iota \circ (\tilde{p}(f \cdot a)\tilde{p}) \\ &= \tilde{p}(f \cdot a)\tilde{p}. \end{aligned}$$

Which shows that $S(\iota)$ coincides with the inclusion of $\tilde{p}SA\tilde{p}$ in SA and therefore $S(\iota)_*$ is an isomorphism. By Bott periodicity we have the following commutative diagram

$$\begin{array}{ccc} K_0(pAp) & \xrightarrow{\iota_*} & K_0(A) \\ \downarrow{\scriptstyle \beta_{pAp}} & & \downarrow{\scriptstyle \beta_A} \\ K_1(S(pAp)) & \xrightarrow{S(\iota)_*} & K_1(SA) \end{array}$$

where the vertical and the bottom arrows are isomorphisms, therefore so is $\iota_* : K_0(pAp) \to K_0(A)$. $\square$

4 K-theory for Cuntz-Pimsner algebras

In this chapter we will present a six-term exact sequence that will allow us to compute the K-theory for some Cuntz-Pimsner algebras. This sequence is essentially the same sequence presented by Pimsner in [17] although with different morphisms that are easier to compute. Our results are based on the sequence introduced in [9] which is defined for Hilbert modules with a finite orthonormal Parseval frame. We show that the hypothesis of orthonormality is not needed. Throughout this chapter B will be a separable unital C*-algebra and (E, φ) will be a B-B correspondence where φ is unital and faithful. The left action of B on E will be denoted by $a \cdot x := \varphi(a)(x)$ for $a \in B$ and $x \in E$.

4.1 Parseval frames and Cuntz-Pimsner algebras

To obtain our desired six-term exact sequence we use the six-term exact sequence induced from

$$0 \longrightarrow J_E \longrightarrow \mathcal{T}_E \longrightarrow \mathcal{O}_E \longrightarrow 0 \ .$$

We show that there exists a commuting square

$$\begin{array}{ccc} K_i(B) & \longrightarrow & K_i(B) \\ \downarrow & & \downarrow \\ K_i(J_E) & \longrightarrow & K_i(\mathcal{T}_E) \end{array} \tag{4.1}$$

where $i = 0, 1$ and the vertical arrows are isomorphisms. In order to do this we need a Parseval frame on our Hilbert B-module E. Using our Parseval frame we construct an isomorphism between B and a full corner algebra of J_E, hence there is an isomorphism between the K-theory of B and J_E. The isomorphism between the K-theory of B and $\mathcal{T}_E$ will be induced from the natural inclusion of B in $\mathcal{T}_E$. Most of the results in this section can be found in [9], we simply adapt them and their proofs to our setting, which is slightly more general.

Definition 4.1.1. Let E be a Hilbert B-module. We say that

$$\{x_1, \ldots, x_N\} \subset E$$

is a *finite Parseval frame* for E if it satisfies the following reconstruction formula

$$x = \sum_{i=1}^{N} x_i \langle x_i, x\rangle, \quad \forall x \in E.$$

Remark 4.1.2. Note that when E has a finite Parseval frame, then $\mathcal{K}(E) = \mathcal{L}(E)$. This is because

$$x = \mathrm{id}(x) = \sum_{i=1}^{N} x_i \langle x_i, x\rangle = \sum_{i=1}^{N} |x_i\rangle\langle x_i|(x)$$

and hence $\mathrm{id} = \sum_{i=1}^{N} |x_i\rangle\langle x_i| \in \mathcal{K}(E)$. Since $\mathcal{K}(E)$ is an ideal then $\mathcal{K}(E) = \mathcal{L}(E)$. In our setting we have $J_E = B$ since φ is injective.

Before constructing our isomorphism we will require some previous lemmas:

Lemma 4.1.3. *Suppose that E is a right-Hilbert module over a unital C^*-algebra B such that (E, φ) is a B-B correspondence* *with φ unital, and that $\{x_j \ : 1 \le j \le N\}$ is a finite Parseval frame for E. Let (π, t, C) be a Toeplitz representation of E. Then $q := \sum_{j=1}^{N} t(x_j)t(x_j)^*$ is a projection that commutes with every $\pi(a)$.*

Proof. Fix $a \in B$, then by the reconstruction formula we get

$$\begin{aligned} q\pi(a)q &= \sum_{j,k=1}^{N} t(x_j)t(x_j)^*\pi(a)t(x_k)t(x_k)^* \\ &= \sum_{j,k=1}^{N} t(x_j)\pi(\langle x_j, a\cdot x_k\rangle)t(x_k)^* \\ &= \sum_{k=1}^{N}\left(\sum_{j=1}^{N} t(x_j\cdot\langle x_j, a\cdot x_k\rangle)t(x_k)^*\right) \\ &= \sum_{k=1}^{N} t(a\cdot x_k)t(x_k)^* \\ &= \pi(a)q. \end{aligned}$$

By taking $a = 1$ we get that $q^2 = q$ and since q is self-adjoint it is a projection. Now $q\pi(a) = (\pi(a^*)q)^* = (q\pi(a^*)q)^* = q\pi(a)q = \pi(a)q$. □

From now on we assume that our Hilbert B-module E has a finite Parseval frame $\{x_j : 1 \le j \le N\}$. We write Q for the quotient map from $\mathcal{T}_E \to \mathcal{O}_E$, and $(\pi, t, \mathcal{T}_E)$ for the universal Toeplitz covariant representation of E in $\mathcal{T}_E$ (refered as $(i_A, i_E, \mathcal{T}_E)$ in Chapter 3).

Lemma 4.1.4. *Define $\Omega : B \to \mathrm{M}_N(B)$ by $\Omega(a) = (\langle x_j, a\cdot x_k\rangle)_{j,k}$. Then Ω is a homomorphism of C^*-algebras.*

Proof. It is straightforward to see that Ω is linear. Take $a, b \in B$ then

$$\begin{aligned}\Omega(a)\Omega(b) &= (\langle x_j, a \cdot x_k\rangle)_{j,k}(\langle x_r, b \cdot x_s\rangle)_{r,s} \\ &= \left(\sum_{i=1}^{N} \langle x_j, a \cdot x_i\rangle\langle x_i, b \cdot x_s\rangle\right)_{j,s} \\ &= \left(\langle x_j, a \cdot \sum_{i=1}^{N} x_i\langle x_i, b \cdot x_s\rangle\rangle\right)_{j,s} \\ &= \left(\langle x_j, ab \cdot x_s\rangle\right)_{j,s} = \Omega(ab)\end{aligned}$$

and

$$\begin{aligned}\Omega(a)^* &= (\langle x_j, a \cdot x_k\rangle)^*_{j,k} \\ &= (\langle x_k, a \cdot x_j\rangle^*)_{j,k} \\ &= (\langle a \cdot x_j, x_k\rangle)_{j,k} \\ &= (\langle x_j, a^* \cdot x_k\rangle)_{j,k} = \Omega(a^*)\end{aligned}$$

hence Ω is a *-homomorphism. □

From Remark 2.1.5 we know that if $(\pi, t, \mathcal{T}_E)$ is the universal Toeplitz representation of E then

$$\mathcal{T}_E = \overline{\operatorname{span}}\{t^{\otimes k}(x)t^{\otimes l}(y)^* \; : \; k, l \geq 0, \, x \in E^{\otimes k}, \, y \in E^{\otimes l}\}$$

and that $q := \sum_{j=1}^{N} t(x_j)t(x_j)^*$ is a projection that commutes with every $\pi(a)$.

Lemma 4.1.5. *With the preceding notation, we have:*

1. $1 - q = 1 - \sum_{j=1}^{N} t(x_j)t(x_j)^*$ *is a full projection in* $\ker Q$ *(i.e.* $\overline{\operatorname{span}} \ker Q(1-q)\ker Q = \ker Q$*);*

2. $(1-q)t^{\otimes k}(x) = 0$ *for all* $x \in E^{\otimes k}$ *with* $k \geq 1$*; and*

3. $\ker Q = \overline{\operatorname{span}}\{t^{\otimes k}(x)(1-q)t^{\otimes l}(y)^* \ : \ k, l \geq 0,\ x \in E^{\otimes k},\ y \in E^{\otimes l}\}$.

The following proof can also be found in [9, Lemma 3.2].

Proof. 1. From Remark 4.1.2 we have $\varphi(a) \in \mathcal{K}(E)$ for all $a \in B$ and therefore

$$(\pi, t)^{(1)}(\varphi(a)) = \sum_{j=1}^{N} t(\varphi(a)x_j)t(x_j)^* = \pi(a)q.$$

Since φ is unital, it follows that $\mathcal{T}_E$ is unital and $\pi(1) = 1$. Now, since $J_E = B$ and $\mathcal{T}(J_E) = \ker Q$ is generated by elements of the form $\pi(a) - (\pi, t)^{(1)}(\varphi(a))$ with $a \in J_E$, we have

$$Q(1-q) = Q(\pi(1) - \pi(1)q) = Q(\pi(1) - (\pi, t)^{(1)}(\varphi(1))) = 0$$

hence $1 - q$ belongs to $\ker Q$. Once again, since $\ker Q$ is generated by elements of the form

$$\begin{aligned} \pi(a) - (\pi, t)^{(1)}(\varphi(a)) &= \pi(a) - \pi(a)q \\ &= \pi(a)(1-q) \\ &= \underbrace{\pi(a)(1-q)}_{\in \ker Q}(1-q)\underbrace{(1-q)}_{\in \ker Q} \end{aligned}$$

it follows that $\pi(a) - (\pi, t)^{(1)}(\varphi(a)) \in \ker Q(1-q)\ker Q$, in other words $1 - q$ is a full projection.

2. We begin with $k = 1$, take $x \in E^{\otimes 1} = E$ then

$$qt(x) = \sum_{j=1}^{N} t(x_j)t(x_j)^* t(x) = t\left(\sum_{j=1}^{N} x_j \langle x_j, x \rangle\right) = t(x)$$

hence $(1-q)t(x) = 0$. Now for $k > 1$ an elementary tensor $x = x_1 \otimes \cdots \otimes x_k \in E^{\otimes k}$ we have

$$(1-q)t^{\otimes k}(x) = (1-q)t(x_1)\cdots t(x_k) = 0$$

by linearity and continuity we have the result for arbitrary $x \in E^{\otimes k}$.

3. By (1) we have $\ker Q = \mathcal{T}_E(1-q)\mathcal{T}_E$, remembering the structure of $\mathcal{T}_E$ and using the previous item, $\ker Q$ is spanned by elements of the form

$$\begin{aligned} t^{\otimes k}(x)\pi(a)^*(1-q)\pi(b)t^{\otimes l}(y)^* &= t^{\otimes k}(x)\pi(a^*b)(1-q)t^{\otimes l}(y)^* \\ &= t^{\otimes k}(x \cdot a^*b)(1-q)t^{\otimes l}(y)^* \end{aligned}$$

for $x \in E^{\otimes k}$, $y \in E^{\otimes l}$ and $a, b \in B$. We can conclude the desired result.

□

Lemma 4.1.6. *There is a homomorphism $\rho : B \to \ker Q$ such that $\rho(a) = \pi(a)(1-q)$, and ρ is an isomorphism of B onto $(1-q)\ker Q(1-q)$.*

Proof. Due to item (3) of the previous lemma and the fact that $\pi(a)$ commutes with q for all $a \in B$, we have

$$\rho(a) = \pi(a)(1-q) = (1-q)\pi(a)(1-q) \in \ker Q$$

therefore ρ is a well defined *-homomorphism. Moreover

$$(1-q)\pi(a)(1-q) = (1-q)^2\pi(a)(1-q)^2 \in (1-q)\ker Q(1-q)$$

therefore ρ is a *-homomorphism from B to $(1-q)\ker Q(1-q)$. Now, since

$$(1-q)\ker Q(1-q) = \overline{\operatorname{span}}\{(1-q)t^{\otimes k}(x)(1-q)t^{\otimes l}(y)^*(1-q)\}$$

which, due to item (2) from the previous lemma, is equal to

$$\overline{\text{span}}\{(1-q)\pi(a)(1-q)\pi(b^*)(1-q),\, a,b\in B\}$$

$$=\overline{\text{span}}\{\pi(ab^*)(1-q),\, a,b\in B\}\subset\rho(B)$$

we conclude that ρ is surjective.

To see that ρ is injective we recall the Fock representation $(\varphi_\infty, T, \mathcal{L}(F_E))$ introduced in Section 2.1. Due to the universal property we have a *-homomorphism $f:\mathcal{T}_E\to\mathcal{L}(F_E)$ such that $f\circ\pi=\varphi_\infty$ and $f\circ t=T$. Now we take $a\in B$ such that $\rho(a)=0$, we would have

$$0=f(\rho(a))=f(\pi(a)(1-q))=\varphi_\infty(a)\left(\text{id}-\sum_{j=1}^{N}T(x_j)T(x_j)^*\right).$$

By the definition of T we have $T(x)^*(b)=0$ for all $b\in B$ and $x\in E$. This means that for all $b\in B$ we have

$$0=\varphi_\infty(a)\left(\text{id}-\sum_{j=1}^{N}T(x_j)T(x_j)^*\right)(b)=\varphi_\infty(a)(b)=ab$$

and therefore we conclude $a=0$. □

Consider the following matrix in $\mathrm{M}_2(\mathrm{M}_3(B))$,

$$\begin{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}&\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}\\\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{pmatrix}&\begin{pmatrix}d_{11}&d_{12}&d_{13}\\d_{21}&d_{22}&d_{23}\\d_{31}&d_{32}&d_{33}\end{pmatrix}\end{pmatrix}$$

it can naturally be seen as a matrix in $\mathrm{M}_6(B)$. It is easy to verify that the matrix above is unitary equivalent (via elementary operations) to the following matrix in $\mathrm{M}_3(\mathrm{M}_2(B))$ when both are seen in $\mathrm{M}_6(B)$:

$$\begin{pmatrix} \begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix} & \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix} & \begin{pmatrix} a_{13} & b_{13} \\ c_{13} & d_{13} \end{pmatrix} \\ \begin{pmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{pmatrix} & \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} & \begin{pmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{pmatrix} \\ \begin{pmatrix} a_{31} & b_{31} \\ c_{31} & d_{31} \end{pmatrix} & \begin{pmatrix} a_{32} & b_{32} \\ c_{32} & d_{32} \end{pmatrix} & \begin{pmatrix} a_{33} & b_{33} \\ c_{33} & d_{33} \end{pmatrix} \end{pmatrix}$$

This can be generalized for arbitrary matrices in $\mathrm{M}_r(\mathrm{M}_N(B))$ and $\mathrm{M}_N(\mathrm{M}_r(B))$, as shown in the following lemma:

Lemma 4.1.7. *Suppose that B is a C^*-algebra, $r \geq 1$ and $N \geq 2$ are integers, and*

$$\{b_{j,s;k,t} \ : 0 \leq j, k < N \text{ and } 1 \leq s, t \leq r\}$$

is a subset of B. For m, n satisfying $1 \leq m, n \leq rN$, we define

$$c_{m,n} = b_{j,s;k,t} \quad \text{where } m = (s-1)N + j \text{ and } n = (t-1)N + k,$$
$$d_{m,n} = b_{j,s;k,t} \quad \text{where } m = (j-1)r + s \text{ and } n = (k-1)r + t.$$

Then there is a scalar unitary permutation matrix U such that the matrices $C := (c_{m,n})$ and $D := (d_{m,n})$ are related by $C = UDU^*$.

Proof. For $1 \leq p, q \leq rN$, define

$$u_{p,q} = \begin{cases} 1 & \text{if there exists } k, t \text{ such that } p = (t-1)N + k \\ & \text{and } q = (k-1)r + t \\ 0 & \text{otherwise} \end{cases}$$

It can be verified that every column only has one 1, so $U := (u_{p,q})$ is a unitary permutation matrix. Doing very careful matrix multiplications we show that $CU = UD$. □

Before jumping into the existence of our commuting square we introduce the following special matrix in $\mathrm{M}_{rN}(\mathcal{T}_E)$:

$$T = \begin{pmatrix} t(x_1)1_r & \cdots & t(x_N)1_r \\ 0_r & \cdots & 0_r \\ \vdots & \cdots & \vdots \end{pmatrix}.$$

Lemma 4.1.8. *The matrix T is a partial isometry such that*

$$\begin{aligned} TT^* &= (q1_r) \oplus 0_{r(N-1)} \\ TT^*(\pi(a) \oplus b) &= (\pi(a) \oplus b)TT^* \\ T^*(\pi(a) \oplus 0_{r(N-1)}) &= T^*(\pi(a) \oplus b) \end{aligned}$$

for all $a \in \mathrm{M}_r(B)$ and $b \in \mathrm{M}_{r(N-1)}(\mathcal{T}_E)$.

Proof. First we have

$$\begin{aligned} TT^* &= \begin{pmatrix} t(x_1)1_r & \cdots & t(x_N)1_r \\ 0_r & \cdots & 0_r \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} t(x_1)^*1_r & 0_r & \cdots \\ \vdots & \vdots & \vdots \\ t(x_N)^*1_r & 0_r & \cdots \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^N t(x_i)t(x_i)^* \cdot 1_r & 0_r & \cdots & 0_r \\ \vdots & & \ddots & \vdots \\ 0_r & & \cdots & 0_r \end{pmatrix} \\ &= (q1_r) \oplus 0_{r(N-1)} \end{aligned}$$

moreover, due to item (2) from Lemma 4.1.5 we have $qt(x_j) = t(x_j)$

for all j and therefore

$$\begin{aligned}
TT^*T &= ((q1_r) \oplus 0_{r(N-1)})T \\
&= \begin{pmatrix} qt(x_1)1_r & \cdots & qt(x_N)1_r \\ \vdots & \ddots & \vdots \\ 0_r & \cdots & 0_r \end{pmatrix} \\
&= \begin{pmatrix} t(x_1)1_r & \cdots & t(x_N)1_r \\ 0_r & \cdots & 0_r \\ \vdots & \cdots & \vdots \end{pmatrix} = T
\end{aligned}$$

which means that T is a partial isometry. If we take $a \in \mathrm{M}_r(B)$ then

$$\begin{aligned}
TT^*(\pi(a) \oplus b) &= ((q1_r) \oplus 0_{r(N-1)})(\pi(a) \oplus b) \\
&= (q1_r\pi(a) \oplus 0_{r(N-1)}) \\
&= (\pi(a)q1_r \oplus 0_{r(N-1)}) \\
&= (\pi(a) \oplus b)((q1_r) \oplus 0_{r(N-1)}) \\
&= (\pi(a) \oplus b)TT^*.
\end{aligned}$$

Finally, we do the following:

$$\begin{aligned}
T^*(\pi(a) \oplus b) &= \begin{pmatrix} t(x_1)^*1_r & 0_r & \cdots \\ \vdots & \vdots & \vdots \\ t(x_N)^*1_r & 0_r & \cdots \end{pmatrix} \begin{pmatrix} \pi(a) & 0_{r,r(N-1)} \\ 0_{r(N-1),r} & b \end{pmatrix} \\
&= \begin{pmatrix} t(x_1)^*\pi(a)1_r & 0_r & \cdots \\ \vdots & \vdots & \vdots \\ t(x_N)^*\pi(a)1_r & 0_r & \cdots \end{pmatrix} \\
&= \begin{pmatrix} t(x_1)^*1_r & 0_r & \cdots \\ \vdots & \vdots & \vdots \\ t(x_N)^*1_r & 0_r & \cdots \end{pmatrix} \begin{pmatrix} \pi(a) & 0_{r,r(N-1)} \\ 0_{r(N-1),r} & 0_{r(N-1)} \end{pmatrix} \\
&= T^*(\pi(a) \oplus 0_{r(N-1)}).
\end{aligned}$$

□

We will use T to help us understand better how does Ω_* acts on $K_i(B)$. The following Lemma allows us to rewrite matrices of the form $\pi(\Omega(a))$ for $a \in \mathrm{M}_r(B)$ in a much simpler form.

Lemma 4.1.9. *Let $a \in \mathrm{M}_r(B)$ then $\pi(\Omega(a))$ is unitarily equivalent to $T^*(\pi(a) \oplus 0_{r(N-1)})T$.*

Proof. Using the identification of $\mathrm{M}_N(\mathrm{M}_r(B))$ with $\mathrm{M}_{rN}(B)$ we can look at $\Omega(a)$ as the following matrix in $\mathrm{M}_{rN}(B)$:

$$\begin{pmatrix} \langle x_1, a_{11}x_1\rangle & \cdots & \langle x_1, a_{11}x_N\rangle & & \langle x_1, a_{1r}x_1\rangle & \cdots & \langle x_1, a_{1r}x_N\rangle \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \langle x_N, a_{11}x_1\rangle & \cdots & \langle x_N, a_{11}x_N\rangle & & \langle x_N, a_{1r}x_1\rangle & \cdots & \langle x_N, a_{1r}x_N\rangle \\ & \vdots & & \ddots & & \vdots & \\ \langle x_1, a_{r1}x_1\rangle & \cdots & \langle x_1, a_{r1}x_N\rangle & & \langle x_1, a_{rr}x_1\rangle & \cdots & \langle x_1, a_{rr}x_N\rangle \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \langle x_N, a_{r1}x_1\rangle & \cdots & \langle x_N, a_{r1}x_N\rangle & & \langle x_N, a_{rr}x_1\rangle & \cdots & \langle x_N, a_{rr}x_N\rangle \end{pmatrix}.$$

Hence we can write $\pi \circ \Omega(p) = (c_{m,n})$ where

$$c_{(s-1)N+j,(t-1)N+k} = b_{j,s;t,k}$$

for $b_{j,s;t,k} = \pi(\langle x_j, a_{st}x_k\rangle)$, we take $C = (c_{m,n}) = \pi \circ \Omega(p)$. Conveniently this has the same form as our matrix C in Lemma 4.1.7. On the other hand, by taking $D = (d_{m,n})$ as in Lemma 4.1.7 (by this we mean $d_{m,n} = b_{j,s;t,k}$ in the same manner) we have

$$D = T^*(\pi(a) \oplus 0_{r(N-1)})T$$

and by Lemma 4.1.7 C and D are unitarily equivalent. □

We can now show the existence of the commuting square 4.1. The result for K_0 is proved in the same way as in [9], however to prove the result for K_1 we do some slight adjustments to compensate for the lack of orthonormality. We begin with the case of K_0:

Proposition 4.1.10. *The following diagram commutes:*

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\operatorname{id} - \Omega_*} & K_0(B) \\ \downarrow \rho_* & & \downarrow \pi_* \\ K_0(\ker Q) & \xrightarrow{\iota_*} & K_0(\mathcal{T}_E) \end{array} \tag{4.2}$$

Proof. Since B is unital, due to Remark 3.3.2, we can look at $K_0(B)$ as $K_{00}(B)$. In this case, taking $p = (p_{st}) \in \mathcal{P}_r(B)$ we have

$$\begin{aligned} \rho_*([p]) &= [(\rho(p_{st}))] \\ &= [(\pi(p_{st})(1-q))] \\ &= [(\pi(p_{st}))] - [(\pi(p_{st})q)] \\ &= \pi_*([p]) - [\pi(p) \cdot q1_r] \\ \pi_* \circ (\operatorname{id} - \Omega_*)([p]) &= \pi_*([p]) - \pi_* \circ \Omega_*([p]) \end{aligned}$$

therefore we only need to show $\pi_* \circ \Omega_*([p]) = [\pi(p) \cdot q1_r]$ to obtain the commutativity of our diagram. Note that $\Omega_*([p])$ is seen as an element in $K_0(B)$, not in $K_0(\mathrm{M}_N(B))$ and hence $\Omega(p)$ is taken as a matrix in $\mathrm{M}_{rN}(B)$. By Lemma 4.1.9 it is unitarily equivalent to

$$T^*(\pi(p) \oplus 0_{r(N-1)})T.$$

Now consider $V = (\pi(p) \oplus 0_{r(N-1)})T$, which is a partial isometry since

$$\begin{aligned}
VV^*V &= (\pi(p) \oplus 0_{r(N-1)})TT^*(\pi(p) \oplus 0_{r(N-1)})V \\
&= (\pi(p) \oplus 0_{r(N-1)})(q1_r \oplus 0_{r(N-1)})(\pi(p) \oplus 0_{r(N-1)})V \\
&= (\pi(p) \oplus 0_{r(N-1)})(q1_r \oplus 0_{r(N-1)})(\pi(p) \oplus 0_{r(N-1)})T \\
&= ((\pi(p) \cdot q1_r \cdot \pi(p)) \oplus 0_{r(N-1)})T \\
&= ((\pi(p) \cdot q1_r) \oplus 0_{r(N-1)})T \\
&= (\pi(p) \oplus 0_{r(N-1)})(q1_r \oplus 0_{r(N-1)})T \\
&= (\pi(p) \oplus 0_{r(N-1)})TT^*T \\
&= (\pi(p) \oplus 0_{r(N-1)})T = V.
\end{aligned}$$

The following two projections

$$\begin{aligned}
VV^* &= (\pi(p) \oplus 0_{r(N-1)})TT^* = ((\pi(p) \cdot q1_r) \oplus 0_{r(N-1)}) \\
V^*V &= T^*(\pi(p) \oplus 0_{r(N-1)})T
\end{aligned}$$

are equivalent projections, therefore

$$\begin{aligned}
[\pi(p) \cdot q1_r] &= [(\pi(p) \cdot q1_r) \oplus 0_{r(N-1)}] \\
&= [T^*(\pi(p) \oplus 0_{r(N-1)})T] \\
&= [\pi \circ \Omega(p)] \\
&= \pi_* \circ \Omega_*([p]).
\end{aligned}$$

This means that our diagram commutes. □

In the case of K_1 we take a slightly different approach than the one presented in [9]. When E has a orthonormal frame the matrix T is an isometry and therefore the matrix in [9, Lemma 3.6] is unitary, we adapt that construction to the case when T is simply a partial isometry:

Lemma 4.1.11. *Suppose that S is a partial isometry in a unital C^*-algebra B. Then*

$$U := \begin{pmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{pmatrix}$$

is a unitary element of $\mathrm{M}_2(B)$ *and its class in* $K_1(B)$ *is the identity.*

Proof. Since S is a partial isometry then SS^* and S^*S are projections, moreover $SS^*S = S$ and $S^*SS^* = S^*$. We quickly verify that U is unitary:

$$\begin{aligned} U^*U &= \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right)^* \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} S^* & 1-S^*S \\ 1-SS^* & S \end{smallmatrix}\right) \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} S^*S+(1-S^*S)^2 & S^*(1-SS^*)+(1-S^*S)S^* \\ (1-SS^*)S+S(1-S^*S) & (1-SS^*)^2+SS^* \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} UU^* &= \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right) \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right)^* \\ &= \left(\begin{smallmatrix} S & 1-SS^* \\ 1-S^*S & S^* \end{smallmatrix}\right) \left(\begin{smallmatrix} S^* & 1-S^*S \\ 1-SS^* & S \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} SS^*+(1-SS^*)^2 & S(1-S^*S)+(1-SS^*)S \\ (1-SS^*)S^*+S^*(1-SS^*) & (1-SS^*)^2+S^*S \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right). \end{aligned}$$

Now let $\mathcal{P} = C^*(v)$ be the universal C*-algebra generated by a partial isometry. By theorem 2.1 and 3.1 in [4] we have that $K_1(\mathcal{P}) = 0$, hence we have

$$\left[\begin{pmatrix} v & 1-vv^* \\ 1-v^*v & v^* \end{pmatrix}\right]_1 = [1]_1, \quad \text{in } K_1(\mathcal{P})$$

By constructing an homomorphism $\pi_S : \mathcal{P} \to B$ such that $\pi_S(v) = S$ we have

$$\left[\begin{pmatrix} S & 1 - SS^* \\ 1 - S^*S & S^* \end{pmatrix}\right]_1 = (\pi_S)_*\left(\left[\begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}\right]_1\right)$$
$$= (\pi_S)_*([1]_1) = [1]_1, \quad \text{in } K_1(B)$$

□

Remark 4.1.12. Note that when S is an isometry, our matrix coincides with the one presented in [9, Lemma 3.6].

Proposition 4.1.13. *The following diagram commutes:*

$$\begin{array}{ccc} K_1(B) & \xrightarrow{\operatorname{id} - \Omega_*} & K_1(B) \\ \downarrow \rho_* & & \downarrow \pi_* \\ K_1(\ker Q) & \xrightarrow{\iota_*} & K_1(\mathcal{T}_E) \end{array} \quad . \tag{4.3}$$

Proof. Sometimes throughout this proof we will use $\mathcal{T}$ to refer to $\mathcal{T}_E$. Recall from 3.4.8 that $K_1(B) \cong \mathcal{U}_\infty(B)/\sim_1$. Using Proposition 3.10, for any $u \in \mathcal{U}_r(B)$ we have

$$\begin{aligned} \iota_* \circ \rho_*([u]_1) &= (\iota \circ \rho)_*([u]_1) \\ &= [(\iota \circ \rho)(u - 1_r) + 1_r]_1 \\ &= [\pi(u - 1_r)(1_{\mathcal{T}} - q)1_r + 1_r]_1 \\ &= [\pi(u)(1_{\mathcal{T}} - q)1_r + q1_r]_1. \end{aligned}$$

To compute $\pi_* \circ (\operatorname{id} - \Omega_*)$ we proceed in a similar way, take $u \in \mathcal{U}_r(B)$ then

$$\begin{aligned} \pi_* \circ \Omega_*([u]_1) &= (\pi \circ \Omega)_*([u]_1) \\ &= [\pi(\Omega(u - 1_r)) + 1_{rN}]_1 \end{aligned}$$

and by Proposition 3.5.4 we have

$$\pi_* \circ \Omega_*([u]_1) = [\pi(\Omega(u + 1_r)) - 1_{rN}]_1. \tag{4.4}$$

Now by Lemma 4.1.9 we have a unitary equivalence between $\pi(\Omega(u+1_r))$ and $T^*(\pi(u+1_r) \oplus 0_{r(N-1)})T$. Taking $W \in \mathfrak{U}_{rN}(\mathbb{C})$ as the unitary such that

$$\pi(\Omega(u + 1_r)) = W^*T^*(\pi(u + 1_r) \oplus 0_{r(N-1)})TW$$

and using it in (4.4) we have

$$\begin{aligned} \pi_* \circ \Omega_*([u]_1) = & \\ &= [\pi(\Omega(u + 1_r)) - 1_{rN}]_1 \\ &= [W^*T^*(\pi(u + 1_r) \oplus 0_{r(N-1)})TW - 1_{rN}]_1 \\ &= [W^*(T^*(\pi(u + 1_r) \oplus 0_{r(N-1)})T - 1_{rN})W]_1 \end{aligned}$$

since $\sim_1$ preserves unitary equivalence,

$$= [T^*(\pi(u + 1_r) \oplus 0_{r(N-1)})T - 1_{rN}]_1$$

using 4.1.8 we can replace $0_{r(N-1)}$ by $2 \cdot 1_{r(N-1)}$,

$$= [T^*(\pi(u + 1_r) \oplus 2 \cdot 1_{r(N-1)})T - 1_{rN}]_1$$

and since π is a unital *-homomorphism,

$$\begin{aligned} &= [T^*(\pi(u) \oplus 1_{r(N-1)})T + T^*(1_r \oplus 1_{r(N-1)})T - 1_{rN}]_1 \\ &= [T^*(\pi(u) \oplus 1_{r(N-1)})T + T^*T - 1_{rN}]_1. \end{aligned}$$

Note that since we have a multiplicative structure on $K_1(B)$ for any two group homomorphisms $f, g : K_1(B) \to K_1(B)$ the group homomorphism $f - g : K_1(B) \to K_1(B)$ will be defined as $(f - g)(m) = f(m)(g(m))^{-1}$ for any $m \in K_1(B)$, hence we conclude that

$$\pi_* \circ (\mathrm{id} - \Omega_*)([u]_1) = \\ [\pi(u) \oplus 1_{r(N-1)}]_1([T^*(\pi(u) \oplus 1_{r(N-1)})T + T^*T - 1_{rN}]_1)^{-1}. \tag{4.5}$$

From Proposition 3.4.6 we know that we can look at $\iota_* \circ \rho_*([u]_1)$ as an element in $\mathrm{GL}_\infty(B)/\sim_1$ via the isomorphism $(\phi_1)^{-1}$. This means that we can multiply $(\phi_1)^{-1} \circ \iota_* \circ \rho_*([u]_1)$ with invertible matrices homotopic to 1_{rN} without modifying its class in $\mathrm{GL}_\infty(B)/\sim_1$. Moreover, it is easy to verify that any matrix of the form

$$\begin{pmatrix} 1_{rN} & X \\ 0_{rN} & 1_{rN} \end{pmatrix}, \quad \begin{pmatrix} 1_{rN} & 0_{rN} \\ X & 1_{rN} \end{pmatrix} \tag{4.6}$$

has trivial class in $\mathrm{GL}_\infty(B)/\sim_1$. Using Lemma 4.1.11 we have

$$\left[\begin{pmatrix} T & 1_{rN} - TT^* \\ 1_{rN} - T^*T & T^* \end{pmatrix}\right]_1 = [1]_1 \tag{4.7}$$

therefore we can also multiply by this matrix without modifying the class in $\mathrm{GL}_\infty(B)/\sim_1$. Recall that T is a partial isometry, this will mean 4 things to us

1. T^*T is a projection,
2. TT^* is a projection,
3. $TT^*T = T$,
4. $T^*TT^* = T^*$.

We will use $\pi(u)'$ to denote $\pi(u)\oplus 1_{r(N-1)}$ and do the following calculations in $\mathrm{GL}_\infty(B)/\sim_1$:

$$\begin{aligned}
&(\phi_1)^{-1}\circ\iota_*\circ\rho_*([u]_1) = \\
&\qquad = [\pi(u)(1_{\mathcal{J}}-q)1_r + q1_r]_1 \\
&\qquad = [(\pi(u)(1_{\mathcal{J}}-q)1_r + q1_r)\oplus 1_{r(N-1)}]_1 \\
&\qquad = [\pi'(u)(1_{rN}-TT^*)+TT^*]_1
\end{aligned}$$

multiplying by (4.7) we have:

$$\begin{aligned}
&= \left[\begin{pmatrix} \pi'(u)(1_{rN}-TT^*)+TT^* & 0_{rN} \\ 0_{rN} & 1_{rN} \end{pmatrix}\right]_1 \left[\begin{pmatrix} T & 1_{rN}-TT^* \\ 1_{rN}-T^*T & T^* \end{pmatrix}\right]_1 \\
&= \left[\begin{pmatrix} \pi'(u)(T-TT^*T)+TT^*T & \pi'(u)(1_{rN}-TT^*)^2+TT^*-(TT^*)^2 \\ 1_{rN}-T^*T & T^* \end{pmatrix}\right]_1
\end{aligned}$$

using that T is a partial isometry,

$$= \left[\begin{pmatrix} T & \pi'(u)(1_{rN}-TT^*) \\ 1_{rN}-T^*T & T^* \end{pmatrix}\right]_1$$

mutiplying by a matrix of the form (4.6) on the right,

$$\begin{aligned}
&= \left[\begin{pmatrix} T & \pi'(u)(1_{rN}-TT^*) \\ 1_{rN}-T^*T & T^* \end{pmatrix}\right]_1 \left[\begin{pmatrix} 1_{rN} & T^*\pi'(u) \\ 0_{rN} & 1_{rN} \end{pmatrix}\right]_1 \\
&= \left[\begin{pmatrix} T & TT^*\pi'(u)+\pi'(u)(1_{rN}-TT^*) \\ 1_{rN}-T^*T & T^*\pi'(u)-T^*TT^*\pi'(u)+T^* \end{pmatrix}\right]_1
\end{aligned}$$

using that T is a partial isometry and Lemma 4.1.8,

$$= \left[\begin{pmatrix} T & \pi'(u) \\ 1_{rN}-T^*T & T^* \end{pmatrix}\right]_1$$

doing the elementary operation of switching columns and multiplying by another matrix of the form (4.6) on the right,

$$\begin{aligned}
&= \left[\begin{pmatrix} \pi'(u) & T \\ T^* & 1_{rN}-T^*T \end{pmatrix}\right]_1 \left[\begin{pmatrix} 1_{rN} & -\pi'(u^*)T \\ 0_{rN} & 1_{rN} \end{pmatrix}\right]_1 \\
&= \left[\begin{pmatrix} \pi'(u) & 0_{rN} \\ T^* & -T^*\pi'(u^{-1})T+1_{rN}-T^*T \end{pmatrix}\right]_1
\end{aligned}$$

mutiplying by a matrix of the form (4.6) on the left,

$$= \left[\begin{pmatrix} 1_{rN} & 0_{rN} \\ -T^*\pi'(u^*) & 1_{rN} \end{pmatrix}\right]_1 \left[\begin{pmatrix} \pi'(u) & 0_{rN} \\ T^* & -T^*\pi'(u^{-1})T+1_{rN}-T^*T \end{pmatrix}\right]_1$$
$$= \left[\begin{pmatrix} \pi'(u) & 0_{rN} \\ 0_{rN} & -T^*\pi'(u^*)T+1_{rN}-T^*T \end{pmatrix}\right]_1$$

multiplying by $1_{rN} \oplus (-1_{rN})$, which has trivial class due to Example 3.4.5, we have

$$= \left[\begin{pmatrix} \pi'(u) & 0_{rN} \\ 0_{rN} & -T^*\pi'(u^*)T+1_{rN}-T^*T \end{pmatrix}\right]_1 \left[\begin{pmatrix} 1_{rN} & 0_{rN} \\ 0_{rN} & -1_{rN} \end{pmatrix}\right]_1$$
$$= \left[\begin{pmatrix} \pi'(u) & 0_{rN} \\ 0_{rN} & T^*\pi'(u^*)T-1_{rN}+T^*T \end{pmatrix}\right]_1$$
$$= [\pi(u) \oplus 1_{r(N-1)}]_1[T^*\pi'(u^*)T + T^*T - 1_{rN}]_1$$
$$= [\pi(u) \oplus 1_{r(N-1)}]_1([T^*\pi'(u)T + T^*T - 1_{rN}]_1)^{-1}.$$

Since $\pi(u) \oplus 1_{r(N-1)}$ and $T^*\pi'(u)T - T^*T + 1_{rN}$ are both unitary matrices, we have

$$\begin{aligned}\iota_* \circ \rho_*([u]_1) &= \phi_1([\pi(u) \oplus 1_{r(N-1)}]_1([T^*\pi'(u)T + T^*T - 1_{rN}]_1)^{-1}) \\ &= [\pi(u) \oplus 1_{r(N-1)}]_1([T^*\pi'(u)T + T^*T - 1_{rN}]_1)^{-1}.\end{aligned}$$

Finally due to our previous calculation in (4.5) we have:

$$\begin{aligned}\iota_* \circ \rho_*([u]_1) &= [\pi(u) \oplus 1_{r(N-1)}]_1([T^*\pi'(u)T + T^*T - 1_{rN}]_1)^{-1} \\ &= \pi_* \circ (\mathrm{id} - \Omega_*)([u])\end{aligned}$$

which means our diagram commutes. □

From the sequence

$$0 \longrightarrow \ker Q \longrightarrow \mathcal{T}_E \longrightarrow \mathcal{O}_E \longrightarrow 0$$

applying (3.21) we obtain the following six-term exact sequence:

$$\begin{array}{ccccc}
K_0(\ker Q) & \xrightarrow{\iota_*} & K_0(\mathcal{T}_E) & \xrightarrow{Q_*} & K_0(\mathcal{O}_E) \\
\delta_1 \uparrow & & & & \downarrow \delta_0 \\
K_1(\mathcal{O}_E) & \xleftarrow[Q_*]{} & K_1(\mathcal{T}_E) & \xleftarrow[\iota_*]{} & K_1(\ker Q)
\end{array}$$

combining this with the results from Proposition 4.1.10 and 4.1.13 we obtain the following diagram:

$$\begin{array}{ccccc}
K_0(B) & \xrightarrow{\mathrm{id}-\Omega_*} & K_0(B) & & \\
\downarrow \rho_* & & \downarrow \pi_* & & \\
K_0(\ker Q) & \xrightarrow{\iota_*} & K_0(\mathcal{T}_E) & \xrightarrow{Q_*} & K_0(\mathcal{O}_E) \\
\delta_1 \uparrow & & & & \downarrow \delta_0 \\
K_1(\mathcal{O}_E) & \xleftarrow[Q_*]{} & K_1(\mathcal{T}_E) & \xleftarrow[\iota_*]{} & K_1(\ker Q) \\
 & & \pi_* \uparrow & & \rho_* \uparrow \\
 & & K_1(B) & \xleftarrow[\mathrm{id}-\Omega*]{} & K_1(B).
\end{array}$$

In order to obtain our desired six-term exact sequence we just need for ρ_* and π_* to be isomorphisms. The fact that π_* is an isomorphism comes from the fact B is separable and [9, Theorem 4.4], where Pimsner used KK-theory to construct an inverse for π_*. Whether or not his approach can be done using simply K-theory is not as important, this is because our homomorphism π_* is constructed with K-theory. To verify that ρ_* is an isomorphism we use the following commutative diagram

$$\begin{array}{ccc}
(1-q)\ker Q(1-q) & \xrightarrow{\iota} & \ker Q \\
\rho \uparrow & \nearrow_{\rho} & \\
B & &
\end{array} .$$

Since $\ker Q$ is separable (because B and E are) and $(1-q)$ is a full projection, from Proposition 3.5.11 and Corollary 3.6.10 we have that ι_* is an isomorphism, and since $\rho : B \to (1-q)\ker Q(1-q)$ is also an isomorphism we conclude that so is $\rho_* : K_i(B) \to K_i(\ker Q)$. Finally we have the following result:

Theorem 4.1.14. *The following six-term exact sequence is exact:*

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\mathrm{id}-\Omega_*} & K_0(B) & \xrightarrow{(j_B)_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle(\rho_*)^{-1}\circ\delta_1}\uparrow & & & & \downarrow{\scriptstyle(\rho_*)^{-1}\circ\delta_0} \\ K_1(\mathcal{O}_E) & \xleftarrow[(j_B)_*]{} & K_1(B) & \xleftarrow[\mathrm{id}-\Omega_*]{} & K_1(B) \end{array} \tag{4.8}$$

where $j_B = Q \circ \pi$.

Example 4.1.15. Consider $B = \mathbb{C}$ and $E = \mathbb{C}^n$ with $n \geq 2$ with the canonical $\mathbb{C}$-$\mathbb{C}$ correspondence structure, we know from a previous example that $\mathcal{O}_E \cong \mathcal{O}_n$. Since B is a separable unital C*-algebra and E has the canonical base as a Parseval frame we can use the previous theorem to calculate the K-theory of $\mathcal{O}_n$. Using the fact that $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$ the sequence (4.8) becomes:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\mathrm{id}-\Omega_*} & \mathbb{Z} & \xrightarrow{(j_B)_*} & K_0(\mathcal{O}_n) \\ {\scriptstyle(\rho_*)^{-1}\circ\delta_1}\uparrow & & & & \downarrow{\scriptstyle(\rho_*)^{-1}\circ\delta_0} \\ K_1(\mathcal{O}_n) & \xleftarrow[(j_B)_*]{} & 0 & \xleftarrow[\mathrm{id}-\Omega_*]{} & 0 \end{array}.$$

This means that we only need to compute $\mathrm{coker}(\mathrm{id}-\Omega_*)$ and $\ker(\mathrm{id}-\Omega_*)$ to obtain the K-theory for the Cuntz algebra. First we will take a look at how does Ω acts in this scenario, take

$\{e_1, \dots, e_n\}$ the canonical base in $\mathbb{C}^n$ and $z \in \mathbb{C}$, then

$$\Omega(z) = \begin{pmatrix} \langle e_1, ze_1\rangle & \cdots & \langle e_1, ze_n\rangle \\ \vdots & \ddots & \vdots \\ \langle e_n, ze_1\rangle & \cdots & \langle e_n, ze_n\rangle \end{pmatrix} = \begin{pmatrix} z & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z \end{pmatrix} = z \cdot 1_n \in \mathrm{M}_n(\mathbb{C}).$$

Taking $z = 1$ we have $\Omega(1) = 1_n$, since $[1] - [0]$ is the generator of $K_0(\mathbb{C})$ and $[1_n] = n[1]$ we have $\Omega_*([1] - [0]) = n([1] - [0])$. Therefore

$$(\mathrm{id} - \Omega_*)([1] - [0]) = (1 - n)([1] - [0])$$

meaning $\ker(\mathrm{id} - \Omega_*) = 0$ and $(\mathrm{id} - \Omega_*)(\mathbb{Z}) = (n - 1)\mathbb{Z}$. We can conclude that $K_1(\mathcal{O}_n) = 0$ and $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$.

4.2 An application to Vector Bundles

Consider X a compact Hausdorff topological space and $V \xrightarrow{\pi} X$ a complex vector bundle of rank r. From Appendix B we know that we can see $\Gamma(V)$ as a right-module over $C(X)$, we will define an inner-product that will make into a Hilbert $C(X)$-module. Take $\mathcal{U} = \{U_1, \dots U_m\}$ a trivializing finite open cover of X, this means that for all $j = 1, \dots, m$ there exists a local trivialization $f_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}^r$. Consider $\{\psi_1, \dots, \psi_m\} \subset C(X)$ a partition of unity subordinate to $\mathcal{U}$, we define the following inner product in V_x:

$$\langle u, v\rangle_x = \sum_{j\colon x \in U_j} \psi_j(x) \langle f_j(u), f_j(v)\rangle$$

where $u, v \in V_x$ and the inner-product on the right is the canonical one in $\mathbb{C}^r$. We now have a continuous family of inner-products $\{\langle -, - \rangle\}_x$, this means that if $\xi, \eta \in \Gamma(V)$ then $x \mapsto \langle \xi(x), \eta(x) \rangle_x$ is a continuous complex function. Indeed, $f_j \circ \xi$ and $f_j \circ \eta$ are both continuous functions in U_j, therefore $y \mapsto \langle f_j(\xi(y)), f_j(\eta(y)) \rangle$ is a continuous complex function in U_j. Take $x \in X$ and consider the open set

$$U = (\cap_{x \in U_j} U_j) \cap (\cap_{x \notin U_j} (\operatorname{supp} \psi_j)^c)$$

for all $y \in U$, since $\psi_j(y) = 0$ for all j such that $x \notin U_j$ we have

$$\begin{aligned}\langle \xi(y), \eta(y) \rangle_y &= \sum_{y \in U_j} \psi_j(y) \langle f_j(\xi(y)), f_j(\eta(y)) \rangle \\ &= \sum_{x \in U_j} \psi_j(y) \langle f_j(\xi(y)), f_j(\eta(y)) \rangle\end{aligned}$$

and therefore $y \mapsto \langle \xi(y), \eta(y) \rangle_y$ coincides with the sum of the continuous functions $y \mapsto \psi_j(y) \langle f_j(\xi(y)), f_j(\eta(y)) \rangle$ where j is such that $x \in U_j$. This means $x \mapsto \langle \xi(x), \eta(x) \rangle_x$ is globally continuous. We can define now a inner-product over $C(X)$ on $\Gamma(V)$ by

$$\langle \xi, \eta \rangle (x) := \langle \xi(x), \eta(x) \rangle_x$$

where $\xi, \eta \in \Gamma(V)$. This turns $\Gamma(V)$ into a Hilbert $C(X)$-module. Moreover, defining $\varphi : C(X) \to \mathcal{L}(\Gamma(V))$ by $\varphi(f)(\xi) := \xi \cdot f$, $(\Gamma(V), \varphi)$ becomes a $C(X) - C(X)$ correspondence.

According to the Serre-Swan theorem (see [11, page 59]) for any vector bundle V, the space $\Gamma(V)$ is a finitely generated projective module over $C(X)$. Since we're looking to apply the results from the previous section, a finite number of generators isn't enough, we need a finite Parseval frame. To construct this Parseval frame we will use the inner-product we just defined. For each U_j, the local trivializations induce $s_{j1}, \ldots, s_{jr}$ linearly independent sections. We can assume these are orthonormal, otherwise we do a Gram-Schmidt process using our inner-product (the divisions in this process will not cause any trouble as $\langle s_{jk}(x), s_{jk}(x)\rangle_x > 0$ for all $x \in U_j$ due to the sections being linearly independant). Since our sections are orthonormal we have

$$\begin{aligned} \xi(x) &= \sum_{k=1}^{r} s_{jk}(x) \langle s_{jk}(x), \xi(x)\rangle_x \\ &= \sum_{k=1}^{r} s_{jk} \langle s_{jk}, \xi\rangle (x) \end{aligned} \tag{4.9}$$

for any $\xi \in \Gamma(V)$ and $x \in U_j$, hence $\xi = \sum_{k=1}^{r} s_{jk} \langle s_{jk}, \xi\rangle$ in U_j. We define the global sections $\xi_{jk} \in \Gamma(V)$ by

$$\xi_{jk}(x) = \begin{cases} s_{jk}(x)\sqrt{\psi_j(x)}, & x \in U_j \\ 0, & x \notin U_j \end{cases}$$

which are easily shown to be continuous.

Proposition 4.2.1. *The continuous sections $\xi_{jk} : X \to V$ with $j = 1, \ldots, m$ and $k = 1, \ldots, r$ form a Parseval frame for $\Gamma(V)$ as a Hilbert $C(X)$-module.*

Proof. Take $\xi \in \Gamma(V)$ and fix $x \in X$. Since $\xi_{jk}(x) = 0$ whenever

$x \notin U_j$, we have

$$\sum_{j=1}^{m}\sum_{k=1}^{r}\xi_{jk}\left\langle \xi_{jk},\xi\right\rangle(x) =$$

$$= \sum_{x\in U_j}\sum_{k=1}^{r}\xi_{jk}(x)\left\langle \xi_{jk}(x),\xi(x)\right\rangle_x$$

$$= \sum_{x\in U_j}\sum_{k=1}^{r}s_{jk}(x)\sqrt{\psi_j(x)}\left\langle s_{jk}(x)\sqrt{\psi_j(x)},\xi(x)\right\rangle_x$$

$$= \sum_{x\in U_j}\psi_j(x)\sum_{k=1}^{r}s_{jk}(x)\left\langle s_{jk}(x),\xi(x)\right\rangle_x$$

by (4.9) this is equal to $\sum_{x\in U_j}\psi_j(x)\xi(x) = \xi(x)$. We can conclude that

$$\sum_{j=1}^{m}\sum_{k=1}^{r}\xi_{jk}\left\langle \xi_{jk},\xi\right\rangle = \xi.$$

□

This means that for any vector bundle, $\Gamma(V)$ is a Hilbert $C(X)$-module with finite Parseval frame. It is important to note that in very few situations will $\Gamma(V)$ posses an orthonormal frame. This is because if $\{\xi_i\}_{i=1}^n$ is an orthonormal frame then $\langle \xi_i,\xi_i\rangle(x) = \langle \xi_i(x),\xi_i(x)\rangle_x = 1$ for all $x\in X$, therefore the section ξ_i never vanishes. This would mean that $\{\xi_i\}_{i=1}^n$ are linearly independent sections over X, which only exist for trivial bundles. Moreover, there exist some non-trivial bundles that do not admit any non-vanishing section. The following is an example of this:

Example 4.2.2. Consider $X = \mathbb{C}P^1 \simeq S^2$ and consider the Bott bundle:

$$V = \{(x,v)\in X\times\mathbb{C}^2 : v\in x\}$$

We know that $\Gamma(V)$ is a $C(X)$-bimodule. Moreover, consider the following sections in $\Gamma(V)$,

$$\begin{aligned}\xi_1 : X = \mathbb{C}P^1 &\to V\\ [z:w] &\mapsto \left([z:w], \frac{\overline{z}}{|z|^2+|w|^2}(z,w)\right)\\ \xi_2 : X = \mathbb{C}P^1 &\to V\\ [z:w] &\mapsto \left([z:w], \frac{\overline{w}}{|z|^2+|w|^2}(z,w)\right)\end{aligned}$$

It can easily be verified that they form a non-orthogonal frame for $\Gamma(V)$. This means we can apply the results from the previous section and obtain the following sequence:

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\mathrm{id}-\Omega_*} & K_0(C(X)) & \longrightarrow & K_0(\mathcal{O}_E)\\ \uparrow & & & & \downarrow\\ K_1(\mathcal{O}_E) & \longleftarrow & K_1(C(X)) & \xleftarrow[\mathrm{id}-\Omega_*]{} & K_1(C(X))\end{array}$$

where $E = \Gamma(V)$.

To obtain a clear picture of $K_i(C(X))$, consider the following split exact sequence:

$$0 \longrightarrow C_0(\mathbb{C}) \underset{\sigma}{\overset{\alpha}{\rightleftarrows}} C(X) \xrightarrow{\beta} \mathbb{C} \longrightarrow 0$$

where

$$\alpha(f)([z:w]) = \begin{cases} f(z/w) & ,w \neq 0\\ 0 & ,w = 0\end{cases}$$

$\beta(g) = g([1:0])$ and $\sigma(g)(z) = g([z:1]) - g([1:0])$. It is straightforward to verify that these three *-homomorphisms make the sequence above split exact. From the half-exactness of both K_0 and K_1 we obtain an isomorphism between $K_i(C(X))$ and

$K_i(C_0(\mathbb{C})) \oplus K_i(\mathbb{C})$ given by

$$(\sigma_* \oplus \beta_*) : K_i(C(X)) \to K_i(C_0(\mathbb{C})) \oplus K_i(\mathbb{C}).$$

Since $K_0(C_0(\mathbb{C})) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$ and $K_1(C_0(\mathbb{C})) = K_1(\mathbb{C}) = 0$ we conclude that $K_0(C(X)) \simeq \mathbb{Z}^2$ and $K_1(C(X)) = 0$. Moreover we can obtain the two generators of $K_0(C(X))$ via the isomorphism $(\sigma_* \oplus \beta_*)$, which turn out to be $[p_{\mathrm{Bott}}] - [1]$ and $[1] - [0]$, where

$$p_{\mathrm{Bott}} := \begin{pmatrix} \frac{|z|^2}{|z|^2+|w|^2} & \frac{\overline{w}z}{|z|^2+|w|^2} \\ \frac{w\overline{z}}{|z|^2+|w|^2} & \frac{|w|^2}{|z|^2+|w|^2} \end{pmatrix} \in \mathrm{M}_2(C(X)).$$

The first generator comes from $K_0(C_0(\mathbb{C}))$ and the second one comes from $K_0(\mathbb{C})$. Using this information, our six-term exact sequence becomes

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{\mathrm{id} - \Omega_*} & \mathbb{Z}^2 & \longrightarrow & K_0(\mathcal{O}_E) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_E) & \longleftarrow & 0 & \xleftarrow[\mathrm{id} - \Omega_*]{} & 0 \end{array}$$

and due to the exactness of the sequence we have $K_0(\mathcal{O}_E) \simeq \operatorname{coker}(\mathrm{id} - \Omega_*)$ and $K_1(\mathcal{O}_E) \simeq \ker(\mathrm{id} - \Omega_*)$. Therefore, in order to calculate the K-theory for this Cuntz-Pimsner algebra we only need to know how $\mathrm{id} - \Omega_*$ acts on the two generators of $K_0(C(X))$. Luckily we also have $\Omega(1) = p_{\mathrm{Bott}}$, hence:

$$\begin{aligned} \mathrm{id} - \Omega_*([1] - [0]) &= [1] - [0] - ([\Omega(1)] - [\Omega(0)]) \\ &= [1] - [0] - ([p_{\mathrm{Bott}}] - [0]) \\ &= [1] - [p_{\mathrm{Bott}}] = -([p_{\mathrm{Bott}}] - [1]). \end{aligned}$$

Evaluating the other generator is not as straightforward:

$$\Omega(p_{\mathrm{Bott}}) = \begin{pmatrix} p_{\mathrm{Bott}} \cdot \frac{|z|^2}{|z|^2+|w|^2} \cdot 1_2 & p_{\mathrm{Bott}} \cdot \frac{\overline{w}z}{|z|^2+|w|^2} \cdot 1_2 \\ p_{\mathrm{Bott}} \cdot \frac{w\overline{z}}{|z|^2+|w|^2} \cdot 1_2 & p_{\mathrm{Bott}} \cdot \frac{|w|^2}{|z|^2+|w|^2} \cdot 1_2 \end{pmatrix}$$

which satisfies $\Omega(p_{\text{Bott}})([1:0]) = 1 \oplus 0_3$. Now, we utilize the isomorphism $(\sigma_* \oplus \beta_*)$ and note that

$$\begin{aligned}\beta_*(\Omega_*([p_{\text{Bott}}] - [1])) &= \beta_*([\Omega(p_{\text{Bott}})] - [p_{\text{Bott}}]) \\ &= [\Omega(p_{\text{Bott}})([1:0])] - [p_{\text{Bott}}([1:0])] \\ &= [1 \oplus 0_3] - [1 \oplus 0] = [1] - [1] = 0\end{aligned}$$

which means $\Omega_*([p_{\text{Bott}}] - [1]) = r([p_{\text{Bott}}] - [1])$ for some integer $r \in \mathbb{Z}$. Therefore $\mathrm{id} - \Omega_*([p_{\text{Bott}}] - [1]) = (1-r)([p_{\text{Bott}}] - [1])$ where r is an integer that will not be calculated. This roughly translates to $\mathrm{id} - \Omega_*(m,n) = ((1-r)m - n, 0)$ and therefore

$$\ker \mathrm{id} - \Omega_* = \{(m,n) \in \mathbb{Z}^2 | \, n = m(1-r)\} = \langle (1, 1-r) \rangle.$$

This means that $\ker \mathrm{id} - \Omega_* \simeq \mathbb{Z}$. Moreover, since $\mathrm{id} - \Omega_*(m,n) = ((1-r)m - n, 0)$ we have that

$$\mathrm{im}(\mathrm{id} - \Omega_*) = \mathbb{Z} \oplus 0 \simeq \mathbb{Z}$$

and therefore $\mathrm{coker}\, \mathrm{id} - \Omega_* = \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \oplus 0 \simeq \mathbb{Z}$. We can conclude that $K_0(\mathcal{O}_E) \simeq K_1(\mathcal{O}_E) \simeq \mathbb{Z}$.

Conclusion

In this work we saw how the to obtain a K-theory six-term exact sequence for certain classes of Cuntz-Pimsner algebras arising from Hilbert modules over a separable unital C*-algebra that have a finite Parseval frame. Moreover we saw that these class of algebras naturally arise when dealing with topological vector bundles over compact spaces. Even thought the sequence was already known since the introduction Cuntz-Pimsner algebras, we managed to obtain the sequence using basic K-theory rather than KK-theory.

The original sequence only required for the Hilbert module to be countably generated, this raises the question: can our method be adapted to this case? By Kasparov's stabilization theorem we know that every countably generated Hilbert module over a unital algebra has a countable Parseval frame. This means we only need to explore how can we adapt our results when the Hilbert module has a countable infinite Parseval frame. The first difficulty would be the fact that in this case not every adjointable operator is compact and therefore $\ker Q$ would be quite different. The second difficulty would be adapting our *-homomorphism Ω, as matrices are not big enough to encode all of the information of an infinite frame. We look forward to exploring all of this in a future work.

Appendix

APPENDIX A – Functors and Categories

Category theory can be seen as a language that allows us to describe many similar constructions and concepts that we encounter in different areas of mathematics. When dealing with categories we will encounter very large "sets" and quickly realize naive set-theory is not enough to handle these concepts. It is because of this that we need a distinction between "sets" and "classes", the latter usually corresponding to those very large "sets". In [1, page 13] there is a brief treatment of sets, classes and their differences.

Definition A.1. A category $\mathfrak{C}$ consists of a class of objects denoted by $\mathrm{Obj}(\mathfrak{C})$, a class of morphisms denoted by $\mathrm{Mor}(\mathfrak{C})$, two functions $t, s : \mathrm{Mor}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{C})$, a partial function called composition $\circ : \mathrm{Mor}(\mathfrak{C}) \times \mathrm{Mor}(\mathfrak{C}) \to \mathrm{Mor}(\mathfrak{C})$ which assigns, to any $f, g \in \mathrm{Mor}(\mathfrak{C})$ such that $t(f) = s(g)$, their composite morphism $g \circ f \in \mathrm{Mor}(\mathfrak{C})$ and an identity function $\mathrm{id} : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Mor}(\mathfrak{C})$ which assigns to each object A its identity morphism id_A such that

1. $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$ for all $g, f \in \mathrm{Mor}(\mathfrak{C})$ that can be composed,

2. $s(\mathrm{id}_A) = A$ and $t(\mathrm{id}_A) = A$ for all $A \in \mathrm{Obj}(\mathfrak{C})$,

3. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $s(h) = t(g)$ and $s(g) = t(f)$,

4. if $s(f) = A$ and $t(f) = B$ then $1_B \circ f = f$ and $f \circ 1_A = f$.

For any $A, B \in \mathrm{Obj}(\mathfrak{C})$, we denote

$$\mathrm{Mor}_{\mathfrak{C}}(A, B) := \{f \in \mathrm{Mor}(\mathfrak{C}),\ s(f) = A \quad t(f) = B\}.$$

Sometimes to express that a morphism f belongs to $\mathrm{Mor}_{\mathfrak{C}}(A, B)$ we use the notation $A \xrightarrow{f} B$ or $f : A \to B$.

The definition of category is very generic and it should come as no surprise that we might already know plenty of categories:

Example A.2. 1. The category **Set** which has sets as objects, $\mathrm{Mor}_{\mathbf{Set}}(A, B)$ is the set of functions from A to B, id_A is the identity function on A and the composition is as usual.

2. The category **Top** which has topological spaces as objects, $\mathrm{Mor}_{\mathbf{Top}}(A, B)$ is the set of continuous functions from A to B, id_A is the identity function on A and the composition is as usual.

3. The category $\mathbf{Vec}_{\mathbb{K}}$ which has finite dimensional vector spaces over $\mathbb{K}$ as objects, $\mathrm{Mor}_{\mathbf{Vec}_{\mathbb{K}}}(V, W)$ is the set of linear transformations from V to W, id_V is the identity function on V and the composition is as usual.

4. The category **Grp** which has groups as objects, where $\mathrm{Mor}_{\mathbf{Grp}}(A, B)$ is the set of group homomorphisms from A to B, id_A is the identity function on A and the composition is as usual.

5. The category **C*-alg** which has C*-algebras as objects, where $\mathrm{Mor}_{\mathbf{Grp}}(A, B)$ is the set of *-homomorphisms from A

to B, id_A is the identity function on A and the composition is as usual.

6. The category **Mat** which has natural numbers as objects, $\mathrm{Mor}_{\mathbf{Mat}}(m,n)$ is the set of real $m \times n$ matrices, $\mathrm{id}_n : n \to n$ is the $n \times n$ unit matrix 1_n and composition is defined by $A \circ B = BA$.

7. Given any category $\mathfrak{C}$ we can construct the category $\mathfrak{C}^{\mathrm{op}}$ where the objects and the morphisms are the same but for any $f \in \mathrm{Mor}(\mathfrak{C}^{\mathrm{op}})$, $s^{\mathrm{op}}(f) = t(f)$, $t^{\mathrm{op}}(f) = s(f)$ and $f \circ^{\mathrm{op}} g = g \circ f$. This means $\mathrm{Mor}_{\mathfrak{C}^{\mathrm{op}}}(A,B) = \mathrm{Mor}_{\mathfrak{C}}(B,A)$. We call this the *opposite* category of $\mathfrak{C}$.

Definition A.3. Fix a category $\mathfrak{C}$ and $f \in \mathrm{Mor}_{\mathfrak{C}}(A,B)$. We say f is an *isomorphism* if there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Whenever an isomorphism exists between two objects we say they are *isomorphic*.

It's very common to form links between two categories in order to ease their study. One way of doing this is via functors.

Definition A.4. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. A *functor* F from $\mathfrak{C}$ to $\mathfrak{D}$ is a function that assigns to each object $A \in \mathrm{Obj}(\mathfrak{C})$ an object $F(A) \in \mathrm{Obj}(\mathfrak{D})$ and to any morphism $A \xrightarrow{f} A'$ a morphism $F(A) \xrightarrow{F(f)} F(A')$ in such a way that composition and identity morphisms are preserved.

Proposition A.5. *All functors preserve isomorphisms.*

Definition A.6. Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor.

1. F is *faithful* if the restriction

$$F : \mathrm{Mor}_{\mathfrak{C}}(A,A') \to \mathrm{Mor}_{\mathfrak{D}}(F(A),F(A'))$$

is injective for all $A, A' \in \mathrm{Obj}(\mathfrak{C})$.

2. F is *full* if the restriction

$$F : \mathrm{Mor}_{\mathfrak{C}}(A, A') \to \mathrm{Mor}_{\mathfrak{D}}(F(A), F(A'))$$

is surjective for all $A, A' \in \mathrm{Obj}(\mathfrak{C})$.

3. F is an *isomorphism* if it is full, faithful and its restriction $F : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{D})$ is bijective.

We can also compose two functors, and its composition will also be a functor. Moreover, composition will preserve all three of the properties above. Two categories will be isomorphic if there exists an isomorphism between them, isomorphic categories are essentially the same. Looking for isomorphisms between categories can be quite restrictive, this is why we require a weaker concept.

Definition A.7. A functor $F : \mathfrak{C} \to \mathfrak{D}$ is an *equivalence* if it is full, faithful and for every object $B \in \mathrm{Obj}(\mathfrak{D})$ there exists $A \in \mathrm{Obj}(\mathfrak{C})$ such that $F(A)$ and B are isomorphic. In this case we say $\mathfrak{C}$ and $\mathfrak{D}$ are equivalent categories.

Example A.8.

1. The category **Mat** is equivalent to $\mathbf{Vec}_{\mathbb{R}}$ by assigning to each n the vector space $\mathbb{R}^n$ and to each $n \times m$ matrix the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ that assigns to each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the $1 \times m$ matrix $[x_1, \ldots, x_n]A$ considered as an m-tuple in $\mathbb{R}^m$.

2. The Gelfand representation theorem can be used to show that the category of commutative unital C*-algebras is equivalent to the category of compact Hausdorff spaces.

APPENDIX B – Vector Bundles

A topological vector bundle is essentially a family of vector spaces parametrized by a topological space in a nice way. A topological vector bundle is itself a topological space. Most of the topological spaces we encounter in noncommutative geometry are at the very least locally compact and Hausdorff. We will focus on the scenario when X is a compact Hausdorff space. Since X is compact it is normal and paracompact which means we can use Urysohn's Lemma and Tietze extension Theorem, moreover we have at our disposal partitions of unity. The results presented here are part of a brief introduction to vector bundles found in [11, Chapter 2].

Definition B.1. Let $\mathcal{U} = \{U_\lambda,\ \lambda \in \Lambda\}$ be an open cover of X. A partition of unity subordinate to $\mathcal{U}$ is a family of continuous functions $\{\psi_\lambda : U_\lambda \to [0,1]\}_{\lambda\in\Lambda}$ such that:

1. $\operatorname{supp}\psi_\lambda \subset U_\lambda$,
2. for each $x \in X$ there are finitely many $\lambda \in \Lambda$ such that $\psi_\lambda(x) \neq 0$,
3. $\sum_{\lambda\in\Lambda} \psi_\lambda(x) = 1$ for all $x \in X$.

Theorem B.2. *Let X be a topological space. X is paracompact and Hausdorff if and only if every open cover admits a subordinate partition of unity.*

Partitions of unity are a very useful tool when trying to "glue" many functions together, for example we can build an "inner-product" in a vector bundle by gluing the inner products in each locally trivial neighborhood of the bundle.

Definition B.3. A complex vector bundle of rank n over X is a topological space E together with a continuous surjection $\pi : E \to X$ such that for every $x \in X$,

1. $E_x := \pi^{-1}(x)$ is a complex vector space,

2. there exists an open neighborhood U_x of x and an homeomorphism $\Phi : \pi^{-1}(U_x) \to U_x \times \mathbb{C}^n$ that makes the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_x) & \xrightarrow{\quad\Phi\quad} & U_x \times \mathbb{C}^n \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U_x & \end{array}$$

   and for every $y \in U_x$, $\Phi|_{E_y}$ is a vector space isomorphism between E_y and $\{y\} \times \mathbb{C}^n$.

Remark B.4. The space $X \times \mathbb{C}^n$ has a natural structure of vector bundle over X, we call bundles of this type *trivial*. The definition of above implies that all vector bundles are *locally trivial*.

Definition B.5. Given $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X$ two vector bundles, we say a continuous function $\tau : E \to E'$ is a *bundle map* if $\pi' \circ \tau = \pi$ and $\tau|_{E_x} : E_x \to E'_x$ is a linear map for all $x \in X$. If there exists a bijective bundle map between two vector bundles we say they are isomorphic.

Definition B.6. Given two vector bundles over X, $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} X$ we can define the Whitney sum of them as

$$E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2,\, \pi_1(v_1) = \pi_2(v_2)\} \subset E_1 \times E_2$$

with the induced topology. The projection $\pi : E_1 \oplus E_2 \to X$ is defined as $\pi(v_1, v_2) := \pi_1(v_1) = \pi_2(v_2)$. In this case we have $(E_1 \oplus E_2)_x \cong (E_1)_x \oplus (E_2)_x$.

Proposition B.7. *Let E be a vector bundle over X. There exists another vector bundle E' over X such that $E \oplus E' \cong X \times \mathbb{C}^n$ for some n.*

Definition B.8. Given a complex vector bundle E over X, a section on E is a continuous function $\xi : X \to E$ such that $\xi(x) \in E_x$ for all $x \in X$. We denote the complex vector space of sections on E by $\Gamma(E)$.

It is not hard to verify that $\Gamma(E_1 \oplus E_2) \cong \Gamma(E_1) \oplus \Gamma(E_2)$ for any two vector bundles.

Remark B.9. The set $\Gamma(E)$ has a natural $C(X)$-module structure, take $f \in C(X)$ and $\xi \in \Gamma(E)$ we define

$$(\xi \cdot f)(x) = f(x)\xi(x)$$

for all $x \in X$. It is not hard to verify that $\xi \cdot f \in \Gamma(E)$. In fact if E' is another bundle over X and $\tau : E \to E'$ is a bundle map, then we can define $\Gamma(\tau) : \Gamma(E) \to \Gamma(E')$ as $\Gamma(\tau)(\xi) = \tau \circ \xi$. It can also be shown that for any $f \in C(X)$ we have $\Gamma(\tau)(\xi \cdot f) = \Gamma(\tau)(\xi) \cdot f$ and $\Gamma(\mathrm{id}_E) = 1_{\Gamma(E)}$, therefore any bundle map induces a module homomorphism.

We can look at vector bundles over X as a category where the morphisms are the bundle maps. In this setting we have:

Proposition B.10. Γ *defines a functor from the category of vector bundles over X to the category of $C(X)$-modules which is full and faithful.*

Theorem B.11 (Serre-Swan)**.** *The functor Γ from the category of vector bundles over a compact space X to the category of finitely generated projective modules over $C(X)$ is an equivalence of categories.*

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