voltando atraz:

$$\text{rot}\,\vec{v} = f(\rho)\,2\vec{c} + \frac{df}{d\rho}\,\frac{(\vec{c}\wedge\vec{r})\wedge\vec{c}}{\rho}\wedge(\vec{c}\wedge\vec{r}) = 0$$

$$= 2f.\vec{c} + \frac{1}{\rho}\frac{df}{d\rho}\left\{(\vec{c}\wedge\vec{r})^2\vec{c} - \underbrace{\left[(\vec{c}\wedge\vec{r})\times\vec{c}\right]}_{=0}(\vec{c}\wedge\vec{r})\right\}$$

$$= 2f.\vec{c} + \rho\frac{df}{d\rho}\vec{c} = 0 \qquad \vec{c}\neq 0 \rightarrow$$

$$2f + \rho\frac{df}{d\rho} = 0 \rightarrow \frac{df}{f} = -\frac{2d\rho}{\rho}$$

$$lf = l\frac{k}{\rho^2} \rightarrow \boxed{f(\rho) = \frac{k}{\rho^2}}$$ sendo k const.te arbitrária.

Aplicação do divergente:

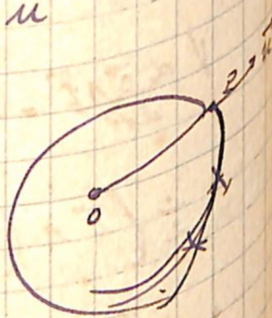
Determinar $\varphi(\vec{r})$ para que $\text{div}\,\varphi(r).\vec{r} = 0$ $\quad \vec{r} = P-O$

$$r = |P-O|$$

$$\text{div}\,\varphi(r)\,\vec{r} = \varphi\,\text{div}\,\vec{r} + \text{grad}\,\varphi\times\vec{u}$$

$$\text{grad}\,\varphi(r) = \frac{d\varphi}{dr}\,\text{grad}\,r.$$

$$\text{grad}\,|\vec{r}| = \left(\lim_{h\to 0}\frac{h}{h}\right)\frac{\vec{r}}{r} = \frac{\vec{r}}{r}$$



$$\boxed{\text{div}\,\vec{r} = I_1\left(\frac{d\vec{r}}{dP}\right)}$$

$$I_1(\sigma) = (\vec{i}\times\sigma\vec{i} + \vec{j}\times\sigma\vec{j} + \vec{k}\times\sigma\vec{k}),$$ sendo

no caso $I_1(\sigma) = 3$ e $\text{div}\,\vec{r} = 3$

donde: $\text{div}\,\varphi(r)\,\vec{r} = 3\varphi(\vec{r}) + \frac{d\varphi}{dr}\,\frac{\vec{r}}{r}\times\vec{r} = 0$

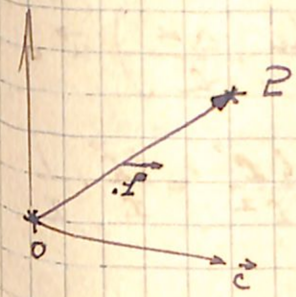
$$3\varphi + r\frac{d\varphi}{dr} = 0 \qquad \text{(integrando)}$$

$$\boxed{\varphi = \frac{k}{r^3}}$$

18.8.46.

Cálculo do divergente:–

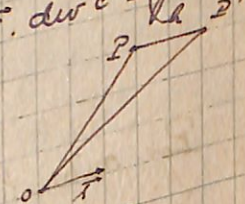
$\text{div}\,\vec{c}\wedge(\vec{r}\wedge\vec{c})$ $\qquad |\vec{c}| = c^2$, $d\vec{c} = 0$, $\vec{r} = P-O.$



$$\vec{c}\wedge(\vec{r}\wedge\vec{c}) = (\vec{c}\times\vec{c})\,\vec{r} - (\vec{c}\times\vec{r})\,\vec{c} \rightarrow$$

$$\text{div}(\ldots) = \text{div}\,c^2\vec{r} - \text{div}\,(\vec{c}\times\vec{r})\,\vec{c} =$$

$$= c^2\,\text{div}\,\vec{r} - \vec{c}\times\vec{r}.\,\text{div}\,\vec{c} - \text{grad}\,\vec{c}\times\vec{r}\times\vec{c}$$



$$\text{div}\,\vec{r} = I_1\left(\frac{d\vec{r}}{dP}\right)$$

$$\frac{d\vec{r}}{dP}.\vec{s} = \lim_{h\to 0}\frac{h\vec{s}}{h} = \vec{s}$$

No caso, $d\vec{r}/dP = 1$

$$\text{div}\,\vec{r} = I_1 = (\vec{i}\times\sigma\vec{i} + \vec{j}\times\sigma\vec{j} + \vec{k}\times\sigma\vec{k}) = 3 \qquad \sigma = 1$$

$$c^2\,\text{div}\,\vec{r} = 3c^2$$

$\vec{c}\times\vec{r}\;\text{div}\,\vec{c} = 0$ $\quad$ pois $\text{div}\,\vec{c} = 0$

grad$(\vec{c}\times\vec{r})$ podemos desenvolver o produto escalar ou aplicar a definição de grad.

$$\text{grad}\,(\vec c\times\vec r) = \text{grad}\, r \cdot \cos\alpha + r\,\underset{(1)}{\text{grad}}\cos\alpha +$$

$$+ \cos\alpha\,\underset{(2)}{\text{grad}}\, r =$$

$$= \overset{(2)}{\frac{\cos\alpha}{r}}\vec r + r\,\text{sen}\,\alpha\,\text{grad}\,\alpha = \frac{\cos\alpha}{r}\vec r - \frac{1}{r}\,\text{sen}\,\alpha\,\frac{i\vec r}{r}$$

$\vec n = i\vec r/|\vec r|$ (i = operador vetorial)

$$\text{grad}\,\alpha = \left(\frac{d\alpha}{dP}\, i\,\frac{\vec r}{r}\right) i\,\frac{\vec r}{r}$$

$$\frac{d\alpha}{dP} = \lim\frac{\Delta\alpha}{dP} = \left(\lim_{h\to 0}\frac{\text{arctg}\,\frac{h}{r}}{h}\right)\vec u = \left(\lim_{h\to 0}\frac{\Delta\alpha}{h}\right)\vec u =$$

$$= \left(\lim_{h\to 0}\frac{\Delta\alpha}{r\,\text{tg}\,\Delta\alpha}\right)\vec n = \frac{\vec n}{r} = \frac{i\vec r}{r^2} = \frac{i\vec r}{r}\cdot\frac{1}{r}$$

$$\text{grad}\,(\vec c\times\vec r) = \frac{\vec r}{r}\left(\cos\alpha - i\,\text{sen}\,\alpha\right) =$$

$$= \frac{\vec r}{r}e^{-i\alpha} = \vec c$$

Por outro lado, $df = \text{grad}\, f\times dP$

$$\text{grad}\,(\vec c\times\vec r)\cancel{\times dP} = \vec c\times\cancel{dP}$$

$$\text{grad}\,(\vec c\times\vec r) = \vec c$$

finalmente –

$$\text{div}\,\vec c\wedge(\vec r\wedge\vec c) =$$

$$= 3|\vec c|^2 - |\vec c|^2 = 2|\vec c|^2$$

Operadores simples.
$$\left\{\begin{array}{l}\text{grad}\, f(P) = \vec v\\ \text{rot.}\,\vec u(P) = \vec w\\ \text{div.}\,\vec u(P) = \varphi(P)\end{array}\right.$$ (se for nula, o campo é solenoidal)

Operadores superpostos

$$\text{rot. grad}\, f(P) = 2V\left(\frac{d\,\text{grad}\, f(P)}{dP}\right)$$

O laplaciano é uma aplicação destes operadores: $\nabla_2 f(P) = \text{div. grad}\, f(P)$

índice (2) para indicar f. o operador é de segunda ordem.

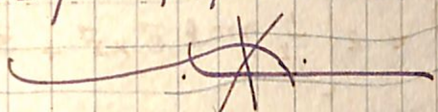
Camargueano (?!?!):

$$\Delta'_2\vec u(P) = \text{grad div}\,\vec u - \text{rot. rot}\,\vec u.$$

Determinar $f(\rho)$ de maneira que:

$\Delta_2 f(\rho) = 0$ (função harmônica da Física)

$\rho = |\vec c\wedge\vec r|$ $|\vec c| = 1$ $d\vec c = 0$ $\vec r = P - O$



26. 8. 46. falta exerc. sobre Δ_2 (Lapl.)

Calcular: $\Delta'_2\,\frac{(\vec c\wedge\vec r)\wedge\vec r}{\vec r\times\vec r}$ $|\vec c| = 1$ e fixo, $\vec r = P - O$.

façamos $\vec r\times\vec r = r^2$

Das combinações possíveis dos operadores simples:

1) ~~grad rot~~ grad div ~~rot div~~ 2) div grad 3) div rot gradgrad 4) rot rot s/ rot grad div div

Destes, $\begin{cases} \text{div. rot} = 0 \\ \text{rot. grad} = 0 \end{cases}$

Restam três tiram-pé:

$\Delta_2 \equiv$ div. grad. (aplicado a campo escalar)

$\Delta_2' \equiv$ grad div − rot. rot (aplicado a campo vetorial)

$$\Delta_2' \frac{(\vec{c}\wedge\vec{r})\wedge\vec{r}}{r^2} = \text{grad div} \frac{(\vec{c}\wedge\vec{r})\wedge r}{r^2} - \text{rot rot} \frac{(\vec{c}\wedge\vec{r})}{r}$$

$$\text{div}\frac{(\vec{c}\wedge\vec{r})\wedge\vec{r}}{r^2} = \frac{1}{r^2}\,\text{div}(\vec{c}\wedge\vec{r})\wedge\vec{r} + \underbrace{\left(\text{grad}\frac{1}{r^2}\times[(\vec{c}\wedge\vec{r})\wedge\vec{r}]\right)}_{=0}$$

$$\text{div}(\vec{c}\wedge\vec{r})\wedge\vec{r} = -(\vec{c}\wedge\vec{r})\times \cancel{\text{rot}\,\vec{r}} + \cancel{\vec{r}\times\text{rot}(\vec{c}\wedge\vec{r})} =$$

$(\text{rot}\,\vec{c}\wedge\vec{r} = 2\vec{c})$ $(\text{rot}\,\vec{r} = 0)$

$$= \vec{r}\times 2\vec{c}$$

$$\text{grad}\frac{1}{r^2} = -\frac{2}{r^3}\,\text{grad}\,\vec{r} = -\frac{2}{r^4}\cdot\vec{r}$$

então: $\text{div}\frac{(\vec{c}\wedge\vec{r})\wedge\vec{r}}{r^2} = \boxed{\frac{2\vec{c}\times\vec{r}}{r^2} - \frac{2}{r^4}}$

$$\text{grad}\, 2\frac{\vec{c}\times\vec{r}}{r^2} = 2\left[\frac{1}{r^2}\,\text{grad}\,\vec{c}\times\vec{r} + \vec{c}\times\vec{r}\,\text{grad}\frac{1}{r^2}\right] =$$

$$= \boxed{2\left[\frac{\vec{c}}{r^2} - \frac{2\vec{c}\times\vec{r}}{r^4}\cdot\vec{r}\right]}$$ (1ª parcela de Δ_2')

Passemos ao rotacional:

$$\text{rot}\frac{(\vec{c}\wedge\vec{r})\wedge\vec{r}}{r^2} =$$

Pela 3ª propr.: $(\text{rot}\, f\,\vec{u})$

$$= \frac{1}{r^2}\,\text{rot}\left[(\vec{c}\wedge\vec{r})\wedge\vec{r}\right] + \text{grad}\frac{1}{r^2}\wedge\left[(\vec{c}\wedge\vec{r})\wedge\vec{r}\right]$$

$$= \frac{1}{r^2}\,\text{rot}\left[(\vec{c}\times\vec{r})\,\vec{r} - \frac{1}{r^2}\,\text{rot}\, r^2\vec{c}\right] - \frac{2}{r^4}\left[(\vec{c}\times\vec{r})\vec{r} - r^2\vec{c}\right] =$$

$$= \frac{1}{r^2}\left[\vec{c}\times\vec{r}\;\underbrace{\text{rot}\,\vec{r}}_{=0} + \text{grad}(\vec{c}\times\vec{r})\wedge\vec{r}\right] -$$

$$-\frac{1}{r^2}\left[r^2\underbrace{\text{rot}\,\vec{c}}_{=0} + \text{grad}\, r^2\wedge\vec{c}\right] + \frac{2}{r^2}\,\vec{r}\wedge\vec{c} =$$

$$= \frac{1}{r^2}\,\vec{c}\wedge\vec{r} - \cancel{\frac{1}{r^2}\,2\vec{r}\wedge\vec{c}} + \cancel{\frac{2}{r^2}\,\vec{r}\wedge\vec{c}} =$$

$$= \boxed{\frac{1}{r^2}\,\vec{c}\wedge\vec{r}}$$

aplicando rot. novamente:

$$\text{rot}\frac{1}{r^2}\,\vec{c}\wedge\vec{r} = \frac{1}{r^2}\,\text{rot}(\vec{c}\wedge\vec{r}) + \text{grad}\frac{1}{r^2}\wedge(\vec{c}\wedge\vec{r})$$

$$= \frac{2\vec{c}}{r^2} - \frac{2}{r^3}\cdot\frac{\vec{r}}{|\vec{r}|}\wedge(\vec{c}\wedge\vec{r}) =$$

$$= \frac{2\vec{c}}{r^2} - \frac{2}{r^4}\left[r^2\vec{c} + (\vec{r}\times\vec{c})\vec{r}\right] =$$

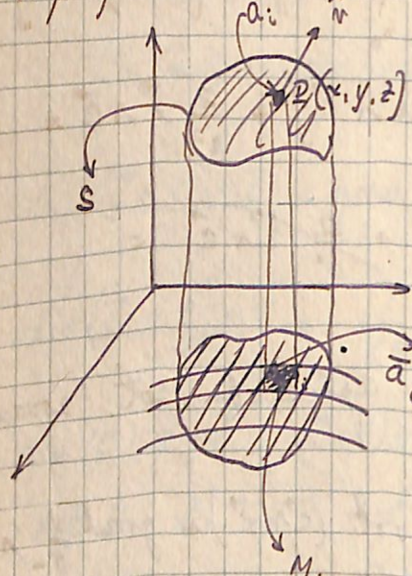
$$\frac{2(\vec{r}\times\vec{c})\vec{r}}{r^4}$$

$$\Delta_2'(\ldots) = \boxed{2\frac{\vec{c}}{r^2} - 6\frac{\vec{r}\times\vec{c}}{r^4}\,\vec{r}}$$

28. 8. 46

Integrais de superfície.

as coord. de $f(x, y, z)$ devem ser coord. de 1 superfície.



Consideramos um elemento σ_i sobre a superfície.

Temos:

$$\lim_{\substack{\sigma_i \to 0 \\ (n \to \infty)}} \sum_1^n f(x_i, y_i, z_i)\sigma_i = I_S = \iint_S f(x, y, z)\, dS$$

Onde $dS = \sqrt{EG - F^2}\, du\, dv$

$$P = O + \varphi(u, v)$$

e $\begin{cases} E = P'_u \times P'_u \\ F = P'_u \times P'_v \\ G = P'_v \times P'_v \end{cases}$ ou $dS = \sqrt{A^2 + B^2 + C^2}\, du\, dv$

ou ainda $dS = \dfrac{dx\, dy}{\cos \gamma} = \dfrac{dx\, dz}{\cos \beta}$ - - - -

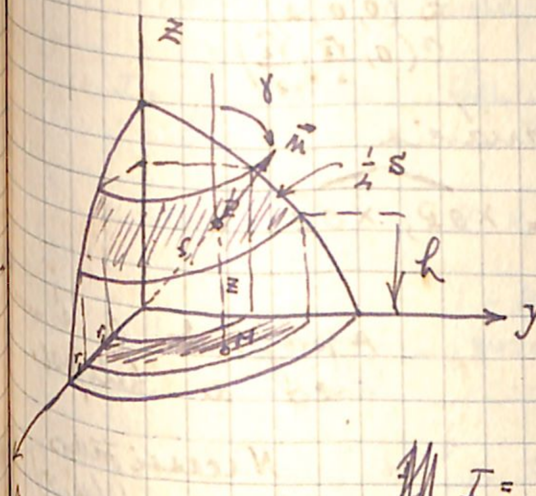
quando $z = \varphi(x, y)$

$dS = \sqrt{1 + p^2 + q^2}\, dx\, dy$ $\quad \begin{cases} p = \dfrac{\partial z}{\partial x} \\ q = \dfrac{\partial z}{\partial y} \end{cases}$

e mais $dS = |P'_u \wedge P'_v|\, du\, dv$.

Exercicios

Calcular $I = \iint_S z\, dS$ sendo S a zona de superf. esferica $x^2 + y^2 + z^2 = r^2$ com as bases nos planos $z = h$ e $z = k > h$



Então:

$$I = \iint_\Omega z \frac{dx\, dy}{\cos \gamma}$$

pela figura observamos que $\cos \gamma = z/r$ donde:

$$I = r \iint_\Omega dx\, dy = r\pi\left[r_1^2 - r_2^2\right] = \pi r\left[(r^2 - h^2) - (r^2 - k^2)\right] =$$

$$= \underline{\underline{\pi r (k^2 - h^2)}}$$

$I = \pi r (k^2 - h^2)$

Pela integral: $\iint_\Omega dx\, dy$

$x = \rho \cos \theta$ $\quad |J| = \rho$ →

$y = \rho \sin \theta$

$$\iint_{\Omega'} \rho\, d\rho\, d\theta = \int_0^{2\pi} d\theta \int_{r_2}^{r_1} \rho\, d\rho = \frac{2\pi}{2}(r_1^2 - r_2^2) =$$

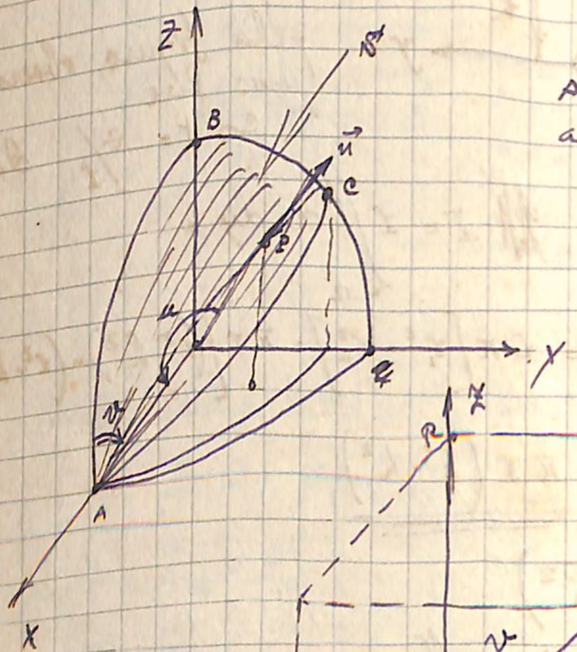
$$= \pi (k^2 - h^2)$$

Calcular

$$I = \iint_S x \, dS$$ extendida ao triang. esferico situado na esfera $x^2+y^2+z^2=1$ e vertices $A(1,0,0)$, $B(0,0,1)$, $C\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

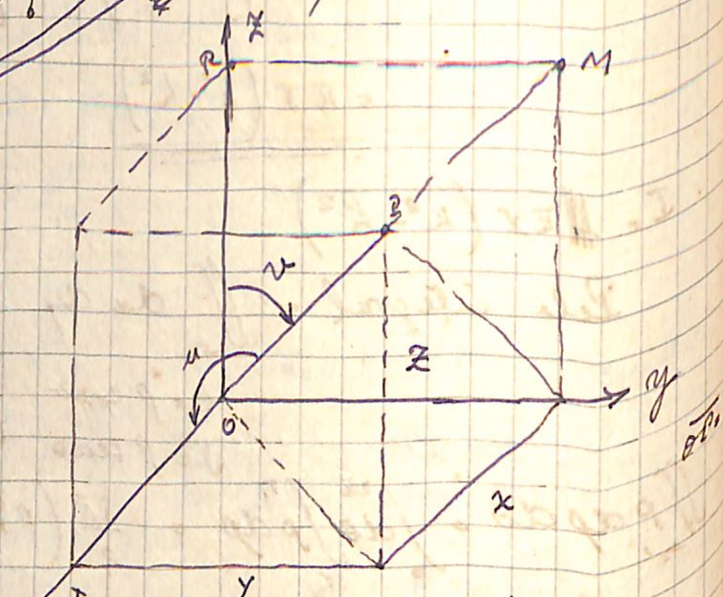
Com a mudança de variáveis

$u = \widehat{OP, Ox}$ $v = \widehat{xOP, xOz}$



A proj. do Δ é um arco de helipse.

Necessitamos de $x = f_1(u,v)$
$y = f_2(u,v)$
$z = f_3(u,v)$



$\left(Z\hat{O}M = \vec{v} \,!!!\right)$

$$\boxed{x = \cos u}$$

$$\begin{cases} y = \overline{OM} \cdot \operatorname{sen} v \\ z = \overline{OM} \cdot \cos v \end{cases}$$

mas $\overline{OM} = \overline{TP}$ donde:

$\overline{OM} = \operatorname{sen} u$

y ~~= ...~~

$$\begin{cases} \boxed{y = \operatorname{sen} u \cdot \operatorname{sen} v} \\ \boxed{z = \operatorname{sen} u \cdot \cos v} \end{cases}$$

Voltemos à integral:

$$I = \iint_\Omega \cos u \cdot \frac{du\,dv}{\cos \gamma} |J|$$

$$\cos \gamma = z/r = z = \operatorname{sen} u \cdot \cos v$$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\operatorname{sen} u & 0 \\ \cos u \operatorname{sen} v & \operatorname{sen} u \cos v \end{vmatrix} =$$

$$= |\operatorname{sen}^2 u \cos v|$$

$$I = \iint_\Omega \cos u \cdot \operatorname{sen} u \cdot du\,dv =$$

$$= \int_0^{\pi/2} \cos u \cdot \operatorname{sen} u \, du \int_0^{\pi/4} dv = \frac{\pi}{4}\left[\frac{\operatorname{sen}^2 u}{2}\right]_0^{\pi/2} = \frac{\pi}{8}$$

Projetando sobre o plano yOz, obtemos $1/8$ de circulo —

$$I = \iint_S x \frac{dy\,dz}{\cos \alpha} = \iint x \frac{dy\,dz}{\frac{x}{r}} = \iint dy\,dz =$$

$$= \pi/8$$

4.9.46.

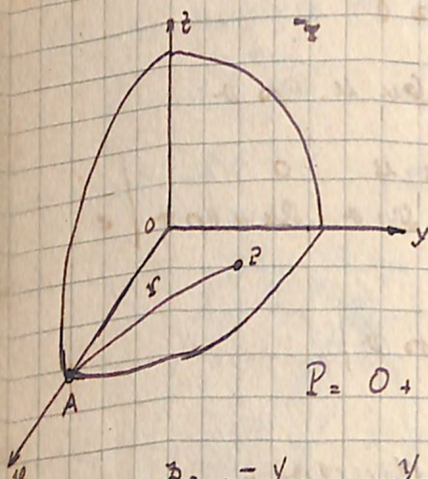
Calcular a integral de superf.:

$I = \iint_S \frac{ds}{r}$ extendida à porção de superficie da esfera $x^2+y^2+z^2=1$

$x, y, z \geq 0$

onde $r = \overline{AP}$

$A \equiv (1,0,0)$ $P \equiv (x, y, z)$



$r = \sqrt{(x-1)^2+y^2+z^2}$

$ds = \sqrt{1+p^2+q^2}\, dz\, dy$

~~p = O + x~~ $p = \frac{\partial x}{\partial y}$

$q = \frac{\partial x}{\partial z}$

$P = O + \sqrt{1-(y^2+z^2)}\,\vec{i} + y\vec{j} + z\vec{k}$

$p = \frac{-y}{\sqrt{\cdots}} = -\frac{y}{x}$ $q = \frac{-z}{x}$

$ds = \sqrt{1+\frac{y^2}{x^2}+\frac{z^2}{x^2}}\, dy\, dz = \frac{1}{x}\, dy\, dz$

mais ainda:

$r = \sqrt{2-2x} = \sqrt{2(1-x)}$ logo:

$I = \iint_\Omega \frac{dy\, dz}{x\sqrt{2(1-x)}} = \frac{1}{\sqrt{2}} \iint \frac{dy\, dz}{\sqrt{1-y^2-z^2}\sqrt{1-\sqrt{1-y^2+z^2}}} =$

Projetemos sobre o plano (x, z)

$p = \frac{\partial y}{\partial x} = -\frac{x}{y}$ $q = \frac{\partial y}{\partial z} = -\frac{z}{y}$

$ds = \sqrt{1+p^2+q^2}\, dx\, dy = \sqrt{1+\frac{x^2}{y^2}+z^2/y^2}\, dx\, dz =$

$= \frac{1}{y}\, dx\, dz$

$I = \iint \frac{dx\, dz}{\sqrt{1-x^2-z^2}\,\sqrt{2-2x}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{1-x}} \underbrace{\int_0^{\sqrt{1-x^2}} \frac{dz}{\sqrt{1-x^2-z^2}}}_{I_1}$

$I_1 = \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-\left(\frac{z}{\sqrt{1-x^2}}\right)^2}} \frac{dz}{\sqrt{1-x^2}} = \left[\text{arc sen} \frac{z}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}}$

$= \pi/2$

$I = \frac{\pi}{2\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{1-x}} = -\frac{\pi}{2\sqrt{2}} \int_0^1 (1-x)^{-1/2}(-dx) =$

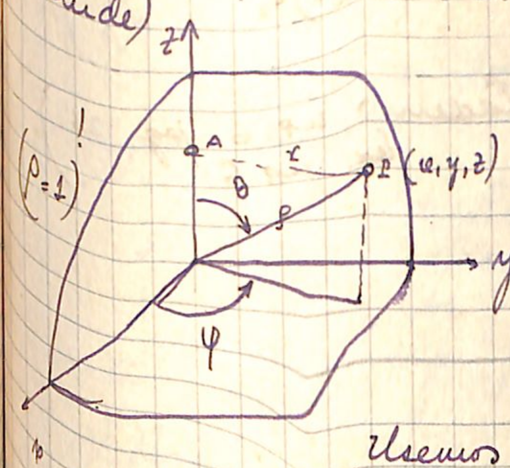
$= -\frac{\pi}{\sqrt{2}} \left[(1-x)^{1/2}\right]_0^1 = \frac{\pi}{\sqrt{2}} = \boxed{\frac{\pi\sqrt{2}}{2}}$ Independendo do ponto P.

Calcular:

$I = \iint_\sigma \frac{d\sigma}{r^4}$ $r = \overline{AP}$ $A \equiv (0,0,h)$ $h<1$

$P \equiv (x, y, z)$

sendo σ toda a esfera $x^2+y^2+z^2=1$ usando como parametros φ (longit.), θ (colatitude)



$P = O + x\vec{i} + y\vec{j} + z\vec{k}$

$x = \text{sen}\,\theta \cos\varphi$

$y = \text{sen}\,\theta\, \text{sen}\,\varphi$

$z = \cos\theta$

$P = O + \text{sen}\,\theta \cos\varphi\, \vec{i} + \text{sen}\,\theta\, \text{sen}\,\varphi\, \vec{j} + \cos\theta\, \vec{k}$

Usemos

$d\sigma = \sqrt{EG-F^2}\, d\theta\, d\varphi$

$\begin{cases} P'_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\text{sen}\,\varphi\,\vec{j} - \text{sen}\,\theta\,\vec{k} \\ P'_\varphi = -\text{sen}\,\theta\,\text{sen}\,\varphi\,\vec{i} + \text{sen}\,\theta\cos\varphi\,\vec{j} \end{cases}$

$E = P'_\theta \times P'_\theta = 1$ $G = P'_\varphi \times P'_\varphi = \text{sen}^2\theta$

$F = P'_\theta \times P'_\varphi = 0$ donde.

$d\sigma = \sqrt{\operatorname{sen}^2\theta}\, d\theta\, d\varphi = \operatorname{sen}\theta\, d\theta\, d\varphi$

Determinemos r: pode-se fazer por G. An.

$r = \sqrt{x^2+y^2+(z-h)^2}$; mas pela fig.:

$r^2 = h^2 + 1 - 2h\cos\theta$

$I = \iint_\Omega \frac{\operatorname{sen}\theta\, d\theta\, d\varphi}{(1+h^2-2h\cos\theta)^2} =$

estudar esta variação!

$= \int_0^{2\pi} d\varphi \underbrace{\int_0^{\pi} \frac{\operatorname{sen}\theta\, d\theta}{(1+h^2-2h\cos\theta)^2}}_{I_1}$

$I_1 = \int_0^{\pi} \frac{\operatorname{sen}\theta\, d\theta}{(1+h^2-2h\cos\theta)^2} = \frac{1}{2h}\int_0^{\pi} (1+h^2-2h\cos\theta)^{-2}\, d(2h\operatorname{sen}\theta) =$

$= -\frac{1}{2h}\left[\frac{1}{1+h^2-2h\cos\theta}\right]_0^{\pi} = \frac{2}{(1-h^2)^2}$

$I = \frac{4\pi}{(1-h^2)^2}$ (Caderno nº 6 pg 23. Ex. nº 6)

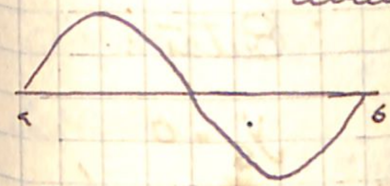
20.8.46. Dr. Breves.

Cálculo de areas e volumes p/ integrais.

1) Areas.

Uma area pode ser calculada por integral simples ou dupla.

$\int_a^b y\,dx$ — área relativa, isto é, a parte situada acima do eixo dos x é posit. e a parte situada abaixo, negativa.



Ex: $\int_0^{\pi} \cos x\, dx = 0$

Para se obter a área em val. absoluto deve-se substituir a expressão anter. por $\int_a^b |y|\,dx$

Tambem se pode calcular area por integral dupla:

$\iint dx\,dy$ neste caso a área é sempre absoluta, desde que considere-se a variação de x e de y sempre no sentido crescente.

2) Volumes — Valem observações análogas. Assim: $\iint z\,dx\,dy$ conduz sempre a um volume relativo, isto é, para z posit. é posit. e p/ z neg. é neg.

O vol. absoluto é dado por $\iint |z|\,dx\,dy$

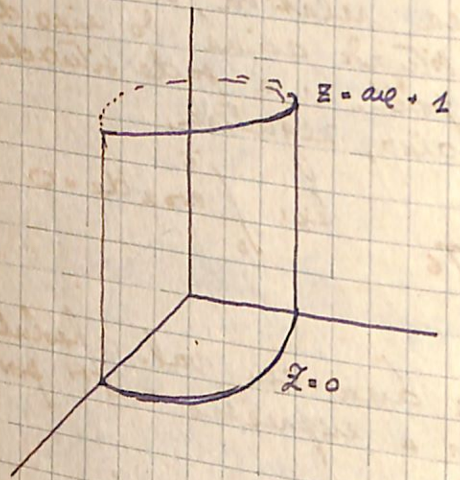
$\iiint dx\,dy\,dz$ tambem fornece volume, será absoluto se considerarmos x, y, z sempre em sentido crescente.

Calcular o baricentro do volume limitado pelo cilindro

$x^2+y^2=1$ e pos planos

$z=0$

$z=ax+1.$



Entao:

$y_G=0$

$x_G=\frac{1}{V}\int x\,dV$

$z_G=\frac{1}{V}\int z\,dV$

Mas no caso: $V=\pi r^2 h=\pi$.

donde $x_G=\frac{1}{V}\int x\,dv=\pi^{-1}\int x\,dv$

Calculemos

$I_1=\iiint x\,dx\,dy\,dz=\int x\,dv \rightarrow$ por mudança para coordenadas cilindricas:

$x=u\cos v$ $\quad y=u\,\text{sen}\,v$ $\quad z=z$

$|J|=u$

$I_1=\iiint u^2\cos v\,du\,dv\,dz=$

$=\int_0^1 u^2\,du\int_0^{2\pi}\cos v\,dv\int_0^{au\cos v+1}dz=$

$=\int_0^1 u^2\,du\int_0^{2\pi}(au\cos^2 v+\cancel{\cos v})\,dv=$

$=\int_0^1 u^2\,du\int_0^{2\pi}\left[au\left(\frac{1}{2}+\frac{1}{2}\cos 2v\right)dv\right]=$

$=a\pi\int_0^1 u^3\,du=a\pi\left[\frac{u^4}{4}\right]_0^1=\frac{a\pi}{4}$

donde: $X_G=\frac{1}{4}a$

Calculemos $\int z\,dv=\iiint z\,dx\,dy\,dz=$

$=\iiint z\,u\,du\,dv\,dz=\int_0^1 u\,du\int_0^{2\pi}dv\int_0^{au\cos v+1}z\,dz=$

$=\frac{1}{2}\int_0^1 u\,du\int_0^{2\pi}(au\cos v+1)^2\,dv=$

$=\frac{1}{2}\int_0^1 u\,du\int_0^{2\pi}\left[a^2u^2\left(\frac{1}{2}+\frac{1}{2}\cos 2v\right)+2au\cos v+1\right]dv=$

$=\frac{2\pi}{2}\int_0^1(a^2u^2+1)\,u\,du=\pi\left[\frac{1}{8}a^2u^4+\frac{1}{2}u^2\right]_0^1=$

$=\frac{\pi}{8}a^2+\frac{\pi}{2}$ portanto

$Z_G=\frac{a^2}{8}+\frac{1}{2}$

27.8.46

Baricentro da sup. do paraboloide $z = x^2 + 2y^2$ int.^{r} ao cilindro $4x^2 + 16y^2 = 1$

$x, y, z \geq 0$

$$ds = \sqrt{1 + 4x^2 + 16y^2}\; dx\, dy$$

$2x = u \cos v \qquad J = \frac{1}{8} u$

$4y = u \operatorname{sen} v$

$$ds = \frac{1}{8} \sqrt{1+u^2}\, u\, du\, dv$$

$$S = \frac{1}{8} \int_0^{\pi/2} dv \int \sqrt{1+u^2}\, u\, du = \frac{\pi}{48} \left(2^{3/2} - 1\right)$$

Daqui: $x_G = \dfrac{\int x\, ds}{S}$

$y_G = \dfrac{\int y\, ds}{S}$

$z_G = \dfrac{\int z\, ds}{S}$

$$\int x\, ds = \frac{1}{16} \int_0^{\pi/2} \cos v\, dv \int_0^1 u^2 \sqrt{1+u^2}\, du$$

Integremos por funções hiperbolicas: — notemos que:

$\operatorname{sen} h\, x = \dfrac{e^x - e^{-x}}{2}$

$\cos h\, x = \dfrac{e^x + e^{-x}}{2}$

$\dfrac{d \operatorname{sen} h x}{dx} = D \operatorname{sen} h\, x = \cos h\, x$

$D \cos h\, x = \operatorname{sen} h\, x$

Todas formas da trigon. hip. obtem-se das formas da trigon. circular, substituindo $\cos x$ por $c.h\, x$ e $\operatorname{sen} x$ por $i \operatorname{sen} h\, x$.

No caso seria:

$$\begin{cases} u = s h\, x & \rightarrow 1 + u^2 = c.h\, x \\ du = ch\, x\, dx \end{cases}$$

$u = 0 \rightarrow \operatorname{sen} h\, x = 0$ para $x = 0$

$u = 1 \rightarrow \dfrac{e^x - e^{-x}}{2} = 1 \quad \left\{ e^x - \dfrac{1}{e^x} = 2 \right.$

$e^{2x} - 1 = 2e^x$ ou $(e^x)^2 - 2e^x - 1 = 0$

donde: $e^x = 1 \pm \sqrt{2}$ ou

$x = l(1 + \sqrt{2})$

logo: $I = \displaystyle\int_0^{l(1+\sqrt{2})} sh^2 x . ch^2 x\, dx = \frac{1}{4} \int_0^{l(1+\sqrt{2})} 4 sh^2 x . ch\, x\, dx =$

$= \dfrac{1}{4} \displaystyle\int_0^{l(1+\sqrt{2})} sh^2 2x\, dx$

Observemos que, de: $\operatorname{sen}^2 x = 1/2 - 1/2 \cos 2x$

obtemos fica $-sh^2 x = 1/2 - 1/2\, ch 2x$

$$I_1 = \frac{1}{4} \int_0^{l(1+\sqrt{2})} \left(-\frac{1}{2} + \frac{1}{2} ch 4x\right) dx =$$

$$= \frac{1}{4} \left[-\frac{x}{2} + \frac{1}{8} sh 4x\right]_0^{l(1+\sqrt{2})} = \frac{1}{8} \left[\frac{1}{4} sh 4 l(1+\sqrt{2}) - l(1+\sqrt{2})\right]$$

voltando atrás:

$$x_G = \frac{1}{128 . S} \left[\frac{1}{4} sh 4 l(1+\sqrt{2}) - l(1+\sqrt{2})\right]$$

$y_G = \frac{\int y\,ds}{s}$

$$\int y\,ds = \frac{1}{32}\int_0^{\pi/2} \operatorname{sen} v \cdot dv \int_0^1 u^2\sqrt{1+u^2}\,du$$

$$y_G = -\frac{1}{256\,s}\left[\ \dots\ \right]$$

$z_G = \frac{\int z\,ds}{s}$ $\qquad z = x^2 + 2y^2$

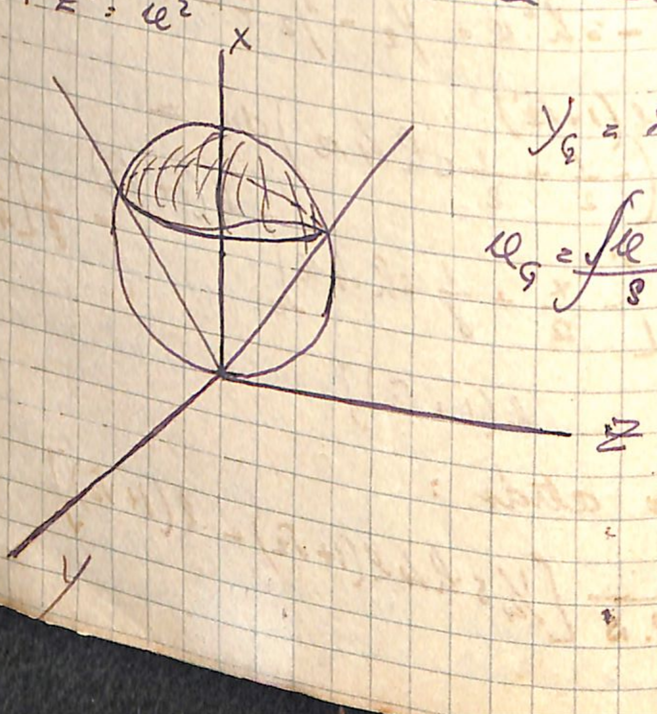
$$z = \frac{1}{4}u^2\left[\cos^2 v + \frac{1}{2}\operatorname{sen}^2 v\right] = \frac{1}{4}u^2\left[1 - \frac{1}{2}\operatorname{sen}^2 v\right]$$

$$\int z\,ds = \frac{1}{32}\int_0^{\pi/2}\left(1-\frac{1}{2}\operatorname{sen}^2 v\right)dv \int u^3\sqrt{1+u^2}\,du$$

fazer $1+u^2 = t^2$

$$\boxed{z_G = \frac{\pi}{640}\left(\sqrt{2}+1\right)}$$

Ex. Determ. o baric. da superf. esferica $x^2+y^2+z^2 = 2x$, interna ao cone $y^2+7z^2 = x^2$



$$y_G = z_G = 0$$

$$x_G = \frac{\int x\,ds}{s}$$

empregue-se coordenadas polares no espaço.

no caso: $x = \rho\cos\theta$

$y = \rho\operatorname{sen}\theta\cos\varphi$

$z = \rho\operatorname{sen}\theta\operatorname{sen}\varphi$

$x^2+y^2+z^2 = \rho^2 = 2\rho\cos\theta$ logo,

$\rho = 2\cos\theta$ e

$$\vec{P} = O + 2\cos^2\theta\,\vec{i} + 2\operatorname{sen}\theta\cos\theta\cos\varphi\,\vec{j} + 2\operatorname{sen}\theta\cos\theta\operatorname{sen}\varphi\,\vec{k}$$

esta é a repres. param. da superf. da esfera em fç. dos parametros θ e φ

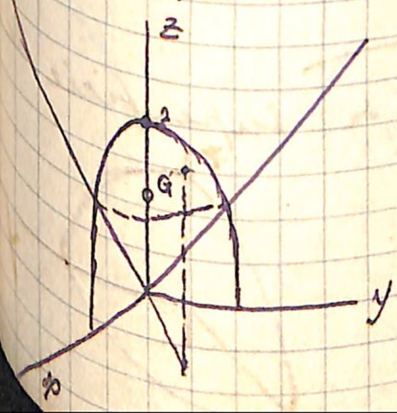
$$ds = \left|P'_\theta \wedge P'_\varphi\right| d\theta\,d\varphi = 2\operatorname{sen}2\theta\,d\theta\,d\varphi$$

$\varphi \to$ de 0 a 2π

$\theta \to$ de 0 a $\bar{\theta}$ (ef. da junção inters. das 2 superf.)

8.9.946.

Consid. o cone $a^2x^2+y^2 = z^2$ e o paraboloide: $a^2x^2+y^2 = 2-z$, determinar o baric. do volume limitado pelo cone e pelo paraboloide. $(z \geq 0)$ O cone é elítico.



Neste caso: $x_G = y_G = 0$ e

$$z_G = \frac{1}{V}\int z\,dV$$

$$V = \iiint_\Omega dx\,dy\,dz$$

Façamos mudanças para

coordenadas polares cilíndricas (polares)

$x = \rho\cos\theta \qquad y = \rho\,\mathrm{sen}\,\theta \qquad z = z$

$|J| = \rho/a$

$$V = \iiint \rho/a \; d\rho\, d\theta\, dz$$

Ficamos com:

$\begin{cases} \text{cone: } \rho^2 = z^2 \\ \text{parabol.: } \rho^2 = 2 - z \end{cases}$

$$V = \frac{1}{a}\int_0^1 \rho\, d\rho \int_0^{2\pi} d\theta \int_\rho^{2-\rho^2} dz = \frac{1}{a}\int_0^1 \rho\, d\rho \int_0^{2\pi} (2-\rho^2-\rho)\, d\theta =$$

$$= \frac{2\pi}{a}\int_0^1 (2\rho - \rho^3 - \rho^2)\, d\rho = \frac{2\pi}{a}\left[\frac{2\rho^2}{2} - \frac{\rho^4}{4} - \frac{\rho^3}{3}\right]_0^1$$

$= \frac{5\pi}{6a}$ Voltemos a $Z_G = \frac{1}{V}\int z\, dv =$

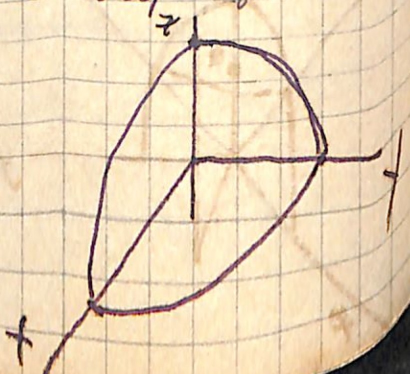
$$= \frac{6a}{5\pi}\int_0^1 \rho\, d\rho \int_0^{2\pi} d\theta \int_\rho^{2-\rho^2} z\, dz = \frac{6a}{5\pi}\int_0^1 \rho\, d\rho \int_0^{2\pi} \left[\frac{z^2}{2}\right]_\rho^{2-\rho^2} =$$

$$= \frac{12\pi a}{2{,}5\pi}\int_0^1 \rho(4 + \rho^4 - 5\rho^2)\, d\rho = \frac{12a}{10}\left[2\rho^2 + \frac{\rho^5}{5} - \frac{5\rho^3}{3}\right]_0^1$$

$$= \frac{12a}{10}\cdot\frac{11}{12} = \boxed{\frac{11}{10}a}$$

Achar o baricentro da superfície de um oitante de esfera.

$\boxed{x^2 + y^2 + z^2 = a^2}$



Devido à equação da esfera ser simétrica em relação aos três eixos, devemos ter:

$X_G = Y_G = Z_G$ então, p. ex.

$$X_G = \frac{1}{S}\int x\, ds \qquad S = \pi a^2/2$$

$$I_1 = \int_\Omega x\, ds = \int x\sqrt{EG - F^2}\, d\theta\, d\varphi$$

fazendo $\begin{cases} x = \rho\,\mathrm{sen}\,\theta\cos\varphi \\ y = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi \\ z = \rho\cos\theta \end{cases} \qquad \rho = a$

$P = O + a\,\mathrm{sen}\,\theta\cos\varphi\,\vec{i} + a\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\,\vec{j} + a\cos\theta\,\vec{k}$

$P'_\theta = a\cos\theta\cos\varphi\,\vec{i} + a\cos\theta\,\mathrm{sen}\,\varphi\,\vec{j} - a\,\mathrm{sen}\,\theta\,\vec{k}$

$P'_\varphi = -a\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\,\vec{i} + a\cos\theta\cos\varphi\,\vec{j}$.

$$I_1 = \int_{\Omega_1} x\,|P'_\theta \wedge P'_\varphi|\, d\theta\, d\varphi$$

$$|P'_\theta \wedge P'_\varphi| = a^2\,\mathrm{sen}\,\theta\sqrt{\mathrm{sen}^2\theta\cos^2\varphi + \mathrm{sen}^2\theta\,\mathrm{sen}^2\varphi + \cos^2\theta} = a^2\,\mathrm{sen}\,\theta$$

$$I_1 = a^3\iint \mathrm{sen}^2\theta\cos\varphi\, d\theta\, d\varphi = a^3\int_0^{\pi/2}\mathrm{sen}^2\theta\, d\theta\int_0^{\pi/2}\cos\varphi\, d\varphi =$$

$$= \frac{a^3}{2}\cdot\frac{\pi}{2} = \frac{a^3\pi}{4}$$

$$X_G = \frac{a^3\pi}{4}\Big/\frac{\pi a^2}{2} = \frac{2a^3\pi}{4\pi a^2} = \frac{a}{2}$$

9.9.46 Outra na provinha.

Sejam três funções $P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$ contínuas e q/ a prim.ª derivada p.º menos, contínua.

e sejas a superf. $\begin{cases} x = \varphi_1(u,v) \\ y = \varphi_2(u,v) \\ z = \varphi_3(u,v) \end{cases}$

Por definição:

$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, dS$ é uma integral de superfície.

Outra notação:

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = (p.\ def.) =$$

$$= \iint_D P_1 \frac{d(y,z)}{d(u,v)} + Q_1 \frac{d(z,x)}{d(u,v)} + R_1 \frac{d(x,y)}{d(u,v)}$$

$$\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \quad (s/ val. abs.) \ldots$$

e $P_1 = P[\varphi_1(u,v), \varphi_2, \varphi_3]$
$Q_1 = Q[\varphi_1, \varphi_2, \varphi_3]$ ...

estendida ao domínio de variação de u e v.

Sinal - não se toma $\cos\alpha$, $\cos\beta$, $\cos\gamma$ em valor absoluto. Deve-se sempre convencionar qual das normais se usa.

Calcular:

$$I = \iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \quad \text{(Lotion pg 107)}$$

sendo S: $x = (a + b\cos u)\cos v$
$y = (a + b\cos u)\,\mathrm{sen}\, v$ $\quad z = b\,\mathrm{sen}\, u$

a e b n.º naturais, $0 \le u, v \le 2\pi$ (D)

$$I = \iint_D P_1 \frac{d(y,z)}{d(u,v)} + Q_1 + \ldots$$

① $$\frac{d(y,z)}{d(u,v)} = \begin{vmatrix} -b\,\mathrm{sen}\, u\,\mathrm{sen}\, v & (a+b\cos u)\cos v \\ b\cos u & 0 \end{vmatrix} =$$

$$= -(a + b\cos u)\, b\cos u \cdot \cos v$$

② $$\frac{d(z,x)}{d(u,v)} = \begin{vmatrix} b\cos u & 0 \\ -b\,\mathrm{sen}\, u\cos v & -(a+b\cos u)\,\mathrm{sen}\, v \end{vmatrix} =$$

$$= -(a + b\cos u)\, b\cos u \cdot \mathrm{sen}\, v$$

③ $$\frac{d(x,y)}{d(u,v)} = \cancel{\;} = -b(a+b\cos u)\,\mathrm{sen}\, u\cos^2 v - b(a + b\cos u)\,\mathrm{sen}\, u\,\mathrm{sen}^2 v$$

$$I = \iint_D \Big[-b(a + b\cos u)^2 \cos u \cos^2 v - b(a+b\cos u)^2 \cos u\,\mathrm{sen}^2 v - b^2(a + b\cos u)\,\mathrm{sen}^2 u \Big]\, du\, dv$$

$$= -b \iint_D (a + b\cos u)\left[(a + b\cos u)\cos u + b\,\mathrm{sen}^2 u\right] du\, dv$$

$$= -b \iint_D (a + b\cos u)(a\cos u + b)\, du\, dv$$

$$= -b \iint_D \left(\cancel{a\cos u}\; a^2 \cos u + b^2\cos u \ldots + ab\cos^2 u + ab\right] du\, dv$$

$$= -b \iint_D \left[(a^2 + b^2)\cos u + ab(\cos^2 u + 1)\right] du\, dv$$

$$= -b \int_0^{2\pi} dv \int_0^{2\pi} \Big[\underbrace{(a^2 + b^2)\cos u}_{=0} + ab(\cos^2 u + \ldots$$

$= -2\pi a b^2 \int (\cos^2 u + 1)\, du = -2\pi a b^2 \int \left(\frac{3}{2} + \frac{1}{2}\cos 2u\right) du =$

$= -\boxed{6 a b^2 \pi^2}$

Calcular: $I = \iint_S \frac{k(x\cos\alpha + y\cos\beta) + T\cos\gamma}{x^2+y^2}\, dS$

Sendo $k = \frac{y^2}{x}$ $T = \cancel{\quad} = 2x + \frac{y}{z}$

Sendo

S a porção de superf. $x^2 + y^2 = 4z^2$ (com c/ na orig.)

$x > 0 \quad y > 0$

$1/4 \le z \le 1/2$

pode-se fazer $x \equiv u$ e $y \equiv v$, ou outras.

10.9.46

Chama-se momento de inércia de 1 dist.º de pontos materiais $P_i(m_i)$ em rel. a um eixo à expressão

$I = \sum m_i d_i^2$ sendo d_i as dist. de P_i ao eixo.

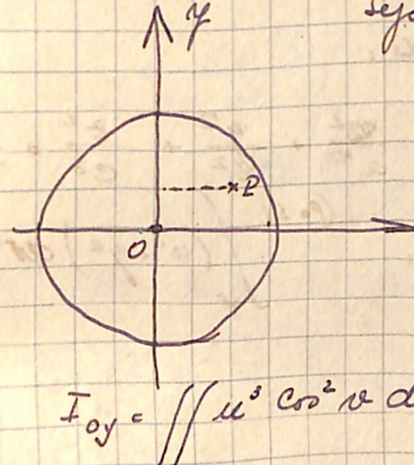
No caso de distribuição cont. de massa, a somatória deve ser substituída por $\int$.

— x —

Exercícios:

Momento de inércia de um círc.º em relação a um diâmetro.

Seja em rel. a Oy:



$I_{Oy} = \int_S x^2\, ds$

$x^2 + y^2 = R^2$ (círc.º)

Substituamos:

$\begin{cases} x = u\cos v \\ y = u \operatorname{sen} v \end{cases} \quad |J| = u$

$I_{Oy} = \iint u^3 \cos^2 v\, du\, dv = \int_0^{2\pi} \cos^2 v\, dv \int_0^R u^3\, du =$

$= \frac{R^4}{4}\int_0^{2\pi} \cos^2 v\, dv = \frac{R^4}{4}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2v\right) dv =$

$= \cancel{\frac{R^4}{4}\int \frac{dv}{2}} + \frac{R^4}{4\cdot 2}\left[\frac{v}{2} + \frac{\operatorname{sen} 2v}{4}\right]_0^{2\pi} =$

$= \frac{R^4 \cdot 2\pi}{8} = \boxed{\frac{\pi R^4}{4}}$ mas $\rho\, ds = dm$ logo:

$I = \frac{mR^2}{4} = \pi R^2 \cdot \frac{R^2}{4}$

Cálculo do momento de inercia do mesmo circulo em rel. a um eixo vertical:

$$I_{oz} = \int (x^2+y^2)\,ds = 2I_{oy} = \frac{mR^2}{2}$$

Mom.to de inercia de um helipsoide em relação a um dos eixos.

Equação do helipsoide:

$\frac{z^2}{c^2}+\frac{y^2}{b^2}=1$ (elipse.) $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$I = \int_V \ell^2\,dm \overset{(\rho=1)}{=} \int_V (x^2+y^2)\,dv$$

$\frac{dm}{dv} = 1$

$$I = \iiint_E (x^2+y^2)\,dx\,dy\,dz$$

$$\begin{cases} x = au\cos v \\ y = bu \operatorname{sen} v \\ z = z \end{cases} \quad |J| = a.b.u$$

$$I = \int dz \int a^3 b u^3 \cos^2 v\,du\,dv + I_2$$

$u^2 + z^2/c^2 = 1 \qquad u = \sqrt{1-\frac{z^2}{c^2}}$

$$I_1 = a^3 b \int_{-c}^{c} dz \int_0^{2\pi} \cos^2 v\,dv \int_{-\sqrt{1-\frac{z^2}{c^2}}}^{+\sqrt{\cdots}} u^3\,du =$$

(ρ, π) ou $(0 \to \sqrt{\cdots})$ engano nos extr

$$= \frac{\pi a^3 b}{4}\int_{-c}^{+c} dz\; 2\left(1-\frac{z^2}{c^2}\right)^2 =$$

$$= \frac{\pi a^3 b}{2}\int_{-c}^{+c}\left(1-z^2/c^2\right)^2 dz =$$

$$= \frac{\pi a^3 b}{2}\left[z + \frac{z^5}{5c^4} - \frac{2}{3}\,\frac{z^3}{3c^2}\right]_{-c}^{c}$$

$$= \frac{\pi a^3 b}{2}\left(2c + \frac{2c^5}{5c^4} - \frac{4c^3}{3c^2}\right) =$$

$$= \frac{\pi a^3 b}{2}\left(\frac{30c+6c-20c}{15}\right) =$$

$$= \frac{16\pi a^3 bc}{2.15} = \frac{8}{15}\pi a^3 bc$$

De maneira análoga calcularíamos

$I_2 = \frac{8\pi}{15} b^3 . a . c$ finalmente:

$$I = I_1 + I_2 = \boxed{\frac{8}{15}\pi abc\left[a^2+b^2\right]}$$

mas $V_{elipsoide} = \frac{4}{3}\pi abc$ logo:

$$\boxed{I_z = \frac{2}{5}\,m\,(a^2+b^2)}$$

($I_z = 1/5\, m\,(a^2+b^2)$ c/ correção de extr.)

Caso da esfera.

$$x^2+y^2+z^2 = R^2$$

$$I = \int \ell^2\,dm = \iiint (x^2+y^2)\,dx\,dy\,dz$$

$$\left.\begin{array}{l} x = u \,\mathrm{sen}\,\theta . \cos\varphi \\ y = u \,\mathrm{sen}\,\theta . \,\mathrm{sen}\,\varphi \\ z = u \cos\theta \end{array}\right\} \quad |J| = u^2 \,\mathrm{sen}\,\theta$$

$$I = \iiint u^4 \,\mathrm{sen}^3\theta \, du \, d\theta \, d\varphi$$

$$= \int_0^a u^4 du \cdot \underbrace{\int_0^{\pi} \mathrm{sen}^3\theta \, d\theta}_{I_0} . \int_0^{2\pi} d\varphi =$$

$$= \frac{1}{5} a^5 . \frac{4}{3} . 2\pi = \frac{2a^2}{5} \underbrace{\left(\frac{4}{3}\pi a^3\right)}_{M} = \frac{2}{5} M a^2$$

$$I_0 = \int_0^{\pi} \mathrm{sen}\,\theta (1 + \cos^2\theta) = \left(-\cos\theta + \frac{1}{3}\,\mathrm{sen}^3\theta\right)_0^{\pi} = \frac{4}{3}$$

Recapitulação.

9. 10. 946

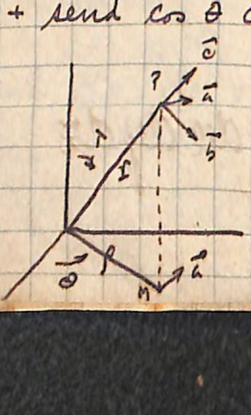
Calcular div $\vec{u}$, sendo

$\vec{u} = \lambda \vec{a} + r^2 \,\mathrm{sen}\,\theta . \vec{b} + \mathrm{sen}\,\lambda \cos\theta \, \vec{c}$, e

$P(r, \theta, \lambda)$ e mais $\vec{a}, \vec{b}, \vec{c}$, os vetores normais às sup. $\theta = c^{te}$, $\lambda = c^{te}$ e $r = c^{te}$.

Então:

$$\mathrm{div}\,\vec{u} = \lambda \,\mathrm{div}\,\vec{a} + \mathrm{grad}\,\lambda \times \vec{a} +$$
$$+ r^2 \,\mathrm{sen}\,\theta \,\mathrm{div}\,\vec{b} + \mathrm{grad}\, r^2 \,\mathrm{sen}\,\theta \times \vec{b} +$$
$$+ \mathrm{sen}\,\theta \cos\theta \,\mathrm{div}\,\vec{c} + \mathrm{grad}\,\mathrm{sen}\,\lambda \cos\theta \times \vec{c}$$



Valem as rel.:

$\vec{a} = \vec{c} \wedge \vec{b}$

$\vec{b} = \vec{a} \wedge \vec{c}$

$\vec{c} = \vec{b} \wedge \vec{a}$ pois

$\vec{a}, \vec{b}, \vec{c}$ formam um triedro triortogonal

Calc. os gradientes:

1) $\mathrm{grad}\,\lambda = \lim_{h' \to 0} \left(\frac{\Delta\lambda}{h'}\right) \vec{b} = \frac{\vec{b}}{r}$

2) $\mathrm{grad}\, r^2 \,\mathrm{sen}\,\theta = r^2 \,\mathrm{grad}\,\mathrm{sen}\,\theta + \mathrm{sen}\,\theta \,\mathrm{grad}\, r^2 =$

$= r^2 \cos\theta \,\mathrm{grad}\,\theta + 2r \,\mathrm{sen}\,\theta \,\mathrm{grad}\, r$.

$\mathrm{grad}\,\theta = \frac{\vec{a}}{\rho} = \frac{\vec{a}}{r \,\mathrm{sen}\,\lambda}$

$\mathrm{grad}\, r = \vec{c}$

$\mathrm{grad}\, r^2 \,\mathrm{sen}\,\theta = r^2 \cos\theta . \frac{\vec{a}}{r \,\mathrm{sen}\,\lambda} + 2r \,\mathrm{sen}\,\theta \,\vec{c}$

3) $\mathrm{grad}\,\mathrm{sen}\,\lambda . \cos\theta =$

$= \mathrm{sen}\,\lambda \,\mathrm{grad}\cos\theta + \cos\theta \,\mathrm{grad}\,\mathrm{sen}\,\lambda =$

$= -\mathrm{sen}\,\lambda \,\mathrm{sen}\,\theta \,\mathrm{grad}\,\theta + \cos\theta \cos\lambda \,\mathrm{grad}\,\lambda =$

$= -\cancel{\mathrm{sen}\,\lambda}\,\mathrm{sen}\,\theta \frac{\vec{a}}{r \,\cancel{\mathrm{sen}\,\lambda}} + \cos\theta \cos\lambda \frac{\vec{b}}{r}$

$= -\frac{\mathrm{sen}\,\theta}{r} \vec{a} + \frac{\cos\theta \cos\lambda}{r} \vec{b}$

Calc. dos div.

$\mathrm{div}(\vec{v_1} \wedge \vec{v_2}) = \vec{v_2} \times \mathrm{rot}\,\vec{v_1} - \vec{v_1} \times \mathrm{rot}\,\vec{v_2}$

donde: $\mathrm{div}\,\vec{a} = \mathrm{div}(\vec{c} \wedge \vec{b}) =$

$\vec{b} \times \mathrm{rot}\,\vec{c} - \vec{c} \times \mathrm{rot}\,\vec{b}$.

$\mathrm{rot}\,\vec{c} = \mathrm{rot}\,\mathrm{grad}\, r = 0$

$\vec{b} = r \,\mathrm{grad}\,\lambda \quad \mathrm{rot}\,\vec{b} = \mathrm{rot}\, r \,\mathrm{grad}\,\lambda =$

$= r \underbrace{\mathrm{rot}\,\mathrm{grad}\,\lambda}_{=0} + \mathrm{grad}\, r \wedge \mathrm{grad}\,\lambda =$

$$= \vec{c} \wedge \frac{\vec{b}}{r} = \frac{\vec{a}}{r} \therefore$$

$$\text{div } \vec{a} = \vec{c} \times \frac{\vec{a}}{r} = 0$$

Por outro caminho (definição):

$$\text{rot } \vec{b} = 2V \frac{d\vec{b}}{dP} = 2V \frac{d(r \text{ grad } \lambda)}{dP}$$

$$\frac{d f \vec{u}}{dP} = f \frac{d\vec{u}}{dP} \vec{s} + H(\text{grad } f, \vec{u}) \vec{s}$$

$$\frac{d(r \text{ grad } \lambda)}{dP} \vec{s} = \cancel{R} \frac{d \text{ grad } \lambda}{dP} \vec{s} + H(\text{grad } r, \text{gr}$$

$$= r \frac{d \frac{\vec{b}}{r}}{dP} \vec{s} + H\left(\vec{c}, \frac{\vec{b}}{r}\right) \vec{s} \quad \dots$$

$$\text{rot } [(\vec{c} \wedge \vec{r}) \wedge \vec{r}] = 2V \frac{d [(\vec{c} \wedge \vec{r}) \wedge \vec{r}]}{dP} =$$

$$= 2V \left[(\vec{c} \wedge \vec{r}) \wedge \frac{d\vec{r}}{dP} + \frac{d(\vec{c} \wedge \vec{r})}{dP} \wedge \vec{r} \right] =$$

$$\left(\text{sabendo } f \cdot \frac{d\vec{r}}{dP} \vec{s} = \vec{s}\right)$$

$$= 2(\vec{c} \wedge \vec{r}) + 2V \left(\frac{d\vec{c} \wedge \vec{r}}{dP} \wedge \vec{r} \right) =$$

$$= 2(\vec{c} \wedge \vec{r}) + 2V \left[\left(\vec{c} \wedge \frac{d\vec{r}}{dP} \right) \wedge \vec{r} \right] =$$

$$= 2(\vec{c} \wedge \vec{r}) + 2V \left[\underbrace{(\vec{c} \cancel{\times \vec{r}}) \frac{d\vec{r}}{dP} - \left(\frac{d\vec{r}}{dP} \times \vec{r}\right) \vec{c}}_{\text{não existe vetor}} \right]$$

$$H(\vec{a}, \vec{b}) \vec{v} = (\vec{a} \times \vec{v}) \vec{b}$$

port.

$$\text{rot } [(\vec{c} \wedge \vec{r}) \wedge \vec{r}] = 2(\vec{c} \wedge \vec{r}) + 2VH(\vec{r}, \vec{c}) =$$

$$= 2(\vec{c} \wedge \vec{r}) + (\vec{c} \wedge \vec{r}) = 3(\vec{c} \wedge \vec{r})$$

pois $VH(\vec{a}, \vec{b}) = \frac{1}{2} \vec{a} \wedge \vec{b}$

14.10.946.

Questões da provinha.

$$\Delta'_2 \, e^{\rho^2} . \vec{c} \wedge \vec{r}$$

$$\vec{c} = c^{te} \qquad \rho = |\vec{c} \wedge \vec{r}| \qquad r = P - O.$$

$$\Delta'_2 \equiv \text{grad div} - \text{rot. rot.}$$

$$\text{div } e^{\rho^2} . \vec{c} \wedge \vec{r} = e^{\rho^2} \text{div } \vec{c} \wedge \vec{r} + \text{grad } e^{\rho^2} \times \vec{c} \wedge \vec{r}$$

$$\text{div } \vec{c} \wedge \vec{r} = \vec{r} \times \underbrace{\cancel{\text{rot } \vec{c}}}_{=0} - \vec{c} \times \underbrace{\cancel{\text{rot}} \, \vec{r}}_{=0}$$

$$\text{grad } e^{\rho^2} = 2\rho e^{\rho^2} \text{grad } \rho =$$

$$\text{grad } \rho = \frac{(\vec{c} \wedge \vec{r}) \wedge \vec{c}}{\rho} \rightarrow$$

$$\text{grad } e^{\rho^2} = 2 e^{\rho^2} . (\vec{c} \wedge \vec{r}) \wedge \vec{c}$$

$$\text{div } e^{\rho^2} . \vec{c} \wedge \vec{r} = \text{grad } e^{\rho^2} \times \vec{c} \wedge \vec{r} =$$

$$= 2 e^{\rho^2} (\vec{c} \wedge \vec{r}) \wedge \vec{c} \times (\vec{c} \wedge \vec{r}) = 0$$

$$\text{rot } e^{\rho^2} \vec{c} \wedge \vec{r} = e^{\rho^2} \text{rot } \vec{c} \wedge \vec{r} + \text{grad } e^{\rho^2} \wedge (\vec{c} \wedge \vec{r})$$

$$\text{rot } \vec{c} \wedge \vec{r} = 2\vec{c}$$

$$\operatorname{rot} e^{\rho^2}(\vec{c} \wedge \vec{r}) = 2e^{\rho^2}\vec{c} + 2e^{\rho^2}(\vec{c} \wedge \vec{r}) \wedge \vec{c} \wedge (\vec{c} \wedge \vec{r})$$

$$= 2e^{\rho^2}\left[\vec{c} + (\vec{c} \wedge \vec{r}) \wedge \vec{c} \wedge (\vec{c} \wedge \vec{r})\right] c$$

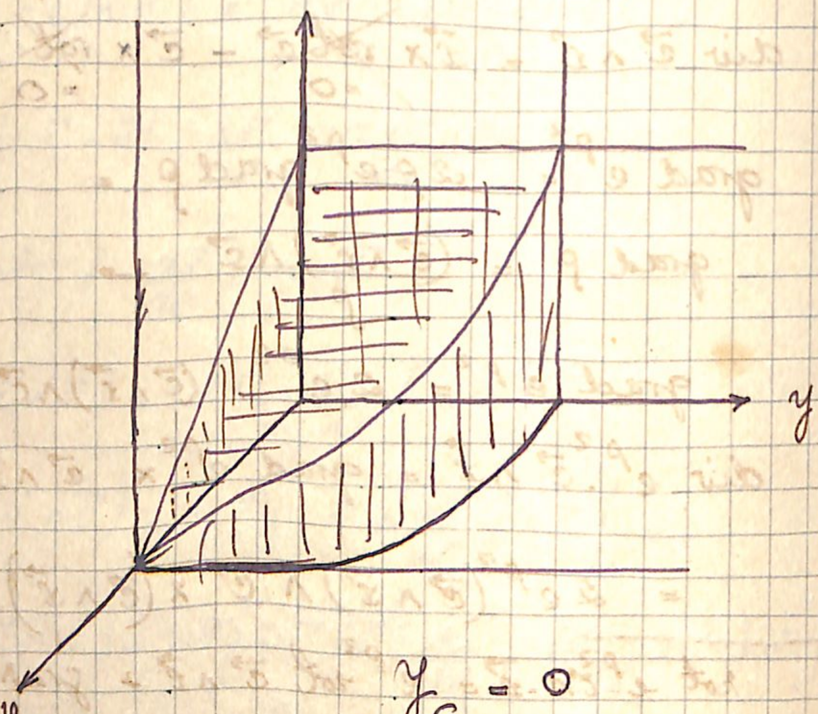
$$= 2e^{\rho^2}\left[\vec{c} + \rho^2 \vec{c} - 0\right] =$$

$$= 2e^{\rho^2}\vec{c}\,(1+\rho^2)$$

R.: $4e^{\rho^2}(1+\rho^2)\,|\vec{c}|^2 \cdot \vec{c} \wedge \vec{r}$

Calc. o centro de gravidade do vol. lim. pelo cil. $x^2+y^2=r^2$, e os planos

$ax + rz = ar$ e $z = 0$.



$y_G = 0$

$$x_G = \frac{1}{V}\int x\,dv \qquad z_G = \frac{1}{V}\int z\,dv$$

$$V = \pi r^2 . a$$

$$x_G = \frac{1}{\pi r^2 a}\int x\,dx\,dy\,dz$$

Questão de outra turma:

$$I = \iint z\,dS$$

sendo $S \begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = \rho e^{\theta} \end{cases}$

$\begin{cases} 0 \le \theta \le \pi/2 \qquad 0 \le \rho \le 1 \end{cases}$ este já é o campo.

15. 10. 946 — Dr. Breves.

Desenv. em serie arctg x.

Consideremos a identidade:

$$\frac{1}{1+x^2} = \frac{1-(-x^2)^n + (-x^2)^n}{1-(-x^2)} = \frac{1-(-x^2)^n}{1-(-x^2)} + \frac{(-x^2)^n}{1+x^2}$$

Ora, $\dfrac{1-y^n}{1-y} = 1 + y + y^2 + y^3 + \dots + y^{n-1}$

logo $\dfrac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + \dots + (-x^2)^n +$

$+ \dfrac{(-x^2)^n}{1+x^2}$. Integrando-se membro a membro:

$$\int_0^x \frac{dx}{1+x^2} = \sum_{i=0}^{n-1} \int_0^x (-x^2)^i dx + \int_0^x \frac{(-x^2)^n}{1+x^2} dx$$

$$\operatorname{arctg} x = \sum_0^{n-1} \frac{(-x^2)^{i+1}}{i+1} + R_n$$

$$\operatorname{arctg} x = \sum_0^{n-1} (-1)^i \frac{x^{2i+1}}{2i+1} + R_n$$

Calculemos $\lim_{n\to\infty} R_n$

$$R_n = \int_0^x \frac{(-x^2)^n}{1+x^2} dx = (-1)^n \int_0^x \frac{1}{1+x^2} \cdot x^{2n} dx$$

Aplicando o 2.º teorema da media (ver)

$$R_n = (-1)^n \frac{1}{1+\theta x^2} \int_0^x x^{2n} dx$$

$0 \leq \theta \leq 1$

$$R_n = (-1)^n \frac{1}{1+\theta x^2} \cdot \frac{x^{2n+1}}{2n+1}$$

Será $\lim_{n\to\infty} R_n = 0$ para $\boxed{|x| \leq 1}$

Conclusão:

arctg x é desenvolvível em série de potências no intervalo fechado $-1, +1$ e o desenvolvimento é:

$$\operatorname{arctg} x = \sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} =$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Em particular para $x = 1$ resulta

$$\operatorname{arctg} 1 = \pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

Exercício c/ sol.:

Provar q. $\pi^2/6 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} + \dots$

Consid. a série

$$f(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \dots + \frac{x^n}{n^2} + \dots$$

O raio de convergência desta série é:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 1$$

portanto a série é convergente no int. aberto $-1, +1$ e unif. e conv. em qq. int. fechado inter. a este.

Raciocinando apenas neste ult.º intervalo, pode-se derivar a série termo a termo.

Derivando, obtem-se

(C) $f'(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} + \dots$

portanto $x f'(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \frac{x^n}{n} + \dots$

derivando novamente —

$x f''(x) + f'(x) = 1 + x + x^2 + \dots + x^n + \dots$

q. é uma série geometrica e convergente porto, sendo

$$x f''(x) + f'(x) = \frac{1}{1-x}$$

Chegamos a uma eq. difer., donde

(1) $x f''(x) + f'(x) = \frac{1}{1+x}$

$$\frac{f''(x)}{f'(x)} = -\frac{1}{x}$$ port.to $l f'(x) = -l x + c.^{te}$ ou

$$f'(x) = \frac{c}{x}$$

determinemos c p.ª q. a

eq. não homogênea seja verificada — para isto consideremos c variável.

$$f''(x) = \frac{c'}{x} \text{ donde em } (1)$$

$$c' = \frac{1}{1-x} \rightarrow c = -l(1-x) + c_1$$

portº $f'(x) = \frac{c_1}{x} - \frac{l(1-x)}{x}$

notando q. $f'(0) = 1$ (C)

conclue-se que $c_1 = 0$ portº

$$f'(x) = -\int \frac{l(1-x)}{x}\,dx \rightarrow$$

~~f(x)~~ = notando q. $f(0)=0$

$$f(x) = -\int_0^x \frac{l(1-x)}{x}\,dx$$

mas a serie primitiva é para $x=1$ →

$$f(1) = -\int_0^1 \frac{l(1-x)}{x}\,dx$$

n. a calcular.

Funções elementares no campo complexo.

Por definição $e^z = 1 + z + z^2/2! + \dots + z^n/n! + \dots = \sum \frac{z^n}{n!}$

$\cos z = 1 - z^2/2! + z^4/4! - \dots + (-1)^n z^{2n}/2n! + \dots =$

$$= \sum (-1)^n z^{2n}/2n!$$

$\operatorname{sen} z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \dots =$

$$= \sum (-1)^n z^{2n+1}/(2n+1)! -$$

Estas definições podem ser dadas depois de se verificar q. as series são conv. em todo o plano complexo.

Dessas def. res.:

$$\cos(-z) = \cos z$$

$$\operatorname{sen}(-z) = -\operatorname{sen} z$$

$$e^{iz} = \sum \frac{(iz)^n}{n!} = \sum \frac{(iz)^{2n}}{2n!} + \sum \frac{(iz)^{2n+1}}{(2n+1)!}$$

$$= \sum (-1)^n \frac{z^{2n}}{2n!} + i \sum (-1)^n z^{2n+1}/(2n+1)! \rightarrow$$

$$\boxed{e^{iz} = \cos z + i \operatorname{sen} z}$$

mudando z em $-z$ →

$$\boxed{e^{-iz} = \cos z - i \operatorname{sen} z}$$

16.10.946 Dr. Cerruti

São conhecidos os operadores i e $e^{i\varphi}$.

Componentes simétricas:

$\alpha = e^{i\frac{2\pi}{n}}$ para $n > 0$ (inteiro)

$\alpha^r = e^{i\frac{2\pi}{n}r}$ sendo $r \gtrless 0$ $\underline{\underline{r \leq n}}$

$\alpha^{-r} = e^{-i\frac{2\pi}{n}r}$ (sentido retrogrado de rotação)

Sequência de operadores de ordem r, de índice $\underline{n}$ se chama a um conj. de operadores α

$$S_n^r = \alpha^0, \alpha^{-r}, \alpha^{-2r}, \alpha^{-3r}, \dots \alpha^{-(n-1)r}$$ se diz

O vetor a que aplicamos a sequência, se diz base da sequência $\vec{b}$.

$$\vec{b}^{-r} = \alpha^0\vec{b}, \alpha^{-r}\vec{b}, \alpha^{-2r}\vec{b}, \dots$$

$\alpha_n^0 \vec{b} = \vec{b}$ Soma de sequências

Se $r \neq 0$ ($\neq n$) $\to \sum S_n^r \vec{b} = 0$

e $r = 0$ ou $r = n$ $\to \sum S_n^r \vec{b} = n\vec{b}$.

Produto de sequências de mesmo índice.

$$S_n^r . S_n^p = S_n^{r+p}$$

$$S_n^{n+p} = S_n^n . S_n^p = S_n^p$$

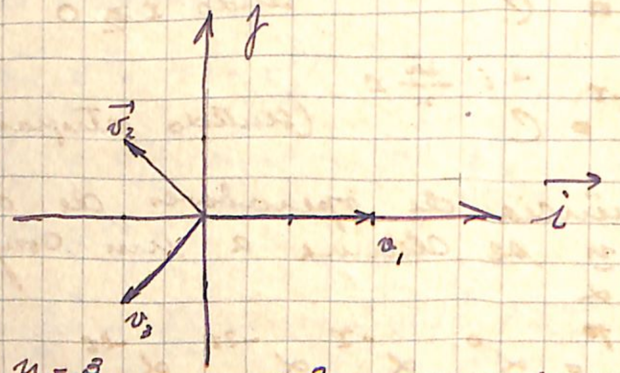
Caso mais comum: $n = 3$

Se temos um sist. de n vetores coplanares de $\vec{R} = \sum^{n}$ ele é equiv. à soma de $(n-1)$ vetores de igual módulo e formando angulos iguais.
Se $R \neq 0$ ter-se-á a soma de n vetores.

Exemplo: —

Sejam dados: $\vec{v}_1 = 2\vec{i}$
$\vec{v}_2 = -\vec{i}+\vec{j}$
$\vec{v}_3 = -\vec{i}-\vec{j}$

Calcular o sistema equivalente de componentes simétricas.
Êles estão contidos no primeiro caso, $\vec{R} = 0$



no caso $n = 3$

$$S_3^2 = \alpha^0; \alpha^{-1}; \alpha^{-2}$$

$$S_3^2 = \alpha^0; \alpha^{-2}; \alpha^{-4}$$

no caso, $\alpha^0 = e^{i\frac{2\pi}{3}} = \cos 120° + i \operatorname{sen} 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\alpha^{-1} = e^{-i\varphi} = e^{-i\frac{2\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i = \alpha^2$$

$$\alpha^{-2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\alpha^{-4} = \alpha^{-1} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\vec{v}_i = \sum_{r=1}^{2} \alpha^{(1-i)r} \vec{u}_r \quad (i = 1,2,3)$$

$i = 1 \to \vec{v}_1 = \alpha^0 \vec{u}_1 + \alpha^0 \vec{u}_2 = 2\vec{i}$

$i = 2 \to \vec{v}_2 = \alpha^{-1} \vec{u}_1 + \alpha^{-2} \vec{u}_2 = -\vec{i} + \vec{j}$

$i = 3 \to \vec{v}_3 = \alpha^{-2} \vec{u}_1 + \alpha^{-4} \vec{u}_2 = -\vec{i} - \vec{j}$

Façamos os coef. de $\vec{u}_1$ serem unitários:

$$2\vec{i} = \alpha^0 \vec{u}_1 + \alpha^0 \vec{u}_2$$

$$-\alpha^1 \vec{i} + \alpha^1 \vec{j} = \alpha^0 \vec{u}_1 + \alpha^{-1} \vec{u}_2$$

$$-\alpha^2 \vec{i} - \alpha^2 \vec{j} = \alpha^0 \vec{u}_1 + \alpha^{-2} \vec{u}_2$$

mas $S_3^1 = \alpha^0; \alpha^{-1}; \alpha^{-2}$, portanto $\sum S_3^1 = 0$

Somando membro a membro e aplicando α^1_0, α^2 nos vetores $\vec{i}, \vec{j}, \vec{k}$ ...

$$3\vec{u}_1 = 2\vec{i} + \frac{1}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j} - \frac{1}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j} + \frac{1}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{i} = (3-\sqrt{3})\vec{i}$$

$$\vec{u}_1 = \left(1 - \frac{\sqrt{3}}{3}\right)\vec{i}$$

$S_3' \vec{\mu_1} = \alpha^0 \vec{\mu_1}, \; \alpha^{-1}\vec{\mu_1}, \; \alpha^{-2}\vec{\mu_2}$

$= \left(1-\frac{\sqrt{3}}{3}\right)\vec{i}, \; -\frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right)\vec{i} - i\frac{\sqrt{3}}{2}\left(1-\frac{\sqrt{3}}{3}\right)\vec{i}$

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$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^4}{4!} + \dots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \frac{z^{2n}}{2n!} + \dots$

$\operatorname{sen} z = z - \frac{z^3}{3!} + \frac{z^4}{4!} - \dots + (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!} + \dots$ p. defin.

Das definições resulta

$\cos z = \cos(-z) \quad \operatorname{sen} z = -\operatorname{sen}(-z)$

$\cos z + i \operatorname{sen}(z) = e^{iz}$

$\cos z - i \operatorname{sen} z = e^{-iz}$

Destas relações resultam as formulas de Euler:

$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \operatorname{sen} z = \frac{e^{iz} - e^{-iz}}{2i}$

A derivadas destas três funções e^z, $\cos z$ e $\operatorname{sen} z$, têm valores análogos ao do campo real.

C. efeito, derivando as séries correspondentes:

$\frac{d}{dz} e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^{n-1}}{(n-1)!} + \dots = e^z$

$\frac{d}{dz} \cos z = -z + \frac{z^3}{3!} - \frac{z^5}{5!} + \dots = -\operatorname{sen} z$

$\frac{d}{dz} \operatorname{sen} z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + \dots = \cos z$

Para completar a analogia com o campo real, resta provar a propriedade: Aplicando o teorema

$e^z \cdot e^w = e^{z+w}$.

de produto de series, ou efetuando:

$$e^z e^w = \left(1+z+\frac{z^2}{2!}+\dots+\frac{z^n}{n!}+\dots\right)\left(1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+\dots+\frac{w^n}{n!}+\dots\right) =$$

$$= (a_1 + a_2 + \dots + a_n + \dots)$$

onde:

$$a_1 = 1.1 = 1 \qquad a_2 = 1.w + z.1 \qquad a_3 = 1\frac{w^2}{2!} + zw + \frac{z^2}{2!}.1$$

. . . . . .

$$a_n = 1.\frac{w^n}{n!} + z.\frac{w^{n-1}}{(n-1)!} + \dots + \frac{z^i}{i!}\frac{w^{n-i}}{(n-i)!} + \dots + \frac{z^n}{n!}.1 =$$

$$= \sum_{i=0}^{n} \frac{1}{i!(n-i)!}\, w^{(n-i)}\, z^i = \sum_0^n \frac{1}{n!}\,\frac{n!}{i!(n-i)!}\, z^i w^{n-i}$$

$$= \frac{1}{n!}\sum_0^n \binom{n}{i} z^i w^{n-i} = \frac{(z+w)^n}{n!}$$

form. do Binom. de Newton

portanto, $e^z e^w = \sum \frac{(z+w)^n}{n!} = e^{z+w}$

Consequências:

$\cos(z+w) = \cos z.\cos w - \operatorname{sen} z.\operatorname{sen} w$

$\operatorname{sen}(z+w) = \operatorname{sen} z.\cos w + \cos z.\operatorname{sen} w$.

Ora, $\cos z.\cos w = \frac{e^{iz}+e^{-iz}}{2}\cdot\frac{e^{iw}+e^{-iw}}{2} =$

$$= \frac{1}{4}\left[e^{i(z+w)} + e^{-i(z+w)} + e^{i(z-w)} + e^{-i(z-w)}\right]$$

$\operatorname{sen} z.\operatorname{sen} w = \frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{e^{iw}-e^{-iw}}{2i} =$

$$= -\frac{1}{4}\left[e^{i(z+w)} + e^{-i(z+w)} - e^{i(z-w)} - e^{-i(z-w)}\right]$$

Por subtração:

$$\cos z.\cos w - \operatorname{sen} z.\operatorname{sen} w = \frac{1}{2}\left[e^{i(z+w)} + e^{-i(z+w)}\right] =$$

$= \cos(z+w)$

Verifica-se tambem que $\operatorname{sen} 0 = 0$, $\cos 0 = 1$

$$\operatorname{sen}\pi/2 = \frac{e^{\pi/2\, i} - e^{-\pi/2\, i}}{2} = 1 \qquad \cos\pi/2 = 0$$

Podemos nestes casos igualar as series de termos reais com as de termos imaginários, e supondo conhecidas as somas das de termos reais, temos as somas " " " imag.

Funções hiperbolicas.

Por definição: $\operatorname{sh} z = \frac{\operatorname{sen} iz}{i}$, ou, aplicando a formula de Euler:

$$\operatorname{sh} z = \frac{1}{i}\,\frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -\frac{1}{2}\left(e^{-z} - e^{z}\right) \rightarrow$$

$$\operatorname{sh} z = \frac{e^z - e^{-z}}{2}$$ e tambem p. defin.:

$$\operatorname{ch} z = \cos iz \overset{\text{Euler}}{=} \frac{e^{i(iz)} + e^{-i(iz)}}{2} =$$

$$= \frac{e^z + e^{-z}}{2}$$

Observação - Das defin. de sh e ch resulta que todas as formulas racionais da trigonometria hiperbolica obtem-se das formulas correspondentes de trigon. circulares, substituindo-se

$\cos = ch$ e $sen = i\,sh$.

Senão vejamos:

$\cos^2 x + sen^2 x = 1$

$(ch\,x)^2 + (i\,sh\,x)^2 = 1 \therefore ch^2 x - sh^2 x = 1$

ou

$sen\,2x = 2\,sen\,x \cdot \cos x$

$\boxed{sh\,2x = 2\cancel{i}\,sh\,x \cdot ch\,x}$

Derivadas —

Falta 1 aula p. fui a pesca.

Provar que

$$\frac{\pi}{2} = 1 + \binom{\frac12}{1} - \binom{\frac12}{2}\frac{1}{3} + \binom{\frac12}{3}\frac{1}{5} - \dots + (-1)^{n-1}\binom{\frac12}{n}\frac{1}{2n-1} + \dots$$

Consideremos a serie:

$$S(x) = 1 + \binom{\frac12}{1}x - \binom{\frac12}{2}\frac{x^3}{3} + \binom{\frac12}{3}\frac{x^5}{5} - \dots + (-1)^{n-1}\binom{\frac12}{n}\frac{x^{2n-1}}{2n-1}$$

Mas pela semelhança com as series de potencia: $a_0 + a_1 x + a_2 x^2 + \dots$, temos

$b_0 + b_1 x + b_2 x^3 + b_3 x^5 + \dots$ ~~onde se~~

$= \frac{1}{x}\left[b_0 x + b_1 x^2 + b_2 x^2 + \dots\right] =$

façamos $x^2 = y \rightarrow \frac{1}{\sqrt{y}}\left[b_0\sqrt{y} + b_1 y + b_2 y^2 + b_3 y^3 + \dots\right]$

é serie de potencia.

Qual o raio de convergência da serie proposta:

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\binom{\frac12}{n}\frac{1}{2n-1}}{\binom{\frac12}{n+1}\frac{1}{2n+1}} = \lim \frac{\binom{\frac12}{n}}{\binom{\frac12}{n+1}}\cdot\frac{2n+1}{2n-1}$$

$$= \lim \frac{n+1}{\frac12 - n}\cdot\frac{2n+1}{2n-1} = \lim \frac{(1+1/n)(2+1/n)}{(1-1/2n)(2-1/n)} = 1$$

Portanto, a série é definida no intervalo aberto $-1 < x < 1$, onde ela é convergente, absoluta e uniformemente.

Derivemo-la —

$$S'(x) = \binom{\frac12}{1} - \binom{\frac12}{2}x^2 + \binom{\frac12}{3}x^4 + \dots + (-1)^{n-1}\binom{1/2}{n}x^{2n-2} + \dots$$

$$x^2 S'(x) = \binom{\frac12}{1}x^2 - \binom{\frac12}{2}x^4 + \dots + (-1)^{n-1}\binom{1/2}{n}x^{2n} + \dots =$$

$= 1 - (1-x^2)^{1/2}$ pelo binomio de Newton.

$$(1-x^2)^{1/2} = 1 - \binom{1/2}{1}x^2 + \binom{1/2}{2}x^4 + \dots$$

Portanto —

$$S'(x) = \frac{1-(1-x^2)^{1/2}}{x^2}$$ e por integração:

$$S(x) = \int_0^x \frac{1-(1-x^2)^{1/2}}{x^2}\,dx = \int_0^x \frac{dx}{x^2} - \int_0^x \frac{\sqrt{1-x^2}}{x^2}\,dx =$$

$$= \left[\frac{1}{x}\right]_0^x - \left[-\operatorname{cotg} t\right]_0^t + 1$$

$$S(x) = -\frac{1}{x} + \frac{(1-x^2)^{1/2}}{x} + \operatorname{arcsen} x + 1 +$$

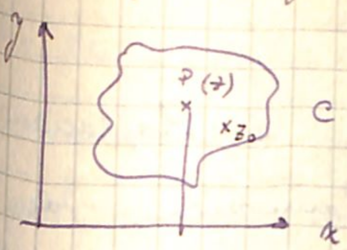
$$+ \lim_{x \to 0}\left[\frac{1}{x} - \frac{(1-x^2)^{1/2}}{x}\right] =$$

$$= -1 + \pi/2 + 1 =$$

$$\boxed{= \pi/2}$$

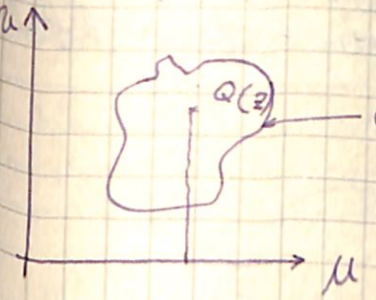
28.10.46 Dr. Cer.

Função complexa de variável complexa



C

$Z = x + iy$



c'

$Z = f(z) = \mu + iv$

$\mu = \mu(x, y)$ $v = v(x, y)$

$f(z)$ é cont. no ponto z_0 se:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

<u>Derivadas</u> – idêntica defin.:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{df}{dz}\right)_{z = z_0}$$ $z \to z_0$ de qq. maneira

Teor.ª imp.ª:– (parte vaga do exame)

As conds nec. e suf. p.ª haver a deriv. são –

1) $f(z)$ seja definida no ponto

2) $f(z)$ " cont. " "

3) $\frac{\partial u}{\partial x}$ $\frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x}$ existam e sejam cont.

4) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Se a fç. satisfaz a estas 4 cond. ela é chamada holomorfa.

Se a fç. é holom.:

$$\Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$ é o Laplaciano, se $\Delta_2 u$ é nulo, a fç. u é harmonica. Idem para v.

Se u e v forem ~~harmonicas~~ holomorfos, ~~a fç. é holomorfa.~~ elas são harmonicas.

Problemas -

① Dada $f(z)$, verificar se é holomorfa; e se suas componentes são harmonicas, sendo holomorfa.

② Construir uma função holomorfa $f(z)$, dada uma das componentes, u ou v. Devemos achar v ou u. (Estudar!)

Exercicios: —

Dada - $z^3 - 3z^2 - z = f(z)$

$z = x + iy$, verif. se é holomorfa.

$z^2 = x^2 + 2ixy - y^2$

$z^3 = x^3 + 2ix^2y + xy^2 + ix^2y - 2xy^2 - \ldots$

$f(z) = x^3 + 3ix^2y - 3xy^2 + \cancel{ix^2y} - \cancel{2x}y^3 - iy \ldots$

$- (3x^2 + 6ixy - 3y^2) - x - iy =$

$u = x^3 - 3xy^2 - 3x^2 + 3y^2 - x$

$v = 3x^2y - y^3 - 6xy - y$

$u'_x = 3x^2 - 3y^2 - 6x - 1$

$u'_y = -6xy + 6y$

$v'_x = 6xy - 6y$

$v'_y = 3x^2 - 3y^2 - 6x - 1$

Vejamos se são harmonicas -

$u''_x = 6x - 6$ $\quad v''_x \cdot$

$u''_y = -6x + 6$ $\quad v''_y =$

Artificios

Seja $f(z) = \frac{1}{z}$ $\quad z = x + iy$

$$\frac{1}{z} = \frac{1}{x + iy} = u + iv$$

$$\frac{x - iy}{x^2 + y^2} = u + iv$$

$u = \frac{x}{x^2 + y^2}$ $\quad v = \frac{-y}{x^2 + y^2}$

$f(z) = \sqrt{z} = \sqrt{x + iy} = u + iv$

elevando ao quadrado,

$x + iy = u^2 - v^2 + 2iuv$

$x = u^2 - v^2$ $\quad y = 2vu$

daqui tiramos $\underline{u}$ e $\underline{v}$.

$$f(z) = l\,z = l(x + iy)$$

$$e^{f(z)} = (x + iy) = z \quad \text{pela exponencial.}$$

$$e^{u+iv} = x + iy = e^{u} \cdot e^{iv} = x + iy =$$

$$= e^{u} \cdot (\cos v + i \operatorname{sen} v) = x + iy$$

portanto:

$$\left.\begin{aligned} e^{u} \cos v &= x \\ e^{u} \operatorname{sen} v &= y \end{aligned}\right\} \text{div.: } \operatorname{tg} v = \frac{y}{x}$$

$$v = \operatorname{arctg} \frac{y}{x}$$

elevando ao quadrado e somando —

$$e^{2u} = x^2 + y^2 \qquad u = \frac{l(x^2 + y^2)}{2}$$

(2) Construção da função hol.

Dá-se: $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y}$$

Integrando: $v = 3x^2 y - y^3 + 6xy + \varphi(x)$

$$\text{mas } \frac{\partial u}{\partial y} = \cancel{-6xy - 6y} = -\frac{\partial v}{\partial x} =$$

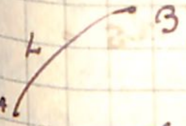
$$= -\cancel{[6xy + 6y} + \varphi'(x)]$$

portanto $\varphi'(x) = 0 \rightarrow \varphi(x) = C$

e vem: $f(z) = z^3 + 3z^2 + C$

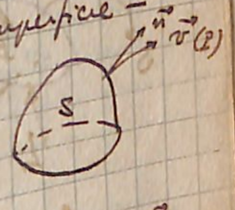
29.10.945

Circulação — Dado $\vec{v}(P)$ definido na linha L, circ. é $I = \int_L \vec{v}(P) \times dP$

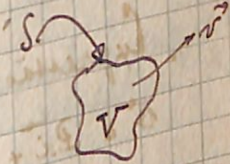


Se tivermos uma superfície —

$$\text{fluxo } \phi = \iint_S \vec{v} \times \vec{n}\, ds$$



Teorema de Gauss —



$$\iiint_V \operatorname{div} \vec{v}\, dV = \iint_S \vec{v} \times \vec{n} \cdot ds$$

Convenciona-se fluxo positivo (quando é a/ precedida do sinal +), de dentro para fora.

$$\vec{v} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\operatorname{div} \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

temos a expressão cartesiana —

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, ds$$

$$= \iint_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy.$$

Caso partic.: $Q = R = 0$

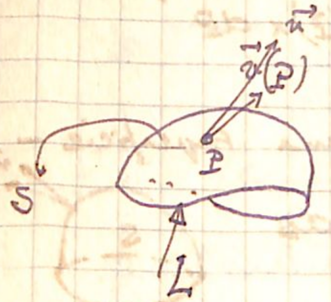
$$\iiint \frac{\partial P}{\partial x}\, dV = \iint_S P\,dy\,dz \quad \text{se } P = x$$

$$V = \iint_S x \cdot dy\,dz, \text{ de maneira identica } \quad V = \iint_S y\,dx\,dz \ldots$$

portanto – $V = \frac{1}{3}\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\, ds$

~~Teorema~~ Vemos daí a transformação de integrais de vol. em int. de superf.

Teor. de Stokes –



$d\Sigma$ –

$$\iint_S \text{rot}\,\vec{v} \times \vec{n}\, ds = \int_L \vec{v} \times d\vec{l}$$

circulação.

Expressão cartesiana –

$\vec{v} = P\vec{i} + Q\vec{j} + R\vec{k}$.

$$\text{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

portº:

$$\iint_S \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \times \vec{n}\, ds = \int_L (P\vec{i} + Q\vec{j} + R\vec{k}) \times d\vec{l}$$

ou

$$\iint \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} ds = \int_L P\,dx + Q\,dy +$$

Tirado no Dep. Mat. (Dr. Ceruti)

... o sistº de vetores $\vec{v}_1 = 2\vec{i} - \vec{j}$, $\vec{v}_2 = -\vec{i} + \vec{j}$, $\vec{v}_3 = -\vec{i}$

... o sistema de compon.es simetricas.

Sol.

Resultte do sistº: $\vec{R} = \sum_1^3 \vec{v}_i = 0$

Teor. 1. pg. 41. nº 33 Calcº vetorial – a resultte sendo ..., o sistema será a soma de 2 sequencias de ...tores $S_3^1 \vec{u}_1$ $S_3^2 \vec{u}_2$

Calcº dos vetores bases $\vec{u}_1, \vec{u}_2$: $\vec{v}_i = \sum_{r=1}^{2} \alpha^{(1-i)r} \cdot \vec{u}_r$

Formamos o sistº de eq.

$$\begin{cases} 2\vec{i} - \vec{j} = \alpha^0 \vec{u}_1 + \alpha^0 \vec{u}_2 & (1) \\ -\vec{i} + \vec{j} = \alpha^{-1} \vec{u}_1 + \alpha^{-2} \vec{u}_2 & (2) \\ -\vec{i} = \alpha^{-2} \vec{u}_1 + \alpha^{-4} \vec{u}_2 & (3) \end{cases}$$

Operando à esquerda dos dois membros de (2) e de (3) respectivate com operadores α^1, α^2, vem:

$$2\vec{i} - \vec{j} = \vec{u}_1 + \vec{u}_2 \quad (1)$$
$$\alpha(-\vec{i} + \vec{j}) = \vec{u}_1 + \alpha^{-1}\vec{u}_2 \quad (2)$$
$$\alpha^2(-\vec{i}) = \vec{u}_1 + \alpha^{-2}\vec{u}_2 \quad (3')$$

Somando membro a membro e observando que ultima coluna dá soma nula, obtemos:

$3\vec{u}_1 = 2\vec{i} - \vec{j} + \alpha(-\vec{i} + \vec{j}) - \alpha^2 \vec{i}$

No caso, $\alpha = e^{i\frac{2\pi}{3}} = \cos 120° + i \,\text{sen}\, 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\alpha^2 = \alpha^{-1} = \cos 120° - i\,\text{sen}\,120° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Aplicando, vem:

$3\vec{u}_1 = (2\vec{i} - \vec{j}) + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(-\vec{i} + \vec{j}) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\vec{i} =$

$= 2\vec{i} - \vec{j} + \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j} =$

$= \left(3 - \frac{\sqrt{3}}{2}\right)\vec{i} + \frac{3}{2}\vec{j} \rightarrow \boxed{\vec{u}_1 = \left(1 - \frac{\sqrt{3}}{6}\right)\vec{i} - \frac{1}{2}\vec{j}}$

De (1) tiramos

$\vec{u}_2 = (2\vec{i} - \vec{j}) - \left(1 - \frac{\sqrt{3}}{6}\right)\vec{i} + \frac{1}{2}\vec{j} =$

$= (1+\sqrt{3}/6)\,\vec{i} - \frac{1}{2}\vec{j} \longleftrightarrow \boxed{\vec{u}_2 = \left(1+\frac{\sqrt{3}}{6}\right)\vec{i} - \frac{1}{2}\vec{j}}$

4) Sequência de vetores S_3^1; S_3^2

A. $S_3^1 = \alpha^0, \alpha^{-1}, \alpha^{-2} = 1, \alpha^2, \alpha$
(seq. de fase negat. ou sim. inversa)
nº 32 pg. 41, in fine.

$$\begin{cases} \vec{u}_1 = (1-\sqrt{3}/6)\,\vec{i} - \frac{1}{2}\vec{j} \\ \alpha^2\vec{u}_1 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\vec{u}_1 = -(1/2+\sqrt{3}/6)\,\vec{i} - (\sqrt{3}/2 \\ \alpha\vec{u}_1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\vec{u}_1 = -(1/2-\sqrt{3}/3)\,\vec{i} + \sqrt{3}/2 \end{cases}$$

Verificação: $\vec{u}_1 + \alpha^2\vec{u}_1 + \alpha\vec{u}_1 = 0$

B. $S_3^2 = \alpha^0, \alpha^{-2}, \alpha^{-4} = 1, \alpha, \alpha^{-1} = 1, \alpha, \alpha^2$
(seq. de fase posit. ou sim. direta)

$\vec{u}_2 = (1+\sqrt{3}/6)\,\vec{i} - 1/2\,\vec{j}$

$\alpha\vec{u}_2 = (-1/2 + i\sqrt{3}/2)\,\vec{u}_2 = -(1/2-\sqrt{3}/6)\,\vec{i} + (\sqrt{3}/2$

$\alpha^2\vec{u}_2 = (-1/2 - i\sqrt{3}/2)\,\vec{u}_2 = -(1/2+\sqrt{3}/2)\,\vec{i} - \sqrt{3}/2\,\vec{j}$.

Verific.: $\vec{u}_2 + \alpha\vec{u}_2 + \alpha^2\vec{u}_2 = 0$

Dado o sist. de vetores $\vec{v}_1 = 2\vec{i}$, $\vec{v}_3 = -2\vec{i} + \vec{j}$; $\vec{v}_4 = -\vec{i} - \vec{j}$ calcular sua componentes simétricas.

Sol. —

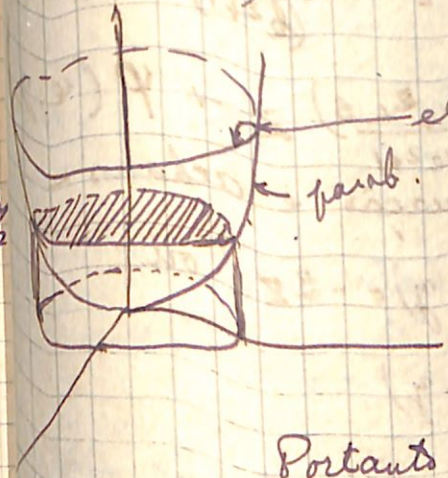
Aplicações — Calcular a circulação do vetor $\vec{v}(P) = (y-1)\,\vec{i} + (2x+1)\,\vec{j} + 2\vec{k}$ ao longo da linha L — $\begin{cases} P = O + x\vec{i} + y\vec{j} + \left(\frac{x^2}{2} + \frac{y^2}{4}\right)\vec{k} \\ y^2 + 2x^2 = 4 \end{cases}$

$$I = \int_L \vec{v} \times dP = \iint_S \operatorname{rot}\vec{v} \times \vec{n}\, ds$$

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y-1) & (2x+1) & 2 \end{vmatrix} = \vec{k}(2-1) = \vec{k}$$

$$\vec{n} = \frac{P'_x \wedge P'_y}{|P'_x \wedge P'_y|}$$

$\vec{n} = \vec{k}$



(observar a intersecção das duas superfícies e considerar a melhor das superfícies — elipse plana)

Portanto —

$$I = \iint_S dx\,dy = \int_0^{\sqrt{2}} dx \int_0^{2\sqrt{1-x^2}} dy = \pi a b = 2\pi\sqrt{2}$$

Exprimir (Potron) $I = \iint_S \left[\frac{x(x^2+3)}{x^2+1}\cos\alpha + \frac{2x^2y(x^2+3)}{(x^2+1)}\cos\beta - 3z\cos\gamma\right] ds$ extendida a S, delimitada por L, por uma integral de forma

$$J = \int_L P(x,y,z)\,dx + Q(x,y,z)\,dy$$

e calc. seu valor no caso partic.

em q. L é $x^2+y^2=1$, situada no plano $z=1$.

pelo teor.ª de Stokes –

$$\iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} dS = \qquad (R \text{ deve ser nul}$$

$$= \int_L P\,dx + Q\,dy$$

Daqui tiramos, igualando os coef. de $\cos\alpha$, $\cos\beta$, $\cos\gamma$:

① $-\frac{\partial Q}{\partial z} = \frac{x(x^2+3)}{x^2+1}$

② $\frac{\partial P}{\partial z} = \frac{2x^2y(x^2+3)}{(x^2+1)^2}$

③ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3z$

$$P = +\int \frac{2x^2y(x^2+3)}{(x^2+1)^2}\,dz = \frac{2x^2y(x^2+3)}{(x^2+1)^2}\,z + \psi(x,y)$$

$$Q = -\int \cdots\,\partial z = \frac{x(x^2+3)}{x^2+1}\,z + \varphi(x,y)$$

vamos agora à 3ª equação e achamos –

$$-3z + \varphi'_x(x,y) - \psi'_y(x,y) = -3z \quad \text{ou}$$

$$\varphi'_x(x,y) = \psi'_y(x,y)$$

Determinar uma função U de u, sendo $u = \frac{x^2+y^2-1}{x}$ e uma fç V de $v = \frac{x^2+y^2+1}{y}$ de maneira q. $U+iV$ seja holomorfa de $z = x+iy$

Ora, U e V devem ser harmonicas, logo –

$\Delta_2 U = \Delta_2 V = 0$

$$\frac{dU}{dx} = \frac{dU}{du}\frac{\partial u}{\partial x} \qquad \frac{d^2U}{dx^2} = \frac{dU}{du}\frac{\partial^2 u}{\partial x^2} + \frac{d^2U}{du^2}\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{dU}{dy} = \frac{dU}{du}\cdot\frac{\partial u}{\partial y} \qquad \frac{d^2U}{dy^2} = \frac{dU}{du}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{d^2U}{du^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2$$

$$U''_x = U'u''_x + U''(u'_x)^2$$

$$U''_y = U'u''_y + U''(u'_y)^2$$

$$U''_x + U''_y = 0$$

ou

$$U'(u''_x + u''_y) + U''(u'^2_x + u'^2_y) = 0$$

$$\frac{U''}{U'} = -\frac{u'_x + u'_y}{u'^2_x + u'^2_y}$$

$$u'_x = \frac{x^2-y^2+1}{x^2} \qquad u''_x = \frac{2(y^2-1)}{x^3}$$

$$u'_y = \frac{2y}{x} \qquad u''_y = \frac{2}{x}$$

$$u''_x + u''_y = \frac{2u}{x^2}$$

$$u''^2_x + u''^2_y = \frac{u^2+4}{x^2}$$

$$\frac{U''}{U'} = -\frac{2u}{u^2+4} \qquad U' = \frac{dU}{dt} = t \quad U' = t'$$

$$\frac{t'}{t} = \frac{-2u}{u^2+4} \qquad l\,t = -l(u^2+4) + l\,c =$$

$$= l\,\frac{c}{u^2+4}$$

$$t = \frac{c}{u^2+4} \qquad U = \int t\,dt = c\int\frac{du}{u^2+4}$$

$$\frac{c}{2}\int\frac{\frac{1}{2}\,du}{1+\left(\frac{u}{2}\right)^2} = \frac{c}{2}\,\mathrm{arctg}\,\frac{u}{2} + C_1$$

c_3

~~Apl~~ De maneira idêntica chegamos a:

$$V = \frac{C'}{2} \cdot \ell \frac{v-2}{v+2} + C_2$$

C_4

Devemos agora aplicar as condições de Riemann - Cauchy.

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$ para podermos calcular as constantes C_1, C_2, C_3 e C_4

Chegamos, efetuando os cálculos a que: $C_1 = C_2$ e

$C_3 = C_4$.

Da-se $u + v = x^3 - 3x y^2 - y^3 + 3x^2 y$ e pede-se u e v de maneira que $f(z) = u + iv$ ($z = x + iy$) seja holomorfa.

Seja $U = u + v$, $V = u - v$ determinemos V, dado U e depois calculemos u e v.

Ora, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

e $u = U - v$ donde -

a) $\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ $\quad$ c) $\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}$

b) $\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ $\quad$ d) $\frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}$

Comparando a e d : $a + d = 0$

$$\begin{cases} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \\ \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0 \end{cases}$$; b com c $\rightarrow$ b - c = 0

$$\frac{\partial U}{\partial x} = 3x^2 - 3y^2 + 6xy = -\frac{\partial V}{\partial y}$$

Integrando -

$$-V = 3y x^2 - \cancel{\frac{3y^3}{3}} + \cancel{\frac{6x y^2}{2}} + \varphi(x)$$

$$\frac{\partial V}{\partial x} = -\cancel{6xy} - \cancel{3y^2} + \varphi'(x) = \frac{\partial U}{\partial y} =$$

$$= -\cancel{6xy} - \cancel{3y^2} + 3x^2$$

Portanto $\varphi'(x) = -3x^2 \rightarrow \varphi(x) = -x^3 + C$

e $V = -\cancel{3y x^2} + \cancel{y^3} + 3y^2 + x^3 + C = u - v$

$U = \cancel{x^3} - 3x y^2 - \cancel{y^3} + \cancel{3x^2 y} = u + v$

Somando:

$$2u = 2x^3 + 3y^2 - 3x y^2 + C$$

$$u = \frac{1}{2}\left[2x^3 + 3y^2(1 - x) + C\right]$$

7.11.46

Calcular $I = \int \frac{dz}{1+z^3}$ $\quad z = x + iy$

ao longo da elipse $2x^2 + y^2 = 4x$

Aplica-se o teor.ª de Cauchy —

$\int f(z) dz$ sendo $f(z)$ holom. nos pontos do contorno e nos s/ interior. $\int_c f(z) dz = 2\pi i \sum_s R(a_s)$

Polos são pontos onde a f. se torna infinita.

Se o polo é simples — $R(a_s) = \lim_{z \to a_s} (z - a_s) \cdot f(z)$

No nosso caso — $z^3 + 1 = 0$ nos dá os polos.

$z^m + 1 = 0$

$$z = \cos (2k+1)\pi/m + i \operatorname{sen} (2k+1)\pi/m = e^{i(2k+1)\pi/m}$$

$m = 3 \begin{cases} k = 0 \\ k = 1 \\ k = 2 \end{cases}$

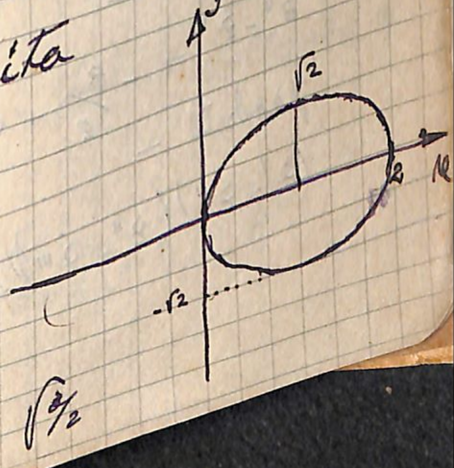
$z' = \cos 60° + i \operatorname{sen} 60° = 1/2 + i\sqrt{3}/2$

$z'' = \cos \pi + i \operatorname{sen} \pi = -1$

$z''' = \cos 5\pi/3 + i \operatorname{sen} 5\pi/3 = 1/2 - i\sqrt{3}/2$

Para estes valores —

$\int_c \frac{dz}{1+z^3}$ é infinita



Elejamos o contorno —

$x = 0 \quad y = 0 \quad x = 1/2 \quad y = \sqrt{3}/2$

Estes valores correspondem à polos interiores à helipse.

Aplicamos o teorema de Cauchy:

$$I = \int_C \frac{dz}{1+z^3} = 2\pi i \sum R(z', z'')$$

$$R(z') = \lim_{z \to z'} \left(\frac{z-z'}{z^3-1}\right) =$$

$$= \lim_{z \to z'} \frac{\cancel{z-z'}}{\cancel{(z-z')}(z-z'')(z-z''')} = \frac{1}{(z'-z'')(z'-z''')}$$

$$R(z''') = \frac{1}{(z'''-z')(z'''-z'')}$$

$$\sum R = \frac{(\cancel{z'-z'''}) + (z'-z'')}{(z'-z'')(z'-z''')(z''-z''')} =$$

$$= \frac{\cancel{z'-z''}}{\cancel{(z'-z''')}} \cdot \frac{1}{(z'-z'')(z''-z''')} =$$

$$\frac{1}{(z'-z'')(z''-z''')}$$

$$z'-z'' = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z''-z''' = -\frac{3}{2} + i\frac{\sqrt{3}}{2}$$ efetuando o produto -

$$(z'-z'')(z''-z''') = \frac{3}{4} - \frac{3}{4}\cdot\sqrt{3}\,i - \frac{\sqrt{3}}{4}\cdot i - \frac{3}{4} = \ldots i\sqrt{3}$$

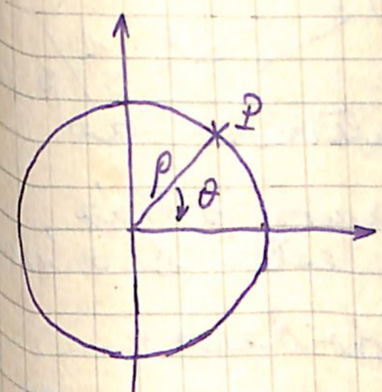
port.° - $\sum R = \frac{1}{i\sqrt{3}} = \frac{i\sqrt{3}}{3}$

$$I = 2\pi i \cdot \frac{\sqrt{3}}{3} i = \boxed{-2\pi\frac{\sqrt{3}}{3}}$$

Calcular -

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta + p^2} \qquad 0 < p < 1$$

Para integrar, consideremos um círculo de raio 1, sobre o plano de Argand e um ponto $P(z = x + iy)$ deste círculo.



Então -

$$z = x + iy = \rho e^{i\theta} =$$
$$= \rho(\cos\theta + i\,\text{sen}\,\theta)$$

$$\begin{cases} \rho = 1 \\ z = e^{i\theta} \quad dz = e^{i\theta} i\,d\theta \end{cases}$$

$$d\theta = \frac{dz}{iz}$$

$$\left.\begin{cases} z = \cos\theta + i\,\text{sen}\,\theta \\ z^{-1} = \cos\theta - i\,\text{sen}\,\theta \end{cases}\right\}$$ daqui -

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$ voltando à integral -

$$I = \int_C \frac{dz}{iz\left[1 - 2p\cdot\frac{1}{2}\left(z + \frac{1}{z}\right) + p^2\right]} =$$

$$= -i \int_C \frac{dz}{z\left[pz^2 - z(p^2+1) + p\right]}$$ e caímos

na integral do exercício anterior — aplicamos o teorema de Cauchy —

$$I = 2\pi i \sum R(a_s)$$

Vejamos os polos —

$$z = \frac{p^2+1 \pm \sqrt{(p^2+1)^2 - 4p^2}}{2p} = \frac{p^2+1 \pm (p^2-1)}{2p} \begin{cases} z' = p \\ z'' = 1/p \end{cases}$$

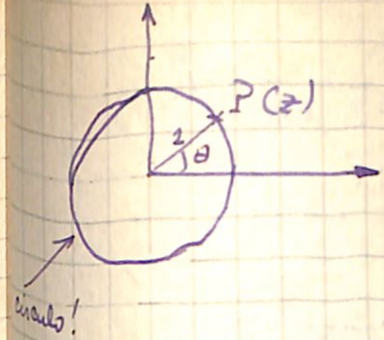
Vejamos se são internos ao círculo que consideramos atráz, ora, $p < 1$, logo este é o único polo que devemos considerar —

$$R(p) = \lim_{z \to p} \frac{(z-p)}{p(z-p)\left(z-\frac{1}{p}\right)} = \frac{1}{p^2-1}$$ logo —

$$I = \frac{2\pi i \cdot i}{p^2-1} = \frac{2\pi}{1-p^2}$$

Calcular $I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \operatorname{sen}^2\theta}$, $a > 0$

ora, $I = \frac{1}{2}\int_0^{2\pi} \frac{a\,d\theta}{a^2+\operatorname{sen}^2\theta}$ ← se a f. é pa



$z = e^{i\theta} = \cos\theta + i\operatorname{sen}\theta$

$dz = e^{i\theta}\, i\, d\theta$

$d\theta = \frac{dz}{iz}$

$z - \frac{1}{z} = 2i\operatorname{sen}\theta$

$$\operatorname{sen}^2\theta = -\left(\frac{z^2-1}{z}\right)^2 \frac{1}{4} = -\frac{(z^2-1)^2}{4z^2}$$

voltando à integral —

$$I = \frac{1}{2}\int_C \frac{a\,dz}{iz\left[a^2 - \frac{(z^2-1)^2}{4z^2}\right]} = \frac{4a}{2i}\int \frac{z\,dz}{4a^2z^2 - (z^2-1)^2}$$

resolvendo a equação biquadrada que aparece no denom., obtemos 4 polos —

$z' = -a + m$ $\quad z'' = -a - m$ $\quad z''' = a + m$

$z'''' = a - m$ $\quad a = \sqrt{a^2-1}$

só ~~...~~ caem dentro do círculo os polos de z' e z''''

Portanto —

$$R(z') = \lim_{z \to z'} (z-z')f(z) = -\frac{1}{2m}$$

$$R(z'''') = \dots = -\frac{1}{2m}$$

$$I = 2\pi i \cdot \frac{4a}{2i}\left(-\frac{1}{2m} - \frac{1}{2m}\right) =$$

$$= \frac{\pi}{\sqrt{a^2+1}}.$$

Dr. Breves.

Séries de funções complexas —

Provar que

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \dots = \frac{\pi}{2\sqrt{3}}$$

$$f(x) = x - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{17}}{17} + \dots$$

será nossa série auxiliar.

$$f'(x) = (1 - x^4) + (x^6 - x^{10}) + (x^{12} - x^{16}) + \dots$$

$$f'(x) = (1 - x^4)(1 + x^6 + x^{12} + \dots)$$

cujo raio de convergência é 1 →

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1$, isto quer dizer que a série é convergente no intervalo $-1 \le x \le$

$$f'(x) = (1 - x^4)\frac{1}{1 - x^6} = \frac{1 + x^2}{1 + x^2 + x^4}$$

portanto – $f(x) = \int \frac{1 + x^2}{x^4 + x^2 + 1}\,dx$

$$\frac{1 + x^2}{1 + x^2 + x^4} = \frac{1 + x^2}{(1 - x + x^2)(1 + x + x^2)} =$$

$$\frac{1}{2} \cdot \frac{(1 - x + x^2) + (1 + x + x^2)}{(1 - x + x^2)\cdot(1 + x + x^2)},$$

$$f(x) = \frac{1}{2}\int \frac{dx}{1 - x + x^2} + \frac{1}{2}\int \frac{dx}{1 + x + x^2} + C$$

$$= \frac{1}{2}\sqrt{\frac{4}{3}}\,\mathrm{arctg}\sqrt{\frac{4}{3}}\left(x - \frac{1}{2}\right) + \frac{1}{2}\sqrt{\frac{4}{3}}\,\mathrm{arctg}\sqrt{\frac{4}{3}}\left(x + \frac{1}{2}\right) + C$$

$$= \frac{1}{\sqrt{3}}\left[\mathrm{arctg}\frac{2x - 1}{\sqrt{3}} + \mathrm{arctg}\frac{2x + 1}{\sqrt{3}}\right] + C$$

ora, $f(0) = 0$ donde –

$$f(0) = \frac{1}{\sqrt{3}}\left[\underbrace{\cancel{\mathrm{arctg}\left(-\frac{1}{\sqrt{3}}\right) + \mathrm{arctg}\frac{1}{\sqrt{3}}}}_{=0}\right] + C$$

$$C = 0$$

portanto $f(x) = \frac{1}{\sqrt{3}}\left[\mathrm{arctg}\frac{2x - 1}{\sqrt{3}} + \mathrm{arctg}\frac{2x + 1}{\sqrt{3}}\right]$

$$\lim_{x \to 1} f(x) = \frac{1}{\sqrt{3}}\left(\mathrm{arctg}\frac{1}{\sqrt{3}} + \mathrm{arctg}\frac{3}{\sqrt{3}}\right) =$$

$$= \frac{1}{\sqrt{3}}\left(\mathrm{arctg}\frac{1}{\sqrt{3}} + \mathrm{arctg}\sqrt{3}\right) = \frac{1}{\sqrt{3}} \cdot \frac{\pi}{2} \text{ c.q.d.}$$

Desenvolver em série —

$$f(x) = \ln\sqrt{1 + x^2} \qquad f'(x) = \frac{2x}{2\left(\sqrt{1 + x^2}\right)^2} =$$

$$= \frac{x}{1 + x^2} = x\,\frac{1}{1 + x^2} =$$

$$= x\,\frac{1 - (-x^2)^n + (-x^2)^n}{1 - (-x^2)} \quad \text{ora,} \quad \frac{1 - t^n}{1 - t} = 1 + t + t^2 + \dots + t^{n-1}$$

$$f'(x) = x\left[1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots + (-x^2)^{n-1} + \frac{(-x^2)^n}{1 - (-x^2)}\right] = x\left[1 - x^2 + x^4 - x^6 + \dots (-1)^{n-1}x^{2(n-1)}\right] +$$

$$+ (-1)^n \frac{x^{2n+1}}{1 + x^2}$$

para $|x| < 1$ —

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (-1)^n \frac{x^{2n+1}}{1+x^2} = 0$$

portanto — $f'(x) = x - x^3 + x^5 - \dots$

O raio de convergencia de $f'(x)$ é 1, então

$$f(x) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} + \dots$$

$f(0) = 0 = C$ port.to —

$$l\sqrt{1+x^2} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} - \dots$$

Desenvolver em serie —

$f(x) = \cos^2 x \qquad f'(x) = -2\cos x \,\text{sen}\, x = -\text{sen}$

Ora, $\text{sen}\, t = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n + \dots$ derivando

$\cos t = a_1 + 2a_2 t + 3a_3 t^2 + \dots + n a_n t^{n-1} \dots$

$-\text{sen}\, t = 2a_2 + 2\cdot 3 a_3 t + 3\cdot 4 a_4 t^2 + \dots\, n(n+1) a_n t^{n-1}$

Ora — $\text{sen}\, t = -(-\text{sen}\, t)$ port.to —

$a_0 = 2a_2 \qquad \text{sen}\, 0 = 0 = a_0$

$a_1 = 2\cdot 3\, a_3 \qquad \cos 0 = 1 = a_1$

$a_2 = 3\cdot 4\, a_4$

$a_3 = \dots \qquad a_0 = 0 = a_2, a_4, a_6, a_8 \dots$

$a_1 = -a_3 \times 6$

$a_1 = -2\cdot 3\cdot a_3$

$a_3 = -4\cdot 5\, a_5$ multiplicando

$a_{n-2} = -(n-1)\, n \cdot a_n$

$a_1 = -n! \cdot a_n = 1$ logo — $a_n = \pm\frac{1}{n!}$

$\text{sen}\, t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots$ portanto —

$$-\text{sen}\, 2x = -2x + \frac{2^3}{3!}x^3 - \frac{2^5}{5!}x^5 + \dots$$

integrando —

$$\cos^2 x = -x^2 + \frac{2^3}{4!}x^3 - \frac{2^5}{6!}x^6 + \dots$$

$f(0) = 1 = C$ donde —

$$\cos^2 x = 1 - x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!} + \dots$$ c.q.d.

Dr. Cerruti.

Calcular: $I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$ $\quad a>0,\ b>0$

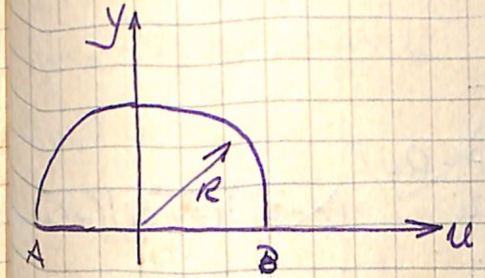
é integral generalizada.

Ora, $\int_{-\infty}^{+\infty} Q(x)\,dx = \int_{-\infty}^{+\infty} \frac{N(x)}{D(x)}\,dx$ os polos devem ser complexos; o grau de D maior que o de N.

Então — $I = 2\pi i \sum R(a_s^+)$

$$\int_\Omega Q(z)\,dz = \int_{AB} Q(z)\,dz + \int_C Q(z)\,dz$$

$$\int_{-R}^{+R} Q(x)\,dx$$



Quando $R \to \infty$ $\int_\Omega Q(z)\,dz = \int_{-\infty}^{+\infty} Q(x)\,dx = 2\pi i \sum R(a_s^+)$. Não deve haver polo real (sobre o eixo Ox) Se houver polos reais simples: —

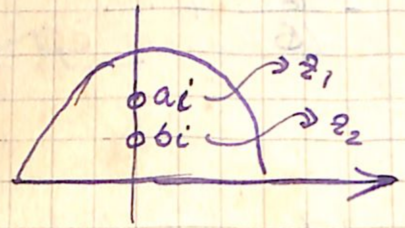
$$\int_{-\infty}^{+\infty} Q(x)\,dx = 2\pi i \sum R(a_s^+) + \pi i \sum R(a_{rs})$$

Consideramos então —

$Q(z) = \frac{1}{(z^2+a^2)(z^2+b^2)^2}$ os polos vemos que

são — $z = \pm ai$ (simples) e $z = \pm bi$ (duplos)

os do semi-plano superior são — $+ai$ e $+bi$



portanto, a integral se transforma em —

$$I = \int_{C+\overline{AB}} \frac{dz}{(z^2+a^2)(z^2+b^2)^2} =$$

$= 2\pi i \sum R(a_s^+)$ aplicando o teorema —

$$I = \lim_{R\to\infty}\left[\int_{\overline{AB}} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} + \cancel{\int_C \frac{dz}{(z^2+a^2)(z^2+b^2)^2}}\right]$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = 2\pi i \sum R(a_s^+)$$

cálculo dos polos —

$$R(z_1) = R(ai) = \lim_{z\to ai} \cancel{(z-ai)} \frac{1}{\cancel{(z-ai)}(z+ai)(z^2+b^2)^2}$$

$$= \frac{1}{2ai(b^2-a^2)^2}$$

caso agora do polo duplo —

em geral, se o polo é de multiplicidade m —

$$R(a) = \lim_{z\to a} \frac{D^{m-1}}{(m-1)!}(z-a)^m f(z) \qquad D \equiv \frac{d}{dz}$$

$$R(bi) = \lim_{z\to bi}\left[\frac{d}{dz}\frac{(z-bi)^2}{(z^2+a^2)(z^2+b^2)^2}\right]\frac{1}{2!} =$$

$$R(bi) = \frac{i}{4}\,\frac{3b^2-a^2}{b^3(a^2-b^2)^2}$$

$$I = 2\pi i\left[\frac{1}{2ai(b^2-a^2)} + \frac{i}{4}\,\frac{3b^2-a^2}{b^3(a^2-b^2)^2}\right] =$$

$$= \boxed{\frac{\pi(a+2b)}{2ab^3(a+b)^2}}$$

Calcular — $I = \int_0^{\pi} \frac{\cos 3x}{\cos x + 2}\,dx$

a função é par, logo —

$I = \frac{1}{2}\int_0^{2\pi} \frac{\cos 3x}{\cos x + 2}\,dx$ para cairmos no caso do círculo unitário.

① $e^{ix} = \cos x + i \operatorname{sen} x$; $e^{i3x} = \cos 3x + i \operatorname{sen} 3x = z^3$ — $dz = e^{ix}\cdot i\,dx$, $dx = \frac{dz}{iz}$

② $1/z = \cos x - i \operatorname{sen} x$; $1/z^3 = \cos 3x - i \operatorname{sen} 3x$

①+② → $\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\cos 3x = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$ suposto $\rho = 1$.

$$I = -\frac{i}{2}\int_C \frac{\frac{1}{2}(z^2 + 1/z^3)}{\frac{1}{2}(z+\frac{1}{z}) + 2}\cdot\frac{dz}{z}$$

Pare! Vai aparecer sujeira na integral!

$$I = \frac{1}{2}\int_0^{2\pi} \frac{e^{i3x} - i\operatorname{sen} 3x}{\cos x + 2}\,dx = \frac{1}{2}\int_0^{2\pi}\frac{e^{i3x}}{\cos x + 2} - \frac{i}{2}\underbrace{\int \frac{\operatorname{sen} 3x\,dx}{\cos x + 2}}_{=0}$$

$+ f(x) = -f(-x)$ ← a fç. é par.

reduzimos o cálculo a —

$I = \frac{1}{2}\int_0^{2\pi} \frac{e^{i3x}\,dx}{\cos x + 2}$ passamos agora ao campo complexo; (circ.º unitario...

$$I = 1/2 \int_C \frac{z^3\,dz}{iz\left[1/2\left(z + 1/z\right) + 2\right]} =$$

$$= -\frac{i}{\cancel{2}}\int_C \frac{z^3\,dz}{z^2 + 4z + 1}$$

Determinação dos polos $- \; z = \begin{cases} |-2+\sqrt{3}| < 1 & \text{(unico p. int. resíduo)} \\ |-2-\sqrt{3}| > 1 \end{cases}$

calculo do resíduo

$$R(z_1) = \lim_{z \to z_1} \frac{(z - z_1)\,z^3}{(z - z_1)(z - z_2)} = \frac{-26 + 15\sqrt{3}}{2\sqrt{3}} \cdot 2i$$

portanto $I = -i \cdot 2\pi i \left(\frac{15 - 26\sqrt{3}}{2\sqrt{3}}\right)$

$$= \pi\left(15 - 26/\sqrt{3}\right)$$

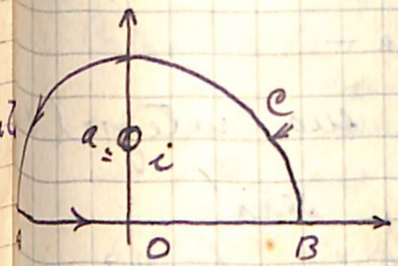
FINIS.

Vaz

calcular $- \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

$Q(z) = \frac{1}{(z^2+1)^3}$ $\quad z = \pm i$ (tripla)

$$\int_C Q(z)\,dz = 2\pi i \sum_s R(a_s)$$



$$\int_C Q(z)\,dz + \int_{AB} Q(z)\,dz =$$

$$= \quad " \quad + \int_{-R}^{R} Q(x)\,dx$$

quando $R \to \infty$

$$\int_\Omega Q(z)\,dz = 0 + \int_{-\infty}^{+\infty} Q(x)\,dx$$

$$I = 2\pi i \sum R(a_s^+)$$

$$R(+i) = \lim_{z \to i} \frac{d^2}{dz^2} (z-i)^3 \cdot \frac{1}{(z^2+1)^3} \cdot \frac{1}{2!} =$$

$$= \lim_{z \to i} \frac{1}{2!}\frac{d^2}{dz^2} \frac{\cancel{(z-i)^3}}{(z+i)^3\cancel{(z-i)^3}} =$$

$$\lim_{z \to i} \frac{12}{(z+i)^5} = \frac{3}{16i}$$

Portanto $\boxed{I = \frac{3\pi}{8}}$

Lema de Jordan − Aplicação.

$I = \int_0^\infty \frac{\cos mx}{a^2 + x^2}\,dx$ $\begin{cases} a > 0 \\ m > 0 \end{cases}$ a função integrando é par,

logo $- \; I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos mx}{a^2 + x^2}\,dx$

$e^{imx} = \cos mx + i \operatorname{sen} mx.$

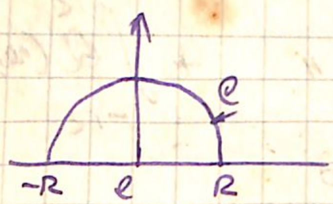
$$I = \tfrac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{imx}}{a^2+x^2}\,dx - \int_{-\infty}^{+\infty} \frac{\operatorname{sen} mx}{a^2+x^2}\,dx$$

pois a função é nula.

Ora, pelo lema de Jordan —

se tivermos $e^{imz} f(z)$, sua integral ao longo do contorno $c+l$, será

$$\int_{c+l} e^{imz} f(z)\,dz = \int_c e^{imz} f(z)\,dz + \int_l e^{imx} f(x)\,dx$$



q.d. $R \to \infty$, diz o lema de Jordan —

$$\int_{c+l} e^{imz} f(z)\,dz = 0 + \int_{-\infty}^{+\infty} e^{imx} f(x)\,dx =$$

$$= 2\pi i \sum_{S} R(a_i^{+})$$

Havendo polos reais simples, acrescenta-se

$$\boxed{\pi i \sum R(a^{r.s})}$$

então, no caso —

$$I = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{e^{imz}}{a^2+z^2}\,dz$$

$a^2+z^2=0 \quad z=\pm ai \quad z' = ai$

$$R = \lim_{z\to ai} (z-ai)\frac{e^{imz}}{a^2+z^2} = \frac{e^{-ma}}{2ai}$$

$$I_1 = \frac{\pi e^{-ma}}{2a}.$$

Calcular $I = \int_{-\infty}^{\infty} \frac{\operatorname{sen} mx}{x(a^2+x^2)^2}\,dx$

Há polos reais simples e polos complexos.

$$I = \frac{1}{i}\int_{-\infty}^{\infty} \frac{e^{imx}}{x(a^2+x^2)^2}\,dx - \frac{1}{i}\int_{-\infty}^{+\infty} \frac{\cos mx}{x(a^2+x^2)^2}\,dx$$

integral ímpar e extremos simétricos.

Consideramos —

$$Q(z) = \frac{e^{imz}}{z(a^2+z^2)^2}$$

$z_1 = 0$

$z_2 = +ai$ (duplo)

$z_3 = -ai$

$$R(z_1) = \lim_{z\to 0} \frac{(z-0)\,e^{imz}}{z(a^2+z^2)^2} = \frac{1}{a^4}$$

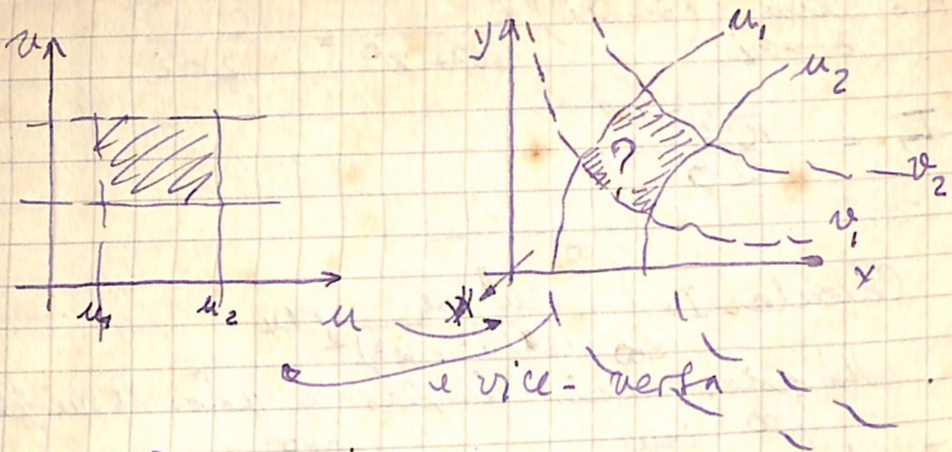
$$R(z_2) = \lim_{z\to ai} \frac{d}{dz}\,\frac{(z-ai)^2\,e^{imz}}{z\underbrace{(a^2+z^2)^2}_{(z-z_2)^2(z+z_2)^2}}$$

$$= \lim_{z\to ai} \frac{d}{dz}\,\frac{e^{imz}}{z(z+z_2)^2} \qquad \text{faça.}$$

Estudar a transformação —

$Z = f(z) = z^2$ e mostrar qual a figura transformada no plano $x\,y$, de um retângulo no plano u, v?

Pergunta-se se a transformação é conforme.



Ora, $\left.\begin{array}{l} Z = u + iv \\ z = x + iy \end{array}\right\} \rightarrow$

$(x+iy)^2 = u + iv$

$\underbrace{x^2 - y^2 + 2ixy} = u + iv$

$u = x^2 - y^2$
$v = 2xy$

$x^2 - y^2 = u_1 \; (u_2) \rightarrow x^2 - y^2 = c^{te}$

$2xy = v_1 \; (v_2) \rightarrow 2xy = c^{te}$ (hipérbole equil.)

Transformação inversa.

Faça!

[illegible] particulares —

a) $a \neq 0 \quad L(y) = P_s(x)$

$\bar{y} = a_0 x^s - a_1 x^{s-1} + \dots + a_s$

b) $a_n = a_{n-1} = a_{n-2} = \dots a_{n-m-1} = 0 \qquad a_{n-m} \neq 0$

$X = P_s(x)$

$\bar{y} = u^m (a_0 u^s + a_1 u^{s-1} + \dots + a_s)$

São nec. m quadraturas.

c) O 2º membro de $L(y)$ é $X = P_s e^{ax}$

$y = z e^{ax}$ integra.

Equações lineares de ordem

Assuntos

(1)

Teorema da invariancia da diferencial –

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ quer x e y sejam variáveis independentes, quer sejam funções de u e v.

Teorema de Euler – $x f'_x + y f'_y + \dots = m f(x, y, z \dots)$

Equações diferenciais –

Eq. dif. lineares – $X_0 y^{(n)} + X_1 y^{(n-1)} + \dots + X_n y = X$

Teorema de Lagrange – a sol. de uma equação linear completa fica reduzida ao conhecimento do sist.ª fund. de soluções e de n quadraturas.

Teorema – e^{rx} é solução $L(y) = 0$ se r fôr raiz não nula da equação caract. $\sum a_{n-i} y^i = 0$