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MATERIA DE QUARKS A TEMPERATURA FINITA

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A minha familia: María E. William F. William T. Gabriel, Jenny e especialmente a meu pequeno amor Joselyn.

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Por esse pão pra comer, por esse chão pra dormir
A certidão pra nascer, e a concessão pra sorrir
Por me deixar respirar, por me deixar existir
Deus lhe pague...

Chico Buarque

RESUMO

A constitução dos objetos estelares continua sendo fonte de especulação. Várias hipóteses têm sido aventadas com relação a estrutura interna dos pulsares. Uma delas baseia-se na conjetura de Bodmer-Witten, de que a altas densidades a matéria mais estável é formada por quarks desconfinados. Nesse caso, os pulsares seriam estrelas quarkiônicas, constituídas tão somente de quarks e léptons.

O modelo mais comumente usado para descrever matéria de quarks é o modelo de sacola de MIT. Outro modelo, um pouco mais sofisticado é o modelo de quarks dependente da densidade ou *Quark mass density dependent model*. Neste trabalho pretendemos comparar as equações de estado desses dois modelos, tanto a temperatura zero, como a temperatura finita. Essas equações de estado servem de *input* para as equações de Tolman-Oppenheimer-Volkoff (TOV), que são equações da relatividade geral para objetos homogêneos e de simetria esférica. Os resultados obtidos são os valores para massas e raios das estrelas.

Palavras-chave: Modelos efeitivos, estrelas de quarks.

ABSTRACT

The constitution of stellar compact objects remains unknown. Many hypotheses have been launched with relation to the internal structure of pulsars. One of them is based on the Bodmer-Witten conjecture, where at high densities the more stable matter is formed by deconfined quarks. One of the most used models is the MIT bag model. Another more refined model is a model where the mass is density dependent, known as quark mass density dependent model (QMDD).

In this work we intend to compare the results of the equations of state (EoS) of these two models, at zero and finite temperature. These equations of state are the input to solve the Tolman-Oppenheimer-Volkoff (TOV) equations of the general relativity for homogeneous compact objects with spherical symmetries. The TOV results are the values of the mass and radius of the star.

Keywords: Efective models, quark stars.

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1 INTRODUCTION

The fundamental theory that governs the strong interactions is the quantum chromodynamics (QCD).

From scattering experiments of leptons on hadrons, we know that there must be a point-like constituent inside the hadrons. This constituents of the hadrons are known as partons. In 1969, Richard Feymann postulated the parton model(FEYMANN R. P.,1969; GREENBER O. W.) and defined the parton as the vibrational energy necessary to accelerate a quark to a speed near to the speed of light. Later he recognized that the partons describe the same objects now known as quarks and gluons. The idea of quark is due independently to Gell'Mann and Zweig in 1964.

The analysis of lepton-hadron scattering show that the quarks are spin $\frac{1}{2}$ particles, (Dirac particles) and that they carry non-integer electric charge.

There are six different "flavors "of quarks, called: up(u), down(d), strange(s), charm(c), bottom(b) and top(t). The electric charges are $+\frac{2}{3}$ for u, c, t and $-\frac{1}{3}$ to d, s, b, respectively.

The masses of different quarks differ widely. Since the quarks are confined inside of the hadrons (at least in normal condition), the mass of the quarks cannot be measured directly but can be inferred indirectly from hadron properties. Most of the hadron mass itself does not originate from in the intrinsic mass of the quark but reside in the kinetic energy of the confined quarks and the fields that bind them together, the so-called glue fields. The typical values for the quarks masses are $m_u \simeq 5$ MeV, $m_d \simeq 10$ MeV, $m_s \simeq 150$ MeV, $m_c \simeq 1500$ MeV, $m_b \simeq 5000$ MeV, $m_t \simeq 40$ GeV (MÜLLER B.,1985).

Quarks are the basic constituents of hadrons, divided usually into mesons (bosons), which comprise one quark and one antiquark and baryons (fermions) made up of three quarks.

One of the main characteristics of QCD is called asymptotic freedom which ensures that, when the separation between two quarks decreases (and the energy of the system increases) the effective coupling strength becomes weaker. At large distances (equivalently low energy), the interaction becomes so strong that quarks and gluons are confined permanently inside of the hadrons. This indirectly says that an isolated quark cannot be detected, or informally, the quarks are forced to live inside of the hadrons (CHUNG K. C., 2001).

However, calculations indicate that at extreme conditions, i.e at sufficiently high temperatures and/or baryon chemical potentials, hadronic matter can suffer phase transition to a new phase of matter containing deconfined quarks and gluons. These conditions may appear for brief moments in ultra-relativistic collisions, but is also probably appearing in stars. The environment of high densities and low temperature exist inside of neutron stars. High temperatures and low baryon chemical potential occurred in the first 10^{-4} seconds after the big bang, and a quark-gluons plasma must have been formed until the temperature dropped to approximately 100 MeV-200 MeV due to the adiabatic expansion of the universe. Hence, the implications of the quark-gluons plasma are important in astrophysics and cosmology (MADSEN J.,2008).

Up, down, and strange quarks or strange quark matter (SQM) can be the true ground state of the quantum chromodynamics. This possibility was first proposed by Bodmer in 1971 (BODMER A. R.,1971), after the idea was retaken by Witten in 1984 (WITTEN E.,1984), and investigation performed by Farhi and Jaffe (FARHI E., JAFFE R.L.,1984) with the "MIT" model at zero temperature supported the Bodmer-Witten idea. The possible existence of stable SQM has multiple consequences for physics and astrophysics. The idea of the existence of strange quark stars is very striking because it is an alternative way to find free quarks.

Two of the phenomenological models used to study SQM are the MIT bag model (CHODOS A., JAFFE R.L., JOHNSON K., THORN C. B., WEISSKOPF V. F.,1974; JOHNSON K.,1975) and the quark mass density dependent model (QMDD) (FOWLER G. N., RAHA S., WEINER R. M.,1981). The first model assumes that hadrons are a kind of bag, and quarks confined inside these bag interact weakly (asymptotic freedom) Fig.1.

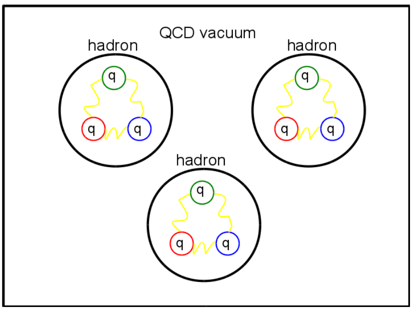

Figura 1: Schematic picture of the hadrons ("bag") in a QCD vacuum. Quark inside the hadron interact weakly.

To create the bags in the QCD vacuum, additional energy is needed, this energy is $E_{\mathscr{B}} = \mathscr{B}V$, where $\mathscr{B}$ is the bag constant, and should be understood as the energy penalty for the deconfined phase.

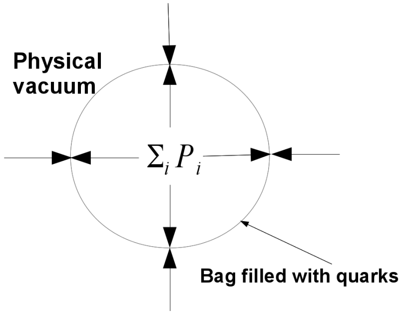

Figura 2: Schematic picture of the MIT bag model. The pressure of quarks in the bag is counterbalanced by the physical vacuum (WEBER F.,1999).

With increasing energy density, a deconfined phase is believed to happen (hadron-quark phase transition). When the transition begins, the bags in the QCD vacuum disappear, and the free quarks are treated as a fermion gas, where the pressure of the quarks are counterbalanced with the QCD vacuum (Fig. 2).

In (FOWLER G. N., RAHA S., WEINER R. M.,1981), a confinement mechanism was introduced by assuming that the quark masses are density dependent. This model, called QMDD, was then applied to describe strange quark matter (CHAKRABARTY S., RAHA S., SINHA B.,1989) and the related quark star properties in (CHAKRABARTY S.,1991a;1991b). Furthemore, the authors pointed out that the results obtained with the QMDD model were quite different from the ones obtained with the MIT model. Subsequently, in (BENVENUTO O. G., LUGONES G. 1995a), the authors claimed that this difference was due to an incorrect thermodynamical treatment of the problem and recalculated the equation of state showing that an extra term is present in the energy density and pressure of the system. This extra term results from the dependence of the quark masses on the baryonic density.

Still another calculation of the equation of state based on thermodynamics and also on the general ensemble theory was obtained in (PENG G. X., CHIANG H. C., ZOU B. S., NING P. Z., LUO S. J.,2000). The authors claim that the extra term reported in (BENVENUTO O. G., LUGONES G. 1995a) is correct in the expression of the pressure, but it should not be present in the energy density equation. In both approaches, the pressure also carries a rearrangement term that appears due to the density dependence, while such a term cancels out in the energy density expressions. In (PENG G. X., CHIANG H. C., ZOU B. S., NING P. Z., LUO S. J.,2000), due to quark confinement and asymptotic freedom, another prescription for the quark masses is used and the values for the quark current masses are somewhat different from the standart ones. Normally $m_u = m_d = 5$ MeV and m_s is of the order of 150 MeV. In (PENG G. X., CHIANG H. C., ZOU B. S., NING P. Z., LUO S. J.,2000), m_u=5 MeV, m_d =10 MeV and m_s = 80 or 90 MeV. Depending on the parametrization used, SQM is completely stable or becomes metastable.

A variation of the QMDD model is a quark mass density - and temperature - dependent model (QMDTD) (ZHANG Y., ZU R. K.,2002). To describe the QCD phase transition, the QMDD model should be extended to QMDTD model assuming that the constant B is a function of the temperature.

Strange quark systems have been studied initially at zero temperature (FARHI E., JAFFE R.L.,1984; TORRES J. R.,2011). Later, works at finite temperature were developed in the frame of the MIT and the QMDD models (REINHARDT H., DANG B. V.,1988; CHMAJ T., SŁOMIŃSKI W.,1989; KETNER CH., WEBER F., WEIGEL M. K., GLENDENNING N. K.,1995, CHAKRABARTY S.,1993; BENVENUTO O. G., LUGONES G.,1995b).

An important ingredient in the strange matter hypothesis is the stability window, identified with the constant values of the model that are consistent with the fact that two-flavor quark matter must be unstable (i.e., its energy per baryon has to be larger than 930 MeV, which is the iron binding energy) and SQM (three-flavor quark matter) must be stable, i.e., its energy per baryon must be lower than 930 MeV. It was also shown in (BENVENUTO O. G., LUGONES G. 1995a) that the zero pressure density does not correspond to the minimum of the energy per baryon, as is normally the case, because of the quark mass density dependence.

If the strange matter hypothesis is true, a new compact object must exist. These objects are called strange stars. The Tolman-Oppenheimer-Volkoff (TOV) equations (TOLMAN R. C.,1939; OPPENHEIMER J. R., VOLKOFF G. M.,1939), describe the profile of static with spherically symmetric stars. To solve the TOV equation, we need the equation of state (EoS), which can be obtained from any of models (MIT, QMDD, QMDTD).

The strange star hypothesis is very speculative but it is difficult to conclusively rule them out. In principle, quark and neutron stars could coexist in the universe (WEBER F.,2005), and under appropriate physical conditions a neutron star could be converted to a quark star. Observationally, to distinguish whether a compact star is a neutron star or a quark star, one has to find a clear observational signature. The unusual small radii appears from observational data support that the compact objects SAX J1808.4 3658, 4U 1728-34, 4U 1820-30, RX J1856.5-3754 and Her X-1 are quark stars rather than neutron stars (LI A., PENG G. X., LU J. F.,2011).

In this work we find and analyze the stability windows related to protoquark stars described by quark matter using two effective models: MIT bag model and QMDD model. Calculate valid equations of state as input to the TOV equations and find the mass radius relation obtained for these coupled differential equations.

This work is presented as follows. In Chapter 1, the MIT bag model is revisited and the equations of state for $T \neq 0$ and $T = 0$ are showed. For finite temperature, we show the curves for two and three flavor matter and appropriate values for the confinement constant are given (stability windows). Finally we show the equation of state for stellar matter on the basis of stability windows.

Chapter 2 is dedicated to the quark mass density dependent model (QMDD). A similar treatment as in chapter 1 for the MIT model is made for the quark mass density dependent model. We show the equation of state for finite temperature and zero temperature, and the stability windows for this model. The gluons contribution is taken into account in this model and the curves for matter with gluons and without them are shown.

Chapter 3 is dedicated to the solution of the TOV equations, using the equation of state found in the previous chapters as input. The maximum masses and radii for quark star for different values of bag constants in the MIT and confinement constant in the QMDD model are obtained.

In Chapter 4 conclusions and the future prospects of this work is given.

In Appendix A the system of units used in this work is given.

In Appendix B a brief introduction to thermodynamics for free Fermi gases is shown

In Appendix C the basic theory of curved space and Einstein's equation which allows to find the Tolman-Oppenheimer-Volkoff equation is reproduced.

2 EQUATION OF STATE (EOS) FOR THE MIT BAG MODEL

The simplest version of the MIT Bag Model bears a phenomenological form that incorporates confinement in quark matter due to a “bag ” of the QCD vacuum. Since they have not been detected experimentally we imagine that quarks are closely confined inside of the hadrons. Within the confinement volume they (quarks) behave as free particles and the simplest way to deal with these particles is by means of a free Fermi gas.

The confinement is described by the bag constant $\mathscr{B}$, which determines the difference between the energy densities of the standard and QCD vacuum. The thermodynamic treatment used to find the equations of state (EoS) for this model is developed in the Appendix B.

At finite temperature, the energy of the system is given by the Helmholtz free energy Eq. (B.8), where $\mathscr{E}$ is the internal energy of the system and $\mathscr{S}$ is the final entropy of the system.

The entropy of a free Fermi gas is given by (SCHMITT A.,2010)

$$\mathscr{S} = -\sum_i \frac{\gamma_i}{2\pi^2}\int_0^\infty dk k^2\{[(1-n_i)\ln(1-n_i)+n_i\ln n_i] + [(1-\overline{n}_i)\ln(1-\overline{n}_i)+\overline{n}_i\ln\overline{n}_i]\}, \tag{2.1}$$

where $\gamma_i = 2_{spin}\times 3_{color}$ is the quark degeneracy and n_i and $\overline{n}_i$ are the Fermi distribution for particles and antiparticles respectively (Eqs. B.31 and B.32). The equation of state (EoS) for the MIT Bag is explicitly given in appendix B. At finite temperature the integrals (B.25), (B.30),(B.36) cannot be solved analytically, and the equation of state for the MIT Bag Model at finite temperature reads:

$$\boxed{\begin{aligned}
\mathscr{P}_i &= \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\int_0^\infty dk\frac{k^4}{\sqrt{k^2+m^2}}(n_i+\overline{n}_i)-\mathscr{B},\\
\varepsilon &= \sum_i \frac{\gamma_i}{2\pi^2}\int_0^\infty k^2\sqrt{k^2+m^2}(n_i+\overline{n}_i)dk+\mathscr{B},\\
\mathscr{F} &= \varepsilon - T\mathscr{S}+\mathscr{B},\\
\rho_{B_i} &= \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\int_0^\infty d^3k k^2(n_i-\overline{n}_i),
\end{aligned}} \tag{2.2}$$

where $\mathscr{F}$ is the free energy density of the system, $\mathscr{B}$ is the bag constant and should be understood as the background energy density inside the hadrons and the net inward pressure exerted on them by the surrounding vacuum

(MINGFENG Z., GUANGZHOU L., ZI Y., YAN X., WENTAO S.,2009), and the quark degeneracy is γ_i.

At $T = 0$ there are no anti-particles and the distribution function n_i is in a good approximation described by a Heaviside step function $\Theta(\mu - k)$:

$$\Theta(\mu - k) = \begin{cases} 1 & \text{if} \quad k \leq \mu, \\ 0 & \text{if} \quad k > \mu. \end{cases} \tag{2.3}$$

For $T = 0$ the chemical potential μ must be identified with the Fermi energy ε_f (the energy of the highest occupied level). Thus, the EoS at $T = 0$ can be simplified and solved analytically.

The equation of state becomes:

$$\boxed{\begin{aligned} \mathscr{P} &= \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\int_0^{k_f} dk \frac{k^4}{\sqrt{k^2+m^2}} - \mathscr{B}, \\ \varepsilon &= \sum_i \frac{\gamma_i}{2\pi^2}\int_0^{k_f} dk k^2\sqrt{k^2+m^2} + \mathscr{B}, \\ \rho_{B_i} &= \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\int_0^{k_f} dk k^2 = \frac{1}{3}\sum_i \frac{\gamma_i}{6\pi^2} k_{f_i}^3, \end{aligned}} \tag{2.4}$$

The equation for pressure and the density energy can be solved analytically using the following integral

$$\int_0^k dk \frac{k^4}{\sqrt{k^2+m^2}}$$
$$= \frac{1}{4}\left[k^3\sqrt{k^2+m^2} - \frac{3}{2}m^2 k\sqrt{k^2+m^2} + \frac{3}{2}m^4 \ln\left(\frac{\sqrt{k^2+m^2}+k}{m}\right)\right] \quad .$$

Substituting this integral into Eq. (1.4) for the pressure and using Eq. (B.10) for the energy density, they become:

$$\mathscr{P} = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\left\{\frac{1}{8}\left[k_i\sqrt{k_i^2+m_i^2}(2k_i^2 - 3m_i^2) + 3m_i^4 \ln\left(\frac{\sqrt{k_i^2+m_i^2}+k_i}{m}\right)\right]\right\} - \mathscr{B}, \tag{2.5}$$

$$\varepsilon = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\left\{\frac{3}{8}\left[k_i\sqrt{k_i^2+m_i^2}(2k_i^2 + m_i^2) - m_i^4 \ln\left(\frac{\sqrt{k_i^2+m_i^2}+k_i}{m}\right)\right]\right\} + \mathscr{B}. \tag{2.6}$$

Defining the variable

$$x_i = \frac{k_i}{m_i}, \tag{2.7}$$

and using the relation $k_i = \sqrt{\mu_i^2 - m_i^2}$, the new variable x_i is (PATHRIA R. K.,1996):

$$x_i = \sqrt{\left(\frac{\mu_i}{m_i}\right)^2 - 1}, \tag{2.8}$$

thus the pressure and energy density in terms of this new variable are:

$$\mathscr{P} = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\frac{m_i^4}{8}\left[\frac{k_i}{m_i}\sqrt{\frac{k_i^2}{m_i^2}+1}\left(2\frac{k_i^2}{m_i^2}-3\right)+3\ln\left(\sqrt{\frac{k_i^2}{m_i^2}+1}+\frac{k_i}{m_i}\right)\right]-\mathscr{B}, \tag{2.9}$$

$$\varepsilon = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\frac{3m_i^4}{8}\left[\frac{k_i}{m_i}\sqrt{\frac{k_i^2}{m_i^2}+1}\left(2\frac{k_i^2}{m_i^2}+1\right)-\ln\left(\sqrt{\frac{k_i^2}{m_i^2}+1}+\frac{k_i}{m_i}\right)\right]+\mathscr{B} \tag{2.10}$$

with the aid of x_i finally we obtain the expressions:

$$\mathscr{P} = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\frac{m_i^4}{8}\left[x_i\sqrt{x_i^2+1}\left(2x_i^2-3\right)+3\ln\left(\sqrt{x_i^2+1}+x_i\right)\right]-\mathscr{B}, \tag{2.11}$$

$$\varepsilon = \frac{1}{3}\sum_i \frac{\gamma_i}{2\pi^2}\frac{3m_i^4}{8}\left[x_i\sqrt{x_i^2+1}\left(2x_i^2+1\right)-\ln\left(\sqrt{x_i^2+1}+x_i\right)\right]+\mathscr{B} \tag{2.12}$$

More compactly, these two equations can be written as follows:

$$\mathscr{P} = \sum_i \frac{\gamma_i m_i^4}{48\pi^2}F(x_i)-\mathscr{B}, \tag{2.13}$$

$$\varepsilon = \sum_i \frac{3\gamma_i m_i^4}{48\pi^2}H(x_i)+\mathscr{B}, \tag{2.14}$$

where the functions $F(x_i)$ and $H(x_i)$ are defined as:

$$F(x_i) = x_i\sqrt{x_i^2+1}\left(2x_i^2-3\right)+3\ln\left(\sqrt{x_i^2+1}+x_i\right), \tag{2.15}$$

$$H(x_i) = x_i\sqrt{x_i^2+1}\left(2x_i^2+1\right)-\ln\left(\sqrt{x_i^2+1}+x_i\right). \tag{2.16}$$

The baryonic number in terms of the x_i variable is given by

$$\rho_{B_i} = \frac{1}{3}\sum_i \frac{\gamma_i}{6\pi^2} m_i^3 x_i^3 . \tag{2.17}$$

Deconfined quark matter is considered by the Bodmer-Witten conjecture and investigations performed by Farhi and Jaffe using this model support the idea in which under extreme conditions (sufficiently high temperature and/or baryon chemical potential) this type of matter should exist and be the ground state of the QCD.

Nuclear matter is a hypothetical system consisting of a large number of protons and neutrons interacting via nuclear force. When nuclear matter is compressed to sufficiently high densities, it is expected, on the basis of the asymptotic freedom of quantum chromodynamics (QCD), that a phase transition occurs to quark matter, which is a degenerate Fermi gas of quarks. In practice, only up, down and strange quarks occur in quark matter, because other quark flavors have masses much larger than the chemical potentials involved.

A compact star or compact object is a star that is a white dwarf, a neutron star, a black hole or our hypothetical strange star. Compact stars form the endpoint of stellar evolution. A star shine and thus loses its nuclear energy reservoir. When a star has exhausted all its energy a stellar death occur, the gas pressure in the hot interior can no longer support the weight of the star and the collapses to a denser star - a compact star. A study of compact objects begins when normal stellar evolutions ends. All these aforementioned objects differ from normal stars in at least two aspect: first, these objects are not burning nuclear fields, and they cannot support themselves against gravitational collapse by means of thermal pressure. The quark stars are supported by the pressure of the quarks.

Our goal at the end of this chapter is to give EoS for our interest compact object strange star made only with *uds* quarks together with leptons which are required for charge neutrality and chemical equilibrium of stellar matter.

We first consider only u and d quarks and leptons. The condition of charge neutrality and baryon number conservation are given by

$$0 = \frac{2}{3}\rho_u - \frac{1}{3}\rho_d - \rho_e - \rho_\mu , \tag{2.18}$$

$$\rho_B = \frac{1}{3}(\rho_u + \rho_d), \tag{2.19}$$

where ρ_e and ρ_μ is the density of electron and muon, respectively.

The condition of the chemical equilibrium comes from the balance (on $\beta-$decay) $n \Leftrightarrow p + e^- + \overline{\nu}_{e^-}$. In this work we only consider deleptonized stellar matter (without neutrinos).

Now, we consider quark matter with u, d and s quarks (which we call strange matter) and electrons. Elementary processes which induce chemical equilibrium are:

$$\mu_d = \mu_u + \mu_e, \quad (2.20)$$
$$\mu_s = \mu_d, \quad (2.21)$$
$$0 = -\frac{1}{3}(\rho_d + \rho_s) + \frac{2}{3}\rho_u - \rho_e - \rho_\mu, \quad (2.22)$$
$$\rho = \frac{1}{3}(\rho_u + \rho_d + \rho_s). \quad (2.23)$$

There may exist states of matter which are stable or meta-stable with large strangeness. With the equations of state it is easy to derive a range for the bag constant, which allows for stable strange matter.

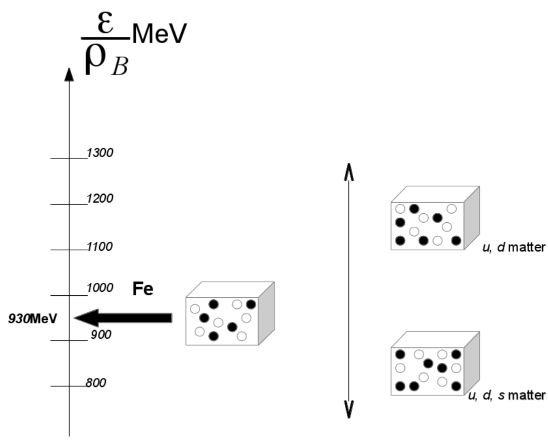

Figura 3: Schematic comparison of the most stable element in nature ^{56}Fe with u, d matter and u, d, s matter (WEBER F.,1999).

The criterion used to find the stability windows in this work is the standard: the upper limit is obtained by the condition that the binding energy, $\mathscr{E}/\rho_B$ of strange matter is smaller than 930 MeV (which is the more stable nucleus in nature, ^{56}Fe) at its point of saturation ($\mathscr{P} = 0$). The lower limit is obtained for the condition that the binding energy of u and d quark matter without strangeness is larger than 930 MeV (has to be less stable than ^{56}Fe). In Fig. 3, we can see a scheme of how to obtain the limit for stable quark matter.

In the remainder of this Chapter, we show the results obtained at finite temperature by the MIT bag model and we change the energy density by the free energy density to analyze the stability windows, since as expected from calculations in the macrocanonical or grand-cannonical ensemble, the quantity related to the thermodynamical potential is the free energy per baryon $F/A = \mathscr{F}/\rho_B = (\varepsilon - T\mathscr{S})/\rho_B$, where $\mathscr{F}$ is the free energy per baryon, ρ_B the baryon density, ε the energy density, T the temperature and $\mathscr{S}$ the entropy density of the system.

All the results were obtained taking into account the following: the quarks have masses $m_u = m_d = 5$ MeV and $m_s = 150$ MeV, we consider matter with identical quark chemical potentials, $\mu_u = \mu_d$ corresponding to symmetric matter in two-flavor matter and $\mu_u = \mu_d = \mu_s$ for three-flavor matter.

Our purpose is to obtain the equation of state (Bag constants) for which the quark matter is stable. In this model, this is obtained at saturation point of the quark matter ($\mathscr{P} = 0$). In the next figures we show values of the free energy density per baryon number for u, d matter as well as for u, d, s matter without electrons for different values of the temperature and the confinement constant.

We analyse Fig. 4 in the context of the condition $\mathscr{F}/\rho_B > 930$ MeV, and Fig. 5 in the context $\mathscr{F}/\rho_B \leq 930$ MeV. This analysis allows us to investigate the stability of the strange quark matter and find possible values for the bag constant. We show the values for which quark matter is stable.

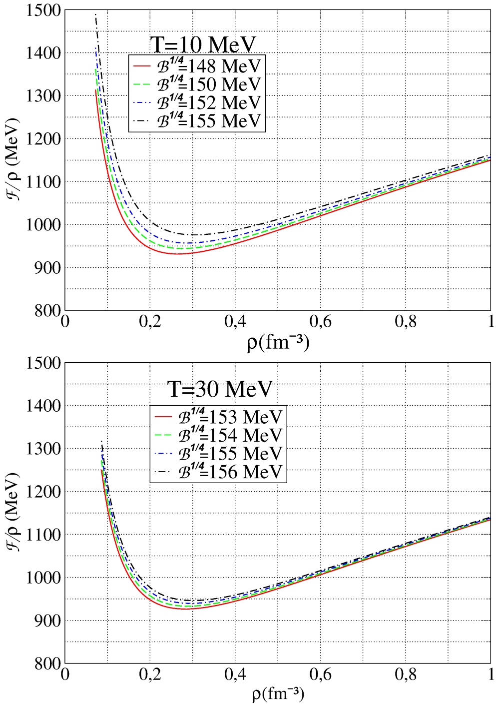

Figura 4: Free energy density per baryonic number density at $T = 10$ MeV and $T = 30$ MeV for the MIT model for u and d quark matter.

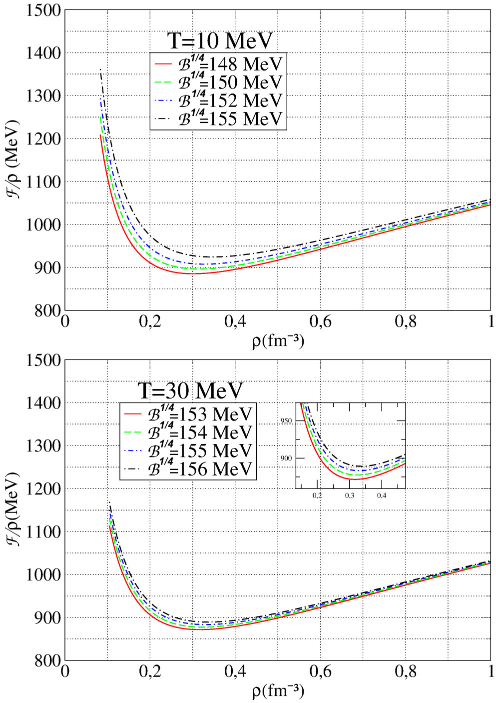

Figura 5: Free energy density per baryonic number density at $T = 10$ MeV and $T = 30$ MeV for the MIT model for u, d and s quark matter and $m_s = 150$ MeV.

If we compare in Fig. 4 the curves for the value of bag constant of 155 MeV at temperature 10 MeV and 30 MeV, the value for the free energy density per baryonic number density at $\mathscr{P} = 0$ is 975.7 MeV and 939.5 MeV respectively. This show that when the temperature is increased the system is more bound. For u, d, s matter the free energy density per baryonic number density should be less than 930 MeV, comparing in Fig. 5 the curves for a bag constant of 155 MeV at temperatures of 10 MeV and 30 MeV the values at zero pressure are 924.3 MeV and 883.3 MeV respectively, again the system is more bound.

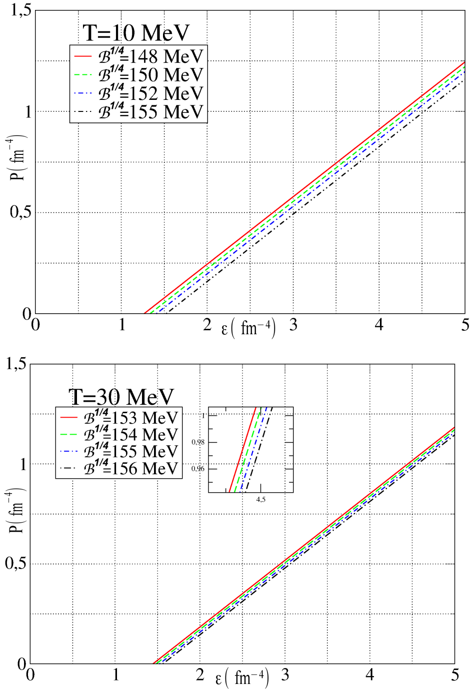

Figura 6: EoS for the MIT model at $T = 10$ MeV and $T = 30$ MeV for u and d quark matter.

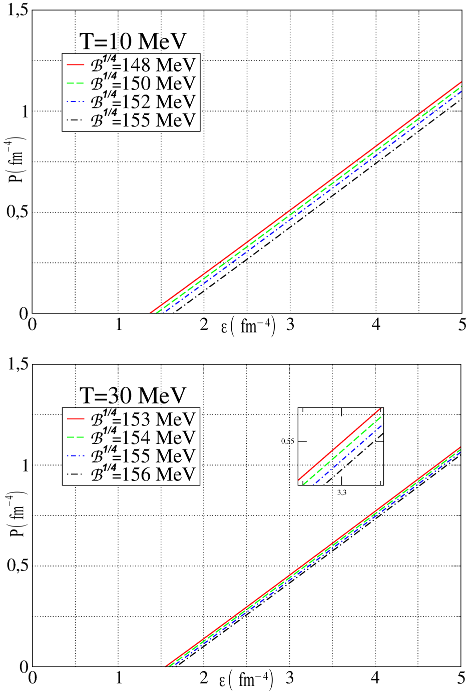

Figura 7: EoS for the MIT model at $T = 10$ MeV and $T = 30$ MeV for u,d and s quark matter and $m_s = 150$ MeV.

In Fig 6 we have the equation of state for two flavor matter, and in Fig. 7 the equation of state for the three flavor matter whose choice of parameters satisfy the stability condition of stable matter. One observes that with the increase of the value of the bag constant at a fixed temperature, the curves shift to the right-hand side, i.e., the equation of state becomes softer. The equation of state for this model presents a linear dependence of the pressure with the energy density.

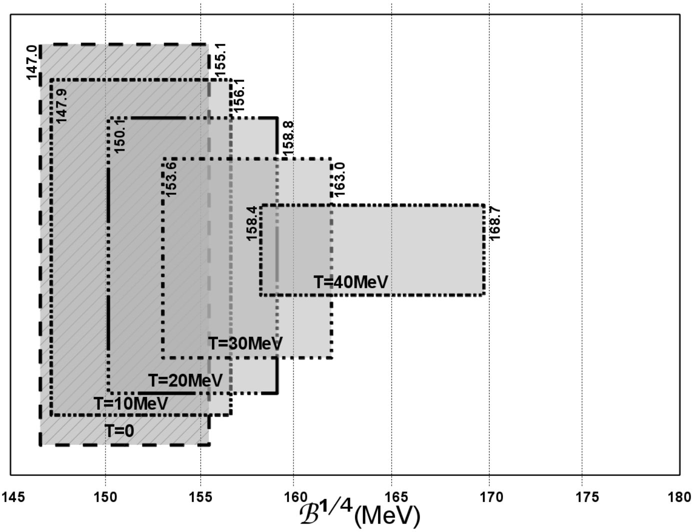

Figura 8: Schematic picture of the stability windows for the MIT bag model.

After the conditions used to search for the stable matter, the values produced for different temperatures are:

$$T = 10 \text{ MeV}, \qquad 147.9 \text{ MeV} < \mathscr{B}^{1/4} < 156.1 \text{ MeV}, \tag{2.24}$$

$$T = 20 \text{ MeV}, \qquad 150.1 \text{ MeV} < \mathscr{B}^{1/4} < 158.8 \text{ MeV}, \tag{2.25}$$

$$T = 30 \text{ MeV}, \qquad 153.6 \text{ MeV} < \mathscr{B}^{1/4} < 163.0 \text{ MeV}. \tag{2.26}$$

$$T = 40 \text{ MeV}, \qquad 158.4 \text{ MeV} < \mathscr{B}^{1/4} < 168.7 \text{ MeV}. \tag{2.27}$$

The scheme of the stability window is presented in Fig. 8.

If one looks at Fig. 1.6, one can see that the lower limit for SU(2) matter, i.e., when only u and d quarks are considered, takes place for $\mathscr{B}^{1/4} = 147$ MeV and $T = 0$. Still, for the zero temperature case, strange matter is stable as far as $\mathscr{B}^{1/4}$ is taken between 147 MeV and 155.1 MeV. As temperature increases, the lower boundary is still the one for zero temperature and quarks u and d, but the upper boundary is defined for strange matter (u,d and s quarks) and the chosen temperature. Hence, any bag value in between 147 MeV and 156 MeV is consistent with a stable strange matter at $T = 10$ MeV. In the same way, for $T = 20$ MeV, bag values can be taken between 147 MeV and

158,8 MeV, for $T = 30$ MeV, 147 MeV $< \mathscr{B}^{1/4} <$ 163 MeV and for $T = 40$ MeV, 147 MeV $< \mathscr{B}^{1/4} <$ 168.7 MeV.

After this short discussion on the stability of strange matter, we are prepared to choose appropriate values for the bag constant and then consider appropriate equations of state for quark matter of compacts objects. This EoS are the input to the Tolman-Oppenheimer-Volkoff (discussed chapter 3), for which we find the profile of the strange stars.

Now, we consider stellar matter within the MIT bag model, i.e., incorporate β-equilibrium and charge neutrality conditions (Eqs 2.18-2.23).

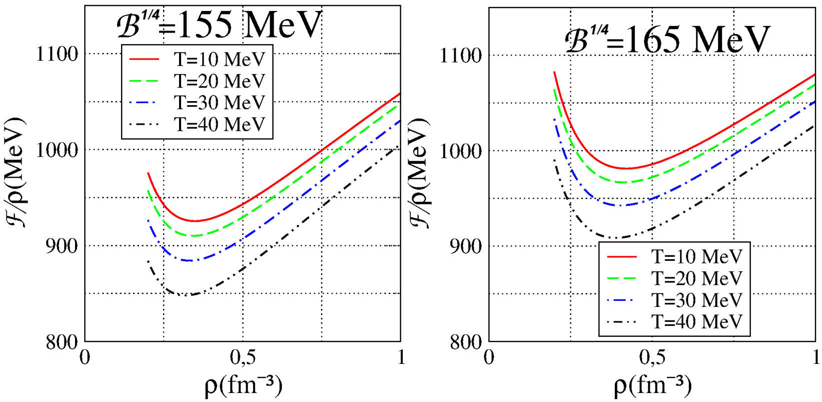

Figura 9: Free energy per baryonic number for the MIT model for u, d and s quark matter with leptons, $m_s = 150$ MeV and bag constants 155 MeV and 165 MeV.

In Figs. 9 and 10, we can see the free energy density per baryonic number density and the equation of state for stellar matter. Comparing the curves in Fig. 9, we note that the system with presence of leptons becomes more bound. In Fig. 10 we can see that the presence of the leptons into the quark matter softens the equation of state for the stars. Equations of state shown in Fig.10 will be used as input to the Tolman-Oppenheimer-Volkoff equations to give the profile of the quark star in Chapter 3.

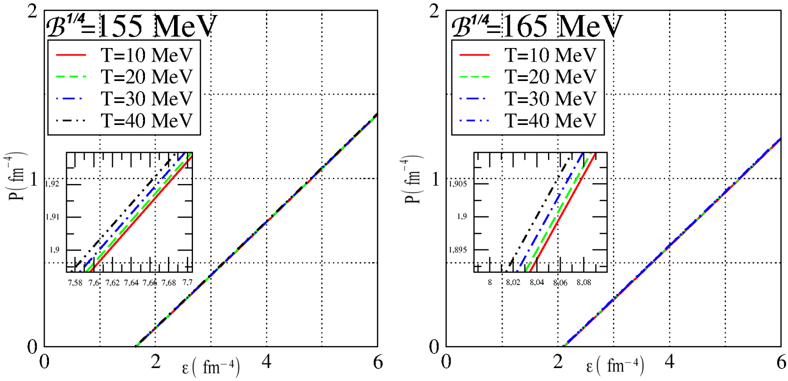

Figura 10: EoS for the MIT model for u, d and s quark matter with leptons, $m_s = 150$ MeV and Bag constant 155 MeV and 165 MeV.

3 EQUATION OF STATE (EOS) FOR THE QUARK MASS DENSITY DEPENDENT MODEL (QMDD)

We assume that strange matter is a Fermi gas mixture of quarks u, d, s, anti-quarks $\overline{u}, \overline{d}, \overline{s}$, where the quark masses are parametrized with the baryonic density ρ_B as (BENVENUTO O. G, LUGONES G.,1995b):

$$m_u = m_d = \frac{C}{3\rho_B}, \tag{3.1}$$

$$m_s = m_{s0} + \frac{C}{3\rho_B}, \tag{3.2}$$

where m_{s0} is the current mass of the strange quark and C is a constant. The baryon density, is defined as

$$\rho_B = \frac{1}{3}(\rho_u + \rho_d + \rho_s). \tag{3.3}$$

Equations (3.1) and (3.2) correspond to a quark confinement mechanism because if the volume of the system tends to infinity and the baryon density goes to zero then m_i tend to infinity according to this equations (YIN S., SU R.).

Just as in the MIT model at finite temperature, the energy of the system for the QMDD model is the Helmholtz free energy and the entropy is given by Eq. (2.1).

In order to find the equation of state at finite temperature, we start from the grand-thermodynamical potential

$$\Omega_{ideal} \equiv \sum_i \Omega_i = \sum_i -\frac{1}{\beta}\frac{\gamma_i V}{(2\pi)^3}\int d^3k\{\ln[1+e^{-\beta(E_i-\mu_i)}] + \ln[1+e^{-\beta(E_i+\mu_i)}]\}, \tag{3.4}$$

where γ_i is the degeneracy factor. The grand-potential for the QMDD model can be written as

$$\Omega_{qmdd} = \sum_i \Omega_i + \Omega_C(\rho_B), \tag{3.5}$$

where $\Omega_C(\rho_B)$ is the potential associated with the term of confinement and the sum runs over $i = u, d, s$.

The total pressure in the QMDD model can be written as

$$\mathscr{P}_{qmdd} = \sum_i \mathscr{P}_{ideal} - B(\rho_B), \tag{3.6}$$

where $B(\rho_B)$ is analogous to the term $\mathscr{B}$ of the MIT bag model. From the basic thermodynamics, the pressure for an ideal system is defined as

$$-\mathscr{P}_{ideal} = \frac{\Omega_{ideal}}{V}. \tag{3.7}$$

using Eqs. (3.6) and (3.7) the pressure for QMDD model can be written as

$$\mathscr{P}_{qmdd} = \sum_{i\bar{i}} \frac{1}{\beta}\frac{\gamma_i}{(2\pi)^3}\int d^3k\{\ln[1+e^{-\beta(E_i-\mu_i)}] + \ln[1+e^{-\beta(E_i+\mu_i)}]\} - B(\rho_B), \tag{3.8}$$

with

$$E_i(k) = \sqrt{k_i^2+m_i^2}. \tag{3.9}$$

Let us considere that

$$\rho_B = \frac{N_B}{V},$$

where N_B is the baryonic mean number and it is fixed. The physical quantities entropy, particle number and pressure, can be found by:

$$\boxed{\mathscr{S} = -\left(\frac{\partial\Omega_{ideal}}{\partial T}\right)_{V,\mu_i}, \quad \mathscr{N} = -\left(\frac{\partial\Omega_{ideal}}{\partial \mu_i}\right)_{T,V}, \quad \mathscr{P} = -\left(\frac{\partial\Omega_{ideal}}{\partial V}\right)_{T,\mu},} \tag{3.10}$$

and can be manipulated to find the physical quantities in the QMDD model.

The pressure can be manipulated as

$$\begin{aligned}\mathscr{P} &= -\left(\frac{\partial\Omega_{ideal}}{\partial V}\right)_{T,\mu_i} = -\left(\frac{\partial\Omega_{ideal}}{\partial(N_B/\rho_B)}\right)_{T,\mu_i}, \\ &= -\frac{1}{N_B}\left(\frac{\partial(\Omega_{ideal})}{\partial(1/\rho_B)}\right)_{T,\mu_i} = \frac{1}{\rho_B V}\left(\frac{\partial\Omega_{ideal}}{\partial(1/\rho_B)}\right)_{T,\mu_i}, \\ &= -\left(\frac{\partial\Omega_{ideal}/\rho_B V}{\partial(1/\rho_B)}\right)_{T,\mu_i} = \frac{1}{V}\left(\frac{\partial\Omega_{ideal}}{\partial(1/\rho_B)}\right)_{T,\mu_i} + \frac{\Omega_{ideal}}{V}.\end{aligned} \tag{3.11}$$

We are going to solve this derivative making a change of variables

$$
\begin{aligned}
x &= \frac{1}{\rho_B} \quad , \rho_B = x^{-1} \\
\left(\frac{\partial \Omega_{ideal}}{\partial(1/\rho_B)}\right) = \frac{\partial \Omega_{ideal}}{\partial x} &= \frac{\partial x^{-1}}{\partial x}\frac{\partial \Omega_{ideal}}{\partial x^{-1}} \\
= -\frac{1}{x^2}\left(\frac{\partial \Omega ideal}{\partial x^{-1}}\right) &= -\rho_B^2\left(\frac{\partial \Omega_{ideal}}{\partial \rho_B}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{P} &= -\frac{\Omega_{ideal}}{V} + \frac{1}{\rho_B}\rho_B^2\left(\frac{\partial \Omega_{ideal}/V}{\partial \rho_B}\right)_{T,\mu_i} \\
&= -\frac{\Omega ideal}{V} + \rho_B\left(\frac{\partial \Omega_{ideal}/V}{\partial \rho_B}\right)_{T,\mu_i}
\end{aligned}
\tag{3.12}
$$

Defining,

$$
\Omega = -\frac{\mathscr{P}}{V} = -\widetilde{\Omega}, \tag{3.13}
$$

$$
\begin{aligned}
\mathscr{P}_{qmdd} &= -\widetilde{\Omega}_{ideal} + \rho_B\left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \rho_B}\right) \\
&= \mathscr{P}_{ideal} + B(\rho_B).
\end{aligned}
\tag{3.14}
$$

We can identify the confinement term as

$$
B(\rho_B) = \rho_B\left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \rho_B}\right)_{T,\mu_i}. \tag{3.15}
$$

If we use the relation $\widetilde{\Omega}_{qmdd} = -\mathscr{P}_{qmdd}$ we identify the grand-potential thermodynamics with the *ansatz* $B(\rho_B)$

$$
\widetilde{\Omega}_{qmdd} = \widetilde{\Omega}_{ideal} - B(\rho_B). \tag{3.16}
$$

From Legendre's transform for the grand-potential thermodynamics

Ω, we have the following relations

$$\begin{aligned}
\mathscr{E}_{qmdd}(\rho_B;\mathscr{S},V,\mathscr{N}) &= \mathscr{S}T - \mathscr{P}_{qmdd}V + \sum_i \mu_i \mathscr{N}_i, \\
-\mathscr{P}_{qmdd}V &= \mathscr{E}_{qmdd}(\rho_B;\mathscr{S},V,\mathscr{N}) - \mathscr{S}T + \sum_i \mu_i \mathscr{N}_i, \\
\Omega_{qmdd} &= \mathscr{E}_{qmdd}(\rho_B;\mathscr{S},V,\mathscr{N}) - \mathscr{S}T + \sum_i \mu_i \mathscr{N}_i, \\
-\ln \mathscr{Z} &= \mathscr{E}_{qmdd}(\rho_B;\mathscr{S},V,\mathscr{N}) - \mathscr{S}T + \sum_i \mu_i \mathscr{N}_i \quad ,
\end{aligned} \tag{3.17}$$

where we use the general relation of thermodynamics $-k \ln \mathscr{Z} = \mathscr{P}V$, where $k = 1$, thus,

$$\Omega_{qmdd} = -\mathscr{P}_{qmdd}V. \tag{3.18}$$

Taken the differential of $\Omega_{qmdd}(\mu_i, T)$

$$\begin{aligned}
d\Omega_{qmdd}(\mu_i,T) &= d\mathscr{E}_{qmdd}(\mathscr{S},V,\mathscr{N}) - d(\mathscr{S}T) - d(\sum_i \mu_i \mathscr{N}_i), \\
d\Omega_{qmdd}(\mu_i,T) &= -\mathscr{S}dT - \mathscr{P}dV - \sum_i \mathscr{N}_i d\mu_i,
\end{aligned} \tag{3.19}$$

the potential density is

$$\begin{aligned}
\frac{\Omega_{qmdd}(\mu_i,T)}{V} &= \frac{\mathscr{E}_{qmdd}}{V} - \frac{\mathscr{S}}{V}T - \sum_i \frac{\mathscr{N}}{V}\mu_i, \\
\widetilde{\Omega}_{qmdd}(\mu_i,T) &= \varepsilon_{qmdd} - sT - \sum_i \rho_{i_{ideal}} \mu_i,
\end{aligned} \tag{3.20}$$

considering the previous relations, the energy density for QMDD model is:

$$\begin{aligned}
\varepsilon_{qmdd} &= \widetilde{\Omega}_{qmdd}(\mu_i,T) + sT + \rho_{i_{ideal}}, \\
\varepsilon_{qmdd} &= \widetilde{\Omega}_{qmdd}(\mu_i,T) + \left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial T}\right)_{\mu_i} T + \left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \mu_i}\right)_T \mu_i,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\varepsilon_{qmdd} &= \widetilde{\Omega}_{ideal} + T\left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial T}\right)_{\mu_i} + \mu_i \left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \mu_i}\right)_T - \rho_B \left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \rho_B}\right), \\
\varepsilon_{qmdd} &= \varepsilon_{ideal} - \rho_B \left(\frac{\partial \widetilde{\Omega}_{ideal}}{\partial \rho_B}\right).
\end{aligned} \tag{3.22}$$

$$\frac{\partial}{\partial m_i}\int d^3k\{\ln[1+e^{-\beta(\sqrt{k^2+m_i^2}-\mu_i)}]+\ln[1+e^{-\beta(\sqrt{k^2+m_{\bar{i}}^2}+\mu_{\bar{i}})}]\}$$

$$\begin{aligned}&=\frac{1}{1+e^{-\beta(\sqrt{k^2+m_i^2}-\mu_i)}}\frac{\partial}{\partial m_i}[e^{-\beta(\sqrt{k^2+m_i^2})-\mu_i}]\\&+\frac{1}{1+e^{-\beta(\sqrt{k^2+m_{\bar{i}}^2}+\mu_{\bar{i}})}}\frac{\partial}{\partial m_{\bar{i}}}[e^{-\beta(\sqrt{k^2+m_{\bar{i}}^2})+\mu_{\bar{i}}}],\end{aligned}$$

solving the derivatives

$$\begin{aligned}\frac{\partial}{\partial m_i}[e^{-\beta(\sqrt{k^2+m_i^2}-\mu_i)}]&=-\beta e^{(\sqrt{k^2+m_i^2}-\mu_i)}\frac{m_i}{\sqrt{k^2+m_i^2}}\\\frac{\partial}{\partial m_{\bar{i}}}[e^{-\beta(\sqrt{k^2+m_{\bar{i}}^2}+\mu_{\bar{i}})}]&=-\beta e^{(\sqrt{k^2+m_{\bar{i}}^2}+\mu_{\bar{i}})}\frac{m_{\bar{i}}}{\sqrt{k^2+m_{\bar{i}}^2}}\end{aligned}$$

$$\begin{aligned}&=\int d^3k\left\{\frac{(-\beta)e^{-\beta(\sqrt{k^2+m_i^2}-\mu_i)}m_i}{(1+e^{-\beta(\sqrt{k^2+m_i^2}-\mu_i)})\sqrt{k^2+m_i^2}}+\frac{(-\beta)e^{-\beta(\sqrt{k_{\bar{i}}^2+m_{\bar{i}}}-\mu_{\bar{i}})}m_{\bar{i}}}{(1+e^{-\beta(\sqrt{k_{\bar{i}}^2+m_{\bar{i}}^2}+\mu_{\bar{i}})})\sqrt{k_{\bar{i}}^2+m_{\bar{i}}^2}}\right\}\\&=\int d^3k\left\{\frac{m_i}{\mathscr{E}_i}\frac{(-\beta)e^{-\beta(\mathscr{E}_i-\mu_i)}}{(1+e^{-\beta(\mathscr{E}_i-\mu_i)})}+\frac{m_{\bar{i}}}{\mathscr{E}_{\bar{i}}}\frac{(-\beta)e^{-\beta(\mathscr{E}_{\bar{i}}+\mu_{\bar{i}})}}{(1+e^{-\beta(\mathscr{E}_{\bar{i}}+\mu_{\bar{i}})})}\right\}\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial\widetilde{\Omega}_{ideal}}{\partial m}\right)_{T,\mu_i}&=\sum_{i\bar{i}}\frac{\gamma_i}{(2\pi)^3}\int d^3k\left\{\frac{m_i}{\mathscr{E}_i}\frac{1}{e^{\beta(\mathscr{E}_i-\mu_i)}+1}+\frac{m_{\bar{i}}}{\mathscr{E}_{\bar{i}}}\frac{1}{e^{\beta(\mathscr{E}_{\bar{i}}+\mu_{\bar{i}})}}\right\}\\&=\sum_i\frac{\gamma_i}{(2\pi)^3}\int d^3k\frac{m_i}{\mathscr{E}_i}\left[n_i+\bar{n}_i\right],\end{aligned} \tag{3.23}$$

where, we made $i=\bar{i}$ for $u,d,s,\bar{u},\bar{d},\bar{s}$, and $n_i(\bar{n}_i)$ are the distribution function of Fermi-Dirac for particles(anti-particles).

In spherical coordinates the differential d^3k is $4\pi k^2dk$, then

$$\left(\frac{\partial\widetilde{\Omega}_{ideal}}{\partial m}\right)_{T,\mu_i} = \sum_i \frac{\gamma_i}{2\pi^2}\int k^2 dk \frac{m_i}{\mathscr{E}_i}\left[n_i+\overline{n}_i\right]. \qquad (3.24)$$

Now, $B(\rho_B)$ can be written as

$$B(\rho_B) = -\frac{C}{3\rho_B}\sum_i \frac{\gamma_i}{2\pi^2}\int dkk^2 \frac{m_i}{\mathscr{E}_i}[n_i+\overline{n}_i]. \qquad (3.25)$$

We finally summarize the equation of state for the QMDD model at finite temperature .

The pressure is given by

$$\mathscr{P}_{qmdd} = \mathscr{P}_{ideal} + B(\rho_B), \qquad (3.26)$$

where the pressure $\mathscr{P}_{ideal}$, is given by

$$\mathscr{P}_{ideal} = \frac{1}{6\pi^2}\sum_i \gamma_i \int_0^\infty dk \frac{k^4}{\sqrt{k_i^2+m_i^2}}[n_i+\overline{n}_i],$$

thus,

$$\mathscr{P}_{qmdd} = \frac{1}{6\pi^2}\sum_i \gamma_i \int_0^\infty dk \frac{k^4}{\sqrt{k_i^2+m_i^2}}[n_i+\overline{n}_i] - \frac{C}{3\rho_B}\sum_i \frac{\gamma_i}{2\pi^2}\int_0^\infty dk \frac{k^2 m_i}{\sqrt{k^2+m_i^2}}[n_i+\overline{n}_i]$$

$$\mathscr{P}_{qmdd} = \frac{1}{2\pi^2}\sum_i \int_0^\infty dk \frac{k^2}{\sqrt{k_i^2+m_i^2}}\left[\frac{k^2}{3} - \frac{Cm_i}{3\rho_B}\right][n_i+\overline{n}_i]. \qquad (3.27)$$

The energy density is

$$\mathscr{E}_{qmdd} = \mathscr{E}_{ideal} - B(\rho_B), \qquad (3.28)$$

where $\mathscr{E}_{qmdd}$ is

$$\mathscr{E}_{ideal} = \frac{1}{2\pi^2}\sum_i \gamma_i \int dkk^2 \sqrt{k_i^2+m_i^2}[n_i+\overline{n}_i] + \frac{C}{3\rho_B}\sum_i \frac{\gamma_i}{2\pi^2}\int dkk^2 \frac{m_i}{\sqrt{k_i^2+m_i^2}}[n_i+\overline{n}_i],$$

thus,

$$\mathscr{E}_{qmdd} = \frac{1}{2\pi^2}\sum_i \gamma_i \int dk k^2 \sqrt{k_i^2+m_i^2}\left[1+\frac{1}{k_i^2+m_i^2}\left(\frac{Cm_i}{3\rho_B}\right)[n_i+\bar{n}_i]\right], \tag{3.29}$$

and the baryonic density

$$\rho_B = \frac{1}{3}\sum_i \rho_i, \quad \text{with} \quad \rho_i = \frac{\gamma_i}{2\pi^2}\int dk k^2 [n_i+\bar{n}_i],$$

thus,

$$\rho_B = \frac{1}{3}\frac{\gamma_i}{2\pi^2}\int dk k^2 [n_i+\bar{n}_i]. \tag{3.30}$$

Collecting the EoS for QMDD at finite temperature we have

$$\boxed{\begin{aligned}
\mathscr{P}_{qmdd} &= \frac{1}{2\pi^2}\sum_i \gamma_i \int_0^\infty dk \frac{k^2}{\sqrt{k_i^2+m_i^2}}\left[\frac{k^2}{3} - C\frac{m_i}{3\rho_B}\right][n_i+\bar{n}_i],\\
\varepsilon_{qmdd} &= \frac{1}{2\pi^2}\sum_i \gamma_i \int_0^\infty dk k^2 \sqrt{k_i^2+m_i^2}\left[1+\frac{1}{k_i^2+m_i^2}\left(C\frac{m_i}{3\rho_B}\right)\right][n_i+\bar{n}_i],\\
\mathscr{F}_{qmdd} &= \varepsilon_{qmdd} - T\mathscr{S},\\
\rho_B &= \frac{1}{3}\frac{\gamma_i}{2\pi^2}\int_0^\infty dk k^2 [n_i-\bar{n}_i],
\end{aligned}} \tag{3.31}$$

where $\mathscr{F}$ is the free energy of the system.

At zero temperature, the Fermi-Dirac distribution becomes a Heaviside step function (Eq. (1.3)), and the equation of state (EoS) for zero temperature can be found with the potential thermodynamics.

$$\widetilde{\Omega}_{qmdd} = \widetilde{\Omega}_{ideal} - B(\rho_B). \tag{3.32}$$

The grand-potential thermodynamics can be expressed in terms of the function $F(x_i)$ (Eq. 2.15),

$$\widetilde{\Omega}_{ideal} = -\sum_i \frac{\gamma_i}{48\pi^2} m_i^4 F(x_i). \tag{3.33}$$

From basic calculus the function $\arg(\sinh(\mathrm{x}))$ can be written as $\arg(\sinh(\mathrm{x})) = \sqrt{x^2+1}+x$, thus the function $F(x)$ can be written as

$$F(x_i) = x_i\sqrt{x_i^2+1}(2x_i^2-3) + 3\arg(\sinh(x_i)), \tag{3.34}$$

where the x parameter is related to the chemical potential and the effective mass by

$$x_i = \sqrt{\left(\frac{\mu_i}{m_i}\right)^2 - 1}. \tag{3.35}$$

The confinement term becomes

$$B(\rho_B) = -C\frac{1}{2\pi^2}\sum_i \gamma_i \int_0^{k_f} dkk^2 \frac{m_i}{k^2+m_i^2}, \tag{3.36}$$

which can be easily calculated .

The pressure for the QMDD model is then calculated from

$$\mathscr{P}_{qmdd} = \mathscr{P}_{ideal} + B(\rho_B), \tag{3.37}$$

and the confinement term is identified as

$$B(\rho_B) = \rho_B \left(\frac{\partial\widetilde{\Omega}}{\partial\rho_B}\right)_{T,\mu}. \tag{3.38}$$

Calculating the derivative and making some algebra, we obtain for the pressure:

$$\mathscr{P}_{qmdd} = \sum_i \frac{\gamma_i m_i^4}{48\pi^2}\left[F(x_i) - \frac{C}{\rho_B}\left(\frac{4}{m_i}\right)G(x_i)\right], \tag{3.39}$$

where the function $G(x_i)$ is defined as

$$G(x_i) = x_i\sqrt{x_i^2+1} - arg(\sinh(x_i)). \tag{3.40}$$

The energy density is found from the following thermodynamic relation $\widetilde{\Omega}_{ideal} = -\mathscr{P}_{ideal}$ and the Eq. (3.24) at $T=0$,

$$\varepsilon_{qmdd} = \widetilde{\Omega}_{ideal} + \left(\frac{\partial\widetilde{\Omega}_{ideal}}{\partial\mu_i}\right)_T \mu_i - \left(\frac{\partial\widetilde{\Omega}_{ideal}}{\partial\rho_B}\right)_{T,\mu_i}\rho_B. \tag{3.41}$$

We calculate the particle density $\rho_{i_{qmdd}}$, using the relation

$$\rho_{i_{qmdd}} = \rho_{i_{ideal}} = -\left(\frac{\partial\widetilde{\Omega}_{i_{ideal}}}{\partial\mu_i}\right)_{T,\rho_B}, \tag{3.42}$$

deriving Eq. (3.36), using the definition for the function $F(x_i)$ and making again some algebra, the particle density becomes

$$\rho_i = \gamma_i \frac{8m_i^3 x_i^3}{48\pi^2}. \tag{3.43}$$

With this result we can calculate the energy density

$$\varepsilon_{qmdd} = \sum_i \frac{\gamma_i m_i^4}{48\pi^2}\left[3H(x_i) + \frac{C}{\rho_B}\left(\frac{4}{m_i}\right)G(x_i)\right], \tag{3.44}$$

where, we define the parametrized function

$$H(x_i) = x_i\sqrt{x_i+1}(1+2x_i^2) - arg(\sinh(x_i)). \tag{3.45}$$

Now, collecting the EoS for density dependent model at zero temperature

$$\boxed{\begin{aligned}
\mathscr{P}_{qmdd} &= \sum_i \frac{\gamma_i m_i^4}{48\pi^2}\left[F(x_i) - \frac{C}{\rho_B}\left(\frac{4}{m_i}\right)G(x_i)\right], \\
\varepsilon_{qmdd} &= \sum_i \frac{\gamma_i m_i^4}{48\pi^2}\left[3H(x_i) + \frac{C}{\rho_B}\left(\frac{4}{m_i}\right)G(x_i)\right], \\
\rho_B &= \frac{1}{3}\sum_i \gamma_i \frac{8m_i^3 x_i^3}{48\pi^2}.
\end{aligned}} \tag{3.46}$$

All the results given in this section are based on the work of Benvenuto and Lugones (BENVENUTO O. G., LUGONES G.,1995a,1995b).

As well as in the previous chapter, here we have the same considerations as in the MIT bag model in order to obtain the stability windows. The quark mass is $m_u = m_d = 5$ MeV, $m_s = 150$ MeV, the chemical potential for su(2) matter are equal ($\mu_u = \mu_d$), and for su(3) matter $\mu_u = \mu_d = \mu_s$.

The stability for the quark matter in the QMDD model is found using the same criterion used in the MIT model, i.e, free energy density at saturation point ($\mathscr{P} = 0$) is greater that 930 MeV for two flavor matter and less than 930 MeV for three flavor matter. In the MIT model, the point where $\mathscr{P} = 0$ coincides with the minimum of the curve, but in the QMDD model this is not the case because of the thermodynamic inconsistency discussed in the introduction. In Fig. 11 we draw the curve of the free energy density versus pressure for u, d matter for a confinement constant of 77 MeVfm^{-3} and a temperature of 10 MeV to illustrate this fact. The zero pressure is away from the minimum of the curve. At first the results are obtained without taking into account the gluons contribution. Afterwards these are taken into account and compared with matter without gluons.

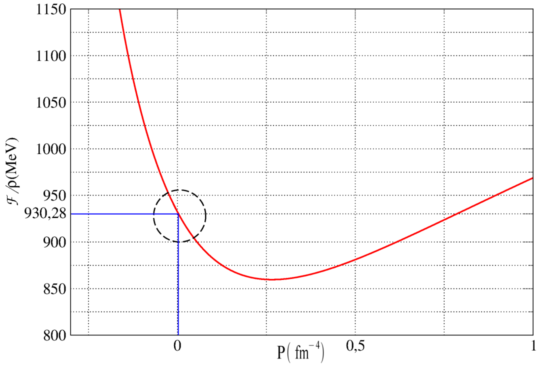

Figura 11: Free energy per pressure. The dot represents the zero pressure point of the quark matter to search for the stability window.

The curves in Figs. 12 and 13 must be understood in the same context as the MIT model, $\mathscr{F}/\rho_B$ must be larger than 930 MeV for *ud* matter and lower than 930 MeV for *uds* matter at the zero pressure point. Figs. 12 and 13 show respectively the free energy density per baryonic number density for the QMDD model without leptons for *ud* quark matter and for *uds* for $T = 10$ MeV and $T = 30$ MeV and different values of the confinement constant.

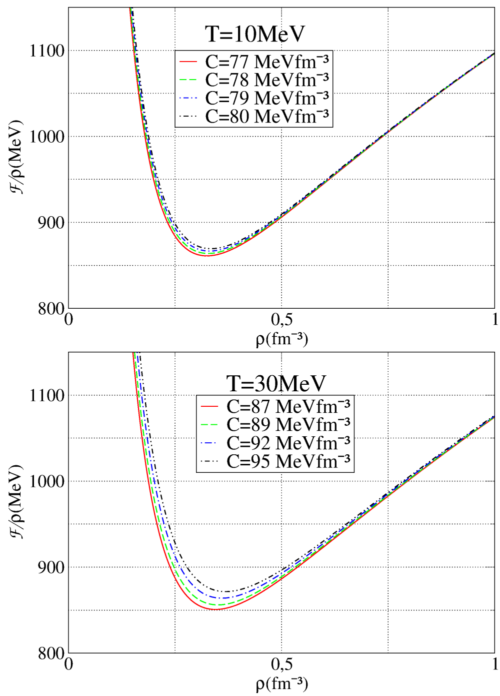

Figura 12: Free energy density per baryonic number density for the QMDD model without leptons for *ud* quark matter for $T = 10$ MeV and $T = 30$ MeV and different values of the confinement constant.

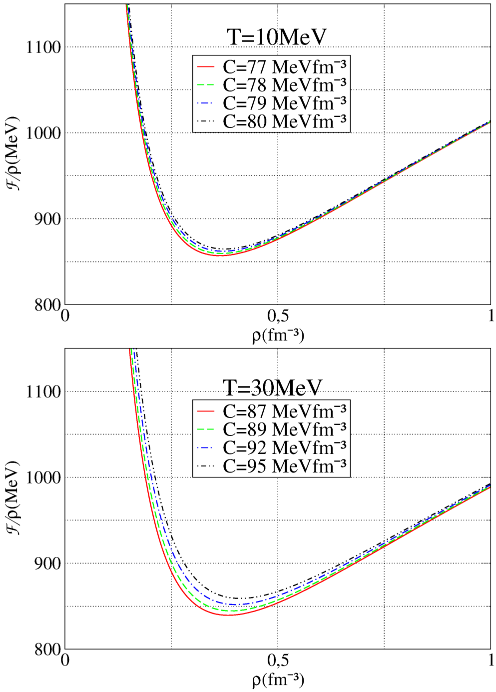

Figura 13: Free energy density per baryonic number density for the QMDD model without leptons for *uds* matter, for $T = 10$ MeV, $T = 30$ MeV and different values of the confinement constant.

In Figs. 14 and 15 we show the equation of state for two and three flavor matter respectively for different temperatures and values of confinement constant. The simple form of this equation changes with respect to the MIT model. Note that the equation of state does not present a linear dependence with the pressure when the energy density increases and at lower pressures the curve is slightly warped.

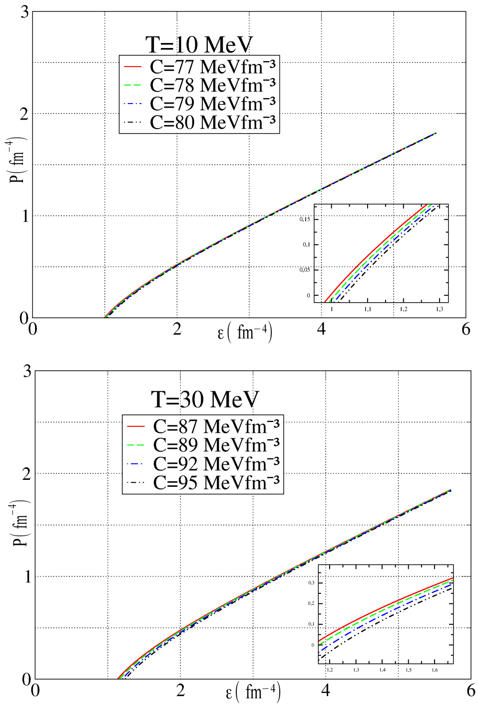

Figura 14: EoS for the QMDD model without leptons. The up curves are to *ud* quark matter for $T = 10$ MeV and $T = 30$ MeV and different values of confinement constant.

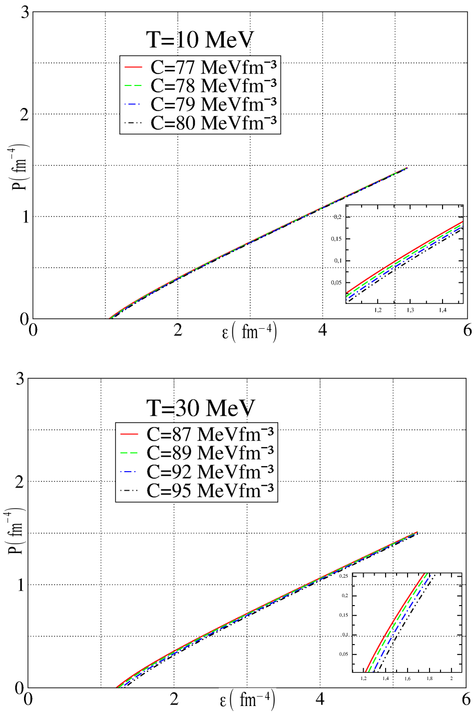

Figura 15: EoS for the QMDD model without leptons for *uds* matter, at $T = 10$ MeV and $T = 30$ MeV and different values of bag constant.

Having values found with the curves of free density energy per baryonic density for the confinement constant where the quark matter is most stable, we can build the stability windows for this model.

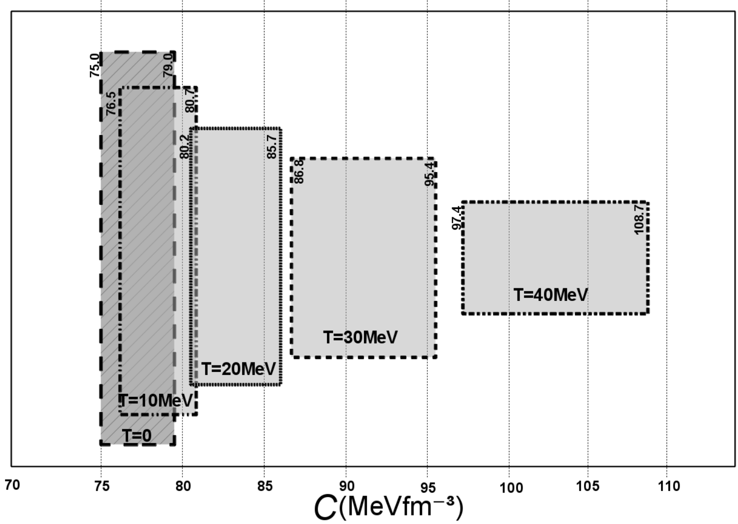

Figura 16: Schematic picture of the stability windows for the QMDD model.

In Fig. 16 we show a scheme of the stability window for the QMDD model. The obtained ranges of the confinement constant are:

$$T = 10\,\text{MeV}, \qquad 76.5\ \text{MeVfm}^{-3} < C < 80.7\ \text{MeVfm}^{-3}, \tag{3.47}$$

$$T = 20\,\text{MeV}, \qquad 80.2\ \text{MeVfm}^{-3} < C < 85.7\ \text{MeVfm}^{-3}, \tag{3.48}$$

$$T = 30\,\text{MeV}, \qquad 86.8\ \text{MeVfm}^{-3} < C < 95.4\ \text{MeVfm}^{-3}, \tag{3.49}$$

$$T = 40\,\text{MeV}, \qquad 97.4\ \text{MeVfm}^{-3} < C < 108.7\ \text{MeVfm}^{-3}. \tag{3.50}$$

For the zero temperature case, strange matter is stable as far as C is taken between 75.0 MeVfm^{-3} and 79.0 MeVfm^{-3}. As temperature increases, the lower boundary is still the one for zero temperature, but the upper boundary is defined for three flavor matter (u,d and s quarks) and the chosen temperature. Hence, any value of the confinement constant between 75.0 MeVfm^{-3} and 80.7 MeVfm^{-3} is consistent with a stable strange matter at $T = 10$ MeV. In the same way, for $T = 20$ MeV, confinement values can be taken between 75.0 MeVfm^{-3} and 85.7 MeVfm^{-3}. For $T = 30$ MeV, 75.0 MeVfm$^{-3} < C <$ 95.4 MeVfm^{-3} and for $T = 40$ MeV, 75.0 MeVfm^{-3} $<$

$C < 108.7$ MeVfm^{-3}.

We again include leptons so that charge neutrality and chemical equilibrium are taken into account.

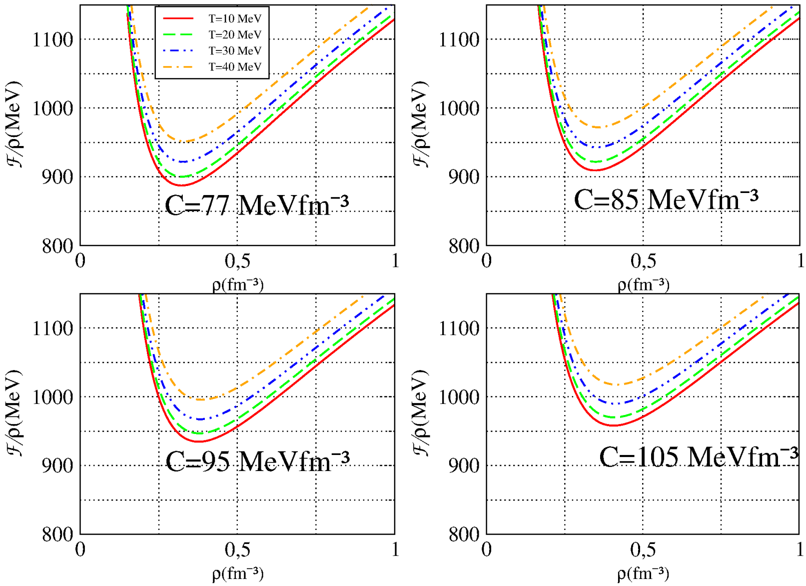

Figura 17: Free energy per baryonic number for the QMDD model for u, d, s quark matter with leptons at different values of confinement constant and temperature, with $m_s = 150$ MeV.

In Fig. 17 we plot the free energy density per baryon for different values of the confinement constant for stellar matter, using $m_s = 150$ MeV as the value of the strange quark mass. If one looks at this figure, we note that the system is less bound when the confinement constant increases.

In Fig. 18, we plot the curves corresponding to the equations of state. Fig. 18 will serve as input for the TOV equations in the next chapter.

With intention of compare with the results obtained in (BENVENUTO O. G, LUGONES G.,1995b), the gluons contributions is taken into account in this model. Analysis is done with the free energy density. The gluons must be considered as an ideal Bose gas of noninteracting bosons. Gluons carry color and spin as well as quarks.

The degeneracy of the gluons is

$$\gamma_g = 2(\text{spin}) \times 8(\text{color}) = 16.$$

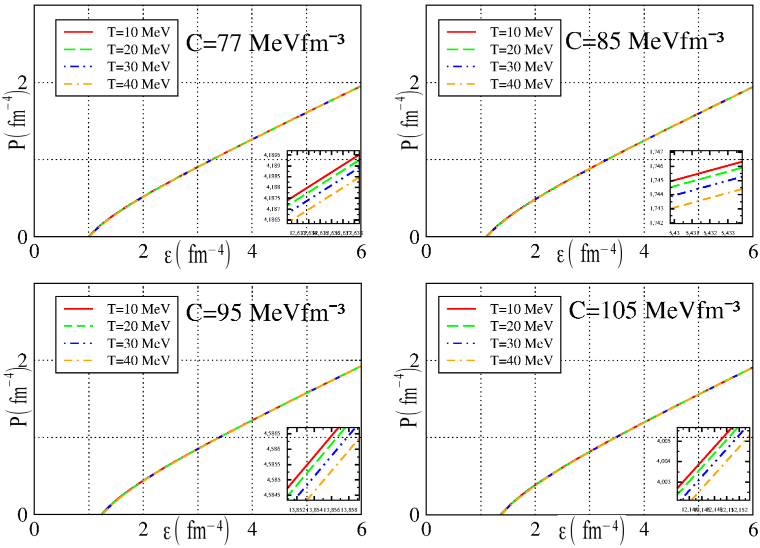

Figura 18: EoS for the QMDD model for u, d, s quark matter with leptons at different values of confinement constant and temperature, with $m_s = 150$ MeV.

It is possible to calculate the contributions to the energy density and pressure using the grand canonical partition function, energy density and pressure for a bosonic system.

$$
\begin{aligned}
q(T,V,z) &= \ln \mathscr{Z} = -\sum_k \ln(1 - ze^{-\beta\varepsilon_k}), \\
\varepsilon(T,V,z) &= \frac{1}{V}\sum_k \frac{\varepsilon_k}{z^{-1}e^{\beta\varepsilon_k} - 1}, \\
\mathscr{P} &= \frac{1}{3}\varepsilon,
\end{aligned}
\tag{3.51}
$$

where $z = e^{\beta\mu}$ is the fugacity of the gas and $\beta = 1/T$.

For large volumes, the sum can be rewritten in terms of integrals as follow:

$$
\sum = \int \frac{d^3\vec{r}d^3\vec{p}}{h^3} = \frac{4\pi V}{h^3}\int_0^\infty \varepsilon^2 d\varepsilon, \tag{3.52}
$$

thus, the grand canonical partition function and energy density can be written as (GREINER W., NEISE L.,1994)

$$\begin{aligned} q(T,V) &= \frac{4\pi V}{(hc)^3}\frac{1}{3}\beta\int_0^\infty d\varepsilon\frac{\varepsilon^3}{e^{\beta\varepsilon}-1}, \\ \varepsilon(T,V) &= \frac{4\pi}{(hc)^3}\int_0^\infty d\varepsilon\frac{\varepsilon^3}{e^{\beta\varepsilon}-1}. \end{aligned} \tag{3.53}$$

This integral can be solved using (GLENDENNING N. K.,2000)

$$\int_0^\infty dx\frac{x^3}{e^x-1} = \frac{\pi^4}{15}, \tag{3.54}$$

knowing that $\mathscr{Z} = \mathscr{P}V/T$ (k =1) and taking into account the degeneracy of gluons. their the energy density and pressure become:

$$\boxed{\begin{aligned} \mathscr{P} &= \frac{8}{45}\pi^2T^4, \\ \varepsilon &= \frac{8}{15}\pi^2T^4. \end{aligned}} \tag{3.55}$$

Let us compare the free energy density curves per barionic density with gluons and without gluons and the EoS for a value of confinement constant of 77 MeVfm^{-3}

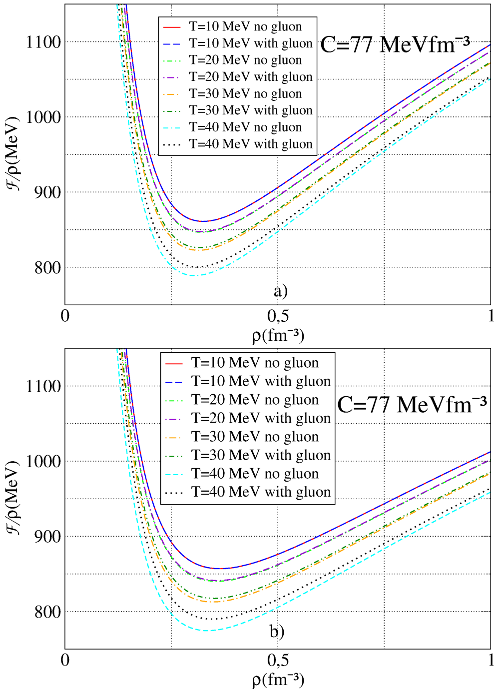

Figura 19: Free energy density per baryonic number density with gluons and without gluons and confinement constant of 77 MeVfm^{-3}. Fig a) *ud* matter. Fig. b) *uds* matter .

In Fig. 19, we display the curves of free energy per baryonic density for a confinement constant of 77 MeVfm^{-3} and 105 MeVfm^{-3} for *ud* and *uds* quark matter respectively at different temperatures. We can see that the effect of the gluons is noticeable only for temperatures between 30 MeV $\sim$ 40 MeV. One can see that the matter with gluons is less bound that matter without gluons.

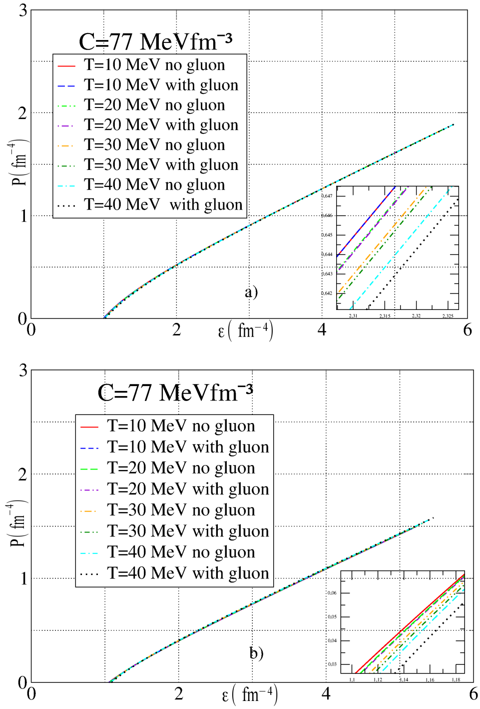

Figura 20: EoS with gluons and without gluons and confinement constant 77 MeVfm^{-3}. Fig. a) *ud* matter. Fig. b) *uds* matter with

In Fig. 20 show the equation of state for *ud* and *uds* matter, where again the effect of gluons are observed only for higher temperature. We conclude at this point that in Figs. 21 for the QMDD model, the equations of state with gluons and without gluons the presence of the leptons (stellar matter) hardens the EoS.

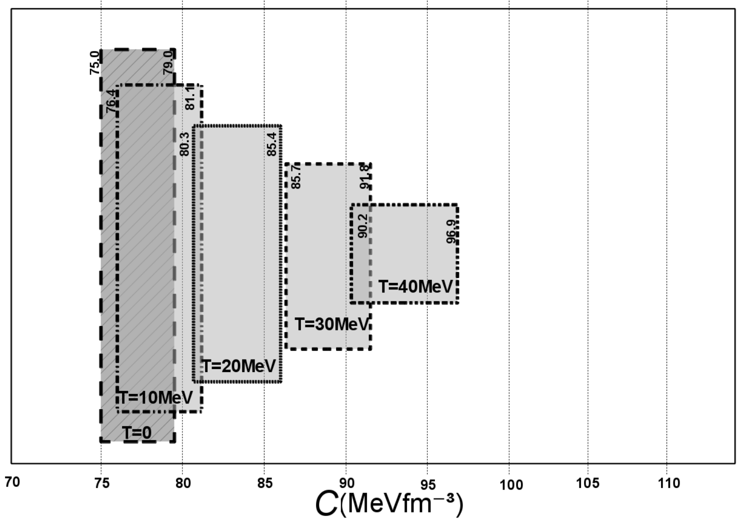

Figura 21: Schematic figure of the stability windows for the QMDD model with gluons.

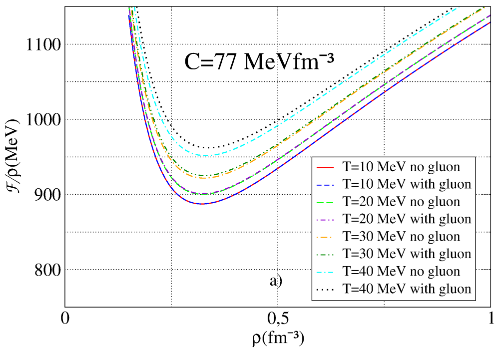

Figura 22: Comparison of free energy density per baryonic number density for stellar matter with gluons and without gluons for a confinement constant 77 MeVfm^{-3}

$$T = 10 \text{ MeV}, \quad 76.4 \text{ MeVfm}^{-3} < C < 81.1 \text{ MeVfm}^{-3},$$

$$T = 20 \text{ MeV}, \quad 80.3 \text{ MeVfm}^{-3} < C < 85.4 \text{ MeVfm}^{-3},$$

$$T = 30 \text{ MeV}, \quad 85.7 \text{ MeVfm}^{-3} < C < 91.8 \text{ MeVfm}^{-3},$$

$$T = 40 \text{ MeV}, \quad 90.2 \text{ MeVfm}^{-3} < C < 96.9 \text{ MeVfm}^{-3}.$$

In Fig. 21 we plot the stability windows when gluons are included. If we compare it with the stability window without gluons (Fig. 16) we can say that for temperatures of 10 MeV $\sim$ 30 MeV, no substantial changes occur, but to temperatures of the order of 40 MeV, the change is more evident.

Finally, we consider stellar matter with gluons within QMDD model. In the description of compact stars, both charge neutrality and β-equilibrium conditions have to be imposed,

$$2\rho_u = \rho_d + \rho_s + (\rho_e + \rho_\mu),$$

$$\mu_s = \mu_u = \mu_d, \quad \mu_e = \mu_u.$$

In Fig. 22 we plot the free energy density per baryonic density and in Fig. 23 the comparison of the equation of state for stellar matter.

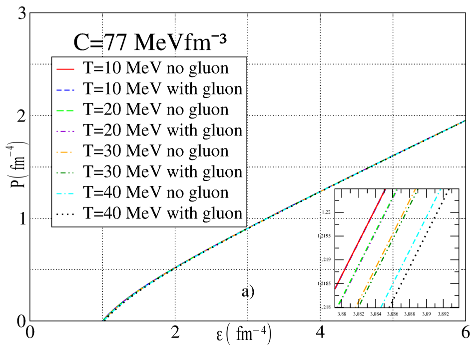

Figura 23: Comparison of EoS for stellar matter with gluons and without gluons for a confinement constant 77 MeVfm^{-3}

From the results shown above, we conclude that the stability window obtained with the free energy per baryon is wider than the one obtained with the binding energy as in (BENVENUTO O. G., LUGONES G.,1995a,1995b), yielding a larger range of possibilities for the confinement constant.

4 TOLMAN-OPPENHEIMER-VOLKOFF (TOV) EQUATION

In general relativity, a model for an isolated star generally consists of a region filled with a fluid, which is technically a perfect fluid. This fluid is a solution of the Einstein's field equation in its interior. In the exterior the solution is the asymptotically flat vacuum.

To calculate the internal structure of a static star with spherical symmetry, we start from the Schwarzschild metric (see Appendix C.3):

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

together with the energy-momentum tensor $T^{\mu\nu}$, parametrized in the form of perfect fluid (YAGI K., HATSUDA T., MIAKE Y.,2005; GLENDENNING N. K.,2000)

$$T^{\mu\nu} = (\mathscr{P} + \varepsilon)u^\mu u^\nu - \mathscr{P}g^{\mu\nu}, \tag{4.2}$$

where u^μ is the fluid four-velocity with the normalization condition

$$g_{\mu\nu}u^\mu u^\nu = 1. \tag{4.3}$$

In general, a perfect fluid is the ideal situation for the perfect isotropy to be maintained according to an observer moving with the same local velocity of the fluid, because the mean free path and the times are short. In this case the energy-momentum tensor of the fluid is diagonal in the local rest frame of the fluid.

$$\mathring{T}^{\mu\nu}(x) = \begin{bmatrix} \varepsilon(x) & 0 & 0 & 0 \\ 0 & \mathscr{P}(x) & 0 & 0 \\ 0 & 0 & \mathscr{P}(x) & 0 \\ 0 & 0 & 0 & \mathscr{P}(x) \end{bmatrix}, \tag{4.4}$$

where $\varepsilon(x)$ is a local energy density and $\mathscr{P}(x)$ is a local pressure and "$\circ$" in the energy- momentum tensor refers to the rest frame of the fluid. This equation is simply the Pascal law, that says that the pressure exerted by the fluid is equal in all directions and is perpendicular to the area on which it acts (YAGI K., HATSUDA T., MIAKE Y.,2005).

We remember that $T^{ij}df_j = \mathscr{P}df_i$ is the ith component of the force acting on a surface element, df, where $T^{ij} = \mathscr{P}\delta^{ij}$.

We assume that the equation of state, which relates the energy density and pressure is

$$\mathscr{P} = \mathscr{P}(\varepsilon). \tag{4.5}$$

Using the result of appendix C, we are prepared to obtain the Tolman-Oppenheimer-

Volkoff (TOV) equation for static relativistic star. Outside the stars, in this region we found that the Einstein's tensor is zero, meaning that the Ricci tensor and the scalar curvature are zero too. Inside to the star, we need both the Ricci tensor and the scalar curvature to build the Einstein's equation.

Using the result obtained in appendix C.3 for $R_{\mu\nu}$, we have:

$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= e^{-2\nu}R_{00} - e^{-2\lambda}R_{11} - \frac{2}{r^2}R_{22}, \\ &= e^{-2\lambda}\left\{-2\frac{d^2\nu}{dr^2} + 2\frac{d\lambda}{dr}\frac{d\nu}{dr} - 2\left(\frac{d\nu}{dr}\right)^2 - \frac{2}{r^2} + \frac{4}{r}\frac{d\lambda}{dr} - \frac{4}{r}\frac{d\nu}{dr}\right\} + \frac{2}{r^2}. \end{aligned} \tag{4.6}$$

For the sake of simplicity, we work with mixed tensor

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}R,$$

We can then write the components of the Einstein's tensor

$$\begin{aligned} G^0_0 &\equiv \frac{e^{-2\lambda}}{r^2}\left(1 - 2r\frac{d\lambda}{dr}\right) - \frac{1}{r^2} = -\frac{d}{dr}[r(1-e^{-2\lambda})], \\ G^1_1 &\equiv \frac{e^{-2\lambda}}{r^2}\left(1 + 2r\frac{d\nu}{dr}\right) - \frac{1}{r^2}, \\ G^2_2 &\equiv e^{-2\lambda}\left(\frac{d^2\nu}{dr^2} + \left(\frac{d\nu}{dr}\right)^2 - \frac{d\lambda}{dr}\frac{d\nu}{dr} + \frac{1}{r}\left(\frac{d\nu}{dr} - \frac{d\lambda}{dr}\right)\right), \\ G^3_3 &= G^2_2. \end{aligned} \tag{4.7}$$

(Remember that λ is a function of r). If we assume that the star is static, then the fluid three-velocity is zero:

$$u^{\mu} = 0 \quad (\mu \neq 0), \qquad u^0 = \frac{1}{\sqrt{g_{00}}}. \tag{4.8}$$

The components of the energy-momentum tensor that are not zero are:

$$T^0_0 = \varepsilon, \quad T^{\mu}_{\mu} = -\mathscr{P} \quad (\mu \neq 0). \tag{4.9}$$

So, the component (00) of Einstein's equation can be written as:

$$r^2 G^0_0 = -\frac{d}{dr}[r(1-e^{-2\lambda})] = -8\pi r^2 G T^0_0 = kr^2\varepsilon(r), \tag{4.10}$$

and can be easily integrated

$$e^{-2\lambda} = 1 + \frac{k}{r}\int_0^r dr\varepsilon(r)r^2, \tag{4.11}$$

whit $k = -8\pi G$ which is the proportionality constant in the Einstein's equation (Eq. C.23).

At zero pressure we define the edge of the star. Zero pressure means that the gravitational attraction of the star cannot support any more overlapping matter. Moreover outside the surface of the star, the Schwarzschild solution (Eq. C.45), should be valid such that $M(r)$ is related to the gravitational mass. The appropriate gravitational mass can be defines as

$$M \equiv M(R), \tag{4.12}$$

considering that R is the radius of the star.

Using Eq. (4.7), now we can write the field equations for spherical symmetries for static stars considering the constant of the Einstein's equation, and noticing that the solution gives a relationship between the mass $M(r)$ on any radial coordinate and the metric $g_{11} \propto \lambda(r)$.

The equations related to (4.7) are

$$\begin{aligned}
G_0^0 &\equiv \frac{e^{-2\lambda}}{r^2}\left(1 - 2r\frac{d\lambda}{dr}\right) - \frac{1}{r^2} = -8\pi G\varepsilon(r), \\
G_1^1 &\equiv \frac{e^{-2\lambda}}{r^2}\left(1 + 2r\frac{d\nu}{dr}\right) - \frac{1}{r^2} = 8\pi G\mathscr{P}(r), \\
G_2^2 &\equiv e^{-2\lambda}\left(\frac{d^2\nu}{dr^2} + \left(\frac{d\nu}{dr}\right)^2 - \frac{d\lambda}{dr}\frac{d\nu}{dr} + \frac{1}{r}\left(\frac{d\nu}{dr} - \frac{d\lambda}{dr}\right)\right) = 8\pi G\mathscr{P}(r), \\
G_3^3 &= G_2^2 = 8\pi G\mathscr{P}(r).
\end{aligned} \tag{4.13}$$

By substituting (C.42) into G_0^0 (Eq. 4.13) and simplifying the notation, we make $G = 1$ (Appendix A), and obtain

$$\frac{dM(r)}{dr} \equiv 4\pi\varepsilon(r)r^2, \tag{4.14}$$

or its integral form

$$M(r) = 4\pi\int_0^r drr^2\varepsilon(r). \tag{4.15}$$

To solve the equations for G_0^0 and G_1^1

$$-2r\frac{d\lambda}{dr} = (1-8\pi r^2\varepsilon)e^{2\lambda} - 1, \tag{4.16}$$

$$2r\frac{d\nu}{dr} = (1+8\pi r^2\mathscr{P})e^{2\lambda} - 1. \tag{4.17}$$

Taking the derivative of the last equation and multiplying by r we have

$$\frac{d}{dr}\left(2r\frac{d\nu}{dr}\right) = r\frac{d}{dr}\left((1+8\pi r^2\mathscr{P})e^{2\lambda}-1\right),$$

$$2r\frac{d\nu}{dr} + 2r^2\frac{d^2\nu}{dr^2} = r\left(2\frac{d\lambda}{dr}e^{2\lambda} + 16\pi r\mathscr{P}e^{2\lambda} + 8\pi r^2\frac{d\mathscr{P}}{dr}e^{2\lambda} + 16\pi r^2\mathscr{P}\frac{d\lambda}{dr}e^{2\lambda}\right),$$

$$2r\frac{d\nu}{dr} + 2r^2\frac{d^2\nu}{dr^2} = \left[2r\frac{d\lambda}{dr}(1+8\pi r^2\mathscr{P}) + \left(16\pi r^2\mathscr{P} + 8\pi r^3\frac{d\mathscr{P}}{dr}\right)\right]e^{2\lambda}. \tag{4.18}$$

Using (4.14) and (4.15) we need:

$$2r^2\frac{d^2\nu}{dr^2} = 1 + \left(16\pi r^2\mathscr{P} + 8\pi r^3\frac{d\mathscr{P}}{dr}\right)e^{2\lambda} - (1+8\pi r^2\mathscr{P})(1-8\pi r^2\varepsilon)e^{4\lambda}. \tag{4.19}$$

Taking the square of the Eq. (4.17) the result is

$$2r^2\left(\frac{d\nu}{dr}\right)^2 = \frac{1}{2}(1+8\pi r^2\mathscr{P})^2e^{4\lambda} - (1+8\pi r^2\mathscr{P})e^{2\lambda} + \frac{1}{2}. \tag{4.20}$$

Equations (4.16 - 4.17) and (4.19 - 4.20) give expressions for $\frac{d\lambda}{dr}$, $\frac{d\nu}{dr}$, $\frac{d^2\nu}{dr^2}$, and $\left(\frac{d\nu}{dr}\right)^2$in terms of the components of the energy momentum tensor and the metric tensor $\mathscr{P}$, $\frac{d\mathscr{P}}{dr}$, ε, and $e^{2\lambda} = g_{11}$. This result can be replaced into G_2^2 (Eq. 4.14), and we obtain after some algebra the result (GLENDENNING N. K.,2000)

$$\frac{d\mathscr{P}}{dr} = -\frac{[\mathscr{P}(r)+\varepsilon(r)][M(r)+4\pi r^3\mathscr{P}(r)]}{r[r-2M(r)]}. \tag{4.21}$$

To obtain the numerical solution of the profile of the star, we solve the differential equation (4.14) and (4.21) together with the equation of state (EoS) Eq. (4.5).

The initial condition for the differential equations are

$$M(0) = 0, \qquad \text{and} \qquad \varepsilon(0) = \varepsilon_{cent}. \tag{4.22}$$

The radius, R, of a star with central energy density ε_{cent} is defined by the condition $\mathscr{P}(r = R) = 0$. Then the total mass is obtained from equation (4.15) with $r = R$.

In the Newtonian limit, we neglect the gravitational radius and pressure , and obtain

$$-\left(\frac{d\mathscr{P}(r)}{dr}\right) = \frac{\varepsilon(r)M(r)}{r^2}, \tag{4.23}$$

which shows a balance between the internal pressure and the gravity acting on a volume element located at a distance r (YAGI K., HATSUDA T., MIAKE Y.,2005; GLENDENNING N. K.,2000).

Next, we show the curve which results from the solution of the TOV equations using the equation of state obtained in Chapters 1 and 2 (Figs. 10 and 18 for matter without gluons and Fig. 23 for matter with gluons) for the MIT and QMDD model respectively.

The accuracy in the profile of the star depends crucially on the precision of the equations of state ($\mathscr{P}$,ε). The more important properties can be obtained for this set of coupled equations and they are the maximum mass and the radius of star.

In Fig. 24 a schematic relationship between the mass of the quark star M and its radius R is shown. TOV equations are curves parameterized by the energy density, where each point represents a possible star in equilibrium.

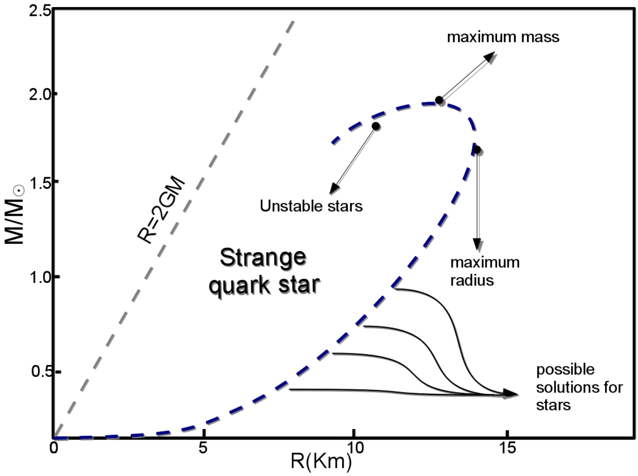

Figura 24: An illustration of the mass-radius relation for quark star. The dashed line shows the Schawarzschild radius ($R = 2GM$).

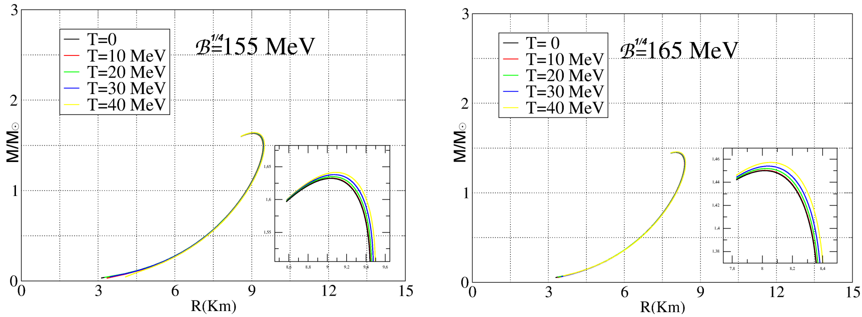

Figura 25: Solution of the TOV equations for the MIT model for the bag constant 155 MeV and 165 MeV without gluons.

T(MeV)	Model	$\mathscr{B}^{1/4}$(MeV)	max. mass ($M_\odot$)	Radius(Km)	Gluons
0	MIT	155	1.63	9.0	
10	MIT	155	1.63	9.0	
20	MIT	155	1.63	9.0	
30	MIT	155	1.64	9.1	
40	MIT	155	1.64	9.1	
0	MIT	165	1.45	8.0	
10	MIT	165	1.45	8.0	
20	MIT	165	1.45	8.0	
30	MIT	165	1.45	8.0	
40	MIT	165	1.46	8.1	

Tabela 1: Profile of the star for the MIT bag model without gluons.

In Fig. 25, we show the numerical solutions of the equations (4.15) and (4.21) for the MIT model using the equation of state obtained in Chapter 1 and displayed in Fig. 10.

Table 1 contains the maximum masses and radii results for different temperatures for the MIT model without gluons. We can see that for a fixed value of the bag constant the maximum masses and radii of the star do not have a meaningful variation with the temperature. However, the maximum masses and radii depends strongly on the bag constants. When we change the value of the bag constant from 155 MeV to 165 MeV both quantities are reduced. Similar behavior was found in (MENEZES D. P., MELROSE D. B.,2005).

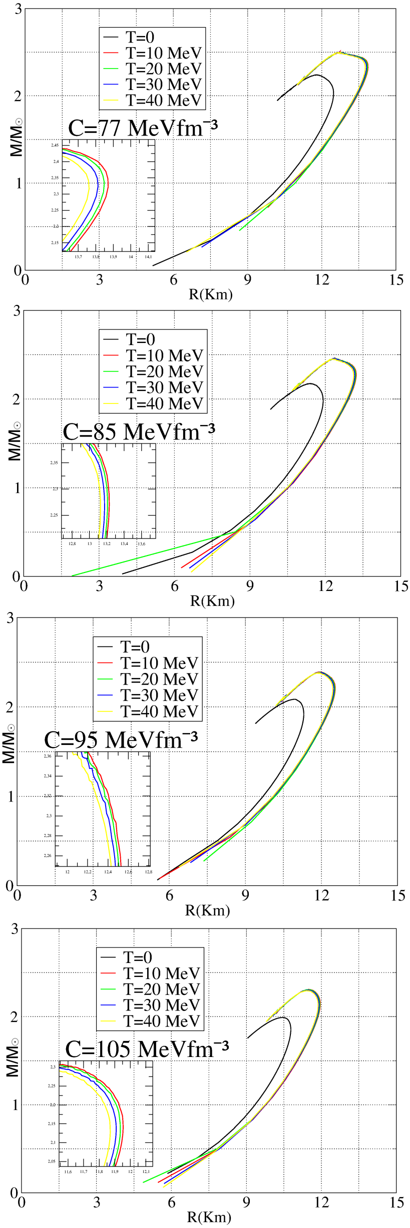

Figura 26: Solution of the TOV equations for the QMDD model for confinement constant of 77 MeVfm^{-3}, 85 MeVfm^{-3}, 95 MeVfm^{-3} and 105 MeVfm^{-3} respectively without gluons.

In Fig. 26 the solution of TOV equations for four values of the confinement constant, namely 77 MeVfm^{-3}, 85 MeVfm^{-3}, 95 MeVfm^{-3} and 105 MeVfm^{-3} without gluons are displayed. In table 2 we note a change in the maximum masses and radii for low values of temperature (0-10 MeV) and we see a decrease of the maximum masses and radii with the increase of the confinement constant.

In the Fig. 27 we display the solution of the TOV equations taking into account the gluons contributions in the quark star. In table 3 we show the values of the maximum masses and radii for the QMDD model with gluons. At zero temperature the gluons do not have any contribution to pressure and energy density (see Eqs. 3.55). Comparing with table 2, we note that the changes in the maximum masses and radii are not significant when the gluons are included. Consequently, for temperatures ranging from 0 to 40 MeV there is no need to consider the gluons contributions.

T(MeV)	Model	C(MeVfm^{-3})	max. mass ($M_\odot$)	Radius(Km)	Gluons
0	QMDD	77	2.2	11.8	
10	QMDD	77	2.5	12.7	
20	QMDD	77	2.5	12.7	
30	QMDD	77	2.5	12.7	
40	QMDD	77	2.5	12.6	
0	QMDD	85	2.2	11.4	
10	QMDD	85	2.5	12.4	
20	QMDD	85	2.5	12.4	
30	QMDD	85	2.5	12.3	
40	QMDD	85	2.5	12.3	
0	QMDD	95	2.1	10.9	
10	QMDD	95	2.4	11.9	
20	QMDD	95	2.4	11.9	
30	QMDD	95	2.4	11.9	
40	QMDD	95	2.4	11.9	
0	QMDD	105	2.0	10.4	
10	QMDD	105	2.3	11.5	
20	QMDD	105	2.3	11.5	
30	QMDD	105	2.3	11.5	
40	QMDD	105	2.3	11.4	

Tabela 2: Profile of the star for the QMDD model without gluons.

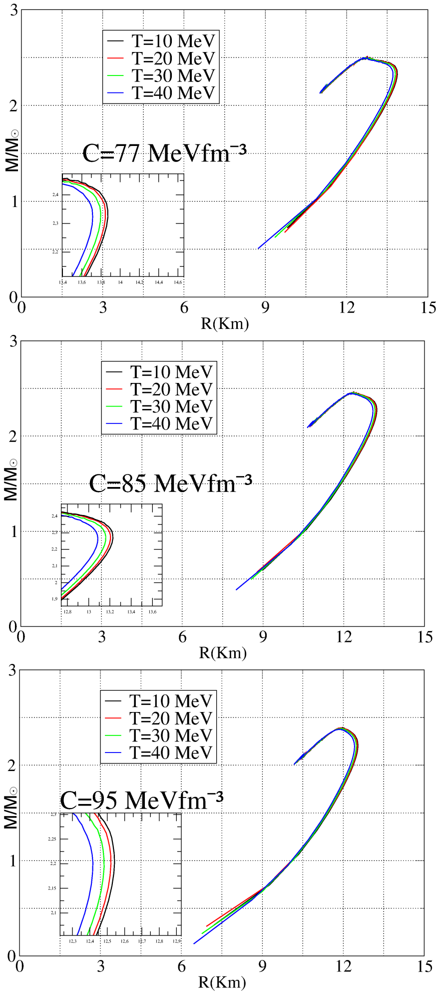

Figura 27: Solution of the TOV equation for the QMDD model for the confinement constant values 77 MeVfm^{-3}, 85 MeVfm^{-3} and 95 MeVfm^{-3} with gluons.

T(MeV)	Model	C(MeVfm^{-3})	max. mass ($M_\odot$)	Radius(Km)	Gluons
0	QMDD	77	2.2	11.8	
10	QMDD	77	2.5	12.7	✓
20	QMDD	77	2.5	12.7	✓
30	QMDD	77	2.5	12.7	✓
40	QMDD	77	2.5	12.8	✓
0	QMDD	85	2.2	11.4	
10	QMDD	85	2.5	12.3	✓
20	QMDD	85	2.5	12.3	✓
30	QMDD	85	2.5	12.3	✓
40	QMDD	85	2.5	12.3	✓
0	QMDD	95	2.1	10.9	
10	QMDD	95	2.3	11.8	✓
20	QMDD	95	2.3	11.9	✓
30	QMDD	95	2.2	11.7	✓
40	QMDD	95	2.2	11.5	✓

Tabela 3: Profile of star for QMDD model with gluons.

Comparing Tables 1-3 we note that the MIT model does not reproduce objects with large masses and it is not appropriated to describe massive pulsars with masses of the order of 2.1 $M_\odot$ (OLIVEIRA J. C., RODRIGUES H., DUARTE S. B.,2001) as the ones recently detected (DEMOREST P., PENNUCCI T., RANSOM S., ROBERTS M., HESSELS J.,2010; ANTONIADIS J, FREIRE P., WEX N., ET.AL.,2013). Nevertheless, pulsar with masses of ~ 1.44 $M_\odot$ (ZHANG C. M., WANG J., ZHAO Y. H., YIN H. X., SONG L. M., MENEZES D. P., WICRAMASINGHE D. T., FERRAIRO L., P.A&A,2011) can be better described with this model.

Recent observations of pulsars PSR J1614-2230 (DEMOREST P., PENNUCCI T., RANSOM S., ROBERTS M., HESSELS J.,2010) and PSR J0348+0432 (ANTONIADIS J., FREIRE P., WEX N., ET.AL.,2013) respectively with masses of 1.97 ± 0.04 $M_\odot$ and 2.01 ± 0.04 $M_\odot$ give a strong constraint on the equations of state for compact stars (WEI W., ZHENG X. P.,2012), which can be better described by the equation of state of the QMDD model due to the dynamic characteristic of the confinement term.

5 CONCLUSIONS

In this work we established conditions within which strange quark matter is stable at finite temperature according to the idea of the Bodmer-Witten conjecture (BODMER A. R.,1971; WITTEN E.,1984), where the strange quark matter is the ground state of strong interaction. In principle, the QCD should contain the answer to the question of whether strange matter is stable or not. Unfortunately, the QCD has not been resolved yet. That is why we need the effective models as an approximation investigate the strange quark matter idea. In this work the strange quark matter was studied via two effective models, i.e., the MIT bag model and QMDD model. This models are characterized by two parameters, namely $\mathscr{B}^{1/4}$ in the MIT model and C in the QMDD model. We found the parameters $\mathscr{B}^{1/4}$ and C for which the strange quark matter is stable.

We start by analyzing the stability windows related to proto-quark stars described by quark matter obtained by this two models. As stated in Chapter 1, instead of considering the binding energy, the quantity that has been studied to obtain the upper limit of the stability window is the free energy density ($\mathscr{F} = \varepsilon - T\mathscr{S}$). Notice that we have used, for two flavor quark matter, the fact that $\mu_u = \mu_d$, which gives symmetric matter ($\rho_u = \rho_d$) and, to be consistent, for strange quark matter we have used $\mu_u = \mu_d = \mu_s$. As the strange quark mass is much larger than the masses of quarks u and d, its relative density is considerably lower. The criterion used to find the stability windows in the MIT model is the standard: the upper limit is obtained by the condition that the free energy density, $\mathscr{F}/\rho_B$ of strange matter is smaller than 930 MeV at its point of saturation ($\mathscr{P} = 0$). The lower limit is obtained for the condition that the free energy density of u and d quark matter without strangeness is large than 930 MeV.

In order to choose adequate values for the confinement constant in the QMDD model (Chapter 2), we display stability windows for finite temperature. The stability for the quark matter in the QMDD model is found using the same criterion used in the MIT model. In the MIT model, the point where $\mathscr{P} = 0$ coincides with lower point of the curve, but in the QMDD model this is not the case due to its intrinsic thermodynamical inconsistency. The gluons contribution as taken into account and the stability windows was analyzed within QMDD model, we found that stability windows analyzed with the free density energy is wider than the analyzed with the density energy.

Another prescription for the QMDD model (PENG G. X., CHIANG H. C., ZOU B. S., NING P. Z., LUO S. J.,2000) shows that this thermodynamical inconsistency can be corrected considering the confinement constant

$B(\rho_B, T)$ as a temperature function.

Within the MIT model, for a fixed value of the bag constant, when the temperature increases the system becomes more bound for *ud* and *uds* matter as can be seen in Fig. 28a). The QMDD model presents a similar behavior. The system is more bound for a fixed value of the confinement constant when increasing the temperature. This can be seen in Fig. 28b).

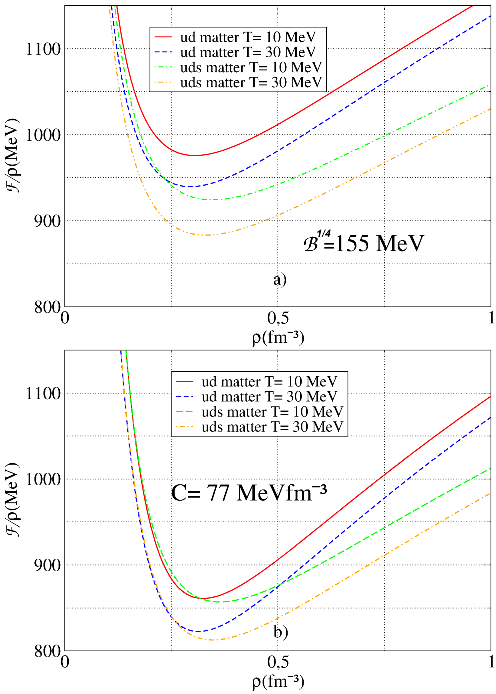

Figura 28: Free energy density per baryonic number density for *ud* and *uds* matter. a) MIT bag model. b) QMDD model

$\beta-$equilibrium and charge neutrality were taken into account in stellar matter. We found that within the MIT model the system is more bound when the temperature increases with a fixed bag constant value as can be seen in Fig. 29a). The leptons in the QMDD model produce the inverse effect. The conditions involving presence of leptons in stellar matter makes the system less bound when increasing the temperature for a fixed confinement constant which is displayed in Fig. 29b).

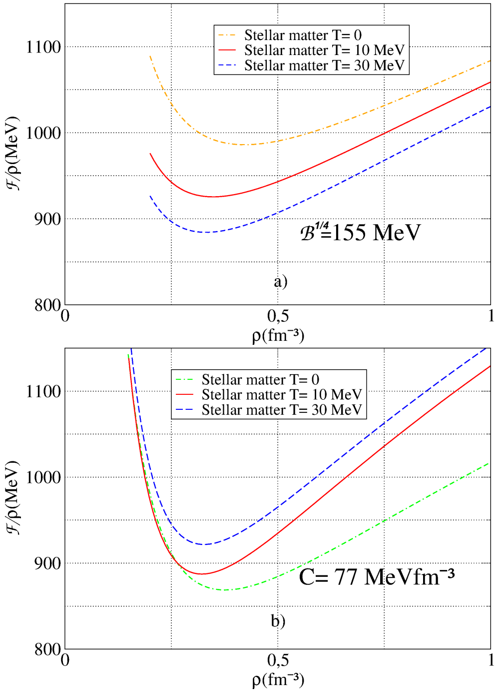

Figura 29: Free energy density per baryonic number density in stellar matter. a) MIT bag model. b) QMDD model.

Appropriate equations of state (EoS) were obtained for quark matter with the necessary leptons to ensure $\beta-$equilibrium and charge neutrality. This equations of state allows to simulate quark stars within this two model.

For the sake of completeness, we have used the EoS calculated in Chapters 1 and 2 as to input to the TOV equations to describe quark stars. The mass radius relation is obtained from the solution of these coupled differential equations. We found in the MIT model that the maximum masses and radii have a strong dependence with the bag constant than with temperature. The maximum masses and radii decrease with the change in the bag constant, as reported in (MENEZES D. P., MELROSE D. B.,2005).

We could see that the QMDD model can reproduce very massive stars of the order of $\sim 2.0\ M_{\odot}$ to $2.5\ M_{\odot}$ (DEMOREST P., PENNUCCI T., RANSOM S., ROBERTS M., HESSELS J.,2010; ANTONIADIS J., FREIRE P., WEX N., ET.AL.,2013).

We conclude that this version of the QMDD model generates larger maximum masses than the MIT model, for all possible parameters inside the stability window. The presence of gluons in the QMDD model does not produce a noticeable effect in the maximum masses and radii of the star, thus we can neglect their contribution.

This work provides a different analysis of the stability windows in strange quark matter for proto-quark stars, where the principal ingredient is the free density energy.

Future prospects in this topic is to study strange quark matter using the prescription given in (ZHANG Y., ZU R. K.,2002), find the stability windows using the free energy density and compare with the stability windows found in this work.

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APPENDIX A – Units

At this point, we want to add a few comments regarding a system of units frequently used in modern physic.

Work in gravitational units G=k=c=1 (where k is a Boltzmann constant) are very useful, since they make computation easier.

To start, we define the gravitational units or geometric units:

$$\begin{aligned} 1 &= \mathrm{c} = 2.9979 \times 10^{10}\ \mathrm{cm/s}, \\ 1 &= \mathrm{G} = 6.6720 \times 10^{-8}\ \mathrm{cm}^3\ \mathrm{g}^{-1}\ \mathrm{s}^{-2}, \\ 1 &= \mathrm{k} = 1.3807 \times 10^{-16}\ \mathrm{erg/K}, \end{aligned} \tag{A.1}$$

where K is the temperature in degrees Kelvin.

In nuclear and particle physics it is more convenient to use units of energy (which are in million electron volts or MeV). Units in GeV are used, since it can be approximated by the nuclear mass$\sim$ 939 MeV. In ergs and another units we have:

$$\begin{aligned} \mathrm{MeV} &= 1.6022 \times 10^{-6}\ \mathrm{ergs} = 1.3234 \times 10^{-55}\ \mathrm{cm}, \\ &= 1.7827 \times 10^{-27}\ \mathrm{g} = 1.1609 \times 10^{10}\ \mathrm{K}. \end{aligned} \tag{A.2}$$

The range of the nuclear force is $\sim 10^{-13}$ cm which is defined as a Fermi.

$$1\ \mathrm{fm} = 10^{-13}\ \mathrm{cm}. \tag{A.3}$$

Two very important constants in physic are the Plank constant $h = 2\pi\hbar$ and the electric charge e.

$$\begin{aligned} &\hbar c = 197.33\ \mathrm{MeVfm}, \\ &e^2 = 1.4400\ \mathrm{MeVfm} = (1.3805 \times 10^{-34}\ \mathrm{cm})^2, \end{aligned} \tag{A.4}$$

and the fine structure constant is

$$\frac{e^2}{\hbar c} = \frac{1}{137}. \tag{A.5}$$

The energy density units can be

$$\frac{\mathrm{MeV}}{\mathrm{fm}^3} = 1.7827 \times 10^{12}\ \mathrm{g/\ cm}^3, \tag{A.6}$$

In nuclear physics a natural choice $\hbar$=c=1. Then, the value of $\hbar$c written above divided by fm^4, is obtained

$$\frac{1}{\text{fm}^4} = 197.33\ \text{MeV fm}^{-3}. \tag{A.7}$$

Moreover,

$$\begin{aligned} \frac{1}{\text{fm}^4} &= 2.6115 \times 10^{-4} \frac{1}{\text{Km}^2}, \\ \frac{\text{MeV}}{\text{fm}^3} &= 1.3234 \times 10^{-6} \frac{1}{\text{Km}^2}. \end{aligned} \tag{A.8}$$

The last unit is appropriate for expressing the energy density and pressure when solving the Tolmann-Oppenheimer- Volkoff (TOV) equation.

When the energy density of a star is expressed in such units and is integrated over the radial coordinate leads to the mass in Km. This can be written in solar mass (GLENDENNING N. K.,2007).

$$\text{M}_{\odot} = 1.4766\ \text{Km} = 1.989 \times 10^{33}\ \text{g} = 1.116 \times 10^{60}\ \text{MeV}. \tag{A.9}$$

APPENDIX B – Basic Thermodynamics for free Fermi gases

For a statistical system in equilibrium with volume V, temperature T and chemical potential μ, we can introduce the grand-canonical density operator, $\hat{\rho}$, the grand-potential partition function $\mathscr{Z}(T,V,\mu)$ and the grand potential $\Omega(T,V,\mu)$, in natural units $k_B = \hbar = c = 1$(see appendix A):

$$\hat{\rho} = \frac{1}{\mathscr{Z}} e^{-(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})\beta}, \tag{B.1}$$

$$\begin{aligned} \mathscr{Z}(T,V,\mu) &= tr[e^{-\beta(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})}] \\ &= \sum_n \langle n|e^{-\beta(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})}|n\rangle, \\ &\equiv e^{-\Omega(T,V,\mu)\beta}. \end{aligned} \tag{B.2}$$

Here $\hat{\mathscr{H}}$ is a Hamiltonian, $\hat{\mathscr{N}}$ is a number operator and $\beta \equiv 1/T$. The trace is taken over a complete set of quantum states labeled by n. Note that $Tr\hat{\rho} = 1$ holds for definition. We may introduce an entropy operator,

$$\hat{S} = -\ln\hat{\rho} \tag{B.3}$$

The thermal average of an arbitral operator $\hat{O}$, is given by $\langle\hat{O}\rangle = Tr[\hat{\rho}\hat{O}]$. Hence, the energy, $\mathscr{E}$, particle number, $\mathscr{N}$, and entropy $\mathscr{S}$ averages are given by

$$\mathscr{E} = \langle\hat{\mathscr{H}}\rangle, \quad \mathscr{N} = \langle\hat{\mathscr{N}}\rangle, \quad \mathscr{S} = \langle\hat{\mathscr{S}}\rangle = -Tr[\hat{\rho}\ln\hat{\rho}]. \tag{B.4}$$

The equations (B.1) and (B.2) combined with (B.4), yield the following:

$$\begin{aligned} \mathscr{S} &= -tr[\hat{\rho}\ln\hat{\rho}] \\ &= -\langle n|\hat{\rho}\ln\hat{\rho}|n\rangle \\ &= -\langle n|\frac{e^{-(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})\beta}}{e^{-\Omega\beta}}\ln\left[\frac{e^{-(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})\beta}}{e^{-\Omega\beta}}\right]|n\rangle \\ &= -\langle n|\frac{e^{-(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})\beta}}{e^{-\Omega\beta}}[\beta(-\hat{\mathscr{H}}+\mu\hat{\mathscr{N}}+\Omega)]|n\rangle \\ &= -\frac{\beta}{e^{-\Omega\beta}}\langle n|e^{-(\hat{\mathscr{H}}-\mu\hat{\mathscr{N}})\beta}(-\hat{\mathscr{H}}+\mu\hat{\mathscr{N}}+\Omega)|n\rangle \\ &= -\frac{\beta}{e^{-\Omega\beta}}[-e^{-\Omega\beta}\mathscr{E}+e^{-\Omega\beta}\mu\mathscr{N}+\Omega e^{-\Omega\beta}] \\ T\mathscr{S} &= \mathscr{E}-\mu\mathscr{N}-\Omega, \end{aligned}$$

together with the definition of pressure $\mathscr{P} = -d\Omega/dV|_{T,\mu}$ yields the thermodynamics relation :

$$\begin{aligned} \Omega(T,V,\mu) &= \mathscr{E} - T\mathscr{S} - \mu\mathscr{N}, && \text{(B.5)} \\ d\Omega &= -\mathscr{S}dT - \mathscr{P}dV - \mathscr{N}d\mu, && \text{(B.6)} \\ d\mathscr{E} &= Td\mathscr{S} - \mathscr{P}dV + \mu d\mathscr{N}. && \text{(B.7)} \end{aligned}$$

The equation (B.7) is called the first law of thermodynamics.

In a system with a fixed number of particles and/or pressure, it is useful to introduce the Helmholtz free energy $F(T,V,N)$, obtained by $\mathscr{L}$egendre transformation of $\Omega(T,V,\mu)$:

$$F(T,V,N) = \Omega + \mu\mathscr{N} = \mathscr{E} - T\mathscr{S}, \tag{B.8}$$

and

$$\Omega = -\mathscr{P}V. \tag{B.9}$$

For a spatially uniform system, we introduce the energy density, $\varepsilon = \mathscr{E}/V$, the number density $n = \mathscr{N}/V$, and the entropy density, $s = \mathscr{S}/V$. Then equations (B.5), (B.6) and (B.7), via the equation (B.9), are rewritten as

$$\begin{aligned} -\mathscr{P} &= \varepsilon - Ts - \mu n && \text{(B.10)} \\ d\mathscr{P} &= sdT + nd\mu && \text{(B.11)} \\ d\varepsilon &= Tds + \mu dn. && \text{(B.12)} \end{aligned}$$

Now, we investigate the basic properties of a Fermi gas such as the pressure, the energy density and the particle number for finite temperature.

The operators $\hat{\mathscr{H}}$ and $\hat{\mathscr{N}}$ in Eq. (B.2) are given by:

$$\begin{aligned} \hat{\mathscr{H}} &= \frac{1}{V}\sum_{\mathbf{k}\lambda}\sqrt{k^2+m^2}[A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} + B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda}] && \text{(B.13)} \\ \hat{\mathscr{N}} &= \frac{1}{V}\sum_{\mathbf{k}\lambda}[[A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} - B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda}]. && \text{(B.14)} \end{aligned}$$

These operators are diagonal matrices in the basis of the baryon and anti-baryon number operator (WALECKA J. D.,2004).

$$A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}> = n_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}>$$

$$B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda}|\overline{n}_{\mathbf{k}\lambda}> = \overline{n}_{\mathbf{k}\lambda}|\overline{n}_{\mathbf{k}\lambda}>$$

The particle number occupation in function of the occupation of the baryons (n_i) and the antibaryons ($\overline{n}_i$) in state i, is:

$$\{n_{\mathbf{k}\lambda}\} \equiv \{n_1, n_2, n_3 \cdots\} \tag{B.15}$$
$$\equiv \{n_i\}, \quad i = 1, 2, 3, \cdots \tag{B.16}$$

The grand-partition fuction is given by:

$$\mathscr{Z} = \prod_i \sum_{n_i} <n_i|e^{-\beta(E_i-\mu_i)n_i}|n_i> \prod_j \sum_{n_j} <\overline{n}_j|e^{-\beta(E_j-\mu_j)n_j}|\overline{n}_j>, \tag{B.17}$$

where,

n_i= number of baryons
$\overline{n}_j$= numbers of anti-baryons
There are just two possible values of the occupation numbers for fermions, $n_i, \overline{n}_i = 0, 1$. Thus

$$\mathscr{Z} = \prod_i [<0|1|0> + <1|e^{-\beta(E_i-\mu_i)}|1>] \prod_j [<0|1|0> + <1|e^{-\beta(E_i+\mu_i)}|1>] \tag{B.18}$$
$$= \prod_i [1 + e^{-\beta(E_i-\mu_i)}] \prod_j [1 + e^{-\beta(E_j+\mu_j)}],$$

Thus the thermodynamical potential is given by

$$\Omega = -\frac{1}{\beta} \ln \left[\prod_i (1 + e^{-\beta(E_i-\mu_i)}) \prod_j (1 + e^{-\beta(E_j+\mu_j)}) \right] \tag{B.19}$$
$$= -\frac{1}{\beta} \sum_i \ln(1 + e^{-\beta(E_i-\mu_i)}) - \frac{1}{\beta} \sum_j \ln(1 + e^{-\beta(E_j+\mu_j)})$$

since that,

$$\sum_{k=1}^{n} \ln f(k) = \ln \left(\prod_{k=1}^{n} f(k) \right).$$

The thermodynamic potential differential is:

$$d\Omega(T, V, \{\mu\}) = \left(\frac{\partial \Omega}{\partial T} \right)_{\mu_i, V} dT + \left(\frac{\partial \Omega}{\partial V} \right)_{T, \mu_i} dV + \left(\frac{\partial \Omega}{\partial \mu_i} \right)_{T, V} d\mu_i, \tag{B.20}$$

where

$$\boxed{\mathscr{S} = -\left(\frac{\partial \Omega}{\partial T} \right)_{\mu_i, V}, \quad \mathscr{P} = -\left(\frac{\partial \Omega}{\partial V} \right)_{T, \mu_i}, \quad \mathscr{N}_i = -\left(\frac{\partial \Omega}{\partial \mu_i} \right)_{T, V}}. \tag{B.21}$$

We take the following limit

$$\sum_i \longrightarrow \frac{\gamma_i V}{(2\pi)^3} \int d^3k \tag{B.22}$$

where γ_i is the degeneracy of system, in Eq. (B.19) which becomes

$$\begin{aligned} \Omega &= \sum_i \Omega_i \\ &= -\frac{1}{\beta} \sum_i \frac{\gamma_i V}{(2\pi)^3} \int d^3k \left[\ln(1+e^{-\beta(E_i-\mu_i)}) + \ln(1+e^{-\beta(E_i+\mu_i)}) \right]. \end{aligned} \tag{B.23}$$

The particle number is;

$$\begin{aligned} N_i &= -\left(\frac{\partial \Omega}{\partial \mu_i}\right)_{T,V} \\ &= \left[\frac{\partial}{\partial \mu_i} \frac{1}{\beta} \sum_i \frac{\gamma_i V}{(2\pi)^3} \int d^3k \left[\ln(1+e^{-\beta(E_i-\mu_i)}) + \ln(1+e^{-\beta(E_i+\mu_i)}) \right] \right] \\ &= \frac{\gamma_i V}{(2\pi)^3} \int d^3k \left[\frac{1}{1+e^{\beta(E_i-\mu_i)}} - \frac{1}{1+e^{\beta(E_i+\mu_i)}} \right] \end{aligned} \tag{B.24}$$

the density is $\rho_i = N_i/V$

$$\rho_i = \frac{\gamma_i}{(2\pi)^3} \int d^3k \left[\frac{1}{1+e^{\beta(E_i-\mu_i)}} - \frac{1}{1+e^{\beta(E_i+\mu_i)}} \right], \tag{B.25}$$

the pressure is obtained directly from the thermodynamic potential $p = -\Omega/V$

$$\mathscr{P} = \sum_i \mathscr{P}_i = \frac{1}{\beta} \sum_i \frac{\gamma_i}{(2\pi)^3} \int d^3k \left[\ln(1+e^{-\beta(E_i-\mu_i)}) + \ln(1+e^{-\beta(E_i+\mu_i)}) \right] \tag{B.26}$$

Taking the firs term (the calculation for the second is equal), we have:

$$\mathscr{P}_i = \frac{1}{\beta} \frac{\gamma_i}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \ln[1+e^{-\beta(E_i-\mu_i)}]. \tag{B.27}$$

In a spherical coordinate system $d^3k = k^2 dk d\Omega = 4\pi k^2 dk$, then

$$
\begin{aligned}
\mathscr{P}_i &= \frac{1}{\beta}\frac{\gamma_i}{(2\pi)^3}4\pi\int_0^\infty dkk^2\ln[1+e^{-\beta(E_i-\mu_i)}] \\
&= \frac{1}{\beta}\frac{\gamma_i}{2\pi^2}\int_0^\infty dkk^2\ln[1+e^{-\beta(E_i-\mu_i)}].
\end{aligned} \tag{B.28}
$$

To integrate by parts, we take $u=\ln[1+e^{-\beta(E_i-\mu_i)}]$ and $dv=\int k^2dk$. Eq (B.28) can then be written as

$$
\frac{1}{\beta}\frac{\gamma_i}{2\pi^2}\int_0^\infty dkk^2\ln[1+e^{-\beta(E_i-\mu_i)}] = \frac{1}{\beta}\frac{\gamma_i}{2\pi^2}\frac{k^3}{3}\ln[1+e^{-\beta(E_i-\mu_i)}]|_0^\infty + \frac{1}{\beta}\frac{\gamma_i}{2\pi^2}\int_0^\infty\frac{k^3}{3}\frac{1}{1+e^{-\beta(E_i-\mu_i)}}\left(\frac{\beta e^{-\beta(E_i-\mu_i)}k}{\sqrt{k^2+m^2}}\right)dk. \tag{B.29}
$$

The first term vanishes. A similar calculation is done for the antiparticles.

Finally

$$
\mathscr{P} = \sum_i \mathscr{P}_i = \frac{1}{3}\frac{1}{2\pi^2}\sum_i\gamma_i\int_0^\infty dk\frac{k^4}{\sqrt{k^2+m^2}}[n_i+\overline{n}_i], \tag{B.30}
$$

with the occupation number for particles and anti-particles give respectively by

$$
n_i = \frac{1}{1+e^{\beta(E_i-\mu_i)}} \tag{B.31}
$$

$$
\overline{n}_i = \frac{1}{1+e^{\beta(E_i+\mu_i)}} \tag{B.32}
$$

The energy density is

$$
\frac{E}{V} = <\hat{\mathscr{H}}> = \frac{1}{V}\frac{\partial}{\partial\beta}(\beta\Omega)+\mu_i\rho_i, \tag{B.33}
$$

with

$$\Omega = -\frac{1}{\beta}\sum_i \frac{\gamma_i V}{(2\pi)^3}\int d^3k[\ln(1+e^{-\beta(E_i-\mu_i)})+\ln(1+e^{-\beta(E_i+\mu_i)})] \quad \text{(B.34)}$$
$$\beta\Omega = -\sum_i \frac{\gamma_i V}{(2\pi)^3}\int d^3k[\ln(1+e^{-\beta(E_i-\mu_i)})+\ln(1+e^{-\beta(E_i+\mu_i)})]$$

$$\frac{\partial}{\partial\beta}(\beta\Omega) = -\sum_i \frac{\gamma_i V}{(2\pi)^3}\int d^3k\left\{\left[\frac{-(E_i-\mu_i)e^{-\beta(E_i-\mu_i)}}{1+e^{-\beta(E_i-\mu_i)}}\right]+\left[\frac{-(E_i+\mu_i)e^{-\beta(E_i+\mu_i)}}{1+e^{-\beta(E_i+\mu_i)}}\right]\right\} \text{(B.35)}$$

Hence, we obtain:

$$\begin{aligned}\varepsilon_i &= \frac{1}{V}\frac{\partial}{\partial\beta}(\beta\Omega)+\mu_i\rho_i \qquad \text{(B.36)}\\ &= \sum_i \frac{\gamma_i}{(2\pi)^3}\int d^3k\left\{\frac{(E_i-\mu_i)}{1+e^{\beta(E_i-\mu_i)}}+\frac{(E_i+\mu_i)}{1+e^{\beta(E_i+\mu_i)}}\right\}+\mu_i\rho_i\\ &= \sum_i \frac{\gamma_i}{(2\pi)^3}\int d^3k\{E_i(n_i+\overline{n}_i)-\mu_i(n_i-\overline{n}_i)\}+\sum_i \frac{\gamma_i}{(2\pi)^3}\int d^3k\mu_i(n_i+\overline{n}_i)\\ &= \sum_i \frac{\gamma_i}{(2\pi)^3}\int d^3kE_i(n_i+\overline{n}_i).\end{aligned}$$

In a spherical coordinate system,

$$\varepsilon_i = \sum_i \frac{\gamma_i}{2\pi^2}\int dkk^2\sqrt{k^2+m^2}(n_i+\overline{n}_i). \quad \text{(B.37)}$$

APPENDIX C – Curved space-time and Einstein equation

C.1 NON-EUCLIDEAN SPACE-TIME

In curvilinear coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, the distance is ds, expressed in terms of a symmetric tensor $g_{\mu\nu}(x)$, which is the metric tensor.

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{C.1}$$

Once the coordinate system is chosen, the associate metric tensor can be calculated. For example, in the Minkowski space, the metric tensor is given by

$$g_{\mu\nu}(x) \to \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{C.2}$$

A contravariant vector $A^\mu(x) = A'^\nu(x')$, transforms under a general coordinate transformation $x^\mu = x^\mu(x')$ as

$$A^\mu(x) = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu(x'). \tag{C.3}$$

The covariants vectors and tensors are defined by lowering the indices with the help of the tensor metric $g_{\mu\nu}$.

$$A_\mu = g_{\mu\nu} A^\nu(x). \tag{C.4}$$

$g_{\mu\nu}$ can be diagonalized, at least locally, and it is a real symmetric matrix. If the diagonalization results gives a positive eigenvalue and three negative eigenvalues, the space-time is the Riemann space, in which the relation $\det g_{\mu\nu}(x) \equiv g(x) < 0$ is satisfied.

We here chosen to define the covariant derivative as:

$$\begin{aligned} \nabla_\lambda A_\mu &= \left(\frac{\partial}{\partial x^\lambda} - \Gamma^\nu_{\lambda\mu} \right) A_\nu, \\ \nabla_\lambda A^\mu &= \left(\frac{\partial}{\partial x^\lambda} + \Gamma^\mu_{\lambda\nu} \right) A^\nu, \end{aligned} \tag{C.5}$$

where ∇_μ and $\frac{\partial}{\partial x_\nu}$ are the same operation if they act on scalar $A_\mu A^\mu$ and $\Gamma^\lambda_{\mu\nu}$ is the Christofell symbol.

Useful relations using the covariant derivative are:

$$\begin{aligned} \nabla_\lambda (A_\mu B_\nu) &= (\nabla_\lambda A_\mu) B_\nu + A_\mu (\nabla_\lambda B_\nu), \\ \nabla_\lambda g_{\mu\nu} &= 0. \end{aligned} \tag{C.6}$$

From the last equation, we can derive an explicit form for the Christoffel symbol, namely

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right), \tag{C.7}$$

and if the space-time is flat all the components of the Christoffel symbol are zero.

The covariant derivative do not commute, namely

$$[\nabla_{\mu}, \nabla_{\nu}] A^{\alpha} = (\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}) A^{\alpha} = R^{\alpha}_{\beta\mu\nu} A^{\beta}, \tag{C.8}$$

where $R^{\alpha}_{\beta\mu\nu}$ is called the *Riemann-Christofell* curvature tensor. For the calculation of $\nabla_{\mu}(\nabla_{\nu} A^{\alpha}) - \nabla_{\nu}(\nabla_{\mu} A^{\alpha})$, the Riemann-Christoffel tensor is:

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}. \tag{C.9}$$

If $R^{\alpha}_{\beta\mu\nu} = 0$, then the space-time is flat.

From the Jacobi identity

$$[\nabla_{\mu}, [\nabla_{\nu}, \nabla_{\lambda}]] + [\nabla_{\nu}, [\nabla_{\lambda}, \nabla_{\mu}]] + [\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]] = 0, \tag{C.10}$$

we obtain

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0, \tag{C.11}$$

and the Bianchi identity

$$\nabla_{\lambda} R^{\alpha}_{\beta\mu\nu} + \nabla_{\mu} R^{\alpha}_{\beta\nu\lambda} + \nabla_{\nu} R^{\alpha}_{\beta\lambda\mu} = 0. \tag{C.12}$$

The Bianchi identity is important for the further developments of the theory of gravity, and allows one to prove Einstein's Equation.

Some symmetry properties follow from the definition of $R^{\alpha}_{\beta\mu\nu}$:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}. \tag{C.13}$$

the Ricci tensor, $R_{\mu\nu}$ and the scalar curvature, R, are simply defined as

$$\begin{aligned} R_{\mu\nu} =& R^{\alpha}_{\mu\nu\alpha} = R_{\mu\nu} \\ R =& R^{\mu}_{\mu}. \end{aligned} \tag{C.14}$$

C.2 THE EINSTEIN'S EQUATION

In order to find the Einstein's equation we multiply the Bianchi identity (Eq. (C.12)) by $g^{\mu\nu}$ and make $\sigma \leftrightarrow \alpha$, we obtain

$$\begin{aligned} &g^{\mu\nu}\nabla_\alpha R^\alpha_{\mu\nu\rho} + g^{\mu\nu}\nabla_\rho R^\alpha_{\mu\alpha\nu} + g^{\mu\nu}\nabla_\nu R^\alpha_{\mu\rho\alpha} = 0, \\ &\nabla_\alpha(g^{\mu\nu}R^\alpha_{\mu\nu\rho}) + \nabla_\rho(g^{\mu\nu}R^\alpha_{\mu\alpha\nu}) + \nabla_\nu(g^{\mu\nu}R^\alpha_{\mu\rho\alpha}) = 0. \end{aligned} \tag{C.15}$$

We look at the terms in the brackets and consider the properties of covariant derivative Eq. (C.6). The first term reads

$$\begin{aligned} g^{\mu\nu}R^\alpha_{\mu\nu\rho} =& g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\nu\rho} = g^{\mu\nu}g^{\alpha\beta}R_{\mu\beta\rho\nu} = g^{\alpha\beta}R^\nu_{\beta\rho\nu} \\ =& g^{\alpha\beta}R^\nu_{\beta\rho\nu} = g^{\alpha\beta}R_{\beta\rho} = R^\alpha_\rho, \end{aligned} \tag{C.16}$$

the second term is

$$\begin{aligned} g^{\mu\nu}R^\alpha_{\mu\alpha\nu} &= -g^{\mu\nu}R^\alpha_{\mu\nu\alpha} = -g^{\mu\nu}R_{\mu\nu} = -R^\mu_\mu \\ &= -R, \end{aligned} \tag{C.17}$$

and the third term

$$g^{\mu\nu}R^\alpha_{\mu\rho\alpha} = g^{\mu\nu}R_{\mu\rho} = R^\nu_\rho. \tag{C.18}$$

Substituting Eq's.(C.16), (C.17) and (C.18) into Eq.(C.15) we obtain

$$0 = \nabla_\alpha R^\alpha_\rho - \nabla_\rho R + \nabla_\nu R^\nu_\rho = 2\nabla_\alpha R^\alpha_\rho - \nabla_\rho R. \tag{C.19}$$

Multiply by $g^{\mu\rho}$ and noticing that

$$\begin{aligned} &g^{\mu\rho}\nabla_\alpha R^\alpha_\rho = \nabla_\alpha(g^{\mu\rho}R^{\alpha\rho}) = \nabla_\alpha R^{\alpha\mu} = \nabla_\nu R^{\mu\nu}, \\ &g^{\mu\rho}\nabla_\rho R = g^{\mu\nu}\nabla_\nu R, \end{aligned} \tag{C.20}$$

replacing in (C.19), the equation becomes

$$\begin{aligned} &2\nabla_\nu R^{\mu\nu} - g^{\mu\nu}\nabla_\nu R = 0, \\ &\nabla_\nu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0. \end{aligned} \tag{C.21}$$

The expression in brackets is the Einstein's equation or *Einstein's curvature tensor*,

$$\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R.} \tag{C.22}$$

In terms of the $G_{\mu\nu}$, the Einstein's equation is written as

$$\boxed{G_{\mu\nu} = 8\pi G T^{\mu\nu}}, \tag{C.23}$$

where G is the gravitational constant (see appendix A, Eq. (A.1)), $T^{\mu\nu}$ is the energy-momentum tensor of matter, radiation and vacuum.

Equations (C.22) and (C.23) are the gravitational source identity, which describe, for each point in space-time, how it is curved by matter (YAGI K., HATSUDA T., MIAKE Y.,2005; GLENDENNING N. K.,2000).

C.3 THE SCHWARZSCHILD METRIC

In this section we intend to find a static solution to Einstein's equation in an isotropic region of the space-time. Such region would be encountered in the interior and exterior of a static star. Under this condition the components of the $g_{\mu\nu}$ are time independent ($x^0 \equiv t$) and $g^{0m} = 0$. Using spherical coordinates $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$, we may express the most general form for the metric as

$$ds^2 = U(r)dt^2 - V(r)dr^2 - W(r)r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{C.24}$$

We can replace r by any function of r without changing the spherical symmetry. We do this in a way that $W(r) = 1$, so ds^2 is now

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{C.25}$$

Comparing with $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, we obtain the metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} e^{2\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{2\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \tag{C.26}$$

According to the Riemann-Christoffel tensor Eq.(C.9), the Ricci tensor can be written as

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{C.27}$$

From Eq. (C.7) we can calculate the Christoffel symbols which are not zero

$$\begin{aligned}
\Gamma^{1}_{00} &= \frac{d\nu}{dr}e^{2(\nu-\lambda)}, & \Gamma^{0}_{10} &= \frac{d\nu}{dr}, \\
\Gamma^{1}_{11} &= \frac{d\lambda}{dr}, & \Gamma^{2}_{12} = \Gamma^{3}_{13} &= \frac{1}{r}, \\
\Gamma^{1}_{22} &= -re^{-2\lambda}, & \Gamma^{3}_{23} &= \cot\theta, \\
\Gamma^{1}_{33} &= r\sin^2\theta e^{-2\lambda}, & \Gamma^{2}_{33} &= -\sin\theta\cos\theta,
\end{aligned} \tag{C.28}$$

replacing in the Ricci tensor

$$\begin{aligned}
R_{00} &= \left(-\frac{d^2\nu}{dr^2} + \frac{d\lambda}{dr}\frac{d\nu}{dr} - \left(\frac{d\nu}{dr}\right)^2 - \frac{2}{r}\frac{d\nu}{dr}\right)e^{2(\nu-\lambda)}; \\
R_{11} &= \frac{d^2\nu}{dr^2} - \frac{d\lambda}{dr}\frac{d\nu}{dr} + \left(\frac{d\nu}{dr}\right)^2 - \frac{2}{r}\frac{d\lambda}{dr}, \\
R_{22} &= \left(1 + r\frac{d\nu}{dr} - r\frac{d\lambda}{dr}\right)e^{-2\lambda} - 1, \\
R_{33} &= R_{22}\sin^2\theta.
\end{aligned} \tag{C.29}$$

In the vacuum space outside a static star, Einstein's equation is

$$\begin{aligned}
G^{\mu\nu} = 0 \Leftrightarrow R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0, \\
R^{\mu\nu} = \frac{1}{2}g^{\mu\nu}R,
\end{aligned} \tag{C.30}$$

multiply by $g_{\alpha\mu}$,

$$\begin{aligned}
g_{\alpha\mu}R^{\mu\nu} &= \frac{1}{2}g_{\alpha\mu}g^{\mu\nu}R, \\
R^{\nu}_{\alpha} &= \frac{1}{2}\delta^{\nu}_{\alpha}R,
\end{aligned} \tag{C.31}$$

by contracting $\alpha = \nu$, we have

$$R = 2R \quad \Rightarrow \quad 2R - R = 0 \quad \Rightarrow \quad R = 0, \tag{C.32}$$

that is, if $G^{\mu\nu} = 0$, it implies that

$$R = 0, \qquad R^{\mu\nu} = 0. \tag{C.33}$$

In the vacuum Einstein's equation can be obtained by making the Ricci tensor null, i.e, $R_{00} = 0$ and $R_{11} = 0$. We then find

$$\begin{aligned} -\nu^{''} + \lambda^{'}\nu^{'} - \nu^{'2} - \frac{2\nu^{'}}{r} &= 0, \\ \nu^{''} - \lambda^{'}\nu^{'} + \nu^{'2} + \frac{2\lambda^{'}}{r} &= 0, \end{aligned} \tag{C.34}$$

where the primes denote differentiation with respect to the r-coordinate. Now, by adding R_{00} and R_{11}, we go to

$$\frac{d\nu}{dr} + \frac{d\lambda}{dr} = 0. \tag{C.35}$$

For $r >> 1$ (r large), the space is not affected by the presence of the star and thereby flat so ν and λ tend to zero, then

$$\lambda + \nu = 0 \quad \Rightarrow \lambda = -\nu, \tag{C.36}$$

using these result in R_{22}, we arrive at

$$\left(1 + 2r\frac{d\nu(r)}{dr}\right) e^{2\nu(r)} = 1, \tag{C.37}$$

which can be written as

$$\frac{d}{dr}(re^{2\nu(r)}) = 1, \tag{C.38}$$

and can be integrate easily

$$\begin{aligned} d(re^{2\nu(r)}) &= dr, \\ re^{2\nu} &= r + 2m, \\ e^{2\nu} &= \frac{r + 2m}{r}, \end{aligned} \tag{C.39}$$

where m is an integration constant which has units of length.

The tensor components $g^{\mu\nu}$ now read:

$$g_{00} \equiv e^{2\nu} = \frac{r+2m}{r},$$
$$g_{00} = 1 + \frac{2m}{r}. \tag{C.40}$$

This result should match the relativistic Newtonian approximation (classical, pre-relativistic) for large values of r. Comparing the relativistic expression just obtained, the expression for which the Newtonian gravitational potential is $\nabla\varphi = -\frac{GM}{r}$:

$$g_{00} = 1 = \frac{2\varphi}{r}. \tag{C.41}$$

The constant m is only the central body mass which produces the gravitational field, Similarly, as $\lambda = -\nu$

$$g_{11} \equiv -e^{2\lambda} = -e^{-2\nu} = -\frac{1}{e^{2\nu}},$$
$$g_{11} = -\left(1 + \frac{2\varphi}{r}\right)^{-1}. \tag{C.42}$$

Substituting these expressions in the line element ds^2, this leads to the following final answer for the metric we are looking for a static spherical symmetric field produced by a spherically symmetric body at rest:

$$ds^2 = \left(1 + \frac{2m}{r}\right) dt^2 - \left(1 + \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{C.43}$$

The constant m have length units and is known as the *geometric mass* of the central body:

$$m = GM. \tag{C.44}$$

This means that the dimensionally correct metric is

$$ds^2 = \left(1 + \frac{2GM}{r}\right) dt^2 - \left(1 + \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (r > R), \tag{C.45}$$

where R denotes the radius of the star. This is a complete derivation to Einstein's equation outside a spherical static star.

The tensor metric is then

$$g_{\mu\nu} = \begin{bmatrix} \left(1+\frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\left(1+\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \end{bmatrix}. \tag{C.46}$$